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# Vector Calculus

## 1.0 INTRODUCTION

In Science and Engineering we often deal with the analysis of forces and velocities and other quantities which are vectors. These vectors are not constants but vary with position and time. Hence they are functions of one or more variables.

Vector Calculus extends the concepts of differential calculus and integral calculus of real functions in an interval to vector functions and thus enabling us to analyse problems over curves and surfaces in three dimension. Vector Calculus finds applications in a wide variety of fields such as fluid flow, heat flow, solid mechanics, electrostatics etc.

In Vector Calculus we deal mainly with two kinds of functions, scalar point functions and vector point functions and their fields.

### 1.1 SCALAR AND VECTOR POINT FUNCTIONS

#### Definition 1.1

If to each point  $P(\vec{r})$  (the point  $P$  with position vector  $\vec{r}$ ) of a region  $R$  in space there is a unique scalar or real number denoted by  $\phi(\vec{r})$ , then  $\phi$  is called a **scalar point function in  $R$** . The region  $R$  is called a **scalar field**.

#### Definition 1.2

If to each point  $P(\vec{r})$  of a region  $R$  in space there is a unique vector denoted by  $\vec{F}(\vec{r})$ , then  $\vec{F}$  is called a **vector point function in  $R$** . The region  $R$  is called a **vector field**.

**Note**

1. In applications, the domain of definition of point functions may be points in a region of space, points on a surface or points on a curve.
2. If we introduce cartesian coordinate system, then  $\vec{r} = \vec{x}\hat{i} + \vec{y}\hat{j} + \vec{z}\hat{k}$  or  $\vec{r} = (x, y, z)$  and instead of  $\vec{F}(\vec{r})$  and  $\phi(\vec{r})$  we can write  $\vec{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$  or  $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$  and  $\phi(\vec{r})$  as  $\phi(x, y, z)$
3. A vector or scalar field that has a geometrical or physical meaning should depend only on the points  $P$  where it is defined but not on the particular choice of the cartesian coordinates. In otherwords, the scalar and vector fields have the property of invariance under a transformation of space coordinates.

**Examples of scalar field**

1. Temperature  $T$  within a body is scalar field, namely temperature field.
2. When an iron bar is heated at one end, the temperature at various points will attain a steady state and the temperature will depend only on the position.
3. The pressure of air in earth's atmosphere is a scalar field called pressure field.
4.  $\phi(x, y, z) = x^3 + y^3 + z^3 - 3xyz$  defines a scalar field.

**Examples of vector field**

1. The velocity of a moving fluid at any instant is a vector point function and defines a vector field.
2. Earth's magnetic field is a vector field.
3. Gravitational force on a particle in space defines a vector field.
4.  $\vec{F}(x, y, z) = x^2\hat{i} - y^2\hat{j} + z\hat{k}$  defines a vector field.

**Note** Vector and scalar functions may also depend on time or on other parameters.

### Definition 1.3 Derivative of a vector function

A vector function  $\vec{f}(t)$  is said to be differentiable at a point  $t$ , if  $\lim_{\Delta t \rightarrow 0} \frac{\vec{f}(t + \Delta t) - \vec{f}(t)}{\Delta t}$  exists. Then it is denoted by  $\frac{d\vec{f}}{dt}$  or  $\vec{f}'$  and is called the derivative of the vector function  $\vec{f}$  at  $t$ .

#### Note

1. If  $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$  then  $\vec{f}(t)$  is differentiable at  $t$  if and only if its components  $f_1(t), f_2(t), f_3(t)$  are differentiable at  $t$  and  $\frac{d\vec{f}(t)}{dt} = f'_1(t)\vec{i} + f'_2(t)\vec{j} + f'_3(t)\vec{k}$
2. If the derivative of  $\frac{d\vec{f}}{dt}$  w.r.to  $t$  exists, it is denoted by  $\frac{d^2\vec{f}}{dt^2}$ . Similarly, we denote higher derivatives.
3. If  $\vec{c}$  is a constant vector, then  $\frac{d\vec{c}}{dt} = \vec{0}$ .  
For  $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$  and  $\frac{d\vec{c}}{dt} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$ .

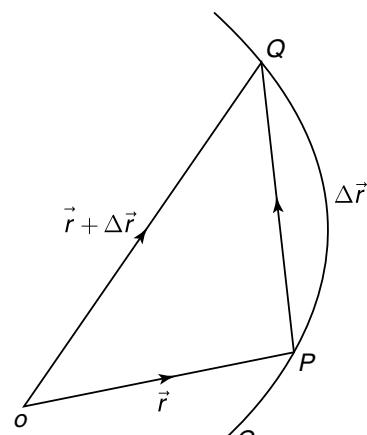
### Geometrical meaning of derivative

Let  $\vec{r}(t)$  be the position vector of a point  $P$  with respect to the origin  $O$ . As  $t$  varies continuously over a time interval  $P$  traces the curve  $C$ . Thus the vector function  $\vec{r}(t)$  represents a curve  $C$  in space.

Let  $\vec{r}$  and  $\vec{r} + \Delta\vec{r}$  be the position vectors of neighbouring points  $P$  and  $Q$  on the curve  $C$ .

$$\begin{aligned}\text{Then } \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \vec{r} + \Delta\vec{r} - \vec{r} \\ &= \Delta\vec{r}\end{aligned}$$

$\therefore \frac{\Delta\vec{r}}{\Delta t}$  is along the chord  $PQ$ .



If  $\lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t}$  exists, it is denoted by  $\frac{d\vec{r}}{dt}$  and  $\frac{d\vec{r}}{dt}$  is in the direction of the tangent at  $P$  to the curve.

If  $\frac{d\vec{r}}{dt} \neq 0$ , then  $\frac{d\vec{r}}{dt}$  or  $\vec{r}'(t)$  is called a tangent vector to the curve  $C$  at  $P$ .

The unit tangent vector at  $P$  is  $= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \hat{u}(t)$ .

Both  $\vec{r}'(t)$  and  $\hat{u}(t)$  are in the direction of increasing  $t$ . Hence their sense depends on the orientation of the curve  $C$ .

## 1.2 DIFFERENTIATION FORMULAE

If  $\vec{f}$  and  $\vec{g}$  are differentiable vector function of  $t$  and  $\phi$  is a scalar function of  $t$  then

$$1. \quad \frac{d}{dt}(\vec{f} \pm \vec{g}) = \frac{d\vec{f}}{dt} \pm \frac{d\vec{g}}{dt}$$

$$2. \quad \frac{d}{dt}(\phi \vec{f}) = \phi \frac{d\vec{f}}{dt} + \frac{d\phi}{dt} \vec{f}$$

$$3. \quad \frac{d}{dt}(\vec{f} \cdot \vec{g}) = \vec{f} \cdot \frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt} \cdot \vec{g}$$

$$4. \quad \frac{d}{dt}(\vec{f} \times \vec{g}) = \vec{f} \times \frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt} \times \vec{g}$$

$$5. \quad \frac{d}{dt}(\vec{f} \cdot \vec{g} \times \vec{h}) = \frac{d\vec{f}}{dt} \cdot \vec{g} \times \vec{h} + \vec{f} \cdot \frac{d\vec{g}}{dt} \times \vec{h} + \vec{f} \cdot \vec{g} \times \frac{d\vec{h}}{dt}.$$

**Note** If  $\vec{f}$  is a continuous function of a scalar  $s$  and  $s$  is a continuous function of  $t$ , then  $\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{ds} \frac{ds}{dt}$ .

6. Let  $\vec{f}(t)$  be a vector function.  $\vec{f}(t)$  changes if its magnitude is changed or its direction is changed or both magnitude and direction are changed. We shall find conditions under which a vector function will remain constant in magnitude or in direction.

(i) Let  $\vec{f}(t)$  be a vector of constant length  $k$ .

$$\text{Then } \vec{f} \cdot \vec{f} = |\vec{f}|^2 = k^2$$

Differentiating w.r.to  $t$ , we get

$$\begin{aligned} & \frac{d\vec{f}}{dt} \cdot \vec{f} + \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \\ \Rightarrow & 2\vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \\ \therefore & \frac{d\vec{f}}{dt} = \vec{0} \quad \text{or} \quad \frac{d\vec{f}}{dt} \text{ is } \perp \text{ to } \vec{f}. \end{aligned}$$

(ii) Let  $\vec{f}(t)$  be a vector function with constant direction and let  $\vec{a}$  be the unit vector in that direction

$$\text{Then } \vec{f}(t) = \phi \vec{a}, \text{ where } \phi = |\vec{f}|$$

$$\therefore \frac{d\vec{f}}{dt} = \frac{d\phi}{dt} \vec{a} + \phi \frac{d\vec{a}}{dt}.$$

But  $\vec{a}$  is a constant vector, since its direction is fixed and magnitude is 1.

$$\begin{aligned} & \therefore \frac{d\vec{a}}{dt} = \vec{0} \\ & \therefore \frac{d\vec{f}}{dt} = \frac{d\phi}{dt} \vec{a} \end{aligned}$$

$$\begin{aligned} & \text{Now } \vec{f} \times \frac{d\vec{f}}{dt} = \phi \vec{a} \times \frac{d\phi}{dt} \vec{a} \\ \Rightarrow & \vec{f} \times \frac{d\vec{f}}{dt} = \phi \frac{d\phi}{dt} \vec{a} \times \vec{a} = \vec{0} \quad (\because \vec{a} \times \vec{a} = \vec{0}) \\ \therefore & \frac{d\vec{f}}{dt} = \vec{0} \quad \text{or} \quad \frac{d\vec{f}}{dt} \text{ is parallel to } \vec{f}. \end{aligned}$$

### 1.3 LEVEL SURFACES

Let  $\phi$  be a continuous scalar point function defined in a region  $R$  in space. Then the set of all points satisfying the equation  $\phi(x, y, z) = C$ , where  $C$  is a constant, determines a surface which is called a **level surface** of  $\phi$ . At every point on a level surface then function  $\phi$  takes the same value  $C$ . If  $C$  is an arbitrary constant, the for different values of  $C$ , we get different level surfaces

of  $\phi$ . No two level surfaces intersect. For, if  $\phi = C_1$  and  $\phi = C_2$  be two level surfaces of  $\phi$  intersecting at a point  $P$ . Then  $\phi(P) = C_1$  and  $\phi(P) = C_2$  and so  $\phi$  has two values at  $P$  which contradicts the uniqueness of value of the function  $\phi$ . So,  $\phi = C_1$  and  $\phi = C_2$  do not intersect.

Thus **only one level surface of  $\phi$  passes through a given point**

For example, if  $\phi(x, y, z)$  represents the temperature of  $(x, y, z)$  in a region  $R$  of space, then the level surfaces of equal temperature are called **isothermal surfaces**.

## 1.4 GRADIENT OF A SCALAR POINT FUNCTION OR GRADIENT OF A SCALAR FIELD

### 1.4.1 Vector differential operator

The symbolic vector  $\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$  is called **Hamiltonian operator** or **vector differential operator** and is denoted by  $\nabla$  (read as del or nabla).

$$\therefore \nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}.$$

It is also known as del operator. This operator can be applied on a scalar point function  $\phi(x, y, z)$  or a vector point function  $\vec{F}(x, y, z)$  which are differentiable functions. This gives rise to three field quantities namely gradient of a scalar, divergence of a vector and curl of a vector function.

### Definition 1.4 Gradient

If  $\phi(x, y, z)$  is a scalar point function continuously differentiable in a given region  $R$  of space, then the gradient of  $\phi$  is defined by  $\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$ .

It is abbreviated as grad  $\phi$ . Thus      grad  $\phi = \nabla\phi$ .

**Note** Since  $\nabla\phi$  is a vector, the gradient of a scalar point function is always a vector point. Thus  $\nabla\phi$  defines a vector field.

Gradient is of great practical importance because some of the vector fields in applications can be obtained from scalar fields and scalar fields are easy to handle.

### 1.4.2 Geometrical meaning of $\nabla\phi$

Let  $\phi(x, y, z)$  be a scalar point function. Let  $\phi(x, y, z) = C$  be a level surface of  $\phi$ . Let  $P$  be a point on this surface with position vector  $\vec{r} = \vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}$ .

Then the differential  $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$  is tangent to the surface at  $P$ .

$$\begin{aligned}\text{Now } \nabla\phi \cdot d\vec{r} &= \left( \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) \\ &= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \\ &= d\phi = 0 \quad [\because \phi = C]\end{aligned}$$

$\therefore \nabla\phi$  is normal to the surface  $\phi(x, y, z) = C$  at  $P$ .

So, a unit normal to the surface at  $P$  is  $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$

There is another unit normal in the opposite direction  $= -\frac{\nabla\phi}{|\nabla\phi|}$ .

### 1.4.3 Directional derivative

The directional derivative of a scalar point function  $\phi$  in a given direction  $\vec{a}$  is the rate of change of  $\phi$  in that direction. It is given by the component of  $\nabla\phi$  in the direction of  $\vec{a}$

$$\therefore \text{the directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}.$$

$$\begin{aligned}\text{Since } \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} &= \frac{|\nabla\phi||\vec{a}|}{|\vec{a}|} \cos\theta, \text{ where } \theta \text{ is the angle between } \nabla\phi \text{ and } \vec{a}. \\ &= |\nabla\phi| \cos\theta\end{aligned}$$

So, the directional derivative at a given point is maximum if  $\cos\theta$  is maximum. i.e.,  $\cos\theta = 1 \Rightarrow \theta = 0$ .

$\therefore$  the maximum directional derivative at a point is in the direction of  $\nabla\phi$  and the maximum directional derivative is  $|\nabla\phi|$ .

#### Note

1. The directional derivative is minimum when  $\cos\theta = -1 \Rightarrow \theta = \pi$   
 $\therefore$  the minimum directional derivative is  $-|\nabla\phi|$
2. In fact, the vector  $\nabla\phi$  is in the direction in which  $\phi$  increases rapidly i.e., outward normal and  $-\nabla\phi$  points in the direction in which  $\phi$  decreases rapidly.

### 1.4.4 Equation of tangent plane and normal to the surface

◆ (i) **Equation of tangent plane**

Let  $A$  be a given point on the surface  $\phi(x, y, z) = C$ .

Let  $\vec{r}_0 = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$  be the position vector of  $A$ .

Let  $P$  be any point on the tangent plane to the surface at the point  $A$  and let  $\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$  be the position vector of  $P$ .

Then  $\nabla\phi$  at  $A$  is normal to the surface and  $\vec{r} - \vec{r}_0$  lies on the tangent plane at  $A$ .

$\therefore$  the equation of the tangent plane at the point  $A$  is  $(\vec{r} - \vec{r}_0) \cdot \nabla\phi = 0$

**Note** The cartesian equation of the plane at the point  $A (x_0, y_0, z_0)$  is

$$(x - x_0) \frac{\partial\phi}{\partial x} + (y - y_0) \frac{\partial\phi}{\partial y} + (z - z_0) \frac{\partial\phi}{\partial z} = 0$$

The partial derivatives are evaluated at the point  $(x_0, y_0, z_0)$ .

◆ (ii) **Equation of the normal at the point  $A$**

Let  $A$  be a given point on the surface  $\phi (x, y, z) = C$  and let  $\vec{r}_0 = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}$  be the position vector of  $A$ .

