Unit -2: ELECTROMAGNETISM

Field:

A field is a region of space at every point of which a function is defined. The function must be single valued, continuous and must have continuous first derivative with respect to space coordinates in the region.

Vector Field: -

The space or regions in which the magnitude or direction or both of the vector quantities change with position and time are called vector fields. The vector quantities like electric field (E), magnetic induction (B), magnetic intensity (H), electric current density (J) are functions of position and time as well as have directions.

Scalar Field: -

The space or regions in which the magnitude of the scalar quantities change with position and time are called scalar fields. The scalar quantities like electrostatic potential (V) and electric charge density (σ) are functions of position and time.

Vector Calculus

The various electromagnetic laws are expressed in terms of the derivatives of these vector and scalar fields with respect to space and time. The derivatives of these fields with respect to space are expressed in terms of (i) gradient, (ii) divergence and (iii) curl operators.

<u>Vector differential Operators:</u>

Del ($\vec{\nabla}$) Operator: It is vector operator defined as $\vec{\nabla}= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Gradient of a scalar Field: -

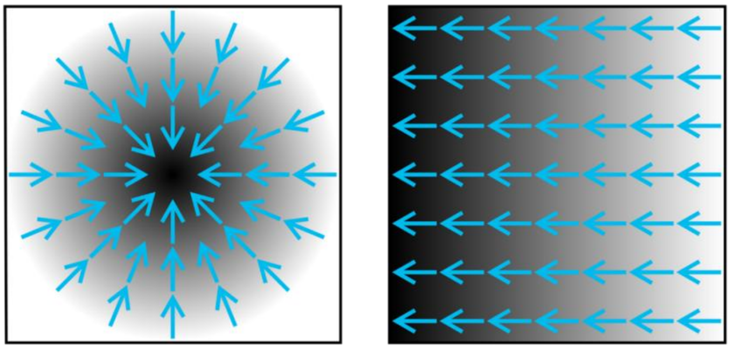
If f (x, y,z) is a single valued, continuous and differentiable scalar field, then its gradient is given by grad f

$$\text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Note that the input, f for the gradient is a **scalar-valued function**, while the output, $\vec{\nabla} f$, is a **vector-valued function**.

Physical Significance:

The gradient of a scalar function is a vector, whose magnitude at any point is equal to the maximum rate of change of scalar function with respect to the space variables and has the direction of that change.



The gradient, represented by the blue arrows, denotes the direction of greatest change of a scalar function. The values of the function are represented in greyscale and increase in value from white (low) to dark (high).

Divergence:

The ***divergence*** of a vector field $F(x, y, z)$ is the scalar-valued function

$$\text{divF} = \vec{\nabla} \cdot \vec{\boldsymbol{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note that the input, **F**, for the divergence is a vector-valued function, while the output, ($\vec{\nabla} \cdot \vec{\boldsymbol{F}}$), is a scalar-valued function. **If the divergence of a vector field is zero then it is called a solenoidal vector.**

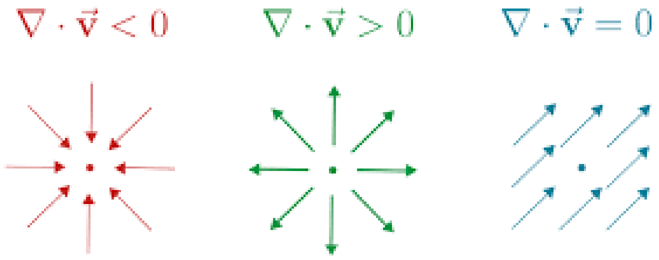
Physical significance of Divergence:

Divergence of a vector field at a point is defined as the total outward flux of that field per second per unit volume through a closed surface drawn around that point.

If $\vec{F}$ represents the velocity of a moving fluid at any point P, then divF gives the rate at which the fluid is diverging per unit volume from the point P.

If div F > 0 at any point P, then either the fluid is expanding or its density at P is decreasing with time or the point P is a source of the fluid. If div $\vec{F} < 0$, then either fluid is contracting or its density is increasing at P or the point is a *sink.*

If div$\vec{\boldsymbol{F}} = 0$, then flux of $\vec{\boldsymbol{F}}$ entering any element of space is exactly balanced by the flux leaving it. Then there is no source or sink in the field nor its density is changing (i.e the fluid is incompressible).



Curl

The ***curl*** of a vector field $F(x, y, z)$ is the vector field

$$\text{curlF} = \vec{\nabla} \times \vec{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\hat{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\hat{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\hat{\mathbf{k}}$$

Note that the input,$\vec{F}$, for the curl is a vector-valued function, and the output, $\vec{\nabla} \times \vec{F}$, is a again a vector-valued function.

Physical significance of Curl:

The curl of a vector field is connected with rotational properties of the vector field and justifies the name *rotation* used for curl. If $\vec{\nabla} \times \vec{\mathbf{F}} = 0$, the field F is termed as *irrotational*.

Some Properties of $\vec{\nabla}$ operator:

(i) ***Curl of gradient of scalar field*** i.e $\vec{\nabla} \times \vec{\nabla}\emptyset = 0$

(ii) ***Divergence of gradient of a scalar field*** i.e $\vec{\nabla}.(\vec{\nabla}\emptyset) = \nabla^2\emptyset$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$= Laplacian operator

(iii) ***Curl of Curl of a vector*** : $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$

(iv) ***Divergence of cross product of two vectors:***: $\vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$

(v) $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$

Stokes Theorem:-

It states that the surface integral of the curl of a vector field $\vec{A}$ over any surface S equal to the line integral of vector field $\vec{A}$over the boundary of that surface, i.e.

$$\int_S (\vec{\nabla} \times \vec{A}).ds = \oint \vec{A}.d\vec{l}$$

Using this theorem, we can convert a surface integral of the curl of a vector field to line integral of the vector field and vice versa. For a closed surface,$\oint \vec{A}.d\vec{l} = 0$ .Hence the surface integral of the curl of a vector field over a closed surface vanishes.

Gauss Divergence Theorem:

Statement: It states that the surface integral of a vector field $\vec{A}$ over any closed surface S is equal to the volume integral of the divergence of that vector field over the volume elcosed by the surface 'S' i.e

$$\oint_S \vec{A}.\overrightarrow{ds} = \int_V (\vec{\nabla}.\ \vec{A})\, dV$$

Electric displacement:- The electric displacement at a point is related to the electric field by the relation

$$\vec{D} = \varepsilon\vec{E}$$

where, ε is the electric permittivity of the medium. In vacuum (free space),

$$\vec{D} = \varepsilon_0\vec{E}$$

The S.I. unit of $\vec{D}$ is Coulomb/m^2.

