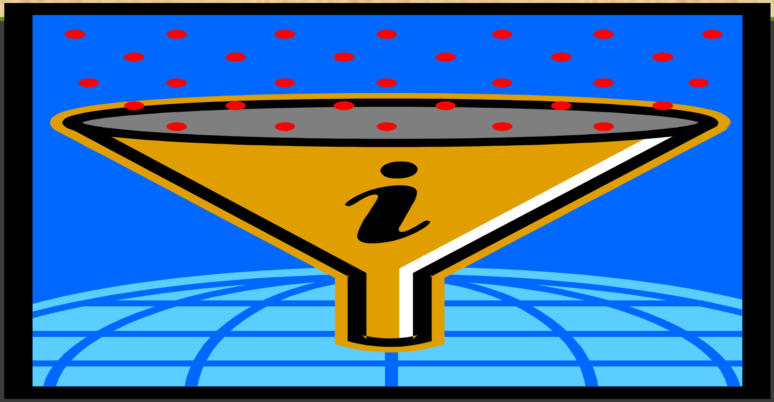
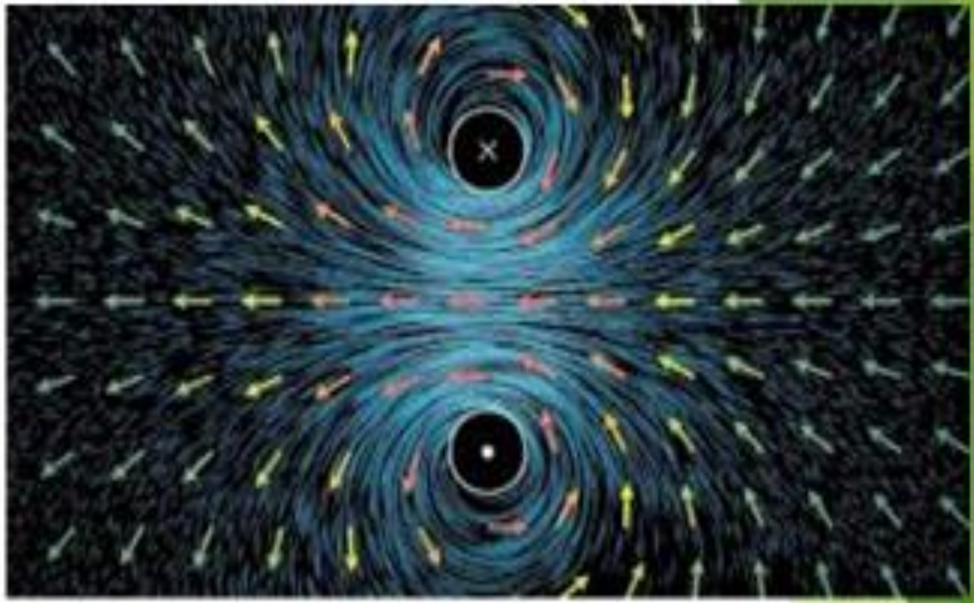
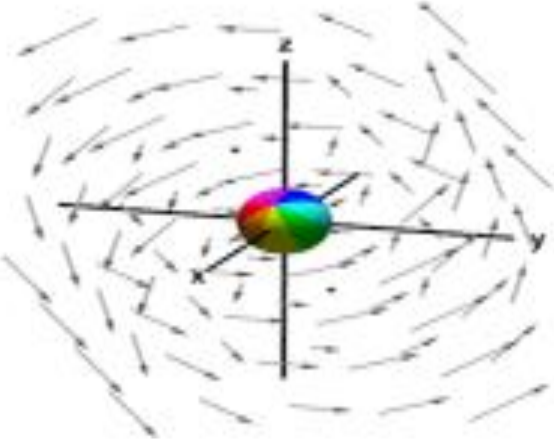
MATHEMATICS - I