Let  $\vec{r}$  be the position vector of any point  $P$  on the normal at the point  $A$ . Then  $\vec{r} - \vec{r}_0$  is parallel to the normal at the point  $A$ .

$\therefore$  the equation of the normal at the point  $A$  is  $(\vec{r} - \vec{r}_0) \times \nabla\phi = 0$ .

The cartesian equation of the normal at the point  $A$  is

$$\frac{x - x_0}{\frac{\partial\phi}{\partial x}} = \frac{y - y_0}{\frac{\partial\phi}{\partial y}} = \frac{z - z_0}{\frac{\partial\phi}{\partial z}},$$

where the partial derivatives are evaluated at  $(x_0, y_0, z_0)$ .

### 1.4.5 Angle between two surfaces at a common point

We know that the angle between two planes is the angle between their normals.

We define angle between two surfaces at a point of intersection  $P$  is the angle between their tangent planes at  $P$  and hence the angle between their normals at  $P$ .

The angle between two surfaces  $f(x, y, z) = C_1$  and  $g(x, y, z) = C_2$  at a common point  $P$  is the angle between their normals at the point  $P$ .

The normal at  $P$  to the surface  $f(x, y, z) = C_1$  is  $\nabla f$ .

The normal at  $P$  to the surface  $g(x, y, z) = C_2$  is  $\nabla g$ .

If  $\theta$  is the angle between the normals at the point  $P$ , then

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|}$$

- (i) If  $\theta = \frac{\pi}{2}$ , then the normals are perpendicular and  $\cos \theta = 0$

$$\Rightarrow \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|} = 0$$

$$\Rightarrow \nabla f \cdot \nabla g = 0$$

$\therefore$  if two surfaces are orthogonal at the point  $P$  then  $\nabla f \cdot \nabla g = 0$

Conversely, if  $\nabla f \cdot \nabla g = 0$ , then  $\theta = \frac{\pi}{2}$  i.e., they are orthogonal.

- (ii) If  $\theta = 0$ , the normals at the common point coincide.

$\therefore$  the two tangent planes coincide and the surfaces touch at the common point.

#### 1.4.6 Properties of gradients

If  $f$  and  $g$  are scalar point functions which are differentiable.

1.  $\nabla C = 0$ , where  $C$  is constant.
2.  $\nabla(Cf) = C\nabla f$ , where  $C$  is a constant.
3.  $\nabla(f \pm g) = \nabla f \pm \nabla g$
4.  $\nabla(fg) = f\nabla g + g\nabla f$
5.  $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$  if  $g \neq 0$

**Proof 1.**  $\nabla C = 0$ ,  $C$  is constant.

$$\text{We know } \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \quad (1)$$

$$= \sum \vec{i} \frac{\partial \phi}{\partial x}$$

$$\therefore \nabla C = \sum \vec{i} \frac{\partial C}{\partial x}$$

$$= 0 \left[ \because C \text{ is a constant } \frac{\partial C}{\partial x} = 0, \frac{\partial C}{\partial y} = 0, \frac{\partial C}{\partial z} = 0 \right] \blacksquare$$

**2.**  $\nabla C\phi = C\nabla\phi$

**Proof**

$$\begin{aligned}\nabla C\phi &= \sum \vec{i} \frac{\partial}{\partial x} (C\phi) \\ &= C \sum \vec{i} \frac{\partial \phi}{\partial x} \\ &= C\nabla\phi\end{aligned}\quad [\text{using (1)}] \quad \blacksquare$$

**3.**  $\nabla(f \pm g) = \nabla f \pm \nabla g$

**Proof** Now  $\nabla(f \pm g) = \sum \vec{i} \frac{\partial}{\partial x} (f \pm g)$  [using (1)]

$$\begin{aligned}&= \sum \left[ \vec{i} \frac{\partial f}{\partial x} \pm \vec{i} \frac{\partial g}{\partial x} \right] \\ &= \sum \vec{i} \frac{\partial f}{\partial x} \pm \sum \vec{i} \frac{\partial g}{\partial x} \\ &= \nabla f \pm \nabla g \\ \therefore \quad \nabla(f \pm g) &= \nabla f \pm \nabla g\end{aligned}\quad \blacksquare$$

**4.**  $\nabla(fg) = f\nabla g + g\nabla f$

**Proof** We have  $\nabla(fg) = \sum \vec{i} \frac{\partial}{\partial x} (fg)$

$$\begin{aligned}&= \sum \vec{i} \left[ f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right] \\ &= \sum \vec{i} \left( f \frac{\partial g}{\partial x} \right) + \sum \vec{i} \left( g \frac{\partial f}{\partial x} \right) \\ &= f \sum \vec{i} \frac{\partial g}{\partial x} + g \sum \vec{i} \frac{\partial f}{\partial x} \\ &= f\nabla g + g\nabla f\end{aligned}\quad \blacksquare$$

$\therefore \quad \nabla(fg) = f\nabla g + g\nabla f$

**5.**  $\nabla \left( \frac{f}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2}$

**Proof** We have  $\nabla \left( \frac{f}{g} \right) = \sum \vec{i} \frac{\partial}{\partial x} \left( \frac{f}{g} \right)$

$$\begin{aligned}
 &= \sum \vec{i} \left[ \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} \right] \\
 &= \frac{1}{g^2} \left[ g \sum \vec{i} \frac{\partial f}{\partial x} - f \sum \vec{i} \frac{\partial g}{\partial x} \right] \\
 &= \frac{g \nabla f - f \nabla g}{g^2} \\
 \therefore \quad \nabla \left( \frac{f}{g} \right) &= \frac{g \nabla f - f \nabla g}{g^2} \quad \blacksquare
 \end{aligned}$$

## WORKED EXAMPLES

### Example 1

Find  $\text{grad } \phi$  for the following functions.

- (i)  $\phi(x, y, z) = 3x^2y - y^3z^2$  at the point  $(1, -2, 1)$
- (ii)  $\phi(x, y, z) = \log(x^2 + y^2 + z^2)$  at the point  $(1, 2, 1)$ .

### Solution

(i) Given  $\phi(x, y, z) = 3x^2y - y^3z^2$

We know  $\text{grad } \phi = \nabla \phi$

$$= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Differentiating  $\phi$  partially w.r. to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 6xy, \quad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2z^2, \quad \frac{\partial \phi}{\partial z} = -2y^3z$$

**At the point  $(1, -2, 1)$ ,**  $\frac{\partial \phi}{\partial x} = 6 \cdot 1(-2) = -12$   
 $\frac{\partial \phi}{\partial y} = 3 \cdot 1^2 - 3 \cdot (-2)^2 1^2 = 3 - 12 = -9$   
 $\frac{\partial \phi}{\partial z} = -2 \cdot (-2)^3 \cdot 1 = 16$

$\therefore$  at the point  $(1, -2, 1)$ ,  $\nabla \phi = -12\vec{i} - 9\vec{j} + 16\vec{k}$

(ii) Given  $\phi(x, y, z) = \log(x^2 + y^2 + z^2)$

We know  $\text{grad } \phi = \nabla \phi$

$$= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get,

$$\frac{\partial \phi}{\partial x} = \frac{1}{x^2 + y^2 + z^2} \cdot 2x$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2 + y^2 + z^2} \cdot 2y$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{x^2 + y^2 + z^2} \cdot 2z$$

**At the point (1, 2, 1),**  $\frac{\partial \phi}{\partial x} = \frac{2 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$

$$\frac{\partial \phi}{\partial y} = \frac{2 \cdot 2}{1^2 + 2^2 + 1^2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{\partial \phi}{\partial z} = \frac{2 \cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \text{at the point (1, 2, 1), } \text{grad } \phi = \frac{1}{3} \vec{i} + \frac{2}{3} \vec{j} + \frac{1}{3} \vec{k}$$

$$= \frac{1}{3} [\vec{i} + 2 \vec{j} + \vec{k}]$$

## Example 2

Find  $\nabla \phi$  at the point (1, 1, 1) if  $\phi(x, y, z) = x^2y + y^2x + z^2$ .

### Solution

Given  $\phi(x, y, z) = x^2y + y^2x + z^2$ .

We know  $\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 2xy + y^2, \quad \frac{\partial \phi}{\partial y} = x^2 + 2xy, \quad \frac{\partial \phi}{\partial z} = 2z$$

**At the point (1, 1, 1),**  $\frac{\partial \phi}{\partial x} = 2 \cdot 1 \cdot 1 + 1^2 = 3$

$$\frac{\partial \phi}{\partial y} = 1^2 + 2 \cdot 1 \cdot 1 = 3$$

$$\frac{\partial \phi}{\partial z} = 2 \cdot 1 = 2$$

$\therefore$  at the point (1, 1, 1)  $\nabla \phi = 3\vec{i} + 3\vec{j} + 2\vec{k}$

### Example 3

If  $\phi = x + xy^2 + yz^2$  find grad  $\phi$ .

#### Solution

Given  $\phi = x + xy^2 + yz^2$

We know  $\text{grad } \phi = \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively,

we get  $\frac{\partial \phi}{\partial x} = 1 + y^2, \quad \frac{\partial \phi}{\partial y} = 2xy + z^2, \quad \frac{\partial \phi}{\partial z} = 2yz$

$\therefore \nabla \phi = (1 + y^2)\vec{i} + (2xy + z^2)\vec{j} + 2yz\vec{k}$

### Example 4

Find the directional derivative of  $\phi(x, y, z) = x^2yz + 4xz^2$  at the point (1, -2, -1) in the direction of the vector  $2\vec{i} - \vec{j} - 2\vec{k}$ . [AU 2008, 2009]

#### Solution

Given  $\phi(x, y, z) = x^2yz + 4xz^2$

We know  $\text{grad } \phi = \nabla \phi$

$$= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial \phi}{\partial y} = x^2z, \quad \frac{\partial \phi}{\partial z} = x^2y + 8xz$$

**At the point  $(1, -2, -1)$ ,**  $\frac{\partial \phi}{\partial x} = 2 \cdot 1(-2)(-1) + 4(-1)^2 = 8$

$$\frac{\partial \phi}{\partial y} = 1^2 \cdot (-1) = -1$$

$$\frac{\partial \phi}{\partial z} = 1^2(-2) + 8 \cdot 1(-1) = -2 - 8 = -10$$

**∴ at the point  $(1, -2, -1)$ ,**  $\nabla \phi = \vec{i} - \vec{j} - 10\vec{k}$

Given direction is  $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

**∴ the directional derivative of  $\phi$  at the point  $(1, -2, -1)$  in the direction of**

$$\begin{aligned}\vec{a} \text{ is } &= \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{\sqrt{4+1+4}} \\ &= \frac{16+1+20}{\sqrt{9}} = \frac{37}{3}\end{aligned}$$

### Example 5

If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $r = |\vec{r}|$  prove that (i)  $\nabla r = \frac{\vec{r}}{r}$ ,

(ii)  $\nabla r^n = nr^{n-2}\vec{r}$ , (iii)  $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$  (iv)  $\nabla(\log r) = \frac{\vec{r}}{r^2}$

[AU 2000, 2003, 2006]

### Solution

Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

and  $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\Rightarrow r^2 = x^2 + y^2 + z^2 \quad (1)$$

$$(i) \quad \nabla r = \frac{\vec{r}}{r}$$

We know

$$\nabla r = \vec{i} \frac{\partial r}{\partial x} + \vec{j} \frac{\partial r}{\partial y} + \vec{k} \frac{\partial r}{\partial z}$$

Differentiating (1) partially w.r.to  $x$ , we get

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned} \therefore \nabla r &= \frac{\vec{i}}{r} + \frac{\vec{j}}{r} + \frac{\vec{k}}{r} \\ &= \frac{1}{r} [\vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}] = \frac{\vec{r}}{r} \end{aligned}$$

$$(ii) \quad \nabla r^n = n r^{n-2} \vec{r}$$

We know

$$\begin{aligned} \nabla r^n &= \vec{i} \frac{\partial}{\partial x}(r^n) + \vec{j} \frac{\partial}{\partial y}(r^n) + \vec{k} \frac{\partial}{\partial z}(r^n) \\ &= \vec{i} \left( n r^{n-1} \frac{\partial r}{\partial x} \right) + \vec{j} \left( n r^{n-1} \frac{\partial r}{\partial y} \right) + \vec{k} \left( n r^{n-1} \frac{\partial r}{\partial z} \right) \\ &= n r^{n-1} \left[ \frac{\vec{x}}{r} \vec{i} + \frac{\vec{y}}{r} \vec{j} + \frac{\vec{z}}{r} \vec{k} \right] \\ &= \frac{n r^{n-1}}{r} [\vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}] = n r^{n-2} \vec{r} \end{aligned}$$

$$(iii) \quad \nabla \left( \frac{1}{r} \right) = -\frac{\vec{r}}{r^3}$$

We know,

$$\begin{aligned} \nabla \left( \frac{1}{r} \right) &= \vec{i} \frac{\partial}{\partial x} \left( \frac{1}{r} \right) + \vec{j} \frac{\partial}{\partial y} \left( \frac{1}{r} \right) + \vec{k} \frac{\partial}{\partial z} \left( \frac{1}{r} \right) \\ &= \vec{i} \left( -\frac{1}{r^2} \frac{\partial r}{\partial x} \right) + \vec{j} \left( -\frac{1}{r^2} \frac{\partial r}{\partial y} \right) + \vec{k} \left( -\frac{1}{r^2} \frac{\partial r}{\partial z} \right) \\ &= -\frac{1}{r^2} \left[ \frac{\vec{x}}{r} \vec{i} + \frac{\vec{y}}{r} \vec{j} + \frac{\vec{z}}{r} \vec{k} \right] \\ &= -\frac{1}{r^3} (\vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}) = -\frac{\vec{r}}{r^3} \end{aligned}$$

$$(iv) \quad \nabla (\log r) = \frac{\vec{r}}{r^2}$$

We know,

$$\nabla (\log r) = \vec{i} \frac{\partial}{\partial x} (\log r) + \vec{j} \frac{\partial}{\partial y} (\log r) + \vec{k} \frac{\partial}{\partial z} (\log r)$$

$$\begin{aligned}
 &= \vec{i} \left( \frac{1}{r} \frac{\partial r}{\partial x} \right) + \vec{j} \left( \frac{1}{r} \frac{\partial r}{\partial y} \right) + \vec{k} \left( \frac{1}{r} \frac{\partial r}{\partial z} \right) \\
 &= \frac{1}{r} \left[ \frac{x}{r} \vec{i} + \frac{y}{r} \vec{j} + \frac{z}{r} \vec{k} \right] = \frac{\vec{r}}{r^2}
 \end{aligned}$$

**Example 6**

**Find the directional derivative of  $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$  at the point  $(1, 1, -1)$  in the direction  $2\vec{i} - \vec{j} - \vec{k}$ .**

**Solution**

Given  $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$

We know  $\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = -4y, \quad \frac{\partial \phi}{\partial z} = 8z$$

**At the point  $(1, 1, -1)$ ,**  $\frac{\partial \phi}{\partial x} = 2 \cdot 1 = 2$   
 $\frac{\partial \phi}{\partial y} = -4 \cdot 1 = -4$   
 $\frac{\partial \phi}{\partial z} = 8(-1) = -8$

$\therefore$  **at the point  $(1, 1, -1)$ ,**  $\nabla \phi = 2\vec{i} - 4\vec{j} - 8\vec{k} = 2[\vec{i} - 2\vec{j} - 4\vec{k}]$

Given direction is  $\vec{a} = 2\vec{i} - \vec{j} - \vec{k}$

$\therefore$  the directional derivative of  $\phi$  at the point  $(1, 1, -1)$  in the direction of  $\vec{a}$  is

$$\begin{aligned}
 &= \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{2[\vec{i} - 2\vec{j} - 4\vec{k}] \cdot [2\vec{i} - \vec{j} - \vec{k}]}{\sqrt{4+1+1}} \\
 &= \frac{2[2+2+4]}{\sqrt{6}} = \frac{16}{\sqrt{6}}
 \end{aligned}$$

**Example 7**

Find the directional derivative of the function  $\phi = xy + yz + zx$  in the direction of the vector  $2\vec{i} + 3\vec{j} + 6\vec{k}$  at the point  $(3, 1, 2)$ .