Electric flux: - The electric flux φ_E linked with a surface S placed in an electric field is the surface integral of the electric field $\vec{E}$ over the surface, i.e.,

$$\varphi_E = \int_S (\overrightarrow{\nabla} \times \overrightarrow{A})$$

The S.I. unit of electric flux Newton (meter)^2 /Coulomb

Magnetic flux: -Magnetic flux φ_B linked with a surface, placed in a magnetic field is the surface integral of magnetic field $\vec{B}$ over the surface; i.e.;

$$\varphi_B = \int_S (\overrightarrow{B}.\ d\overrightarrow{s})$$

The S.I. unit of φ_B is Weber and CGS unit of $\varphi_B$is Maxwell.

1Weber=10^8 Maxwell

Gauss Law of electrostatics:-

"The total electric flux passing through a closed surface is equal to $1/\varepsilon_0$ times the net charge enclosed by the surface", i.e.,

$$\varphi_E = \int_S (\overrightarrow{E}.\ d\overrightarrow{s}) = \frac{q_{net}}{\varepsilon_0}$$

where, net q_{net} = algebraic sum of the charges enclosed by the surface.

The enclosed surface 'S' is known as Gaussian surface.

Notes: -

(i)Here, q_{net} may be +ve, -ve or zero. Accordingly the net electric flux over the area may be outward, inward or zero.

(ii)The electric flux does not depend on the shape and size of the Gaussian surface as long as the charges are enclosed by it.

<u>Limitations of Gauss Law</u>:-

(i) Using Gauss Law, magnitude of electric field at a point can be measured, but the direction cannot be determined. (Electric flux is a scalar quantity).

(ii) This law is applicable only for simple geometrical surface and difficult to apply for arbitrary surface.

<u>Gauss Law in differential form</u>: -

If ρ is the charge density in Gaussian surface, then

$$\sigma = \frac{dq}{dv}$$

$$\Rightarrow dq = \sigma dv$$

$$\therefore q_{net} = \int dq = \int \rho dV \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1)$$

From Gauss Law,

$$\int_S \left(\overrightarrow{E} \cdot d\overrightarrow{s}\right) = \frac{1}{\varepsilon_0} q_{net} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(2)$$

Using eqn. (1) & (2), we have

$$\int_S \left(\overrightarrow{E} \cdot d\overrightarrow{s}\right) = \frac{1}{\varepsilon_0} \int_V \rho dV \ldots\ldots\ldots\ldots\ldots\ldots \;(3)$$

Using Gauss divergence theorem, surface integral of above equation can be converted into volume integral, i.e.,

$$\int_S \left(\overrightarrow{E} \cdot d\overrightarrow{s}\right) = \int_V \vec{\nabla} \cdot \overrightarrow{E} dV \ldots \ldots\ldots\ldots \;(4)$$

From eqn. (3) & (4), we get

$$\int_V \vec{\nabla} \cdot \overrightarrow{E} dV = \frac{1}{\varepsilon_0} \int_V \rho dV$$

$\Rightarrow \int_V \left(\vec{\nabla} \cdot \overrightarrow{E} - \frac{\rho}{\varepsilon_0} dV\right) = 0$, so the integrand must vanish.

$$\Rightarrow \vec{\nabla} \cdot \overrightarrow{E} - \frac{\rho}{\varepsilon_0} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\varepsilon_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(5)$$

The above equation is the differential form of Gauss Law in free space.

For any medium, eqn. (5) can be written as

$$\vec{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\varepsilon} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

Where ε= permittivity of the medium.

<u>Differential form of Gauss Law in terms of displacement vector</u> :-

From eqn.(6) $\vec{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\varepsilon}$

$$\Rightarrow \overrightarrow{\nabla} \cdot \varepsilon \overrightarrow{E} = \rho$$

$$\Rightarrow \vec{\nabla} \cdot \overrightarrow{D} = \rho \;\ldots\ldots\ldots\ldots\ldots\ldots \;(7)$$

Gauss Law in magnetism:-

It states that "The magnetic flux φ_B over a closed surface is always zero" (i.e. ; the magnetic flux leaving a closed surface is the same as that entering the surface as no isolated magnetic pole exists; hence φ_B =0). Therefore,

$$\oint \vec{B} \,.\, d\vec{s} = 0$$

In differential form:-

Using Gauss Divergence Theorem, surface integral of the above equation can be converted into volume integral as

$$\int_S (\vec{B} \,.\, d\vec{s}) = \int_V \vec{\nabla} \,.\, \vec{B}\, \mathrm{dV} = 0$$

$$\Rightarrow \vec{\nabla} \,.\, \vec{B} = 0 \text{.....................} \quad (8)$$

The above equation is the differential form of Gauss law in magnetism.

Faraday's law of electromagnetic induction:-

1st law:-When the magnetic flux linked with a coil changes, an emf is induced between the two ends of the coil and the induced emf exists so long as the change continues.

2nd law:- The induced emf is directly proportional to the negative rate of change of magnetic flux, i.e.,

$$\xi = -\frac{d\varphi}{dt} \text{..................................} \quad (9)$$

Since induced emf is defined as the work done in moving a unit positive charge round the conducting loop, it is equal to the line integral of electric field along the loop, i.e.,

$$\xi = \oint \vec{E}\, \mathrm{dl} \text{...................................} \quad (10)$$

From definition of , the flux linked with the coil is

$$\varphi = \int_S (\vec{B} \,.\, d\vec{s}) \text{..........................} (11)$$

Using eqn.(2) and (3) in eqn. (1), we get

$$\oint \vec{E} \,.\, d\vec{l} = -\frac{\partial}{\partial t}\left(\int_S (\vec{B} \,.\, d\vec{s})\right) \text{.........} (12)$$

Applying Stokes Theorem to the LHS of the above equation, we get

$$\oint \vec{E} \,.\, d\vec{l} = \int_S (\vec{\nabla} \times \vec{E})\mathrm{d}\vec{s} \text{......................} \quad (13)$$

From eqn.(12) & (13), we get

$$\int_S (\vec{\nabla} \times \vec{E})\mathrm{d}\vec{s} = -\frac{\partial}{\partial t}\left(\int_S (\vec{B} \,.\, d\vec{s})\right) = -\int_S \left(\frac{\partial \vec{B}}{\partial t} \,.\, d\vec{s}\right)$$

$$\Rightarrow \int_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right).\mathrm{d}\vec{s} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{...........................} \quad (14)$$

This equation is the differential form of Faraday's law of electromagnetic induction.