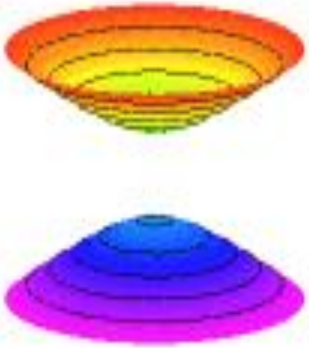
LECTURE - 26- CURL OF A VECTOR FIELD



REAL LIFE APPLICATION







$\nabla \times \mathbf{F}$

curl $(\mathbf{F}) \nabla \times \mathbf{F}$

EXPLANATION

- The curl of a vector field is defined as the vector field having magnitude equal to the maximum "circulation" at each point and to be oriented perpendicularly to this plane of circulation for each point.

- The curl of a vector field measures the tendency for the vector field to swirl around. Imagine that the vector field represents the velocity vectors of water in a lake. If the vector field swirls around, then when we stick a paddle wheel into the water, it will tend to spin. The amount of the spin will depend on how we orient the paddle. Thus, we should expect the curl to be vector valued.

- In vector calculus, the **curl** is a vector operator that describes the infinitesimal circulation of a vector field in three-dimensional Euclidean space. The curl at a point in the field is represented by a vector whose length and direction denote the magnitude and axis of the maximum circulation. The curl of a field is formally defined as the circulation density at each point of the field.

DEFINITION OF CURL

The *curl* of a vector differentiable function

$\vec{v}(x, y, z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is a vector function defined as

$$curl\,\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

AN EXAMPLE ON CURL

Example: Find the curl of $\vec{v} = \frac{1}{2}(x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})$

SOLUTION:

Given vector function is $\vec{v} = \frac{1}{2}(x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})$

$$= \frac{1}{2}(x^2 + y^2 + z^2)\hat{i} + \frac{1}{2}(x^2 + y^2 + z^2)\hat{j} + \frac{1}{2}(x^2 + y^2 + z^2)\hat{k}$$

Let $v_1 = \frac{1}{2}(x^2 + y^2 + z^2), v_2 = \frac{1}{2}(x^2 + y^2 + z^2), v_3 = \frac{1}{2}(x^2 + y^2 + z^2)$

AN EXAMPLE ON CURL

Therefore, $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

By definition, $curl\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)\hat{k}$$

AN EXAMPLE ON CURL

We have

$$\frac{\partial v_1}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{2}(x^2+y^2+z^2)\right)=x$$

$$\frac{\partial v_1}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{2}(x^2+y^2+z^2)\right)=y$$

$$\frac{\partial v_1}{\partial z}=\frac{\partial}{\partial z}\left(\frac{1}{2}(x^2+y^2+z^2)\right)=z$$

$$\frac{\partial v_2}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{2}(x^2+y^2+z^2)\right)=x$$

$$\frac{\partial v_2}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{2}(x^2+y^2+z^2)\right)=y$$

$$\frac{\partial v_2}{\partial z}=\frac{\partial}{\partial z}\left(\frac{1}{2}(x^2+y^2+z^2)\right)=z$$

AN EXAMPLE ON CURL

$$\frac{\partial v_3}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}(x^2 + y^2 + z^2)\right) = x$$

and $\dfrac{\partial v_3}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}(x^2 + y^2 + z^2)\right) = y$

$$\frac{\partial v_3}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{2}(x^2 + y^2 + z^2)\right) = z$$

Therefore, $curl\vec{v} = (y - z)\hat{i} + (z - x)\hat{j} + (x - y)\hat{k}$

PHYSICAL SIGNIFICANCE OF CURL

The curl of a vector function at a point *P* gives the angular velocity at that point of the vector field. A vector function is said to be **irrational** if

$$curl\,\vec{v} = \vec{0}$$

at every point otherwise it is rotational.