**Solution**

Given  $\phi = xy + yz + zx$

We know

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively we get,

$$\frac{\partial\phi}{\partial x} = y + z, \quad \frac{\partial\phi}{\partial y} = z + x, \quad \frac{\partial\phi}{\partial z} = x + y$$

**At the point  $(3, 1, 2)$ ,**  $\frac{\partial\phi}{\partial x} = 1 + 2 = 3$

$$\frac{\partial\phi}{\partial y} = 2 + 3 = 5$$

$$\frac{\partial\phi}{\partial z} = 3 + 1 = 4$$

$\therefore$  **at the point  $(3, 1, 2)$ ,**  $\nabla\phi = 3\vec{i} + 5\vec{j} + 4\vec{k}$

Given direction is  $\vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

$\therefore$  the directional derivative of  $\phi$  at the point  $(3, 1, 2)$  in the direction

$$\begin{aligned} \text{of } \vec{a} \text{ is } &= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{(3\vec{i} + 5\vec{j} + 4\vec{k}) \cdot (2\vec{i} + 3\vec{j} + 6\vec{k})}{\sqrt{4+9+36}} \\ &= \frac{6+15+24}{\sqrt{49}} = \frac{45}{7} \end{aligned}$$

**Example 8**

**Find the directional derivative of the function  $2yz + z^2$  in the direction of the vector  $\vec{i} + 2\vec{j} + 2\vec{k}$  at the point  $(1, -1, 3)$ .**

**Solution**

Given

$$\phi = 2yz + z^2$$

We know

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 2z, \quad \frac{\partial \phi}{\partial z} = 2y + 2z$$

$$\text{At the point } (1, -1, 3), \quad \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 2(3) = 6$$

$$\text{and} \quad \frac{\partial \phi}{\partial z} = 2(-1) + 2 \cdot 3 = 4$$

$$\therefore \text{at the point } (1, -1, 3), \nabla \phi = 6\vec{j} + 4\vec{k}$$

$$\text{Given direction is} \quad \vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$\therefore$  the directional derivative of  $\phi$  at the point  $(1, -1, 3)$  in the direction

$$\begin{aligned} \text{of } \vec{a} \text{ is} &= \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= (6\vec{j} + 4\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{\sqrt{1+4+4}} \\ &= \frac{12+8}{\sqrt{9}} = \frac{20}{3} \end{aligned}$$

**Example 9**

**Find the directional derivative of  $x^3 + y^3 + z^3$  at the point  $(1, -1, 2)$  in the direction of  $\vec{i} + 2\vec{j} + \vec{k}$ .**

**Solution**

Given

$$\phi(x, y, z) = x^3 + y^3 + z^3$$

We know

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Now differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial\phi}{\partial x} = 3x^2, \quad \frac{\partial\phi}{\partial y} = 3y^2, \quad \frac{\partial\phi}{\partial z} = 3z^2$$

**At the point  $(1, -1, 2)$ ,**  $\frac{\partial\phi}{\partial x} = 3 \cdot 1^2 = 3$

$$\frac{\partial\phi}{\partial y} = 3(-1)^2 = 3$$

$$\frac{\partial\phi}{\partial z} = 3 \cdot 2^2 = 12$$

$\therefore$  at the point  $(1, -1, 2)$ ,  $\nabla\phi = 3\vec{i} + 3\vec{j} + 12\vec{k}$

Given direction is  $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$

$\therefore$  the directional derivative of  $\phi$  at the point  $(1, -1, 2)$  in the direction

$$\begin{aligned} \text{of } \vec{a} \text{ is } &= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (3\vec{i} + 3\vec{j} + 12\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + \vec{k})}{\sqrt{1+4+1}} \\ &= \frac{3+6+12}{\sqrt{6}} \\ &= \frac{21}{\sqrt{6}} = 21 \frac{\sqrt{6}}{6} = \frac{7\sqrt{6}}{2} \end{aligned}$$

### Example 10

Find the directional derivative of  $\phi = x^2yz + 4xz^2 + xyz$  at  $(1, 2, 3)$  in the direction of  $2\vec{i} + \vec{j} - \vec{k}$ . [AU 1996]

### Solution

Given

$$\phi = x^2yz + 4xz^2 + xyz$$

We know

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2 + yz$$

$$\frac{\partial \phi}{\partial y} = x^2z + xz$$

$$\frac{\partial \phi}{\partial z} = x^2y + 8xz + xy$$

**At the point (1, 2, 3),**  $\frac{\partial \phi}{\partial x} = 2 \times 1 \times 2 \times 3 + 4 \times 3^2 + 2 \times 3 = 54$

$$\frac{\partial \phi}{\partial y} = 1^2 \times 3 + 1 \times 3 = 6$$

$$\frac{\partial \phi}{\partial z} = 1^2 \times 2 + 8 \times 1 \times 3 + 1 \times 2 = 28$$

$\therefore$  at the point (1, 2, 3),  $\nabla \phi = 54\vec{i} + 6\vec{j} + 28\vec{k}$

Given direction is  $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$\therefore$  the directional derivative of  $\phi$  at the point (1, 2, 3) in the direction

$$\begin{aligned} \text{of } \vec{a} \text{ is } &= \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{(54\vec{i} + 6\vec{j} + 28\vec{k}) \cdot (2\vec{i} + \vec{j} - \vec{k})}{\sqrt{2^2 + 1^2 + 1^2}} \\ &= \frac{108 + 6 - 28}{\sqrt{6}} = \frac{86}{\sqrt{6}} \end{aligned}$$

### Example 11

Find the directional derivative of  $\phi = x^2yz + 4xz^2$  at the point  $P(1, -2, -1)$  in the direction of PQ, where Q is (3, -3, -3). [AU 2008]

### Solution

Given  $\phi = x^2yz + 4xz^2$

We know  $\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial \phi}{\partial y} = x^2z, \quad \frac{\partial \phi}{\partial z} = x^2y + 8xz$$

**At the point  $P(1, -2, -1)$ ,**  $\frac{\partial \phi}{\partial x} = 2 \cdot 1(-2)(-1) + 4(-1)^2 = 8$

$$\frac{\partial \phi}{\partial y} = 1^2(-1) = -1$$

$$\frac{\partial \phi}{\partial z} = 1^2(-2) + 8 \times 1 \times (-1) = -10$$

**∴ at the point  $P(1, -2, -1)$ ,**  $\nabla \phi = 8\vec{i} - \vec{j} - 10\vec{k}$

Given  $Q$  is the point  $(3, -3, -3)$

$$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\begin{aligned} &= 3\vec{i} - 3\vec{j} - 3\vec{k} - [\vec{i} - 2\vec{j} - \vec{k}] \\ &= 2\vec{i} - \vec{j} - 2\vec{k} \end{aligned}$$

Given direction is  $\vec{a} = \overrightarrow{PQ} = 2\vec{i} - \vec{j} - 2\vec{k}$

**∴ the directional derivative of  $\phi$  at  $P$  in the direction of  $\overrightarrow{PQ} = \vec{a}$  is**

$$\begin{aligned} &= \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{\sqrt{4+1+4}} \\ &= \frac{16+1+20}{3} = \frac{37}{3} \end{aligned}$$

### Example 12

**Find a unit normal vector to the surface  $x^3 + y^3 + 3xyz = 3$  at the point  $(1, 2, -1)$ .**

### Solution

The given surface is  $x^3 + y^3 + 3xyz = 3$ , which is taken as  $\phi = C$

$$\therefore \phi = x^3 + y^3 + 3xyz$$

We know that  $\nabla \phi$  is normal to the surface.

So unit normal to the surface is  $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Now  $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively,

we get,  $\frac{\partial\phi}{\partial x} = 3x^2 + 3yz, \quad \frac{\partial\phi}{\partial y} = 3y^2 + 3xz, \quad \frac{\partial\phi}{\partial z} = 3xy$

**At the point (1, 2, -1),**  $\frac{\partial\phi}{\partial x} = 3 \cdot 1^2 + 3 \cdot 2(-1) = -3$

$$\frac{\partial\phi}{\partial y} = 3 \cdot 2^2 + 3 \cdot 1(-1) = 9$$

$$\frac{\partial\phi}{\partial z} = 3 \cdot 1 \cdot 2 = 6$$

$\therefore$  at the point (1, 2, -1),  $\nabla\phi = -3\vec{i} + 9\vec{j} + 6\vec{k}$

$\therefore$  unit normal to the given surface at the point (1, 2, -1) is

$$\vec{n} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{9+81+36}} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{126}}$$

**Note** If the surface equation is written as  $x^3 + y^3 + 3xyz - 3 = 0$ , then we take  $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$ . Here  $C = 0$ .

### Example 13

Find a unit normal to the surface  $xy^3z^2 = 4$  at the point (-1, -1, 2).

### Solution

Given surface is  $xy^3z^2 = 4$

$$\therefore \phi(x, y, z) = xy^3z^2$$

We know  $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = y^3 z^2, \quad \frac{\partial \phi}{\partial y} = 3xy^2 z^2, \quad \frac{\partial \phi}{\partial z} = 2xy^3 z$$

**At the point  $(-1, -1, 2)$ ,**  $\frac{\partial \phi}{\partial x} = (-1)^3 \cdot 2^2 = -4$

$$\frac{\partial \phi}{\partial y} = 3(-1)(-1)^2 \cdot 2^2 = -12$$

$$\frac{\partial \phi}{\partial z} = 2(-1)(-1)^3 \cdot 2 = 4$$

$\therefore$  at the point  $(-1, -1, 2)$ ,  $\nabla \phi = -4\vec{i} - 12\vec{j} + 4\vec{k} = -4(\vec{i} + 3\vec{j} - \vec{k})$

$\therefore$  unit normal to the given surface at the point  $(-1, -1, 2)$  is

$$\begin{aligned}\vec{n} &= \frac{\nabla \phi}{|\nabla \phi|} \\ &= \frac{-4(\vec{i} + 3\vec{j} - \vec{k})}{4\sqrt{1+9+1}} = -\frac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}\end{aligned}$$

### Example 14

Find a unit normal to the surface  $xy^2z^3 = 1$  at the point  $(1, 1, 1)$ .

### Solution

Given  $\phi(x, y, z) = xy^2z^3$

We know,  $\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = y^2 z^3, \quad \frac{\partial \phi}{\partial y} = 2xyz^3, \quad \frac{\partial \phi}{\partial z} = 3xy^2 z^2$$

**At the point  $(1, 1, 1)$ ,**  $\frac{\partial \phi}{\partial x} = 1^2 \cdot 1^3 = 1$

$$\frac{\partial \phi}{\partial y} = 2 \cdot 1 \cdot 1 \cdot 1^3 = 2$$

$$\frac{\partial \phi}{\partial z} = 3 \cdot 1 \cdot 1^2 = 3$$

$\therefore$  at the point  $(1, 1, 1)$ ,  $\nabla\phi = \vec{i} + 2\vec{j} + 3\vec{k}$

$\therefore$  unit normal vector to the given surface at the point  $(1, 1, 1)$  is

$$\begin{aligned}\vec{n} &= \frac{\nabla\phi}{|\nabla\phi|} \\ &= \frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{1+4+9}} = \frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{14}}\end{aligned}$$

### Example 15

**Find a unit normal vector to the surface  $x^2 + y^2 - z^2 = 1$  at the point  $(1, 1, 1)$ .**

### Solution

Given  $\phi(x, y, z) = x^2 + y^2 - z^2$

We know  $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -2z$$

**At the point  $(1, 1, 1)$ ,**  $\frac{\partial\phi}{\partial x} = 2 \cdot 1 = 2, \frac{\partial\phi}{\partial y} = 2 \cdot 1 = 2, \frac{\partial\phi}{\partial z} = -2 \cdot 1 = -2$

$\therefore$  at the point  $(1, 1, 1)$ ,  $\nabla\phi = 2\vec{i} + 2\vec{j} - 2\vec{k} = 2(\vec{i} + \vec{j} - \vec{k})$

$\therefore$  unit normal to the given surface at the point  $(1, 1, 1)$  is

$$\begin{aligned}\vec{n} &= \frac{\nabla\phi}{|\nabla\phi|} \\ &= \frac{2(\vec{i} + \vec{j} - \vec{k})}{2\sqrt{1+1+1}} = \frac{2(\vec{i} + \vec{j} - \vec{k})}{2\sqrt{3}} \\ &= \frac{(\vec{i} + \vec{j} - \vec{k})}{\sqrt{3}}\end{aligned}$$

**Example 16**

Find a unit normal to the surface  $x^2y + 2xz^2 = 8$  at the point  $(1, 0, 2)$ .

**Solution**

Given

$$\phi(x, y, z) = x^2y + 2xz^2$$

We know,

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xy + 2z^2, \quad \frac{\partial\phi}{\partial y} = x^2, \quad \frac{\partial\phi}{\partial z} = 4xz$$

**At the point  $(1, 0, 2)$ ,**

$$\frac{\partial\phi}{\partial x} = 2 \cdot 1 \cdot 0 + 2 \cdot 2^2 = 8, \quad \frac{\partial\phi}{\partial y} = 1^2 = 1$$

$$\frac{\partial\phi}{\partial z} = 4 \cdot 1 \cdot 2 = 8$$

**∴ at the point  $(1, 0, 2)$ ,**

$$\nabla\phi = 8\vec{i} + \vec{j} + 8\vec{k}$$

∴ unit normal vector to the given surface at the point  $(1, 0, 2)$  is

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{64+1+64}} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{129}}$$

**Example 17**

Find the maximum value of the directional derivative of  $\phi = x^3yz$  at the point  $(1, 4, 1)$ .

**Solution**

Given

$$\phi = x^3yz$$

We know,

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

The directional derivative is maximum in the direction of  $\nabla\phi$  and the maximum value  $= |\nabla\phi|$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial\phi}{\partial x} = 3x^2yz, \quad \frac{\partial\phi}{\partial y} = x^3z, \quad \frac{\partial\phi}{\partial z} = x^3y$$

**At the point (1, 4, 1),**  $\frac{\partial \phi}{\partial x} = 3 \cdot 1 \cdot 4 \cdot 1 = 12, \frac{\partial \phi}{\partial y} = 1^3 \cdot 1 = 1, \frac{\partial \phi}{\partial z} = 1^3 \cdot 4 = 4$

**∴ at the point (1, 4, 1),**  $\nabla \phi = 12\vec{i} + \vec{j} + 4\vec{k}$

Maximum value of the directional derivative  $= |\nabla \phi|$

$$\begin{aligned} &= |12\vec{i} + \vec{j} + 4\vec{k}| \\ &= \sqrt{144 + 1 + 16} = \sqrt{161} \end{aligned}$$

### Example 18

In what direction from  $(3, 1, -2)$  is the directional derivative of  $\phi = x^2y^2z^4$  maximum? Find also the magnitude of this maximum.