Ampere's Circuital law:- When current is flowing through a conductor, a magnetic field develops around the conductor. The strength of the magnetic field at any point can be measured by using Biot-Savart's law, i.e.

$$B = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2}$$

Where, i=current through the conductor.
dl=ρa small length of the conductor
$\vec{r}$ =distance of the point, where the magnetic field is to be measured.
Statement of Ampere's law : The line integral of magnetic field B around any closed loop, is equal to μ_0 times the net electric current enclosed by the loop, i.e.,

$$\oint_c \vec{B}.\mathrm{d}\vec{l} = \mu_0 I_{net}$$

Where, c is the closed path enclosing the current and μ_0 is the permeability of the free space. Here, I_{net} is the algebraic sum of the currents through the surface s enclosed by the closed path.

Differential form of Ampere's law:-

If $\vec{J}$ = current density = current per unit area, then,

$$\vec{J} = \frac{dI}{ds}$$

$$\Rightarrow dI = \vec{J}.ds$$

$$\Rightarrow I_{net} = \int dI = \int_S \vec{J}.\mathrm{d}\vec{s} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (15)$$

From Ampere's law, we have

$$\int_C \left(\vec{B}\ .\ d\vec{l}\right) = \mu_0 \int_S I_{net} \ \ldots\ldots\ldots\ (16)$$

Using eqn.(1) in eqn.(2), we get

$$\int_C \left(\vec{B}\ .\ d\vec{l}\right) = \mu_0 \int_S \vec{J}.\mathrm{d}\vec{s} \ \ldots\ldots\ldots\ldots(17)$$

Using Stoke's theorem, line integral of the above equation can be converted into surface integral as follows:

$$\oint_C \vec{B}\ .\ d\vec{l} = \int_S \left(\vec{\nabla} \times \vec{B}\right)\mathrm{d}\vec{s} \ \ldots\ldots\ldots\ (18)$$

From eqn. (17) & (18), we get

$$\int_S \left(\vec{\nabla} \times \vec{B}\right).\mathrm{d}\vec{s} = \mu_0 \int_S \vec{J}.\mathrm{d}\vec{s}$$

$$\Rightarrow \int_S \left(\vec{\nabla} \times \vec{B} - \mu_0\vec{J}\right).\mathrm{d}\vec{s} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{B} - \mu_0\vec{J} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0\vec{J} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Eqn. (19) is the differential form of Ampere's circuital law.

Equation of Continuity

Equation of Continuity: The differential form of law of conservation of charge is called equation of continuity. The law of conservation of charge states that in all processes associated with the motion of charges, total charge in a given volume remains constant also be expressed as charges in a certain volume V can change only if the charge flows into or out of it.

Consider a continuous flow of charges through a surface S enclosing a volume V. The electric current I is the rate of decrease of charge from the volume V through the surface S.

$I = \int_v \frac{\partial q}{\partial t}$ or $\int_S \vec{J}.ds = \frac{\partial}{\partial t}\int_v \rho dV$

where $\vec{J}$ = electric current density and σ =electric charge density

Applying Gauss Divergence Theorem, on LHS of the above equation

$$\int_v (\vec{\nabla}.\vec{J})\, dV = -\frac{\partial}{\partial t}\int_v \rho dV$$

$$\int_v (\vec{\nabla}.\vec{J})\, dV = -\int_v \frac{\partial \rho}{\partial t} dV$$

$$\int_v (\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t})dV = 0$$

$$\int_v (\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t}) = 0$$

This is called as Equation of Continuity

<u>Displacement current and modified Ampere's circuital law</u>:-

According to Ampere's circuital law,

$$\oint \vec{B}.\mathrm{d}\vec{l} = \mu_0 I$$

This means, a magnetic field is produced, when there is a continuous flow of current.

The differential form of Ampere's law is

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$\Rightarrow \vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \vec{J}$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}$

Taking divergence of both sides, we get

$$\vec{\nabla}.(\vec{\nabla} \times \vec{H}) = \vec{\nabla}.\vec{J}$$

But the divergence of curl of a vector is always zero. Therefore.

$$\vec{\nabla}.\vec{J} = 0$$

But the equation of continuity is

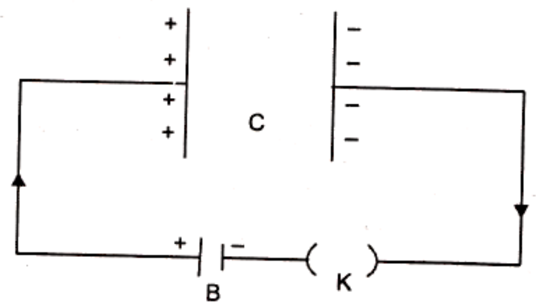
$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

So,

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}$$

$\vec{\nabla}.\vec{J}$ is zero , if we consider charge density is constant or time independent. Thus, we can conclude that Ampere's circuital law is valid only for time independent fields and does not hold good for time varying fields.

Consider a parallel plate capacitor being charged by a cell as shown in the figure. During the process of charging, the electric field between the plates of the capacitor changes with time, i.e.; a time varying electric field exists.



It is found that a magnetic field is also produced between the plates as long as the electric field keeps on changing with time even if no current flows between the plates, i.e.; a time varying electric field also produces a magnetic field. This fact contradicts Ampere's circuital law.

In order to remove this inconsistency, Maxwell modified Ampere's circuital law by introducing the concept of displacement current for the time – varying electric field. Now the modified Ampere's circuital law is

$$\oint \vec{B}.\mathrm{d}\vec{l} = \mu_0(I + I_d)$$

where, I and I_d are called conduction current and displacement current respectively. This modified law is sometimes called Ampere-Maxwell law.

<u>Displacement current</u>:-

If q is the charge on the plates of the capacitor and A is the area of each plate, then the electric field in between the plates is given by

$$\vec{E} = \frac{\sigma}{\varepsilon_0} = \frac{q}{\varepsilon_0 A}$$

$$\therefore \frac{d\vec{E}}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt} = \frac{1}{\varepsilon_0 A} I_d$$

where, $I_d = \frac{dq}{dt}$ = displacement current between the plates.

$$\Rightarrow I_d = \varepsilon_0 A \frac{d\vec{E}}{dt} \text{.................................} (20)$$

Eqn, (20) says that the displacement current exists as long as the electric field changes with time. When the capacitor is fully charged, the electric field becomes constant and hence the displacement current vanishes.

