Video 11.1: Critical points, global and local extrema

Optimization and Computational Linear Algebra for Data Science

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Definitions

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. We say that $x \in \mathbb{R}^n$ is

- a critical point of f if $\nabla f(x) = 0$,
- a *global* minimizer of f if for all $x' \in \mathbb{R}^n$, $f(x) \leq f(x')$,
- a *local* minimizer of f if there exists $\delta > 0$ such that we have $f(x) \leq f(x')$ for all x' verifying $\|x - x'\| \leq \delta$.

Local extrema are critical points

Proposition

$$x \text{ is a local minimizer of } f \implies \nabla f(x) = 0.$$

Proposition

Assume that f is convex. Then

$$\nabla f(x) = 0 \iff x \text{ is a global minimizer of } f.$$

Looking at the Hessian

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Let $x \in \mathbb{R}^n$ be a critical point of f, i.e. $\nabla f(x) = 0$.
Then, if $H_f(x)$ is positive definite (that is, if all the eigenvalues of $H_f(x)$ are strictly positive), then x is a local minimizer of f.

Looking at the Hessian

Proposition

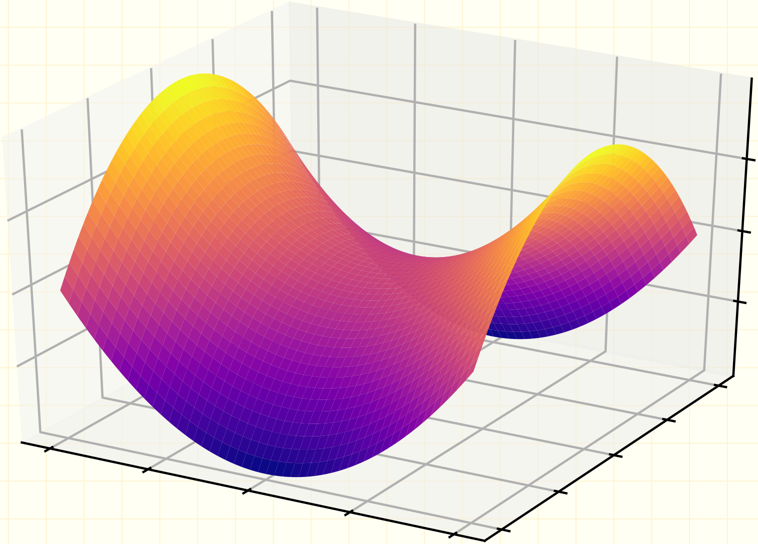
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Let $x \in \mathbb{R}^n$ be a critical point of f, i.e. $\nabla f(x) = 0$.
Then, if $H_f(x)$ is negative definite (that is, if all the eigenvalues of $H_f(x)$ are strictly negative), then x is a local maximizer of f.

Saddle points

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Let $x \in \mathbb{R}^n$ be a critical point of f, i.e. $\nabla f(x) = 0$.
Then, if $H_f(x)$ admits strictly positive eigenvalues and strictly negative eigenvalues, then x is neither a local maximum nor a local minimum. We call x a saddle point.

Saddle points



Example

Study the critical points of $f(x, y) = x^2 + xy^2 - x + 1$.

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