### Solution

Given

$$\phi = x^2y^2z^4$$

We know the directional derivative is maximum in the direction of  $\nabla \phi$  and the maximum value  $= |\nabla \phi|$

Now

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial \phi}{\partial x} = 2xy^2z^4, \quad \frac{\partial \phi}{\partial y} = 2x^2yz^4, \quad \frac{\partial \phi}{\partial z} = 4x^2y^2z^3$$

**At the point (3, 1, -2),**  $\frac{\partial \phi}{\partial x} = 2 \cdot 3 \cdot 1^2 (-2)^4 = 96$

$$\frac{\partial \phi}{\partial y} = 2 \cdot 3^2 \cdot 1^2 (-2)^4 = 288$$

$$\frac{\partial \phi}{\partial z} = 4 \cdot 3^2 \cdot 1^2 (-2)^2 = -288$$

**∴ at the point (3, 1, -2),**  $\nabla \phi = 96\vec{i} + 288\vec{j} - 288\vec{k} = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

The directional derivative is maximum in the direction of  $96(\vec{i} + 3\vec{j} - 3\vec{k})$ .

$$\begin{aligned} \text{Maximum value} &= |\nabla \phi| = |96(\vec{i} + 3\vec{j} - 3\vec{k})| \\ &= 96\sqrt{1+9+9} = 96\sqrt{19} \end{aligned}$$

**Example 19**

In what direction from the point  $(1, 1, -2)$ , is the directional derivative of  $\phi = x^2 - 2y^2 + 4z^2$  maximum? Also find the maximum directional derivative.

**Solution**

Given

$$\phi = x^2 - 2y^2 + 4z^2$$

We know that the directional derivative is maximum in the direction of  $\nabla\phi$ .  
The maximum value =  $|\nabla\phi|$

We have

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -4y, \quad \frac{\partial\phi}{\partial z} = 8z$$

**At the point  $(1, 1, -2)$ ,**  $\frac{\partial\phi}{\partial x} = 2 \cdot 1 = 2$

$$\frac{\partial\phi}{\partial y} = -4 \cdot 1 = -4$$

$$\frac{\partial\phi}{\partial z} = 8(-2) = -16$$

**∴ at the point  $(1, 1, -2)$ ,**  $\nabla\phi = 2\vec{i} - 4\vec{j} - 16\vec{k}$

$$= 2[\vec{i} - 2\vec{j} - 8\vec{k}]$$

**∴ the directional derivative is maximum in the direction of  $2(\vec{i} - 2\vec{j} - 8\vec{k})$**

$$\text{Maximum value} = |\nabla\phi|$$

$$\begin{aligned} &= |2(\vec{i} - 2\vec{j} - 8\vec{k})| \\ &= 2\sqrt{1+4+64} = 2\sqrt{69} \end{aligned}$$

**Example 20**

**What is the greatest rate of increase of  $\phi = xyz^2$  at the point  $(1, 0, 3)$ ?**

**Solution**

Given

$$\phi = xyz^2$$

We know that the greatest rate of increase is the maximum value of the directional derivative and it is equal to  $|\nabla\phi|$

We know,

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

Differentiating  $\phi$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial\phi}{\partial x} = yz^2, \quad \frac{\partial\phi}{\partial y} = xz^2, \quad \frac{\partial\phi}{\partial z} = 2xyz$$

**At the point  $(1, 0, 3)$ ,**

$$\frac{\partial\phi}{\partial x} = 0 \times 3 = 0$$

$$\frac{\partial\phi}{\partial y} = 1 \times 3^2 = 9$$

$$\frac{\partial\phi}{\partial z} = 2 \times 1 \times 0 \times 3 = 0$$

**∴ at the point  $(1, 0, 3)$ ,**

$$\nabla\phi = \vec{0i} + 9\vec{j} + \vec{0k} = 9\vec{j}$$

**∴ the greatest rate of increase is  $|\nabla\phi| = |9\vec{j}| = 9$**

**Example 21**

**Find the angle between the surfaces  $x^2 + y^2 + z^2 = 9$  and  $x^2 + y^2 - z = 3$  at the point  $(2, -1, 2)$ .**

**Solution**

The given surfaces are

$$x^2 + y^2 + z^2 = 9 \tag{1}$$

and

$$x^2 + y^2 - z = 3 \tag{2}$$

$P(2, -1, 2)$  is a common point of (1) and (2)

Let

$$f = x^2 + y^2 + z^2 \text{ and } g = x^2 + y^2 - z$$

Now,  $\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$

Differentiating  $f$  partially w.r.to  $x, y, z$  respectively we get,

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z$$

**At the point  $(2, -1, 2)$ ,**  $\frac{\partial f}{\partial x} = 2 \cdot 2 = 4, \frac{\partial f}{\partial y} = 2(-1) = -2, \frac{\partial f}{\partial z} = 2(+2) = +4$

$\therefore$  **at the point  $(2, -1, 2)$ ,**  $\nabla f = 4\vec{i} - 2\vec{j} + 4\vec{k}$

Now  $\nabla g = \vec{i} \frac{\partial g}{\partial x} + \vec{j} \frac{\partial g}{\partial y} + \vec{k} \frac{\partial g}{\partial z}$

Differentiating  $g$  partially w.r.to  $x, y, z$  respectively, we get

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 2y, \quad \frac{\partial g}{\partial z} = -1$$

**at the point  $(2, -1, 2)$ ,**  $\frac{\partial g}{\partial x} = 2 \cdot 2 = 4, \frac{\partial g}{\partial y} = 2(-1) = -2, \frac{\partial g}{\partial z} = -1$

$\therefore$  **at the point  $(2, -1, 2)$ ,**  $\nabla g = 4\vec{i} - 2\vec{j} - \vec{k}$

If  $\theta$  is the angle between the surfaces (1) and (2) at  $(2, -1, 2)$ , then

$$\begin{aligned} \cos \theta &= \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+16} \cdot \sqrt{16+4+1}} \\ &= \frac{16+4-4}{\sqrt{36}\sqrt{21}} \\ &= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \end{aligned}$$

$\therefore \theta = \cos^{-1} \left( \frac{8}{3\sqrt{21}} \right)$

**Example 22**

Show that the surfaces  $5x^2 - 2yz - 9x = 0$  and  $4x^2y + z^3 - 4 = 0$  are orthogonal at the point  $(1, -1, 2)$ .

**Solution**

The given surfaces are

$$5x^2 - 2yz - 9x = 0 \quad (1)$$

and  $4x^2y + z^3 - 4 = 0 \quad (2)$

Let  $f = 5x^2 - 2yz - 9x$  and  $g = 4x^2y + z^3 - 4$

To prove (1) and (2) cut orthogonally at the point  $(1, -1, 2)$

i.e., to prove  $\nabla f \cdot \nabla g = 0$

Now

$$\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 10x - 9, \quad \frac{\partial f}{\partial y} = -2z \quad \text{and} \quad \frac{\partial f}{\partial z} = -2y$$

$\therefore$

$$\nabla f = (10x - 9)\vec{i} - 2z\vec{j} - 2y\vec{k}$$

and

$$\nabla g = \vec{i} \frac{\partial g}{\partial x} + \vec{j} \frac{\partial g}{\partial y} + \vec{k} \frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2 \quad \text{and} \quad \frac{\partial g}{\partial z} = 3z^2$$

$\therefore$

$$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

**At the point  $(1, -1, 2)$ ,**

$$\nabla f = (10 - 9)\vec{i} - 2 \cdot 2\vec{j} - 2(-1)\vec{k}$$

$\Rightarrow$

$$\nabla f = \vec{i} - 4\vec{j} + 2\vec{k}$$

and

$$\nabla g = 8 \cdot 1 \cdot (-1)\vec{i} + 4 \cdot 1^2\vec{j} + 3 \cdot 2^2\vec{k}$$

$\Rightarrow$

$$\nabla g = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$\therefore$

$$\nabla f \cdot \nabla g = (\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 12\vec{k})$$

$$= -8 - 16 + 24 = 0$$

Hence the two surfaces cut orthogonally at the point  $(1, -1, 2)$ .

**Example 23**

Find  $a$  and  $b$  if the surfaces  $ax^2 - byz = (a + 2)x$  and  $4x^2y + z^3 = 4$  cut orthogonally at the point  $(1, -1, 2)$ . [AU 2005, 2006]

**Solution**

The given surfaces are

$$ax^2 - byz - (a + 2)x = 0 \quad (1)$$

$$4x^2y + z^3 - 4 = 0 \quad (2)$$

Let  $f = ax^2 - byz - (a + 2)x$  and  $g = 4x^2y + z^3 - 4$

Given the surfaces (1) and (2) cut orthogonally at the point  $(1, -1, 2)$ .

$$\therefore \nabla f \cdot \nabla g = 0 \quad (3)$$

Now

$$\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2ax - a - 2, \quad \frac{\partial f}{\partial y} = -bz \quad \text{and} \quad \frac{\partial f}{\partial z} = -by$$

$\therefore$

$$\nabla f = (2ax - a - 2)\vec{i} - bz\vec{j} - by\vec{k}$$

and

$$\nabla g = \vec{i} \frac{\partial g}{\partial x} + \vec{j} \frac{\partial g}{\partial y} + \vec{k} \frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2 \quad \text{and} \quad \frac{\partial g}{\partial z} = 3z^2$$

$\therefore$

$$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

**At the point  $(1, -1, 2)$ ,**

$$\nabla f = (2a - a - 2)\vec{i} - b\vec{j} - b(-1)\vec{k}$$

$\Rightarrow$

$$\nabla f = (a - 2)\vec{i} - 2b\vec{j} + b\vec{k}$$

and

$$\nabla g = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$\therefore$

$$\nabla f \cdot \nabla g = ((a - 2)\vec{i} - 2b\vec{j} + b\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 12\vec{k})$$

$$= -8(a - 2) - 8b + 12b$$

$$= -8a + 4b + 16$$

From (3),  $\nabla f \cdot \nabla g = 0$

$$\Rightarrow -8a + 4b + 16 = 0$$

$$\Rightarrow 2a - b = 4 \quad (4)$$

Since  $(1, -1, 2)$  is a point on the surface  $f = 0$ , we get

$$a + 2b - (a + 2) = 0 \Rightarrow 2b = 2 \Rightarrow b = 1$$

$$\therefore (4) \Rightarrow 2a = 4 + b = 4 + 1 = 5 \Rightarrow a = \frac{5}{2}$$

$$\therefore a = \frac{5}{2}, \quad b = 1$$

### Example 24

**Find the angle between the spheres  $x^2 + y^2 + z^2 = 29$  and  $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$  at the point  $(4, -3, 2)$ .**

### Solution

The given spheres are  $x^2 + y^2 + z^2 = 29 \quad (1)$

and  $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0 \quad (2)$

Let  $f = x^2 + y^2 + z^2$  and  $g = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47$

Let  $\theta$  be the angle between (1) and (2) at the point  $(4, -3, 2)$

Then  $\cos \theta = \frac{\nabla f \cdot \nabla g}{\|\nabla f\| \|\nabla g\|}$

Now  $\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y \quad \text{and} \quad \frac{\partial f}{\partial z} = 2z$$

$$\therefore \nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

and  $\nabla g = \vec{i} \frac{\partial g}{\partial x} + \vec{j} \frac{\partial g}{\partial y} + \vec{k} \frac{\partial g}{\partial z}$

$$\frac{\partial g}{\partial x} = 2x + 4, \quad \frac{\partial g}{\partial y} = 2y - 6 \quad \text{and} \quad \frac{\partial g}{\partial z} = 2z - 8$$

$$\nabla g = (2x + 4)\vec{i} + (2y - 6)\vec{j} + (2z - 8)\vec{k}$$

**At the point  $(4, -3, 2)$ ,**

$$\nabla f = 2 \cdot 4\vec{i} + 2(-3)\vec{j} + 2 \cdot 2\vec{k}$$

$$= 8\vec{i} - 6\vec{j} + 4\vec{k} = 2[4\vec{i} - 3\vec{j} + 2\vec{k}]$$

and

$$\nabla g = (2 \cdot 4 + 4)\vec{i} + (2(-3) - 6)\vec{j} + (2 \cdot 2 - 8)\vec{k}$$

$$= 12\vec{i} - 12\vec{j} - 4\vec{k} = 4[3\vec{i} - 3\vec{j} - \vec{k}]$$

∴

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{\|\nabla f\| \|\nabla g\|}$$

$$= \frac{2(4\vec{i} - 3\vec{j} + 2\vec{k})}{2\sqrt{16+9+4}} \cdot \frac{4(3\vec{i} - 3\vec{j} - \vec{k})}{4\sqrt{9+9+1}}$$

$$= \frac{12+9-2}{\sqrt{29}\sqrt{19}}$$

$$= \frac{19}{\sqrt{29}} \cdot \frac{1}{\sqrt{29}} = \sqrt{\frac{19}{29}}$$

∴

$$\theta = \cos^{-1} \left( \sqrt{\frac{19}{29}} \right)$$

### Example 25

Find the angle between the normals to the surface  $xy = z^2$  at the points  $(1, 4, 2)$  and  $(-3, -3, 3)$ .

### Solution

The given surface is  $xy - z^2 = 0$

$$\therefore \phi = xy - z^2$$

We know  $\nabla \phi$  is normal to the surface at the point  $(x, y, z)$

Let  $\vec{n}_1, \vec{n}_2$ , be the normals to the surface at the points  $(1, 4, 2)$  and  $(-3, -3, 3)$  respectively.

$$\therefore \vec{n}_1 = \nabla \phi \text{ at the point } (1, 4, 2)$$

$$\text{and } \vec{n}_2 = \nabla \phi \text{ at the point } (-3, -3, 3)$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = y, \quad \frac{\partial \phi}{\partial y} = x, \quad \text{and} \quad \frac{\partial \phi}{\partial z} = -2z$$

$$\therefore \nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$\text{At the point } (1, 4, 2), \quad \nabla\phi = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$\therefore \vec{n}_1 = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$\text{At the point } (-3, -3, 3), \quad \nabla\phi = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$\therefore \vec{n}_2 = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

If  $\theta$  is the angle between the normals, then

$$\begin{aligned}\cos\theta &= \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \\ &= \frac{(4\vec{i} + \vec{j} - 4\vec{k}) \cdot (-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{16+1+16} \sqrt{9+9+36}} \\ &= \frac{-12 - 3 + 24}{\sqrt{33} \sqrt{54}} \\ &= \frac{9}{\sqrt{3} \sqrt{11} 3\sqrt{3} \sqrt{2}} = \frac{1}{\sqrt{22}} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)\end{aligned}$$

### Example 26

**Find the directional derivative of the function  $\phi = xy^2 + yz^3$  at the point  $(2, -1, 1)$  in the direction of the normal to the surface  $x\log z - y^2 + 4 = 0$  at the point  $(-1, 2, 1)$ .** [AU 2008]

### Solution

Given  $\phi = xy^2 + yz^3$

$$\begin{aligned}\therefore \nabla\phi &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \\ &= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}\end{aligned}$$

$$\text{At the point } (2, -1, 1), \quad \nabla\phi = (-1)^2\vec{i} + (-4+1)\vec{j} + 3(-1)1^2\vec{k}$$

$$\Rightarrow \nabla\phi = \vec{i} - 3\vec{j} - 3\vec{k}$$

The directional derivative of  $\phi$  in the direction of the normal to the surface  $x\log z - y^2 + 4 = 0$  at the point  $(-1, 2, 1)$  is required.