I_d may be written as

$$I_d = A\frac{d}{dt}\left(\varepsilon_0\vec{E}\right) = A\frac{d\vec{D}}{dt}$$

$$\Rightarrow \frac{I_d}{A} = \frac{d\vec{D}}{dt}$$

$\Rightarrow \overrightarrow{J_d} = \frac{d\vec{D}}{dt}$, where, $\overrightarrow{J_d}$= displacement current density.

Hence, displacement current density is equal to time rate of change of electric displacement. In terms of electric flux (φ_E), I_d can be written as

$$I_d = \varepsilon_0\frac{\partial}{\partial t}\int_A \vec{E}\,.\,d\vec{A} = \varepsilon_0\frac{\partial\varphi_E}{\partial t}$$

Comparison between conduction current and displacement current: -

	Conduction current		Displacement Current
1	It is produced due to actual flow of charge carriers in a conductors	1	It exists in vacuum or any medium without flow of charge carriers even in absence of charge carriers.
2	It depends on the resistance and potential difference of the conductor	2	It depends on electric permittivity of the medium and the rate at which electric field changes with time
3	$I = \frac{dq}{dt}$	3	$I_d = \varepsilon_0 A\frac{d\vec{E}}{dt}$
4	It obeys Ohm's law	4	It does not obey Ohm's law

Differential form of modified Ampere's circuital law: -

$\oint \vec{B}\,.\,\vec{dl} = \mu_o\left(\vec{I} + \overrightarrow{I_d}\right)$ 1

We have found out that the differential form of the equation $\oint \vec{B}\,.\,\vec{dl} = \mu_0\vec{I}$ is $\vec{\nabla} \times \vec{B} = \mu_0\vec{J}$.

Thus from generalization, the differential form of eqn. (1) is

$\vec{\nabla} \times \vec{B} = \mu_0\,(\vec{J} + \overrightarrow{J_d})$

$\Rightarrow \vec{\nabla} \times \frac{\vec{B}}{\mu_0} = (\vec{J} + \overrightarrow{J_d})$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}(\varepsilon_0\vec{E})$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$

Maxwell's electromagnetic equations: -

There are four fundamental equations of electromagnetism known as Maxwell's electromagnetic equations, which may be written in differential form as

1.$\vec{\nabla}.\vec{D} = \rho$ (Differential form of Gauss law in electrostatics) …………………….. (21)

2.$\vec{\nabla}.\vec{B} = 0$ (Differential form of Gauss law in magnetism) ………………………….. (22)

3.$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ (Differential form of Faraday's law of electromagnetic induction) …(23)

4.$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ (Maxwell's modified Ampere's circuital law) ………………… (24)

where, $\vec{D}$ = electric displacement vector

ρ = charge density

$\vec{B}$ = magnetic induction

$\vec{E}$ = electric field

$\vec{H}$ = magnetic intensity

$\vec{J}$ = current density

Eqns. (21) to (24) are the Maxwell's equations in a medium in presence of free charges and currents

Notes: -

(i) Eqns. (22) & (23) do not depend on the medium as well as the presence or absence of the free charges and currents where as eqns. (21) & (24) depend upon the medium and also on the presence of free charges and currents.

(ii) the first two divergence equations do not include time dependence of fields while the last two curl equations involve time dependence of fields.

Maxwell's equations in terms of $\vec{E}$ & $\vec{B}$: -

Since $\vec{D} = \varepsilon\vec{E}$ and $\vec{H} = \frac{\vec{B}}{\mu}$, the Maxwell's equations can be written as

$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon}$……………………………………. (25)

$\vec{\nabla}.\vec{B} = 0$ …………………………………… (26)

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ ………………………… (27)

$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$ ……………………… (28)

Maxwell's equations in vacuum in absence of charges and current: -

For vacuum, $\mu \rightarrow \mu_0$ & $\varepsilon \rightarrow \varepsilon_0$

In absence of charge, $\rho \rightarrow 0$

In absence of current, $J \rightarrow 0$

Hence the Maxwell's equations become:

$\vec{\nabla}.\vec{E} = 0$ …………………..………………………………….. (29)

$\vec{\nabla}.\vec{B} = 0$ ………………… ………………………..….......... (30)

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ ……………… ………………………..... (31)

$\vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$ ………………………………………… (32)

Maxwell's equations in integral form: -

1. $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$

Integrating with respect to volume, we get

$$\int_V \vec{\nabla}.\,\vec{E}\,dV = \frac{1}{\varepsilon_o}\int_v \rho\,dV$$

Applying Gauss divergence theorem to LHS, we get

$$\int_S \vec{E}.\mathrm{d}\vec{s} = \frac{1}{\varepsilon_0}\int_V \rho.\mathrm{dv}$$

2. $\vec{\nabla}.\vec{B} = 0$

Integrating with respect to volume, we get

$$\int_V \vec{\nabla}.\vec{B}\,dV = 0$$

Applying Gauss divergence theorem to LHS, we get

$$\int_s \vec{B}.\mathrm{d}\vec{s} = 0$$

3. $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Integrating with respect to surface, we get

$$\oint(\vec{\nabla} \times \vec{E})\,.\,\overrightarrow{\mathrm{ds}} = -\oint \frac{\partial \vec{B}}{\partial t}\,.\,\overrightarrow{\mathrm{ds}}$$

Applying Stoke's theorem to LHS, we get

$$\int \vec{E}.\vec{dl} = -\frac{\partial}{\partial t}\oint B.\vec{ds}$$

4. $\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Integrating with respect to surface, we get

$$\oint(\vec{\nabla}\ X\ \vec{B}).\vec{ds} = \mu_o\oint_s\ (\vec{J} + \varepsilon_o\frac{\partial\vec{E}}{\partial t}).\vec{ds}$$

Applying Stoke's theorem to LHS, we get

$$\oint_c\ \vec{B}.\vec{dl} = \mu_o\oint_s\ (\vec{J} + \varepsilon_o\frac{\partial\vec{E}}{\partial t}).\vec{ds}$$

Physical significance of Maxwell's equations: -

(i) All the laws incorporated in Maxwell's equations were developed from experimental observations.

(ii) The very existence of electromagnetic wave and its speed of propagation could be proved from Maxwell's equations.

(iii) Maxwell's equations provided a unified description of electric and magnetic phenomena, which were assumed to be independent of each other.

Maxwell's equations and electromagnetic waves

Electromagnetic waves: -

Electromagnetic waves consist of both electric and magnetic fields which oscillates perpendicular to each other and perpendicular to the direction of propagation of wave.