SOME USEFUL RESULTS ON CURL

$$1.\ curl(\vec{u}+\vec{v}) = curl\,\vec{u} + curl\,\vec{v}$$

PROOF:

Let $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\therefore \vec{u}+\vec{v} = \left(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}\right) + \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= (u_1 + v_1)\hat{i} + (u_2 + v_2)\hat{j} + (u_3 + v_3)\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$curl(\vec{u}+\vec{v}) = \vec{\nabla}\times(\vec{u}+\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1+v_1 & u_2+v_2 & u_3+v_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(u_3+v_3) - \frac{\partial}{\partial z}(u_2+v_2)\right]\hat{i}$$

$$+ \left[\frac{\partial}{\partial z}(u_1+v_1) - \frac{\partial}{\partial x}(u_3+v_3)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(u_2+v_2) - \frac{\partial}{\partial y}(u_1+v_1)\right]\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$= \left[\left(\frac{\partial u_3}{\partial y} + \frac{\partial v_3}{\partial y} \right) - \left(\frac{\partial u_2}{\partial z} + \frac{\partial v_2}{\partial z} \right) \right] \hat{i}$$

$$+ \left[\left(\frac{\partial u_1}{\partial z} + \frac{\partial v_1}{\partial z} \right) - \left(\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial x} \right) \right] \hat{j}$$

$$+ \left[\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial x} \right) - \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial y} \right) \right] \hat{k}$$

SOME USEFUL RESULTS ON CURL

$$=\left[\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right)\hat{i}+\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)\hat{j}+\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)\hat{k}\right]$$

$$+\left[\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}\right]$$

$$=curl\,\vec{u}+curl\,\vec{v}$$

SOME USEFUL RESULTS ON CURL

2. $div(curl\vec{v}) = 0$

PROOF:

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

By definition, $curl\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$div\left(curl\vec{v}\right)=\vec{\nabla}\bullet\left(curl\vec{v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)$$

$$+\frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)$$

$$=\frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial x\partial z}+\frac{\partial^2 v_1}{\partial y\partial z}-\frac{\partial^2 v_3}{\partial y\partial x}+\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_1}{\partial z\partial y}$$

$$=\boxed{\frac{\partial^2 v_3}{\partial x\partial y}}-\boxed{\frac{\partial^2 v_2}{\partial z\partial x}}+\boxed{\frac{\partial^2 v_1}{\partial y\partial z}}-\boxed{\frac{\partial^2 v_3}{\partial x\partial y}}+\boxed{\frac{\partial^2 v_2}{\partial z\partial x}}-\boxed{\frac{\partial^2 v_1}{\partial y\partial z}}$$

$$=0$$

SOME USEFUL RESULTS ON CURL

3. $curl(f\vec{v}) = f\, curl\vec{v} + \vec{\nabla} f \times \vec{v}$

PROOF: Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

By definition, $curl\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$f\vec{v} = (fv_1)\hat{i} + (fv_2)\hat{j} + (fv_3)\hat{k}$$

$$curl\left(f\vec{v}\right) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fv_1 & fv_2 & fv_3 \end{vmatrix}$$

$$= \left(\frac{\partial (fv_3)}{\partial y} - \frac{\partial (fv_2)}{\partial z}\right)\hat{i} + \left(\frac{\partial (fv_1)}{\partial z} - \frac{\partial (fv_3)}{\partial x}\right)\hat{j} + \left(\frac{\partial (fv_2)}{\partial x} - \frac{\partial (fv_1)}{\partial y}\right)\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$= \left(f\frac{\partial v_3}{\partial y} + v_3\frac{\partial f}{\partial y} - f\frac{\partial v_2}{\partial z} - v_2\frac{\partial f}{\partial z} \right)\hat{i}$$

$$+ \left(f\frac{\partial v_1}{\partial z} + v_1\frac{\partial f}{\partial z} - f\frac{\partial v_3}{\partial x} - v_3\frac{\partial f}{\partial x} \right)\hat{j}$$