Let

$$f = x \log z - y^2 + 4$$

$\therefore$

$$\begin{aligned}\nabla f &= \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z} \\ &= \log z \vec{i} - 2y \vec{j} + \frac{x}{z} \vec{k}\end{aligned}$$

**At the point  $(-1, 2, 1)$ ,**  $\nabla f = \log 1 \vec{i} - 4 \vec{j} + \left(\frac{-1}{1}\right) \vec{k}$

$$= 0 \vec{i} - 4 \vec{j} - \vec{k} = -4 \vec{j} - \vec{k}$$

$\therefore$

$$\vec{a} = -4 \vec{j} - \vec{k}$$

Required directional derivative  $= \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} - 3 \vec{j} - 3 \vec{k}) \cdot \frac{(-4 \vec{j} - \vec{k})}{\sqrt{16+1}}$$

$$= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

### Example 27

Find  $\phi$  if  $\nabla \phi = (6xy + z^3) \vec{i} + (3x^2 - z) \vec{j} + (3xz^2 - y) \vec{k}$

### Solution

Given

$$\nabla \phi = (6xy + z^3) \vec{i} + (3x^2 - z) \vec{j} + (3xz^2 - y) \vec{k} \quad (1)$$

But

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \quad (2)$$

Equating the coefficients of  $\vec{i}, \vec{j}, \vec{k}$  from (1) and (2), we get

$$\frac{\partial \phi}{\partial x} = 6xy + z^3 \quad (3)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \quad (4)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \quad (5)$$

Integrating (3) partially w.r.to  $x$ , treating  $y$  and  $z$  constant, we get

$$\phi = 3x^2y + z^3x + f_1(y, z) \quad (6)$$

Integrating (4) partially w.r.to  $y$ , we get

$$\phi = 3x^2y - zy + f_2(x, z) \quad (7)$$

Integrating (5) partially w.r.to  $z$ , we get

$$\phi = xz^3 - yz + f_3(x, y) \quad (8)$$

From (6), (7), (8),  $\phi$  is obtained by adding all the terms and an arbitrary constant  $C$ , but omitting  $f_1(y, z)$ ,  $f_2(x, z)$ ,  $f_3(x, y)$  and choosing only one of the repeated terms.

Thus

$$\phi = 3x^2y + xz^3 - yz + C$$

### Example 28

If  $\nabla \phi = 2xyz^3 \vec{i} + x^2z^3 \vec{j} + 3x^2yz^2 \vec{k}$ , then find  $\phi$  if  $\phi(1, -2, 2) = 4$ .

### Solution

Given

$$\nabla \phi = 2xyz^3 \vec{i} + x^2z^3 \vec{j} + 3x^2yz^2 \vec{k} \quad (1)$$

But

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \quad (2)$$

Equating the coefficients of  $\vec{i}, \vec{j}, \vec{k}$ , from (1) and (2), we get

$$\frac{\partial \phi}{\partial x} = 2xyz^3 \quad (3)$$

$$\frac{\partial \phi}{\partial y} = x^2z^3 \quad (4)$$

$$\frac{\partial \phi}{\partial z} = 3x^2yz^2 \quad (5)$$

Integrating (3) partially w.r.to  $x$ , we get

$$\phi = x^2yz^3 + f_1(y, z) \quad (6)$$

Integrating (4) partially w.r.to  $y$ , we get,

$$\phi = x^2 z^3 y + f_2(x, z) \quad (7)$$

Integrating (5) partially w.r.to  $z$ , we get,

$$\phi = x^2 y z^3 + f_3(x, y) \quad (8)$$

From (6), (7), (8),  $\phi$  is obtained by adding all the terms and an arbitrary constant  $C$ , but omitting  $f_1(y, z), f_2(x, z), f_3(x, y)$  and choosing only one of the repeated terms.

Thus

$$\phi = x^2 y z^3 + C$$

Given

$$\phi(1, -2, 2) = 4$$

$\therefore$

$$1 \times (-2) \times 8 + C = 4$$

$\Rightarrow$

$$C = 4 + 16 = 20$$

$\therefore$

$$\phi = x^2 y z^3 + 20$$

### Example 29

Find the equation of the tangent plane and the equation of the normal to the surface  $x^2 - 4y^2 + 3z^2 + 4 = 0$  at the point  $(3, 2, 1)$ .

### Solution

The given surface is  $x^2 - 4y^2 + 3z^2 + 4 = 0$

Let

$$\phi = x^2 - 4y^2 + 3z^2 + 4$$

$\therefore$

$$\begin{aligned} \nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= 2x\vec{i} - 8y\vec{j} + 6z\vec{k} \end{aligned}$$

At the point  $(3, 2, 1)$ ,

$$\nabla \phi = 6\vec{i} - 16\vec{j} + 6\vec{k}$$

We know that the equation of the tangent plane at the point  $(x_0, y_0, z_0)$  is

$$(x - x_0) \frac{\partial \phi}{\partial x} + (y - y_0) \frac{\partial \phi}{\partial y} + (z - z_0) \frac{\partial \phi}{\partial z} = 0$$

Here

$$(x_0, y_0, z_0) = (3, 2, 1)$$

and

$$\frac{\partial \phi}{\partial x} = 6, \quad \frac{\partial \phi}{\partial y} = -16, \quad \frac{\partial \phi}{\partial z} = 6$$

$\therefore$  the equation of the tangent plane is

$$\begin{aligned} & (x-3)6 + (y-2)(-16) + (z-1)6 = 0 \\ \Rightarrow & 3(x-3) - 8(y-2) + 3(z-1) = 0 && [\text{dividing by 2}] \\ \Rightarrow & 3x - 8y + 3z - 9 + 16 - 3 = 0 \\ \Rightarrow & 3x - 8y + 3z + 4 = 0 \end{aligned}$$

The equation of the normal at the point  $(x_0, y_0, z_0)$  is

$$\frac{x - x_0}{\frac{\partial \phi}{\partial x}} = \frac{y - y_0}{\frac{\partial \phi}{\partial y}} = \frac{z - z_0}{\frac{\partial \phi}{\partial z}}$$

The equation of the normal at the point  $(3, 2, 1)$  is

$$\begin{aligned} & \frac{x-3}{6} = \frac{y-2}{-16} = \frac{z-1}{6} \\ \Rightarrow & \frac{x-3}{3} = \frac{y-2}{-8} = \frac{z-1}{3} \end{aligned}$$

### Example 30

**Find the equation of the tangent plane and normal to the surface  $x^2 + y^2 + z^2 = 25$  at the point  $(4, 0, 3)$ .**

### Solution

The given surface is  $x^2 + y^2 + z^2 = 25$

Let  $\phi = x^2 + y^2 + z^2 - 25$

$$\begin{aligned} \therefore \nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \end{aligned}$$

**At the point  $(4, 0, 3)$ ,**

$$\begin{aligned} \nabla \phi &= 8\vec{i} + 0\vec{j} + 6\vec{k} \\ &= 8\vec{i} + 6\vec{k} \end{aligned}$$

The equation of the tangent plane at the point  $(x_0, y_0, z_0)$  is

$$(x - x_0) \frac{\partial \phi}{\partial x} + (y - y_0) \frac{\partial \phi}{\partial y} + (z - z_0) \frac{\partial \phi}{\partial z} = 0$$

Here

$$(x_0, y_0, z_0) = (4, 0, 3)$$

and  $\frac{\partial \phi}{\partial x} = 8, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 6$

$\therefore$  the equation of the tangent plane at the point (4, 0, 3) is

$$(x - 4)8 + (y - 0)0 + (z - 3)6 = 0$$

$$\Rightarrow 4(x - 4) + 3(z - 3) = 0$$

$$\Rightarrow 4x + 3z - 16 - 9 = 0$$

$$\Rightarrow 4x + 3z = 25$$

The equation of the normal at the point (4, 0, 3) is

$$\begin{aligned} \frac{x - x_0}{\frac{\partial \phi}{\partial x}} &= \frac{y - y_0}{\frac{\partial \phi}{\partial y}} = \frac{z - z_0}{\frac{\partial \phi}{\partial z}} \\ \Rightarrow \frac{x - 4}{8} &= \frac{y - 0}{0} = \frac{z - 3}{6} \\ \Rightarrow \frac{x - 4}{4} &= \frac{y}{0} = \frac{z - 3}{3} \end{aligned}$$

### Example 31

If the directional derivative of

$\phi(x, y, z) = a(x + y) + b(y + z) + c(z + x)$  has maximum value 12 at the point (1, 2, 1) in the direction parallel to the line  $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ , find the value of  $a, b, c$ .

### Solution

Given  $\phi = a(x + y) + b(y + z) + c(z + x)$

$$\therefore \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\Rightarrow \nabla \phi = (a+c)\vec{i} + (a+b)\vec{j} + (b+c)\vec{k}$$

We know that the directional derivative is maximum in the direction of  $\nabla \phi$ . But given it is maximum in the direction parallel to the line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}.$$

$$\therefore \frac{a+c}{1} = \frac{a+b}{2} = \frac{b+c}{3} = K$$

$$\Rightarrow a + c = K \quad (1)$$

$$a + b = 2K \quad (2)$$

$$b + c = 3K \quad (3)$$

Adding we get,

$$a + c + a + b + b + c = K + 2K + 3K$$

$$\Rightarrow 2(a + b + c) = 6K$$

$$\Rightarrow a + b + c = 3K \quad (4)$$

$$\text{From (3),} \quad a + 3K = 3K \quad \Rightarrow \quad a = 0$$

$$\text{From (1),} \quad 0 + c = K \quad \Rightarrow \quad c = K$$

$$\text{From (2),} \quad 0 + b = 2K \quad \Rightarrow \quad b = 2K$$

Given the maximum value of directional derivative = 12

$$\Rightarrow |\nabla \phi| = 12$$

$$\Rightarrow \sqrt{(a+c)^2 + (a+b)^2 + (b+c)^2} = 12$$

$$\Rightarrow (a+c)^2 + (a+b)^2 + (b+c)^2 = 144$$

$$\Rightarrow K^2 + 4K^2 + 9K^2 = 144$$

$$\Rightarrow 14K^2 = 144$$

$$\Rightarrow K^2 = \frac{144}{14} \quad \Rightarrow \quad K = \pm \frac{12}{\sqrt{14}}$$

$$\therefore a = 0, \quad b = \pm \frac{24}{\sqrt{14}}, \quad c = \pm \frac{12}{\sqrt{14}}$$

### **Example 32**

If  $\vec{u} = x + y + z$ ,  $\vec{v} = x^2 + y^2 + z^2$ ,  $\vec{w} = xy + yz + zx$ , then show that the vectors  $\nabla u$ ,  $\nabla v$ ,  $\nabla w$  are coplanar.

### **Solution**

$$\text{Given} \quad u = x + y + z, \quad v = x^2 + y^2 + z^2, \quad w = xy + yz + zx$$

$$\text{Now,} \quad \nabla u = \vec{i} \frac{\partial u}{\partial x} + \vec{j} \frac{\partial u}{\partial y} + \vec{k} \frac{\partial u}{\partial z}$$

$$\Rightarrow \nabla u = \vec{i} + \vec{j} + \vec{k}$$

$$\begin{aligned}\nabla v &= \vec{i} \frac{\partial v}{\partial x} + \vec{j} \frac{\partial v}{\partial y} + \vec{k} \frac{\partial v}{\partial z} \\ \Rightarrow \nabla v &= 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \\ \nabla w &= \vec{i} \frac{\partial w}{\partial x} + \vec{j} \frac{\partial w}{\partial y} + \vec{k} \frac{\partial w}{\partial z} \\ \Rightarrow \nabla w &= (y+z)\vec{i} + (z+x)\vec{j} + (x+y)\vec{k}\end{aligned}$$

We know that three vectors  $\vec{a}, \vec{b}, \vec{c}$  are coplanar, if their scalar triple product  $\vec{a} \cdot \vec{b} \times \vec{c} = 0$ .

$\therefore \nabla u, \nabla v, \nabla w$  are coplanar, if  $\nabla u \cdot \nabla v \times \nabla w = 0$

$$\begin{aligned}\text{Now } \nabla u \cdot \nabla v \times \nabla w &= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} \quad R_2 \rightarrow R_2 + R_3 \\ &= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} \\ &= 0 \quad [\text{since } R_1 = R_2]\end{aligned}$$

$\therefore$  the vectors  $\nabla u, \nabla v, \nabla w$  are coplanar.

### EXERCISE 1.1

1. If  $\phi(x, y, z) = 3xz^2y - y^3z^2$ , find  $\nabla\phi$  at the point  $(1, -2, -1)$ .
2. If  $\phi = 2xz - y^2$  find  $\text{grad } \phi$  at the point  $(1, 3, 2)$ .
3. Find the directional derivative of  $\phi = 3x^2 + 2y - 3z$  at the point  $(1, 1, 1)$  in the direction of  $2\vec{i} + 2\vec{j} - \vec{k}$  [AU 2001]
4. Find the directional derivative of  $xyz - xy^2z^2$  at the point  $(1, 2, -1)$  in the direction of the vector  $\vec{i} - \vec{j} - 3\vec{k}$ .