Electromagnetic wave equations for $\vec{E}$&$\vec{B}$: -

The wave equations for electric and magnetic fields can be obtained by decoupling Maxwell's electromagnetic equations. Maxwell's equations in terms of $\vec{E}$&$\vec{B}$ are given by

$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon}$ …………………. (33)

$\vec{\nabla}.\vec{B} = 0$ ………………… (34)

$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$ ………….. (35)

$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$ …… (36)

(a) Wave equations in charge free conducting medium: -

In charge free conducting medium, $\rho \rightarrow 0$; $J \rightarrow 0$

If μ & ε are independent of position and time, then the Maxwell's equations take the form:

$\vec{\nabla}.\vec{E} = 0$ ………………………. (37)

$\vec{\nabla}.\vec{B} = 0$ …………………… (38)

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ………………… (39)

$\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t} = 0$ ……. (40)

Taking curl of eqn. (39), we get

$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$

$\Rightarrow \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$ … (41)

Using eqn. (37) and (40) in the above equation, we get

$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}(\mu\sigma\vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t})$

$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\sigma\frac{\partial \vec{E}}{\partial t}$ …………… (42)

Similarly taking curl of eqn. (40) and using eqn. (38) & (39), we get

$\nabla^2\vec{B} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} = \mu\sigma \frac{\partial \vec{B}}{\partial t}$ …………… (43)

Eqns. (10 & 11) are the wave equations for $\vec{E}$&$\vec{B}$ in charge free conducting medium of conductivity σ. The terms $\mu\sigma\frac{\partial \vec{E}}{\partial t}$ and $\mu\sigma\frac{\partial \vec{B}}{\partial t}$ are the dissipative terms.

(b) Wave equations in charge free non-conducting medium: -

In charge free non-conducting medium σ = 0 (or J=0). The wave equations (42) and (43) become

$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$ ……………… (44)

$\nabla^2\vec{B} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} = 0$ ……………… (45)

Comparing the above two equations with the general wave equation

$\nabla^2\psi - \frac{1}{\upsilon^2}\frac{\partial^2\psi}{\partial t^2} = 0$ ……………… (46)

We get the speed of electromagnetic wave in the medium as

$\upsilon = \frac{1}{\sqrt{\mu\varepsilon}}$ ………………………… (47)

which is less than ‘c’, as $\mu < \mu_0$ & $\varepsilon < \varepsilon_0$.

(c) Wave equations in free space: -

In free space, $\varepsilon = \varepsilon_0$ & $\mu = \mu_0$. In absence of charge and current (ρ = 0 & σ = 0 or J =0).

Thus the wave eqns. (10) and (11) take the from

$\nabla^2\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$ …………… (48)

$\nabla^2\vec{B} - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = 0$ ……..…… (49)

Comparing the above two equations with the general wave equation

$\nabla^2\psi - \frac{1}{\upsilon^2}\frac{\partial^2\psi}{\partial t^2} = 0$

We get the speed of electromagnetic wave in vacuum as

$\upsilon = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

Putting $\mu_0 = 4\pi \times 10^{-7}$Wb $A^{-1}m^{-1}$ and $\varepsilon_0 = 8.85 \times 10^{-12}\frac{(\text{coulomb})^2}{\text{N m}^2}$, we get

$v = 3 \times 10^8$ m/s = speed in vacuum.

The Plane Wave Solution: -

The plane wave solution of the wave eqn. (48) and (49) are given by

$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)}$……...(50)

$\vec{B}(\vec{r},t) = \vec{B}_0 e^{i(\vec{k}.\vec{r}-\omega t)}$………(51)

where, $\vec{E}_0$ and $\vec{B}_0$ are amplitudes which are constant vectors, $\vec{r}$ is the position vector and $\vec{k}$ is the wave propagation vector defined as

$\vec{k} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$

where $\hat{n}$ is the unit vector along $\vec{k}$, ω is the angular frequency and c is the wave velocity.

Show that:

$$(\boldsymbol{i})\vec{\nabla}.\vec{\boldsymbol{E}} = \boldsymbol{i}\vec{\boldsymbol{k}}.\vec{\boldsymbol{E}}$$
$$(\boldsymbol{ii})\vec{\nabla}.\vec{\boldsymbol{B}} = \boldsymbol{i}\vec{\boldsymbol{k}}.\vec{\boldsymbol{E}}$$
$$(\boldsymbol{iii})\,\vec{\nabla}\times\vec{\boldsymbol{E}} = \boldsymbol{i}\vec{\boldsymbol{k}}\times\vec{\boldsymbol{E}}$$
$$(\boldsymbol{iv})\vec{\nabla}\times\vec{\boldsymbol{B}} = \boldsymbol{i}\vec{\boldsymbol{k}}\times\vec{\boldsymbol{B}}$$
$$(i)\vec{\nabla}.\vec{E} = i\vec{k}.\vec{E}$$

From the wave equations (50) and (51), we have:

$$\vec{E}(\vec{r},t) = \overrightarrow{E_0}e^{i(\vec{k}.\vec{r}-\omega t)}$$
$$\vec{B}(\vec{r},t) = \overrightarrow{B_0}e^{i(\vec{k}.\vec{r}-\omega t)}$$
$$\therefore \vec{\nabla}.\vec{E}(\vec{r},t) = \vec{\nabla}.\overrightarrow{E_0}e^{i(\vec{k}.\vec{r}-\omega t)}$$
$$\Rightarrow \vec{\nabla}.\vec{E}(\vec{r},t) = \vec{\nabla}.\overrightarrow{E_0}e^{i(\vec{k}.\vec{r}-\omega t)} + \vec{E}_0.\left[\vec{\nabla}e^{i(\vec{k}.\vec{r}-\omega t)}\right]$$

$\Rightarrow \vec{\nabla}.\vec{E} = \vec{E}_0.i\vec{k}e^{i(\vec{k}.\vec{r}-\omega t)}$ [$\therefore \vec{\nabla}.\vec{E} = 0$ as $\vec{E}_0$ is a constant vector]

$$\Rightarrow \vec{\nabla}.\vec{E} = i\vec{k}.\overrightarrow{E_0}.i\vec{k}e^{i(\vec{k}.\vec{r}-\omega t)}$$
$$\Rightarrow \vec{\nabla}.\vec{E} = i\vec{k}.\vec{E}$$

Similarly, we can get $(ii)\vec{\nabla}.\vec{B} = i\vec{k}.\vec{E}$ **(Homework)**

$(iii)\vec{\nabla} \times \vec{E} = i\vec{k} \times \vec{E}$

L.H.S.= $\vec{\nabla} \times \vec{E}(\vec{r},t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$

