Recitation 12

Carles Domingo

Exercise 4, 2018 review

Let $A \in \mathbb{R}^{n\times n}$ be symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.
Prove that $\|Ax\| \leq \max_i |\lambda_i| \|x\|$ for any $x \in \mathbb{R}^n$.

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Exercise 8, 2018 review

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Consider the optimization problem

$$\begin{aligned} \text{minimize}_{x} \, &\|x\|_2 \\ \text{subject to } &Ax = b \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are fixed and $b \in \mathrm{Im}(A)$.

(a) Prove that any minimizer x^* must belong to $\mathrm{Im}(A^\top)$.

(b) Give a formula for the minimizer x^*, and show it is unique.

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Exercise 10, 2018 review

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times k}$ and define the block matrix $C \in \mathbb{R}^{n\times(m+k)}$ by

$$C = \begin{bmatrix} A & B \end{bmatrix}.$$

Either prove the following statement or give a counterexample:

$$\mathsf{Rank}(C) = \mathsf{Rank}(A) + \mathsf{Rank}(B).$$

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Let $A \in \mathbb{R}^{n \times n}$ be a square matrix with the (unusual) property that the image space (or column space) $\text{Im}(A)$ of A is equal to its kernel (or nullspace) $\text{Ker}(A)$.

(a) What can you say about A^2?

(b) Give an example of such an A.

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Exercise 25, 2018 review

Let $A = U\Sigma V^\top$ denote the Singular Value Decomposition of $A \in \mathbb{R}^{m\times n}$. Let $A' = U\Sigma' V^\top$ where Σ' is obtained from Σ by replacing every entry by zero except for the entry corresponding to the largest singular value.

(a) Show that A' is the best rank 1 approximation of A in the Frobenius norm, meaning that A' is the solution to $\min_{B:\text{rank}(B)=1} \|B - A\|_F$.

(b) Show that A' is the best rank 1 approximation of A in the spectral norm, meaning that A' is the solution to $\min_{B:\text{rank}(B)=1} \|B - A\|$.

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Exercise 0.9, 2019 review

For each of the following statement, say if they are true or false and justify your answer.

- If a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a unique minimizer then f is convex.
- If a continuous function $f : \mathbb{R} \to \mathbb{R}$ is such that there exists x_0 such that f is decreasing on $(-\infty, x_0]$ and increasing on $[x_0, +\infty)$ then f is convex.
- A twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative f' is non-decreasing is convex.

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Exercise 0.10, 2019 review

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex, differentiable function. Assume that there exist $x, y \in \mathbb{R}^n$ such that $\nabla f(x) = \nabla f(y) = 0$. Show that $\nabla f((x+y)/2) = 0$.