5. Find the directional derivative of the function  $\phi = x^2 - y^2 + 2z^2$  at the point  $P(1, 2, 3)$  in the direction of the line PQ where  $Q = (5, 0, 4)$ .
6. Find the unit normal vector to the surface
- $x^2 + 2y^2 + z^2 = 7$  at the point  $(1, -1, 2)$ .
  - $x^2 + y^2 - z^2 = 1$  at the point  $(1, 1, 1)$ .
  - $x^2 + y^2 - z = 1$  at the point  $(1, 1, 1)$ .
  - $x^2 + y^2 = z$  at the point  $(1, 2, 5)$ .
7. Find the angle between the surfaces  $x^2 + y + z = 2$  and  $x \log z = y^2 - 1$  at the point  $(1, 1, 1)$ . **[AU 2001]**
8. Find the angle between the surfaces  $2yz + z^2 = 3$  and  $x^2 + y^2 + z^2 = 3$  at the point  $(1, 1, 1)$ .
9. Find the angle between the surfaces  $xyz = 4$  and  $x^2 + y^2 + z^2 = 9$  at the point  $\vec{i} + 2\vec{j} + 2\vec{k}$ .
10. Find the equation of the tangent plane and normal line to the surface  $xz^2 + x^2y - z + 1 = 0$  at the point  $(1, -3, 2)$ .
11. Find the equation of the tangent plane and normal line to the surface  $2xz^2 - 3xy - 4x = 7$  at the point  $(1, -1, 2)$ .
12. Find the equation of the tangent plane and normal line to the surface  $2z - x^2 = 0$  at the point  $P(2, 0, 2)$ .
13. Find  $\phi$  if
- $\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (8z^3 - 3x^2yz^2)\vec{k}$
  - $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$  if  $\phi(1, -2, 2) = 4$
  - $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$
  - $\nabla\phi = (2xyz + x)\vec{i} + x^2z\vec{j} + x^2y\vec{k}$
  - $\nabla\phi = (y + \sin z)\vec{i} + x\vec{j} + x \cos z\vec{k}$ .
14. Find the angle between the normals to the intersecting surfaces  $xy - z^2 - 1 = 0$  and  $y^2 - 3z - 1 = 0$  at the point  $(1, 1, 0)$ .
15. Find the angle between the normals to the surface  $x^2 = yz$  at the points  $(1, 1, 1)$  and  $(2, 4, 1)$ .
16. Find the values of  $a$  and  $b$  so that the surfaces  $ax^3 - by^2z = (a+3)x^2$  and  $4x^2y - z^3 = 11$  may cut orthogonally at the point  $(2, -1, -3)$

17. The temperature at any point in space is given by  $T = xy + yz + zx$ . Find the direction in which the temperature changes most rapidly from the point  $(1, 1, 1)$  and determine the maximum rate of change.
18. In what direction is the directional derivative of the function  $\phi = x^2 - 2y^2 + 4z^2$  from the point  $(1, 1, -1)$  is maximum and what is its value?
19. Find the maximum value of the directional derivative of the function  $\phi = 2x^2 + 3y^2 + 5z^2$  at the point  $(1, 1, -4)$ .

### ANSWERS TO EXERCISE 1.1

1.  $-6\vec{i} - 9\vec{j} - 4\vec{k}$
2.  $4\vec{i} - 6\vec{j} + 2\vec{k}$
3.  $\frac{19}{3}$
4.  $\frac{29}{\sqrt{11}}$
5.  $\frac{28}{\sqrt{21}}$
6. (i)  $\frac{\vec{i} - 2\vec{j} + 2\vec{k}}{3}$     (ii)  $\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$     (iii)  $\frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$     (iv)  $\frac{2\vec{i} + 4\vec{j} - 5\vec{k}}{3\sqrt{5}}$
7.  $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$
8.  $\cos^{-1}\sqrt{\frac{3}{5}}$
9.  $\cos^{-1}\sqrt{\frac{2}{3}}$
10.  $2x - y - 3z + 1 = 0, \frac{x-1}{-2} = \frac{y+3}{1} = \frac{z-2}{3}$
11.  $7x - 3y + 8z - 26 = 0, \frac{x-1}{7} = \frac{y+1}{-3} = \frac{z-2}{3}$
12.  $2x - z = 2; \frac{x-2}{-2} = \frac{y}{0} = \frac{z-2}{1}$
13. (i)  $\phi = xy^2 - x^2yz^3 + 3y + 2z^4 + c$     (ii)  $\phi = x^2yz^3 + 20$   
 (iii)  $\phi = 3x^2y + xz^3 - yz + c$                   (iv)  $\phi = x^2yz + \frac{x^2}{2} + c$   
 (v)  $\phi = xy + x \sin z + c$
14.  $\cos^{-1}\left(\frac{2}{\sqrt{26}}\right)$
15.  $\cos^{-1}\frac{13}{3\sqrt{22}}$
16.  $a = -\frac{7}{3}, b = \frac{64}{9}$
17.  $\vec{i} + \vec{j} + \vec{k}, 2\sqrt{3}$
18.  $2\vec{i} - 4\vec{j} - 8\vec{k}, 2\sqrt{21}$
19. 1652

## 1.5 DIVERGENCE OF A VECTOR POINT FUNCTION OR DIVERGENCE OF A VECTOR FIELD

### Definition 1.5

If  $\vec{F}(x, y, z)$  be a vector point function continuously differentiable in a region  $R$  of space, then the **divergence of  $\vec{F}$**  is defined by

$$\nabla \cdot \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

It is abbreviated as  $\operatorname{div} \vec{F}$  and thus  $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$

$$\text{If } \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}, \quad \text{then} \quad \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

**If  $\vec{F}$  is a constant vector, then  $\nabla \cdot \vec{F} = 0$  and conversely if  $\nabla \cdot \vec{F} = 0$ , then  $\vec{F}$  is a constant vector.**

**Note** (i) From the definition it is clear that  $\operatorname{div} \vec{F}$  is a scalar point function. So, the divergence of a vector field is a scalar point function. The notation  $\nabla \cdot \vec{F}$  is not a scalar product in the usual sense, since  $\nabla \cdot \vec{F} \neq \vec{F} \cdot \nabla$ . In fact  $\vec{F} \cdot \nabla = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}$  is a scalar operator.

### 1.5.1 Physical Interpretation of divergence

Physical interpretation of divergence applied to a vector field is that it gives approximately the ‘loss’ of the physical quantity at a given point per unit volume per unit time.

- (i) If  $\vec{v}(x, y, z)$  is the moving fluid at a point  $(x, y, z)$ , then the ‘loss’ of the fluid per unit volume per unit time at the point is given by  $\operatorname{div} \vec{v}$ . Thus divergence gives a measure of the outward flux per unit volume of the flow at  $(x, y, z)$ .  
If there is no ‘loss’ of fluid anywhere, then  $\operatorname{div} \vec{v} = 0$  and the fluid is said to be incompressible.
- (ii) If  $\vec{v}$  represents an electric flux,  $\operatorname{div} \vec{v}$  is the amount of electric flux which diverges per unit volume in unit time.
- (iii) If  $\vec{v}$  represents the heat flux,  $\operatorname{div} \vec{v}$  is the rate at which heat is issuing from a point per unit volume.

**Definition 1.6 Solenoidal vector**

If  $\operatorname{div} \vec{F} = 0$  everywhere in a region  $R$ , then  $\vec{F}$  is called a solenoidal vector point function and  $R$  is called a solenoidal field.

## 1.6 CURL OF A VECTOR POINT FUNCTION OR CURL OF A VECTOR FIELD

**Definition 1.7**

If  $\vec{F}(x, y, z)$  be a vector point function continuously differentiable in a region  $R$ , then the curl of  $\vec{F}$  is defined by

$$\nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

It is abbreviated as  $\operatorname{curl} \vec{F}$

Thus,  $\operatorname{curl} \vec{F} = \nabla \times \vec{F}$

If  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ , then

$$\begin{aligned} \operatorname{curl} \vec{F} &= \nabla \times \vec{F} \\ &= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \times (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \\ &= \vec{i} \left[ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] + \vec{j} \left[ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right] + \vec{k} \left[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \end{aligned}$$

This is symbolically written as  $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

If  $\vec{F}$  is a constant vector, then  $\operatorname{curl} \vec{F} = \vec{0}$

### Physical meaning of $\operatorname{curl} \vec{F}$

If  $\vec{F}$  represents the linear velocity of the point  $P$  of a rigid body that rotates about a fixed axis (e.g., top) with constant angular velocity  $\omega$ , then  $\operatorname{curl} \vec{F}$  at  $P$  is equal to  $2\vec{\omega}$ .

If the body is not rotating, then  $\vec{\omega} = \vec{0}$

$$\therefore \text{Curl } \vec{F} = \vec{0}$$

### **Definition 1.8 Irrotational vector field**

Let  $\vec{F}(x, y, z)$  be a vector point function. If  $\text{curl } \vec{F} = \vec{0}$  at all points in a region  $R$ , then  $\vec{F}$  is said to be an irrotational vector in  $R$ . The vector field  $R$  is called an irrotational vector field.

### **Definition 1.9 Conservative vector field**

A vector field  $\vec{F}$  is said to be conservative if there exists a scalar function  $\phi$  such that  $\vec{F} = \nabla\phi$

#### **Note**

1. In a conservative vector field  $\vec{F} = \nabla\phi$   
 $\therefore \nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0} \Rightarrow \vec{F}$  is irrotational.
2. This scalar function  $\phi$  is called the **scalar potential of  $\vec{F}$** .  
**Only irrotational vectors will have scalar potential  $\phi$ .**

## WORKED EXAMPLES

### **Example 1**

**Prove that  $\nabla \times \nabla\phi = \vec{0}$ , where  $\phi$  is a scalar point function.**

#### **Solution**

We have  $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\therefore \nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i} \left[ \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right] - \vec{j} \left[ \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right] + \vec{k} \left[ \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right]$$

$$= 0$$

[Assuming  $\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}$ ,  $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$ ,  $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ ]

$\therefore \nabla \phi$  is always an irrotational vector.

### Example 2

**Find the divergence and curl of the vector  $\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$  at the point (2, -1, 1).** [AU 1996]

### Solution

Given

$$\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$$

$$\begin{aligned} \text{div } \vec{v} &= \nabla \cdot \vec{v} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z) \\ &= yz + 3x^2 + 2xz - y^2 \end{aligned}$$

At the point (2, -1, 1),  $\nabla \cdot \vec{v} = (-1) \cdot 1 + 3 \cdot 4 + 2 \cdot 2 \cdot 1 - (-1)^2 = -1 + 12 + 4 - 1 = 14$

and

$$\begin{aligned} \text{Curl } \vec{v} &= \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y) \right] \\ &\quad - \vec{j} \left[ \frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz) \right] \\ &= \vec{i}[0 - 2yz - 0] - \vec{j}[z^2 - 0 - xy] + \vec{k}[6xy - xz] \\ &= -2yz\vec{i} - (z^2 - xy)\vec{j} + (6xy - xz)\vec{k} \end{aligned}$$

**At the point (2, -1, 1),**

$$\begin{aligned}\nabla \times \vec{v} &= -2(-1)\cdot \vec{i} - (1^2 - 2(-1))\vec{j} + [6 \cdot 2(-1) - 1 \cdot 2]\vec{k} \\ \Rightarrow \nabla \times \vec{v} &= 2\vec{i} - 3\vec{j} - 14\vec{k}\end{aligned}$$

### Example 3

Find the div  $\vec{F}$  and curl  $\vec{F}$ , where  $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$ . [AU 2001]

### Solution

$$\begin{aligned}\text{Given } \vec{F} &= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (x^3 + y^3 + z^3 - 3xyz) \\ &= \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)\end{aligned}$$

$$\begin{aligned}\therefore \text{div } \vec{F} &= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) \\ &= 6x + 6y + 6z = 6(x + y + z)\end{aligned}$$

$$\text{Let } \phi = x^3 + y^3 + z^3 - 3xyz$$

$$\therefore \vec{F} = \text{grad } \phi = \nabla \phi$$

$$\therefore \text{Curl } \vec{F} = \nabla \times \vec{F} = \nabla \times \nabla \phi = 0 \quad [\text{Using example (1)}]$$

### Example 4

Show that the vector  $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$  is irrotational. [AU 2009, 2010]

### Solution

$$\text{Given } \vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$\vec{F}$  is irrotational if  $\text{curl } \vec{F} = \vec{0}$

$$\begin{aligned}\text{Now } \text{curl } \vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}\end{aligned}$$

$$\begin{aligned}
&= \vec{i} \left[ \frac{\partial}{\partial y} (3xz^2 - y) - \frac{\partial}{\partial z} (3x^2 - z) \right] \\
&\quad - \vec{j} \left[ \frac{\partial}{\partial x} (3xz^2 - y) - \frac{\partial}{\partial z} (6xy + z^3) \right] \\
&\quad + \vec{k} \left[ \frac{\partial}{\partial x} (3x^2 - z) - \frac{\partial}{\partial y} (6xy + z^3) \right] \\
&= \vec{i}[-1+1] - \vec{j}[3z^2 - 3z^2] + \vec{k}[6x - 6x] = \vec{0}.
\end{aligned}$$

$\therefore \vec{F}$  is irrotational vector.

### Example 5

Prove that (i)  $\operatorname{div} \vec{r} = 3$ , (ii)  $\operatorname{curl} \vec{r} = \vec{0}$  where  $\vec{r}$  is the position vector of a point  $(x, y, z)$  in space. [AU 2005, 2010]

### Solution

Given  $\vec{r}$  is the position vector of a point  $(x, y, z)$  in space.

$$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$(i) \quad \operatorname{div} \vec{r} = \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1=3$$

$$\begin{aligned}
(ii) \quad \operatorname{Curl} \vec{r} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= \vec{i} \left[ \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x) \right] - \vec{k} \left[ \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right] \\
&= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0] = \vec{0}
\end{aligned}$$

$\therefore \vec{r}$  is an irrotational vector.

**Example 6**

If  $\vec{v} = \vec{w} \times \vec{r}$ , prove that  $\vec{w} = \frac{1}{2} \operatorname{curl} \vec{v}$ , where  $\vec{w}$  is a constant vector and  $\vec{r}$  is the position vector of the point  $(x, y, z)$ .

**Solution**

Given  $\vec{v} = \vec{w} \times \vec{r}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad [\because \vec{r} \text{ is the position vector of } (x, y, z)]$$

and  $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ ,  $w_1, w_2, w_3$  are constants.

$$\begin{aligned} \therefore \vec{w} \times \vec{r} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} \\ &= \vec{i}[w_2z - w_3y] - \vec{j}[w_1z - w_3x] + \vec{k}[w_1y - w_2x] \end{aligned}$$

$$\operatorname{curl} \vec{v} = \nabla \times (\vec{w} \times \vec{r})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2z - w_3y & w_3x - w_1z & w_1y - w_2x \end{vmatrix}$$

$$\begin{aligned} &= \vec{i} \left[ \frac{\partial}{\partial y} (w_1y - w_2x) - \frac{\partial}{\partial z} (w_3x - w_1z) \right] \\ &\quad - \vec{j} \left[ \frac{\partial}{\partial x} (w_1y - w_2x) - \frac{\partial}{\partial z} (w_2z - w_3y) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x} (w_3x - w_1z) - \frac{\partial}{\partial y} (w_2z - w_3y) \right] \end{aligned}$$

$$\begin{aligned} &= \vec{i}[w_1 + w_1] - \vec{j}[-w_2 - w_2] + \vec{k}[w_3 + w_3] \\ &= 2[w_1\vec{i} + w_2\vec{j} + w_3\vec{k}] = 2\vec{w} \end{aligned}$$

$$\therefore \vec{w} = \frac{1}{2} \operatorname{curl} \vec{v}.$$

**Example 7****Find the value of  $a$  if the vector** **$\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} + (axyz - 2x^2y^2)\vec{k}$  is solenoidal.****Solution**Given  $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} + (axyz - 2x^2y^2)\vec{k}$ 

is solenoidal.

$$\therefore \nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(2x^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(axyz - 2x^2y^2) = 0$$

$$\Rightarrow 4xy + 2xy + ayz = 0$$

$$\Rightarrow 6xy + ayz = 0$$

$$\Rightarrow xy(6 + a) = 0$$

$$\Rightarrow (6 + a) = 0 \quad [\because x \neq 0, y \neq 0]$$

$$\Rightarrow a = -6$$

**Example 8****If  $\vec{r}$  is the position vector of a point  $(x, y, z)$  in space and  $\vec{A}$  is a constant vector, prove that  $\vec{A} \times \vec{r}$  is solenoidal.****Solution**Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  [since  $\vec{r}$  is the position vector of  $(x, y, z)$ ]Let  $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ , where  $a_1, a_2, a_3$  are constants.

$$\begin{aligned} \text{Now } \vec{A} \times \vec{r} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \\ &= \vec{i}(a_2z - a_3y) - \vec{j}[a_1z - a_3x] + \vec{k}[a_1y - a_2x] \end{aligned}$$

To prove  $\vec{A} \times \vec{r}$  is solenoidal. i.e., to prove  $\nabla \cdot (\vec{A} \times \vec{r}) = 0$

$$\text{Now } \nabla \cdot (\vec{A} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) \\ = 0 + 0 + 0 = 0$$

$\therefore \vec{A} \times \vec{r}$  is solenoidal.