$\vec{\nabla} \times \vec{E}(\vec{r},t) = \hat{\imath}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$……………….(a)

$\vec{E}(\vec{r},t) = \overrightarrow{E_0} e^{i(\vec{k}.\vec{r}-\omega t)}$, substituting

$$\hat{\imath}E_x + \hat{\jmath}E_y + \hat{k}E_z = \left(\hat{\imath}E_{0x} + \hat{\jmath}E_{0y} + \hat{k}E_{0z}\right)e^{-i\left(\omega t - \vec{k}.\vec{r}\right)}$$

$$\hat{k}.\hat{r} = k_x x + k_y y + k_z z$$

Solving equation (a), we have

$$[\nabla \times \vec{E}(\vec{r},t)]_x = \hat{\imath}\left[E_{0z} i e^{i(\omega t - \vec{k}.\vec{r})} ky - E_{0y} i e^{i(\omega t - \vec{k}.\vec{r})}\right]$$

$$= \hat{\imath}\left[i\{k_y E_z - k_z E_y\}\right]$$

$$= i\left[\vec{k} \times \vec{E}\right]_x$$

$$E_Z = E_{0z} e^{i(\omega t - \vec{k}.\vec{r})}, E_y = E_{0y} e^{i(\omega t - \vec{k}.\vec{r})}$$

Hence, adding all these terms we will get,

$$[\nabla \times \vec{E}(\vec{r},t)]_y = \hat{\jmath}[(k_z E_x - k_x E_z)] = i\left[\vec{k} \times \vec{E}\right]_y$$

$$[\nabla \times \vec{E}(\vec{r},t)]_z = \hat{\jmath}\left[(k_x E_y - k_y E_x)\right] = i\left[\vec{k} \times \vec{E}\right]_z$$

Hence, adding all these terms, we have

$$\vec{\nabla} \times \vec{E} = i\vec{k} \times \vec{E} \quad (52)$$

Similarly, $\vec{\nabla} \times \vec{B} = i\vec{k} \times \vec{B}$ (53)

Transverse Nature of Electromagnetic Waves:

Maxwell's first and second equations in vacuum are:

$$\vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

This shows that E and B both are perpendicular to the direction of propagation vector. This indicates that e.m. waves are transverse in nature.

Using the above equations,

$$i\vec{k}.\vec{E} = 0 \Rightarrow \vec{k}.\vec{E} = 0 \text{………………….(54)}$$

$$i\vec{k}.\vec{B} = 0 \Rightarrow \vec{k}.\vec{B} = 0 \text{…………………..(55)}$$

Further, restrictions are provided by the curl equations:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = -\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Using the results of above problems and equation (52) and (53)

$$i\left(\vec{k} \times \vec{E}\right) = -(-i\omega \vec{B})$$

$$\Rightarrow \vec{k} \times \vec{E} = \omega \vec{B} \ldots\ldots\ldots\ldots(56)$$

$$\text{Similarly,} \quad i(\vec{k} \times \vec{B}) = \mu_0 \epsilon_0 (-i\omega \vec{E})$$

$$\vec{k} \times \vec{B} = -\mu_0 \epsilon_0 \omega \vec{E} \ldots\ldots\ldots\ldots\ldots\ldots(57)$$

From equation (56) it is clear that B is perpendicular to both k and E and according to equation (57) it is clear that E is perpendicular to both k and B. Thus, it can be concluded that both E, B, and k are mutually orthogonal and forms a right handed coordinate system.

Electromagnetic energy density: Poynting Theorem:-

Poynting's theorem is an important concept in electromagnetism that describes the flow of energy in an electromagnetic field.

Statement: -It states that the rate of energy flow through any surface surrounding a volume of space is equal to the negative of the time rate of change of the electromagnetic energy density within that volume, plus the electromagnetic energy flux density passing through the surface.

Mathematically, Poynting's theorem can be expressed as: $\vec{P} = \vec{E} \times \vec{H}$

From Maxwell's time varying equations in differential form, we have

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ……………………… (58)

and $\vec{\nabla} \times \vec{H} = J + \frac{\partial \vec{D}}{\partial t}$ …………………(59)

Taking scalar product of eqn. (58) by $\vec{H}$ and eqn. (59) by $\vec{E}$ and then subtracting, we get

$\vec{H}.(\vec{\nabla} \times \vec{E}) - \vec{E}(\vec{\nabla} \times \vec{H}) = -\vec{H}.\frac{\partial \vec{B}}{\partial t} - \vec{E}.\frac{\partial \vec{D}}{\partial t} - \vec{E}.\vec{J}$……………..(60)

Using the vector identity

$\vec{\nabla}.(\vec{E} \times \vec{H}) = \vec{H}.(\vec{\nabla} \times \vec{E}) - \vec{E}.(\vec{\nabla} \times \vec{H})$, [as $\vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.curl\, \vec{A} - \vec{A}.curl \vec{B}$]

Eqn. (60) becomes

$\vec{\nabla}.(\vec{E} \times \vec{H}) = -\left(\vec{H}.\frac{\partial \vec{B}}{\partial t} + \vec{E}.\frac{\partial \vec{D}}{\partial t}\right) - \vec{E}.\vec{J}$………………..(61)

For a linear medium, $\vec{B} = \mu \vec{H}$ and $\vec{D} = \epsilon \vec{E}$. Therefore

$\vec{H}.\frac{\partial \vec{B}}{\partial t} = \vec{H}.\frac{\partial}{\partial t}(\mu \vec{H}) = \frac{1}{2}\mu \frac{\partial}{\partial t}(\vec{H}.\vec{H}) = \frac{1}{2}\mu \frac{\partial}{\partial t}(H^2) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}.\vec{B}\right)$……….(62(a))

And $\vec{E}.\frac{\partial \vec{D}}{\partial t} = \vec{E}.\frac{\partial}{\partial t}(\varepsilon \vec{E}) = \frac{1}{2}\varepsilon \frac{\partial}{\partial t}(\vec{E}.\vec{E}) = \frac{1}{2}\varepsilon \frac{\partial}{\partial t}(E^2) = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}.\vec{D}\right)$……..(62(b))

Using the above relationship equation (61) becomes;

$$\vec{\nabla}.(\vec{E} \times \vec{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{E}.\vec{D} + \vec{H}.\vec{B})\right] - \vec{J}.\vec{E}$$

Integrating the above equation over a volume V, bounded by a surface S, we have

$$\int_v \vec{\nabla}.(\vec{E}\times\vec{H})\,dV = -\int_V \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{E}.\vec{D}+\vec{H}.\vec{B})\right]dV - \int_V \vec{J}.\vec{E}dV$$

