About the Lagrangian

Optimization and Computational Linear Algebra for Data Science

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Lagrange multipliers

minimize $f(x)$ subject to $g(x) = 0.$

Encoding the constraint in L

We have $L(x, \lambda) = f(x) + \lambda g(x)$, hence

$$\max_{\lambda \in \mathbb{R}} L(x, \lambda) = \begin{cases} f(x) & \text{if} \quad g(x) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Primal and dual problems

- Primal optimization problem:

$$\text{minimize} \quad \max_{\lambda \in \mathbb{R}} L(x, \lambda) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

- Dual optimization problem:

$$\text{maximize} \quad \min_{x \in \mathbb{R}^n} L(x, \lambda) \quad \text{subject to} \quad \lambda \in \mathbb{R}.$$

Saddle point interpretation

Theorem

Assume that:

- $x^\star$ *is a solution to the primal problem.*
- $\lambda^\star$ *is a solution to the dual problem.*
- *Strong duality holds:* $\min_x \max_\lambda L(x, \lambda) = \max_\lambda \min_x L(x, \lambda)$.

Then $(x^\star, \lambda^\star)$ *is a saddle-point of the Lagrangian* L.