### Example 9

Prove that the vector  $\vec{F} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$  is solenoidal.

### Solution

$$\text{Given } \vec{F} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$$

To prove  $\vec{F}$  is solenoidal i.e., to prove  $\nabla \cdot \vec{F} = 0$

$$\text{Now } \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z) \\ = 1+1-2=0$$

$\therefore \vec{F}$  is a solenoidal vector.

### Example 10

Show that  $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$  is irrotational and solenoidal. [AU 2011]

### Solution

$$\text{Given } \vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}.$$

We have to prove  $\vec{F}$  is irrotational and solenoidal.

i.e., to prove  $\nabla \times \vec{F} = \vec{0}$  and  $\nabla \cdot \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix} \\ = \vec{i}(3x - 3x) - \vec{j}[3y - 2z - (-2z + 3y)] + \vec{k}[3z + 2y - (2y + 3z)] \\ = \vec{0}$$

$\therefore \vec{F}$  is irrotational.

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z) \\ &= -2 + 2x + (-2x + 2) = 0 \quad \therefore \vec{F} \text{ is solenoidal.}\end{aligned}$$

**Example 11**

If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $r = |\vec{r}|$ , prove that  $r^n \vec{r}$  is solenoidal if  $n = -3$  and irrotational for all values of  $n$ . [AU 2003, 2004]

**Solution**

Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2 \quad (1)$$

$$\begin{aligned}r^n \vec{r} &= r^n(x\vec{i} + y\vec{j} + z\vec{k}) \\ &= r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}\end{aligned}$$

$$\therefore \operatorname{div}(r^n \vec{r}) = \nabla \cdot (r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) \quad (2)$$

But  $\frac{\partial}{\partial x}(r^n x) = r^n + x \cdot n r^{n-1} \frac{\partial r}{\partial x}$

$$\frac{\partial}{\partial y}(r^n y) = r^n + y \cdot n r^{n-1} \frac{\partial r}{\partial y}$$

$$\frac{\partial}{\partial z}(r^n z) = r^n + z \cdot n r^{n-1} \frac{\partial r}{\partial z}$$

We have,  $r^2 = x^2 + y^2 + z^2, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \frac{\partial}{\partial x}(r^n x) = r^n + n x r^{n-1} \cdot \frac{x}{r} = r^n + n x^2 r^{n-2}$$

$$\frac{\partial}{\partial y}(r^n y) = r^n + n y r^{n-1} \cdot \frac{y}{r} = r^n + n y^2 r^{n-2}$$

and  $\frac{\partial}{\partial z}(r^n z) = r^n + n z r^{n-1} \cdot \frac{z}{r} = r^n + n z^2 r^{n-2}$

Substitute in (2)

$$\begin{aligned}\therefore \operatorname{div}(r^n \vec{r}) &= r^n + nx^2 r^{n-2} + r^n + ny^2 r^{n-2} + r^n + nz^2 r^{n-2} \\ &= 3r^n + nr^{n-2}(x^2 + y^2 + z^2) = 3r^n + nr^{n-2} \cdot r^2 \\ &= 3r^n + nr^n = (n+3)r^n\end{aligned}$$

If  $n = -3$ , then  $\operatorname{div}(r^n \vec{r}) = 0$        $\therefore r^n \vec{r}$  is solenoidal if  $n = -3$

$$\begin{aligned}\text{Now } \nabla \times r^n \vec{r} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x) \right] \\ &= \vec{i} \left( n z r^{n-1} \frac{\partial r}{\partial y} - n y r^{n-1} \frac{\partial r}{\partial z} \right) - \vec{j} \left( n z r^{n-1} \frac{\partial r}{\partial x} - n x r^{n-1} \frac{\partial r}{\partial z} \right) \\ &\quad + \vec{k} \left( n y r^{n-1} \frac{\partial r}{\partial x} - n x r^{n-1} \frac{\partial r}{\partial y} \right) \\ &= \vec{i} \left( n z r^{n-1} \frac{y}{r} - n y r^{n-1} \frac{z}{r} \right) - \vec{j} \left( n z r^{n-1} \cdot \frac{x}{r} - n x r^{n-1} \frac{z}{r} \right) \\ &\quad + \vec{k} \left( n y r^{n-1} \cdot \frac{x}{r} - n x r^{n-1} \cdot \frac{y}{r} \right) \\ &= \vec{i}(nr^{n-2}yz - nr^{n-2}yz) - \vec{j}(nr^{n-2}xz - nr^{n-2}xz) \\ &\quad + \vec{k}(nr^{n-2}xy - nr^{n-2}xy) = \vec{0}\end{aligned}$$

$\therefore \nabla \times (r^n \vec{r}) = \vec{0}$  for all values of  $n$ .

Here  $r^n \vec{r}$  is irrotational for all values of  $n$ .

### Example 12

Show that  $\vec{v} = xyz^2 \vec{u}$  is solenoidal, where

$$\vec{u} = (2x^2 + 8xy^2 z)\vec{i} + (3x^3 y - 3xy)\vec{j} - (4y^2 z^2 + 2x^3 z)\vec{k}. \quad [\text{AU 2008}]$$

### Solution

Given  $\vec{v} = xyz^2 [(2x^2 + 8xy^2 z)\vec{i} + (3x^3 y - 3xy)\vec{j} - (4y^2 z^2 + 2x^3 z)\vec{k}]$ .

We have to prove  $\nabla \cdot \vec{v} = 0$

$$\begin{aligned}
\nabla \cdot \vec{v} &= \frac{\partial}{\partial x} [xyz^2(2x^2 + 8xy^2z)] + \frac{\partial}{\partial y} [xyz^2(3x^3y - 3xy)] \\
&\quad + \frac{\partial}{\partial z} [-xyz^2(4y^2z^2 + 2x^3z)] \\
&= \frac{\partial}{\partial x} [2x^3yz^2 + 8x^2y^3z^3] + \frac{\partial}{\partial y} [3x^4y^2z^2 - 3x^2y^2z^2] \\
&\quad - \frac{\partial}{\partial z} [4xy^3z^4 + 2x^4yz^3] \\
&= 6x^2yz^2 + 16xy^3z^3 + 6x^4yz^2 - 6x^2yz^2 \\
&\quad - (16xy^3z^3 + 6x^4yz^2) \\
&= 0
\end{aligned}$$

Hence  $\vec{v}$  is solenoidal.

### Example 13

Prove that  $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$  is irrotational and find its scalar potential. [AU 2005, 2008]

### Solution

Given

$$\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$$

Now

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}(0 - 0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y \cos x - 2y \cos x) = \vec{0}$$

$\therefore \vec{F}$  is irrotational.

Hence there exist a scalar function  $\phi$  such that  $\vec{F} = \nabla \phi$

$$\Rightarrow (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = y^2 \cos x + z^3 \tag{1}$$

$$\frac{\partial \phi}{\partial y} = 2y \sin x - 4 \quad (2)$$

and

$$\frac{\partial \phi}{\partial z} = 3xz^2 \quad (3)$$

$$\text{Integrating (1) w.r.to } x, \quad \phi = y^2 \sin x + z^3 x + f_1(y, z) \quad (4)$$

$$\text{Integrating (2) w.r.to } y, \quad \phi = y^2 \sin x - 4y + f_2(x, z) \quad (5)$$

$$\text{Integrating (3) w.r.to } z, \quad \phi = xz^3 + f_3(x, y) \quad (6)$$

From (4), (5), (6),  $\phi = y^2 \sin x + xz^3 - 4y + c$  is the scalar potential, where  $c$  is an arbitrary constant.

### Example 14

(i) Find  $a$  such that  $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$  is solenoidal.

[AU 2004, 2012]

(ii) Find  $a, b, c$  if  $(x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$  is irrotational.

### Solution

(i) Let  $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$

Given  $\vec{F}$  is solenoidal.

$\therefore$

$$\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$\Rightarrow$

$$3 + a + 2 = 0 \Rightarrow a = -5$$

(ii) Let  $\vec{F} = (x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$

Given  $\vec{F}$  is irrotational.

$$\therefore \nabla \times \vec{F} = \vec{0} \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\begin{aligned}
&\Rightarrow \vec{i} \left[ \frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z) \right] \\
&\quad - \vec{j} \left[ \frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az) \right] \\
&\quad + \vec{k} \left[ \frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az) \right] = 0 \\
&\Rightarrow \vec{i}(c+1) - \vec{j}(-1-a) + \vec{k}(b-1) = \vec{0} \\
&\Rightarrow (c+1)\vec{i} + (1+a)\vec{j} + (b-1)\vec{k} = \vec{0} \\
&\therefore c+1=0, \quad 1+a=0, \quad b-1=0 \\
&\Rightarrow a=-1, \quad b=1 \text{ and } c=-1
\end{aligned}$$

**Example 15**

Determine  $f(r)$  so that the vector  $f(r)\vec{r}$  is both solenoidal and irrotational.  
[AU 2003, 2008, 2009]

**Solution**

If  $\vec{r}$  is not specified, it will always represent the position vector of any point  $(x, y, z)$ .

$$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\text{and } r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore r^2 = x^2 + y^2 + z^2 \quad (1)$$

$$\begin{aligned}
\therefore f(r)\vec{r} &= f(r)(x\vec{i} + y\vec{j} + z\vec{k}) \\
&= f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}
\end{aligned}$$

Given  $f(r)\vec{r}$  is solenoidal.

$$\therefore \nabla \cdot (f(r)\vec{r}) = 0 \Rightarrow \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) = 0 \quad (2)$$

$$\text{But } \frac{\partial}{\partial x}(f(r)x) = f(r) + xf'(r)\frac{\partial r}{\partial x}$$

$$\frac{\partial}{\partial y}(f(r)y) = f(r) + yf'(r)\frac{\partial r}{\partial y}$$

and

$$\frac{\partial}{\partial z}(f(r)z) = f(r) + zf'(r)\frac{\partial r}{\partial z}$$

Differentiating (1) we get,  $\frac{\partial r}{\partial x} = \frac{x}{r}$ ,  $\frac{\partial r}{\partial y} = \frac{y}{r}$ ,  $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \frac{\partial}{\partial x}(f(r)x) = f(r) + xf'(r)\cdot\frac{x}{r} = f(r) + \frac{x^2}{r}f'(r)$$

$$\text{Similarly, } \frac{\partial}{\partial y}(f(r)y) = f(r) + \frac{y^2}{r}f'(r)$$

$$\text{and } \frac{\partial}{\partial z}(f(r)z) = f(r) + \frac{z^2}{r}f'(r)$$

$$\therefore (2) \Rightarrow f(r) + \frac{x^2}{r}f'(r) + f(r) + \frac{y^2}{r}f'(r) + f(r) + \frac{z^2}{r}f'(r) = 0$$

$$\Rightarrow 3f(r) + \frac{f'(r)}{r}(x^2 + y^2 + z^2) = 0$$

$$\Rightarrow 3f(r) + \frac{f'(r)}{r}\cdot r^2 = 0$$

$$\Rightarrow 3f(r) + rf'(r) = 0$$

$$\Rightarrow \frac{f'(r)}{f(r)} = -\frac{3}{r}, \text{ here } r \text{ is real variable.}$$

Integrating w.r.to 'r', we get  $\int \frac{f'(r)}{f(r)} dr = -3 \int \frac{1}{r} dr$

$$\Rightarrow \log f(r) = -3 \log r + \log c$$

$$\Rightarrow \log f(r) = -\log r^3 + \log c = \log \frac{c}{r^3}$$

$$\therefore f(r) = \frac{c}{r^3}$$

where  $c$  is the constant of integration.

$$\begin{aligned}
 \text{Now } \nabla \times (f(r)\vec{r}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix} \\
 &= \vec{i} \left[ \frac{\partial}{\partial y}(f(r)z) - \frac{\partial}{\partial z}(f(r)y) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(f(r)z) - \frac{\partial}{\partial z}(f(r)x) \right] \\
 &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(f(r)y) - \frac{\partial}{\partial y}(f(r)x) \right] \\
 &= \sum \vec{i} \left[ zf'(r) \cdot \frac{\partial r}{\partial y} - y \cdot f'(r) \cdot \frac{\partial r}{\partial z} \right] \\
 &= \sum \vec{i} \left[ zf'(r) \cdot \frac{y}{r} - y \cdot f'(r) \cdot \frac{z}{r} \right] = \sum \vec{i} f'(r) \left[ \frac{yz}{r} - \frac{yz}{r} \right] = \vec{0}
 \end{aligned}$$

$\therefore f(r)\vec{r}$  is irrotational for all  $f(r)$  and it is solenoidal for  $f(r) = \frac{c}{r^3}$ , where  $c$  is arbitrary constant.

Hence the required function is  $f(r) = \frac{c}{r^3}$ , which is both solenoidal and irrotational.

## EXERCISE 1.2

1. If  $\vec{F} = xy^2 + 2x^2yz\vec{j} - 3yz^2\vec{k}$ , then find  $\operatorname{div} \vec{F}$  and  $\operatorname{curl} \vec{F}$  at  $(1, 1, -1)$ .
2. If  $\vec{F} = x^2\vec{y} + y^2\vec{z} + z^2\vec{x}$  then find  $\operatorname{curl} \operatorname{curl} \vec{F}$ .
3. Find  $\operatorname{div} \vec{F}$  and  $\operatorname{curl} \vec{F}$  at  $(1, 1, 1)$   
if  $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$
4. Show that the following vectors are solenoidal.
  - (i)  $\vec{F} = (2 + 3y)\vec{i} + (x - 2z)\vec{j} + x\vec{k}$
  - (ii)  $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$
  - (iii)  $\vec{F} = 3x^2\vec{y} - 4xy^2\vec{j} + 2xyz\vec{k}$
5. Find the value of  $a$  if  $\vec{F} = ay^4z^2\vec{i} + 4x^3z^2\vec{j} + 5x^2y^2\vec{k}$  is solenoidal.
6. If the vector  $3x\vec{i} + (x + y)\vec{j} - az\vec{k}$  is solenoidal, then find  $a$ .

7. Show that the following vectors are irrotational.

(i)  $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$  [AU 2009]

(ii)  $\vec{F} = (\sin y + z)\vec{i} + (x \cos y - z)\vec{j} + (x - y)\vec{k}$

(iii)  $\vec{F} = (4xy - z^2)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k}$

8. Find the value of  $a$  if  $\vec{F} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$  is irrotational.

9. If  $\vec{F} = (ax^2 + 2y^2 + 1)\vec{i} + (4xy + by^2z - 3)\vec{j} + (c - y^3)\vec{k}$  is irrotational, then find the values of  $a, b, c$ .