$$\left(f\frac{\partial v_2}{\partial x} + v_2\frac{\partial f}{\partial x} - f\frac{\partial v_1}{\partial y} - v_1\frac{\partial f}{\partial y} \right)\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$= f\left[\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}\right]$$

$$+\left[\left(v_3\frac{\partial f}{\partial y}-v_2\frac{\partial f}{\partial z}\right)\hat{i}+\left(v_1\frac{\partial f}{\partial z}-v_3\frac{\partial f}{\partial x}\right)\hat{j}+\left(v_2\frac{\partial f}{\partial x}-v_1\frac{\partial f}{\partial y}\right)\hat{k}\right]$$

$$= f\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix} \quad \boxed{= f\, curl\vec{v}+\vec{\nabla} f\times\vec{v}}$$

SOME USEFUL RESULTS ON CURL

$$4.\ div(\vec{u}\times\vec{v})=\vec{v}.curl\vec{u}-\vec{u}.curl\vec{v}$$

PROOF : Let $\vec{u}=\mathrm{u}_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ and $\vec{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

$$\therefore \vec{u}\times\vec{v}=\left(\mathrm{u}_1\hat{i}+u_2\hat{j}+u_3\hat{k}\right)\times\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \mathrm{u}_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\left(u_2v_3-u_3v_2\right)\hat{i}+\left(u_3v_1-\mathrm{u}_1v_3\right)\hat{j}+\left(\mathrm{u}_1v_2-u_2v_1\right)\hat{k}$$

SOME USEFUL RESULTS ON CURL

$$\therefore\ div(\vec{u}\times\vec{v})=\vec{\nabla}.(\vec{u}\times\vec{v})$$

$$=\frac{\partial}{\partial x}\left(u_2v_3-u_3v_2\right)+\frac{\partial}{\partial y}\left(u_3v_1-\mathrm{u}_1v_3\right)+\frac{\partial}{\partial z}\left(\mathrm{u}_1v_2-u_2v_1\right)$$

$$=u_2\frac{\partial v_3}{\partial x}+v_3\frac{\partial u_2}{\partial x}-u_3\frac{\partial v_2}{\partial x}+v_2\frac{\partial u_3}{\partial x}$$

$$+u_3\frac{\partial v_1}{\partial y}+v_1\frac{\partial u_3}{\partial y}-u_1\frac{\partial v_3}{\partial y}+v_3\frac{\partial u_1}{\partial y}$$

$$+u_1\frac{\partial v_2}{\partial z}+v_2\frac{\partial u_1}{\partial z}-u_2\frac{\partial v_1}{\partial z}+v_1\frac{\partial u_2}{\partial z}$$

SOME USEFUL RESULTS ON CURL

$$= u_1\left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y}\right) + u_2\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + u_3\left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right)$$

$$+ v_1\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + v_2\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + v_3\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$

$$= \left(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}\right) \bullet \left(\left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y}\right)\hat{i} + \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\hat{j} + \left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right)\hat{k}\right)$$

$$+ \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right) \bullet \left(\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\hat{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right)\hat{j} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\hat{k}\right)$$

SOME USEFUL RESULTS ON CURL

$$=\left(u_1\hat{i}+u_2\hat{j}+u_3\hat{k}\right)\bullet\left(-\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}-\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}-\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}\right)$$

$$+\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)\bullet\left(\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right)\hat{i}+\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)\hat{j}+\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)\hat{k}\right)$$

$$=-\left(u_1\hat{i}+u_2\hat{j}+u_3\hat{k}\right)\bullet\left(\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}\right)$$

$$+\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)\bullet\left(\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right)\hat{i}+\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)\hat{j}+\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)\hat{k}\right)$$

$$=\vec{u}.curl\vec{v}-\vec{v}.curl\vec{u}$$

$$=\vec{v}.curl\vec{u}-\vec{u}.curl\vec{v}$$

SOME USEFUL RESULTS ON CURL

5. $curl(grad\ f) = \vec{0}$

PROOF: We know that $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$\Rightarrow curl(grad\ f) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$

