Recitation 10

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Gradient and Hessian

Definition (Gradient and Hessian of a function)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. If it exists, its gradient at a point $x \in \mathbb{R}^n$ is defined as

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix}$$

If it exists, its Hessian at a point $x \in \mathbb{R}^n$ is defined as

$$Hf(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}$$

Convex sets and convex functions

Definition (Convex set)

A set $C \subseteq \mathbb{R}^n$ if for all $x, y \in C$, and all $\alpha \in [0,1]$,

$$\alpha x + (1-\alpha)y \in C.$$

Definition (Convex function (and strictly convex function))

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}^n$ and all $\alpha \in [0,1]$ it holds that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y). \tag{1}$$

It is strictly convex if moreover $\forall \alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y). \tag{2}$$

Convex sets

1. Which of the following sets are convex?

    1. $\{x \in \mathbb{R}^2 : \|x\| = 1\}$
    2. $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$
    3. $\{x \in \mathbb{R}^2 : \|x\| \geq 1\}$
    4. $\{x \in \mathbb{R}^2 : \|x\| < 1\}$
    5. $\{x \in \mathbb{R}^2 : v^\top x \geq a\}$ for fixed $v \in \mathbb{R}^2$ and $a \in \mathbb{R}$.
    6. $\{x \in \mathbb{R}^2 : v^\top x = a\}$ for fixed $v \in \mathbb{R}^2$ and $a \in \mathbb{R}$.
    7. $\{x \in \mathbb{R}^2 : x_2 \geq x_1^2\}$
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Convex functions and convex sets

For $f : \mathbb{R}^n \to \mathbb{R}$, define the epigraph epi$(f) \subset \mathbb{R}^{n+1}$ to be the set of all points above the graph of f:
epi$(f) := \{(x, t) \in \mathbb{R}^{n+1} \mid t \geq f(x)\}$.

1. Prove that f is convex if and only if epi(f) is convex.
2. Prove that if f, g are convex functions, then $h(x) = \max(f(x), g(x))$ is convex.

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Convex functions and Hessians

Reminder: If f is a twice-differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$, f is convex if and only if its Hessian matrix $H_f(x)$ is positive semidefinite at all points $x \in \mathbb{R}^n$.

Reminder: If for all $x \in \mathbb{R}^n$, the Hessian matrix $H_f(x)$ is positive definite, then f is strictly convex. Show that the reverse is not true, i.e. find a strictly convex twice-differentiable function such that the Hessian matrix $H_f(x)$ is not positive definite everywhere.

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Convex functions and minima

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that if for a certain $x \in \mathbb{R}^n$, there exists $\epsilon > 0$ such that $f(x) = \min\{f(y) \mid \|x - y\| < \epsilon\}$, then $f(x) = \min_{y \in \mathbb{R}^n} f(y)$.

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Strictly convex functions & minima

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. Show that there exists a unique $x^\star$ such that $f(x^\star) = \min_{y \in \mathbb{R}^n} f(y)$.

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Gradients and Hessians

Calculate the gradients and the Hessians of the following functions $f : \mathbb{R}^n \to \mathbb{R}$:

1. $f(x) = \|x\|^2$.
2. $f(x) = \|Ax\|^2$.
3. $f(x) = x^\top Ax$.

For the functions 2 and 3, give necessary and sufficient conditions to have convexity and strict convexity.

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