Using Gauss divergence theorem on LHS of the equation, we get,

$$\oint_s (\vec{E}\times\vec{H}).dS = -\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}(\vec{E}.\vec{D}+\vec{H}.\vec{B})\right]dV - \int_V \vec{J}.\vec{E}dV$$

Let, $(\vec{E}\times\vec{H}) = \vec{P}$

And $\frac{1}{2}(\vec{E}.\vec{D}+\vec{H}.\vec{B}) = u$

Now the above equation becomes

$$\oint_S \vec{P}.d\vec{S} = -\frac{\partial}{\partial t}\int_V udV - \int_V \vec{J}.\vec{E}\,dV$$

$-\frac{\partial}{\partial t}\int_V udV = \int_V \vec{J}.\vec{E}\,dV + \oint_S \vec{P}.d\vec{S}$…………………(63)

The vector $\vec{P}$ is called Poynting vector and eqn. (63) is known as Poynting theorem.

Let us interpret the terms of eqn. (63) one by one.

Interpretation of $\frac{\partial}{\partial t}\int_V udV$: −Rate of decrease of energy stored (energy stored in electric field and energy stored in magnetic field) in volume V due to electric and magnetic field

Interpretation of $\int_V \vec{J}.\vec{E}\,dV$:-the rate of energy transferred into the electromagnetic field through the motion of charges in the volume V ,i.e the total power dissipated in a volume V

We can write $\vec{J}.\vec{E} = \sigma\vec{E}.\vec{E} = \sigma E^2 = \frac{l}{RA}\frac{V^2}{l^2} = \frac{V^2}{R}\times\frac{1}{Al} = \frac{P^2}{V} = \frac{Power}{Volume}$

Thus $\vec{J}.\vec{E}$ represents power dissipated per unit volume. Hence $\int_V \vec{J}.\vec{E}\,dV$ represents the total power dissipated in a volume V.

Interpretation of $\oint \vec{P}.\overrightarrow{ds}$

Here, $\vec{P} = \vec{E}\times\vec{H}$ →rate of flow of energy through a closed surface. The direction of energy flow is perpendicular to both $\vec{E}$ and $\vec{H}$. $\oint \vec{P}.\overrightarrow{ds}$: rate of flow of electromagnetic energy through the surface S, enclosing the volume V.

Alternately Poynting theorem states that “The rate of decrease of total electromagnetic energy within a certain volume is equal to the rate of dissipation of energy in that volume plus the rate of energy flowing out of the surface enclosing the volume.”

Poynting Vector:-

Poynting vector is defined as the cross product of electric field and magnetic field, i.e.;

$$\vec{P} = \vec{E} \times \vec{H}$$

Poynting vector measures the electrical power per unit area, because

$$|\vec{E} \times \vec{H}| = \frac{V}{l} \times \frac{I}{l} = \frac{VI}{l^2} = \frac{power}{area}$$

Its direction is perpendicular to both $\vec{E}$ and $\vec{H}$. It is also called flux vector of electromagnetic field.

Waveguides

Waveguide is a hollow metal pipe used for guiding the waves. The electromagnetic waves in such waveguides may be imagined as waves travelling down the guide in a zig-zag path as these waves are repeatedly reflected between opposite walls of the guide.

To function properly , a waveguide must have a certain minimum diameter relative to the wavelength of the signal. If the waveguide is too narrow or the frequency is too low, the electromagnetic waves cannot propagate. There is minimum frequency, known as cut-off frequency for the propagation of the wave, i.e a wave can propagate only if its frequency is larger than the cut-off frequency. The cut-off frequency is decided by the dimension of the waveguide.

Modes in waveguides:

In order to analyse the mode(wave) propagation in the waveguide, Maxwell's equations along with appropriate bounding conditions determined by the properties of the materials and their interface are to be solved. These equations admit multiple solutions, or modes, which are origin functions of the equation system. The propagation of the waveguide modes depends on the operating wavelength and polarization, and shape and size of the waveguide. The longitudinal mode can be realised in a cavity (closed end waveguide), the longitudinal mode is particularly standing wave pattern formed by the waves confined in the cavity. Number of traversed modes can be excited in the waveguide, which ae classified below.

(a) Transverse electric mode: These modes do not have electric field in the direction of propagation. So electric field vector is in transverse direction.

(b) Transverse magnetic modes: These modes have no magnetic field in the direction of propagation. So magnetic field vector is in transverse direction.

(c) Transverse electromagnetic modes: Thes modes have no electric and magnetic fields in the direction of mode of propagation. In hollow waveguide, TEM modes are not possible because as per Maxwell's equation the electric field then must have zero divergence, zero curl and be zero boundaries. This will result in a zero field. TEM modes can propagate in a co-axialcable.

(d)These modes have both electric and magnetic field components in the direction of propagation. The mode for which the cut-off frequency is the minimum is called the fundamental mode.

Examples:

Example-1: Find the gradient of $|\vec{r}|$, where $\vec{r}$ is the position vector.

Solution:

We know that position vector is

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

Now $\vec{\nabla}|\vec{r}| = \vec{\nabla}(x^2+y^2+z^2)^{1/2}$

$$=\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$$

$$=\frac{2x\hat{i}}{2(x^2+y^2+z^2)^{1/2}}+\frac{2y\hat{j}}{2(x^2+y^2+z^2)^{1/2}}+\frac{2z\hat{k}}{2(x^2+y^2+z^2)^{1/2}}$$

$$=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{1/2}}=\frac{\vec{r}}{|\vec{r}|}=\hat{r}$$

Example-2: Find the gradient of $\left(\frac{1}{r}\right)$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution:

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = (x^2+y^2+z^2)^{1/2}$

$\left(\frac{1}{r}\right) = (x^2+y^2+z^2)^{-1/2}$

$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

$$=\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \qquad \text{......................(1)}$$

Now,

$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$= -xr^{-3} = \frac{-x}{r^3}$

Similarly;

$\frac{\partial}{\partial y}\left(\frac{1}{r}\right)=\frac{-y}{r^3}$

And $\frac{\partial}{\partial z}\left(\frac{1}{r}\right)=\frac{-z}{r^3}$

Putting the values in eqn(1) we get,

$$\vec{\nabla}\left(\frac{1}{r}\right)=\hat{i}\frac{-x}{r^3}+\hat{j}\frac{-y}{r^3}+\hat{k}\frac{-z}{r^3}=\frac{-1}{r^3}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$=\frac{-\vec{r}}{r^3}$$

Example-3: If $\emptyset=4xy^2+2yz^2$ then find gradient of the above function at (1,2,3).

