Recitation 11

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Questions: Unconstrained Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Let $x, h \in \mathbb{R}^n$.
True or False.

1. If f is convex, then $f(x+h) > f(x) + \langle \nabla f(x), h \rangle$.
2. If f is strictly convex, then $f(x+h) > f(x) + \langle \nabla f(x), h \rangle$.
3. If f is strongly convex, then $f(x+h) > f(x) + \langle \nabla f(x), h \rangle$.
4. If f is convex, then f cannot have saddle points.
5. If $\nabla f(x) = 0$, then x is a local minimum of f.
6. If $\nabla f(x) = 0$ and $H_f(x) \succeq 0$, then x is a local minimum of f.
7. If $\nabla f(x) = 0$ and $H_f(x) \succ 0$, then x is a local minimum of f.
8. If $\nabla f(x) = 0$ and f is convex, then x is a local minimum of f.

Solutions: Unconstrained Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Let $x, h \in \mathbb{R}^n$.
True or False.

Solution

1. *If f is convex, then $f(x+h) > f(x) + \langle \nabla f(x), h \rangle$. False, consider $f(x) =< x, w >$ for some $w \in \mathbb{R}^n$.*
2. *If f is strictly convex, then $f(x+h) > f(x) + \langle \nabla f(x), h \rangle$. True. (From Lec 9)*
3. *If f is strongly convex, then $f(x+h) > f(x) + \langle \nabla f(x), h \rangle$. True, because strongly convex implies strictly convex*
4. *If f is convex, then f cannot have saddle points. True. If f had a saddle point, we could draw a chord below f on the "negative" side of the saddle.*

Solutions: Unconstrained Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Let $x, h \in \mathbb{R}^n$.
True or False.

Solution

5. *If $\nabla f(x) = 0$, then x is a local minimum of f.*
   False! Consider $f(x) = -x^2$.
6. *If $\nabla f(x) = 0$ and $H_f(x) \succeq 0$, then x is a local minimum of f.*
   False, consider $f(x) = -x^4$.
7. *If $\nabla f(x) = 0$ and $H_f(x) \succ 0$, then x is a local minimum of f.*
   True, this is exactly the condition we need for local minimums!
8. *If $\nabla f(x) = 0$ and f is convex, then x is a local minimum of f.*
   True, f being convex implies the Hessian is PSD everywhere.

Questions: Constrained Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Let $g(x) \leq 0$, and $h(x) = 0$ be two constraints. Let $x \in \mathbb{R}^n$ be in the feasible set.
True or False.

1. If f, g, h are convex, and $\nabla f(x) = 0$, then x is a local minimum.
2. If f, g, h are convex, and $\nabla f(x) = 0$, then x is a global minimum.
3. If x is a local minimum in the feasible set, then $\nabla f(x) = 0$.
4. (Ignoring h) If x is a local minimum in the feasible set, and $g(x) < 0$, then $\nabla f(x) = 0$.

Solutions : Constrained Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Let $g(x) \leq 0$, and $h(x) = 0$ be two constraints. Let $x \in \mathbb{R}^n$ be in the feasible set.
True or False.

Solution

1. *If* f, g, h *are convex, and* $\nabla f(x) = 0$*, then* x *is a local minimum.*
   True! Since x *is in the feasible set and* $\nabla f(x) = 0$ *, and* f *is convex, then* x *is a local minimum.*
2. *If* f, g, h *are convex, and* $\nabla f(x) = 0$*, then* x *is a global minimum.*
   True! Local minimums are global minimums since f *is convex.*
3. *If* x *is a local minimum in the feasible set, then* $\nabla f(x) = 0$*.*
   False, we could be limited by a constraint. Consider $f(x) = x^2$, $0 \geq g(x) = x + 1$.
4. *(Ignoring h) If* x *is a local minimum in the feasible set, and* $g(x) < 0$*, then* $\nabla f(x) = 0$*.*
   True! Since our constraint is not active, the lagrange multiplier is 0.

Question : Constrained Optimization

Let $f(x, y) = x^2 - 10y$. Let $h(x, y) = x^2 + y^2 = 36$

1. Find the minimum of f constrained by $h(x, y) = 0$.

Solution : Constrained Optimization

Let $f(x, y) = x^2 - 10x + y^2 - 4y$. Let $h(x, y) = x^2 + y^2 = 26$

Solution

Since h is an active constraint, we can solve this by considering the following system of equations.

$\nabla f(x, y) + \lambda \nabla g(x, y) = 0$ *and* $h(x, y) = 0$.

Now, $\nabla f(x, y) = \begin{bmatrix} 2x - 10 \\ 2y - 4 \end{bmatrix}$, $\nabla h(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$.

This gives us a system of three equations and three unknowns.

$2x - 10 + \lambda 2x = 0$

$2y - 2 + \lambda 2y = 0$

$x^2 + y^2 = 26$

Solving this system of equations gives two combinations of (λ, x, y)

$(\lambda, x, y) = (0, 5, 1)$, *and* $(\lambda, x, y) = (-2, -5, -1)$.

Note: $\nabla^2 f(x, y) = H_f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, *so both points are local minima.*

f(5,1) = -28, f(-5,-1) = 79