SOME USEFUL RESULTS ON CURL

$$=\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)\hat{i}+\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)\hat{j}+\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)\hat{k}$$

$$=\vec{0}$$

PROBLEMS FOR PRACTICE

1. Find the curl of $\vec{v} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$

SOLUTION

•

Given vector function is $\vec{v} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$

$$= x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$$

$$So\ curl\,\vec{V} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{bmatrix}$$

PROBLEMS FOR PRACTICE

$$= \left(\frac{\partial (xyz^2)}{\partial y} - \frac{\partial (xy^2 z)}{\partial z} \right) \hat{i}$$

$$+ \left(\frac{\partial (x^2 yz)}{\partial z} - \frac{\partial (xyz^2)}{\partial x} \right) \hat{j}$$

$$+ \left(\frac{\partial (xy^2 z)}{\partial x} - \frac{\partial (x^2 yz)}{\partial y} \right) \hat{k}$$

$$= (xz^2 - xy^2)\hat{i} + (x^2 y - yz^2)\hat{j} + (y^2 z - x^2 z)\hat{k}$$

PROBLEMS FOR PRACTICE

2.Examine whether the flow is irrotational or incompressible

SOLUTON:

where the velocity vector is $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow curl\,\vec{v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = \vec{0}$$

So the flow is irrotational.

$$divl\,\vec{v} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

So the flow is not incompressible.

PROBLEMS FOR PRACTICE

3.Find the value of $\vec{\nabla}.(\vec{\nabla} f \times \vec{\nabla} g)$

SOLUTION:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\vec{\nabla} g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$$

PROBLEMS FOR PRACTICE

$$\vec{\nabla}.(\vec{\nabla} f \times \vec{\nabla} g) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} \right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \right)$$

PROBLEMS FOR PRACTICE

$$
\begin{aligned}
&= \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial z}\frac{\partial g}{\partial y} \\
&+ \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial y \partial x} + \frac{\partial^2 f}{\partial y \partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial g}{\partial z} \\
&+ \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial z \partial y} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y}\frac{\partial g}{\partial x}
\end{aligned}
$$

PROBLEMS FOR PRACTICE

$$
\begin{aligned}
&= \boxed{\frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z\partial x}} + \boxed{\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z}} - \boxed{\frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x\partial y}} - \boxed{\frac{\partial^2 f}{\partial z\partial x}\frac{\partial g}{\partial y}} \\
&+ \boxed{\frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x\partial y}} + \boxed{\frac{\partial^2 f}{\partial y\partial z}\frac{\partial g}{\partial x}} - \boxed{\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y\partial z}} - \boxed{\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z}} \\
&+ \boxed{\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y\partial z}} + \boxed{\frac{\partial^2 f}{\partial z\partial x}\frac{\partial g}{\partial y}} - \boxed{\frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z\partial x}} - \boxed{\frac{\partial^2 f}{\partial y\partial z}\frac{\partial g}{\partial x}}
\end{aligned}
$$

$$= 0$$

ASSIGNMENTS

1. Find the $curl$ of $\vec{v} = yz\,\hat{i} + 3zx\,\hat{j} + z\,\hat{k}$ $at\ P(2,3,4)$

2. Show that $div(curl\,\vec{v}) = 0.$

3. Check the flow rotational or irrotational whose velocity vector is

$$\vec{\mathrm{v}} = \sec x\,\hat{i} + \cos ecx\,\hat{\mathrm{j}}$$

4. Find $curl(f\,\vec{u})$ where f=xyz and $\vec{\mathrm{u}} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.

5. Prove that $\vec{\nabla} \times \vec{r} = \vec{0}.$