Solution:

Gradient $\emptyset = \nabla\emptyset$

$$=\hat{i}\frac{\partial\emptyset}{\partial x}+\hat{j}\frac{\partial\emptyset}{\partial y}+\hat{k}\frac{\partial\emptyset}{\partial z}$$

$$=\hat{i}\frac{\partial}{\partial x}(4xy^2+2yz^2)+\hat{j}\frac{\partial}{\partial y}(4xy^2+2yz^2)+\hat{k}\frac{\partial}{\partial z}(4xy^2+2yz^2)$$

$= \hat{\imath}4y^2 + \hat{\jmath}(8xy + 2z^2) + \hat{k}4yz$

$(\vec{\nabla}\emptyset)_{1,2,3} = 16\hat{\imath} + 34\hat{\jmath} + 24\hat{k}$

Example-4: Find the divergence of a position vector.

Solution:

The position vector $\vec{r}$ is given by;

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now div$\vec{r} = \vec{\nabla}.\vec{r} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$

Example-5: Find the curl of a position vector.

Solution:

The position vector $\vec{r}$ is given by;

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Now Curl$(\vec{r}) = \vec{\nabla} \times \vec{r}$

$= (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \times (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$= \hat{\imath}\left[\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right] - \hat{\jmath}\left[\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right] + \hat{k}\left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]$

$= \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(0) = 0$

So $\vec{\nabla} \times \vec{r} = 0$

Thus Position vector is irrotational.

Example-6: Show that the motion describing by a gradient field is irrotational.

Solution:

Let us consider a scalar field $\emptyset(x, y, z)$.

The gradient field is given by;

grad $\emptyset = \hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z} = \vec{F}$ (Say)

Now, curl $\vec{F} = \vec{\nabla} \times \vec{F}$

$= (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \times (\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z})$

$$=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \emptyset}{\partial x} & \frac{\partial \emptyset}{\partial y} & \frac{\partial \emptyset}{\partial z} \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}\frac{\partial \emptyset}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \emptyset}{\partial y}\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}\frac{\partial \emptyset}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \emptyset}{\partial x}\right] + \hat{k}\left[\frac{\partial}{\partial x}\frac{\partial \emptyset}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \emptyset}{\partial x}\right]$$

$\Rightarrow \vec{\nabla} \times \vec{F} = \hat{\imath}(0)+\hat{\jmath}(0)+\hat{k}(0) = 0$

$\Rightarrow \vec{\nabla} \times (\overrightarrow{\nabla \emptyset}) = 0$

$\Rightarrow Curl(\overrightarrow{\nabla \emptyset})=0$

Thus gard $\emptyset$ is irrotaional.

Example-7: Find out the dimensions of poynting vector.

Solution:

Poynting vector (P) is given by $\vec{p}= \vec{E} \times \vec{H}$

Dimension of $\vec{E}$

$\vec{E} = \frac{\vec{F}}{q}$

Thus $[E] = \frac{[MLT^{-2}]}{[AT]} = [MLT^{-3}A^{-1}]$

Dimension of $\vec{H}$, H=$\frac{B}{\mu} = \frac{1}{\mu}\frac{\mu}{4\pi}\int\frac{Idl\ Sin\theta}{r^2}$

Thus $[H] = \frac{[AL]}{[L^2]} = [AL^{-1}]$

Dimension of $[P]$, $[P]= [MLT^{-3}A^{-1}][AL^{-1}]=[MT^{-3}]$

Example-8: Assuming radius of the sum to be 7×10^5 km and the power radiated by it to be 4×10^{26}watt, calculate the value of the pointing vector at the surface of the sum.

Solution:

Given that Power= 4×10^{26}watt

Radius(R)=7×10^5Km=7×10^8m

Poynting Vector(P)=$\frac{Power}{Area}$=$\frac{Power}{4\pi R^2}$ =$\frac{4 \times 10^{26}}{4\pi(7 \times 10^8)^2}$ = 6.5×10^7Watt/m^2

Example-9: Electric field associated with a charge body is given by $\vec{E}$= $4\hat{\imath} + \hat{\jmath} + 7\hat{k}$ and the surface is given by $\vec{S} = 8\hat{\imath} + 3\hat{\jmath}$. Calculate the electricflux coming out of the surface.

Solution:

Let $\emptyset$ = flux coming out if the surface at any constant.

Let $\emptyset$= $\vec{E}.\vec{S}$ =$(4\hat{\imath} + \hat{\jmath} + 7\hat{k}).(8\hat{\imath} + 3\hat{\jmath})$ = 32+3 = 35 Units

Example-10: A charged having charge 5.3$\times$ 10^{-3} C is placed inside cube. Calculate the amount of flux coming out of any side of the cube.

Solution:

According to Gauss law in electrostatic the flux linked with the entire cube is

$\emptyset = \frac{q}{\varepsilon_0}$

Thus flux linked with one face is

$\emptyset = \frac{1}{6}\left(\frac{q}{\varepsilon_0}\right)$

$=> \emptyset = \frac{1}{6}\left(\frac{5.3\times10^{-5}}{8.85\times10^{-12}}\right) = 10^6 Nm^2/coul^2$

Example-11: Medium is characterized by relative permittivity ϵ_r= 45 and relative permeability μ_r=5,Calculate the speed of the electromagnetic wave in the medium and the refractive index of the medium.

Solution:

Speed of em.wave in any medium = $\nu = \frac{1}{\sqrt{\mu\epsilon}}$

Refractive index of the medium $\mu = \frac{Velocity\ \ in\ Vaccum}{Velocity\ \ in\ other\ medium} = \frac{c}{\nu} = \frac{\frac{1}{\sqrt{\mu_0 \in_0}}}{\frac{1}{\sqrt{\mu\epsilon}}} = \sqrt{\mu_r \in_r}$

$$= \sqrt{5 \times 45} = 15$$

$\frac{\mu}{\mu_0} = \mu_r$, $\frac{\in}{\in_0} = \in_r$

Speed of em wave in any medium $\nu = \frac{c}{\mu}$ =0.2$\times$ 10^8m/sec

Example-12: A laser beam from 100 watt source is focused on an area of $10^{-8}m^2$. Evaluate the magnitude of the poynting vector of the area.

Solution:

$|\vec{S}| = \frac{Power}{Area} = \frac{100 watt}{10^{-8}m^2} = 10^{10}\ W/m^2$