10. Show that  $\vec{F} = (2x + 3y + z^2)\vec{i} + (3x + 2y + z)\vec{j} + (y + 2zx)\vec{k}$  is irrotational and hence find its scalar potential.

11. Prove that  $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$  is irrotational and find its scalar potential. [AU 1996]

12. Show that  $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$  is irrotational, find its scalar potential. [AU 2009, 2010]

### ANSWERS TO EXERCISE 1.2

1.  $5; -5\vec{i} - 6\vec{k}$

2.  $2[\vec{z}\vec{i} + \vec{z}\vec{j} + \vec{y}\vec{k}]$

3.  $6; -2\vec{i} + 2\vec{k}$

5.  $a$  can be any real number

6. 4                            8. 2

9.  $a = 3, b = -3, c = 2$

10.  $\phi = x^2 + y^2 + 3xy + yz + z^2x + c$

11.  $\phi = y^2 \sin x + xz^3 - 4y + c$                     12.  $\phi = 3x^2y + xz^3 - yz + c$

## 1.7 VECTOR IDENTITIES

We shall list the vector identities into two categories.

(i)  $\nabla$  operator applied once to point functions.

(ii)  $\nabla$  operator applied twice to point functions.

### TYPE 1.

If  $f$  and  $g$  are scalar point functions we have already proved the following results.

1.  $\nabla c = 0$ , where  $c$  is a constant.
2.  $\nabla(c\phi) = c\nabla\phi$ , where  $c$  is constant.
3.  $\nabla(f \pm g) = \nabla f \pm \nabla g$
4.  $\nabla(fg) = f\nabla g + g\nabla f$
5.  $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

**6. If  $\vec{F}$  and  $\vec{G}$  are vector point functions, then  $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$ .**

**Proof** 
$$\begin{aligned} \nabla \cdot (\vec{F} + \vec{G}) &= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (\vec{F} + \vec{G}) \\ &= \vec{i} \cdot \left( \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial x} \right) + \vec{j} \cdot \left( \frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{G}}{\partial y} \right) + \vec{k} \cdot \left( \frac{\partial \vec{F}}{\partial z} + \frac{\partial \vec{G}}{\partial z} \right) \\ &= \left( \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z} \right) + \left( \vec{i} \cdot \frac{\partial \vec{G}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{G}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{G}}{\partial z} \right) \\ &= \nabla \cdot \vec{F} + \nabla \cdot \vec{G} \end{aligned}$$

Similarly,  $\nabla \cdot (\vec{F} - \vec{G}) = \nabla \cdot \vec{F} - \nabla \cdot \vec{G}$ . ■

**7. If  $f$  is a scalar point function and  $\vec{G}$  is a vector point function, then**

$$\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f(\nabla \cdot \vec{G})$$

**Proof** Let  $\vec{G} = G_1 \vec{i} + G_2 \vec{j} + G_3 \vec{k}$

then  $f\vec{G} = fG_1 \vec{i} + fG_2 \vec{j} + fG_3 \vec{k}$

$$\begin{aligned} \therefore \nabla \cdot (f\vec{G}) &= \frac{\partial}{\partial x}(fG_1) + \frac{\partial}{\partial y}(fG_2) + \frac{\partial}{\partial z}(fG_3) \\ &= f \frac{\partial G_1}{\partial x} + \frac{\partial f}{\partial x}G_1 + f \frac{\partial G_2}{\partial y} + \frac{\partial f}{\partial y}G_2 + f \frac{\partial G_3}{\partial z} + \frac{\partial f}{\partial z}G_3 \\ &= \frac{\partial f}{\partial x}G_1 + \frac{\partial f}{\partial y}G_2 + \frac{\partial f}{\partial z}G_3 + f \left( \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \right) \end{aligned}$$

$$\therefore \nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f(\nabla \cdot \vec{G}) ■$$

8. If  $f$  is a scalar point function and  $\vec{G}$  is a vector point function, then  

$$\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f(\nabla \times \vec{G})$$

**Proof** Let  $\vec{G} = G_1 \vec{i} + G_2 \vec{j} + G_3 \vec{k}$

$$\therefore f\vec{G} = fG_1 \vec{i} + fG_2 \vec{j} + fG_3 \vec{k}$$

$$\begin{aligned} \text{Now } \nabla \times (f\vec{G}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fG_1 & fG_2 & fG_3 \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial}{\partial y}(fG_3) - \frac{\partial}{\partial z}(fG_2) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(fG_3) - \frac{\partial}{\partial z}(fG_1) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(fG_2) - \frac{\partial}{\partial y}(fG_1) \right] \\ &= \vec{i} \left[ f \frac{\partial G_3}{\partial y} + G_3 \frac{\partial f}{\partial y} - f \frac{\partial G_2}{\partial z} - G_2 \frac{\partial f}{\partial z} \right] \\ &\quad - \vec{j} \left[ f \frac{\partial G_3}{\partial x} + G_3 \frac{\partial f}{\partial x} - f \frac{\partial G_1}{\partial z} - G_1 \frac{\partial f}{\partial z} \right] \\ &\quad + \vec{k} \left[ f \frac{\partial G_2}{\partial x} + G_2 \frac{\partial f}{\partial x} - f \frac{\partial G_1}{\partial y} - G_1 \frac{\partial f}{\partial y} \right] \\ &= f \left[ \left( \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} \right) \vec{i} - \left( \frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z} \right) \vec{j} + \left( \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \vec{k} \right] \\ &\quad + \left( \frac{\partial f}{\partial y} G_3 - \frac{\partial f}{\partial z} G_2 \right) \vec{i} - \left( \frac{\partial f}{\partial x} G_3 - \frac{\partial f}{\partial z} G_1 \right) \vec{j} + \left( \frac{\partial f}{\partial x} G_2 - \frac{\partial f}{\partial y} G_1 \right) \vec{k} \\ &= f \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} \end{aligned}$$

$$\therefore \nabla \times (f\vec{G}) = f(\nabla \times \vec{G}) + (\nabla f) \times \vec{G}. \quad \blacksquare$$

9. If  $\vec{F}$  and  $\vec{G}$  are vector point functions, then

$$\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F})$$

**Proof** We know that  $\nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$

$$= \sum \vec{i} \frac{\partial f}{\partial x}$$

$$\therefore \nabla(\vec{F} \cdot \vec{G}) = \sum \vec{i} \frac{\partial}{\partial x} (\vec{F} \cdot \vec{G})$$

$$= \sum \vec{i} \left[ \vec{F} \cdot \frac{\partial \vec{G}}{\partial x} + \vec{G} \cdot \frac{\partial \vec{F}}{\partial x} \right]$$

$$= \sum \left( \vec{F} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{i} + \sum \left( \vec{G} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{i} \quad (1)$$

We know that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$\therefore (\vec{a} \cdot \vec{b})\vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \vec{a} \times (\vec{b} \times \vec{c})$$

$$\therefore \left( \vec{F} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{i} = (\vec{F} \cdot \vec{i}) \left( \frac{\partial \vec{G}}{\partial x} \right) - \vec{F} \times \left( \frac{\partial \vec{G}}{\partial x} \times \vec{i} \right)$$

$$= (\vec{F} \cdot \vec{i}) \left( \frac{\partial \vec{G}}{\partial x} \right) + \vec{F} \times \left( \vec{i} \times \frac{\partial \vec{G}}{\partial x} \right)$$

$$\therefore \sum \left( \vec{F} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{i} = \left( \vec{F} \cdot \sum \vec{i} \frac{\partial}{\partial x} \right) \vec{G} + \vec{F} \times \sum \left( \vec{i} \times \frac{\partial \vec{G}}{\partial x} \right)$$

$$= (\vec{F} \cdot \nabla) \vec{G} + \vec{F} \times (\nabla \times \vec{G}) \quad (2)$$

Interchanging  $\vec{F}$  and  $\vec{G}$ , we get

$$\sum \left( \vec{G} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{i} = (\vec{G} \cdot \nabla) \vec{F} + \vec{G} \times (\nabla \times \vec{F}) \quad (3)$$

Substituting (2) and (3) in (1) we get

$$\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla) \vec{G} + \vec{F} \times (\nabla \times \vec{G}) + (\vec{G} \cdot \nabla) \vec{F} + \vec{G} \times (\nabla \times \vec{F})$$

$$\therefore \nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) \blacksquare$$

**10. If  $\vec{F}$  and  $\vec{G}$  are vector point functions then**

$$\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$$

$$\text{i.e div } \vec{F} \times \vec{G} = \vec{G} \cdot \text{Curl } \vec{F} - \vec{F} \cdot \text{Curl } \vec{G}.$$

[AU 2005, 2008]

**Proof**

$$\begin{aligned}\nabla \cdot (\vec{F} \times \vec{G}) &= \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{F} \times \vec{G}) \\ &= \sum \vec{i} \cdot \left( \frac{\partial \vec{F}}{\partial x} \times \vec{G} + \vec{F} \times \frac{\partial \vec{G}}{\partial x} \right) \\ &= \sum \vec{i} \cdot \left( \frac{\partial \vec{F}}{\partial x} \times \vec{G} \right) + \sum \vec{i} \cdot \left( \vec{F} \times \frac{\partial \vec{G}}{\partial x} \right)\end{aligned}$$

In a scalar triple product  $\cdot$  and  $\times$  can be interchanged.

$\therefore$  we get

$$\nabla \cdot (\vec{F} \times \vec{G}) = \sum \left( \vec{i} \times \frac{\partial \vec{F}}{\partial x} \right) \cdot \vec{G} - \sum \left( \vec{i} \times \frac{\partial \vec{G}}{\partial x} \right) \cdot \vec{F}$$

$\Rightarrow$

$$\nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G} - (\nabla \times \vec{G}) \cdot \vec{F}$$

■

**11. If  $\vec{F}$  and  $\vec{G}$  are vector product functions, then**

$$\nabla \times (\vec{F} \times \vec{G}) = \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F}) + (\vec{G} \cdot \nabla) \vec{F} - (\vec{F} \cdot \nabla) \vec{G}$$

**Proof**

$$\begin{aligned}\nabla \times (\vec{F} \times \vec{G}) &= \sum \vec{i} \times \frac{\partial}{\partial x} (\vec{F} \times \vec{G}) \\ &= \sum \vec{i} \times \left( \frac{\partial \vec{F}}{\partial x} \times \vec{G} + \vec{F} \times \frac{\partial \vec{G}}{\partial x} \right) \\ &= \sum \vec{i} \times \left( \frac{\partial \vec{F}}{\partial x} \times \vec{G} \right) + \sum \vec{i} \times \left( \vec{F} \times \frac{\partial \vec{G}}{\partial x} \right)\end{aligned}$$

We know

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

$\therefore$

$$\begin{aligned}\sum \vec{i} \times \left( \frac{\partial \vec{F}}{\partial x} \times \vec{G} \right) &= \sum \left[ (\vec{i} \cdot \vec{G}) \frac{\partial \vec{F}}{\partial x} - \left( \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{G} \right] \\ &= \sum (\vec{G} \cdot \vec{i}) \frac{\partial \vec{F}}{\partial x} - \sum \left( \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{G}\end{aligned}$$

■

$$\begin{aligned}
&= \vec{G} \cdot \left( \sum \vec{i} \frac{\partial}{\partial x} \right) \vec{F} - \left( \sum \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{G} \\
&= (\vec{G} \cdot \nabla) \vec{F} - (\nabla \cdot \vec{F}) \vec{G}
\end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned}
\sum \vec{i} \times \left( \vec{F} \times \frac{\partial \vec{G}}{\partial x} \right) &= \sum \left[ \left( \vec{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - \sum (\vec{i} \cdot \vec{F}) \frac{\partial \vec{G}}{\partial x} \right] \\
&= \sum \left( \vec{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - \sum (\vec{i} \cdot \vec{F}) \frac{\partial \vec{G}}{\partial x} \\
&= \left( \sum \vec{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - \sum \left( \vec{F} \cdot \vec{i} \frac{\partial}{\partial x} \right) \vec{G} \\
&= (\nabla \cdot \vec{G}) \vec{F} - \vec{F} \cdot \left( \sum \vec{i} \frac{\partial}{\partial x} \right) \vec{G} \\
&= (\nabla \cdot \vec{G}) \vec{F} - (\vec{F} \cdot \nabla) \vec{G}
\end{aligned} \tag{3}$$

Substituting (2) and (3) in (1) we get

$$\nabla \times (\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla) \vec{F} - (\nabla \cdot \vec{F}) \vec{G} + (\nabla \cdot \vec{G}) \vec{F} - (\vec{F} \cdot \nabla) \vec{G}$$

$$\therefore \text{Curl } \vec{F} \times \vec{G} = \vec{F}(\text{div } \vec{G}) - \vec{G}(\text{div } \vec{F}) + (\vec{G} \cdot \nabla) \vec{F} - (\vec{F} \cdot \nabla) \vec{G} \quad \blacksquare$$

### TYPE II – Identities – $\nabla$ Applied Twice

**1. If  $f$  is scalar point function, then  $\text{div grad } f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$**

**Proof** We know,  $\text{grad } f = \nabla f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$

$$\begin{aligned}
\text{div}(\text{grad } f) &= \nabla \cdot \nabla f \\
&= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left( \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z} \right) \\
&= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

$$\therefore \text{div}(\text{grad } f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \blacksquare$$

**Note**  $\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  is a scalar operator called the Laplacian operator.

**2. If  $\vec{F}$  is a vector point function, then  $\operatorname{div} \operatorname{curl} \vec{F} = 0$ .**

[AU 1996, 2005, 2008]

**Proof** Let  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ , where  $F_1, F_2, F_3$  are scalar functions of  $x, y, z$ .

$$\operatorname{Curl} \vec{F} = \nabla \times \vec{F}$$

$$\begin{aligned} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] - \vec{j} \left[ \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right] + \vec{k} \left[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \end{aligned}$$

$$\therefore \operatorname{div} \operatorname{Curl} \vec{F} = \nabla \cdot \nabla \times \vec{F}$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0 \\ &\quad [\text{Assuming } \frac{\partial^2 F_3}{\partial x \partial y} = \frac{\partial^2 F_3}{\partial y \partial x}, \frac{\partial^2 F_2}{\partial x \partial z} = \frac{\partial^2 F_2}{\partial z \partial x}, \frac{\partial^2 F_1}{\partial y \partial z} = \frac{\partial^2 F_1}{\partial z \partial y}] \quad \blacksquare \end{aligned}$$

**3. If  $\vec{F}$  is a vector point function, then**

$$\operatorname{curl} (\operatorname{Curl} \vec{F}) = \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}. \quad [\text{AU 2004, 2006}]$$

**Proof** Let  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ , where  $F_1, F_2, F_3$  are scalar functions.

$$\text{Then} \quad \operatorname{Curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \vec{j} \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \vec{k} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\begin{aligned}
 \therefore \operatorname{Curl} \operatorname{curl} \vec{F} &= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{array} \right| \\
 &= \sum \vec{i} \left[ \frac{\partial}{\partial y} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \frac{\partial}{\partial z} \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \right] \\
 &= \sum \vec{i} \left[ \frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial x} \right] \\
 &= \sum \vec{i} \left\{ \frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} - \left( \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right\} \\
 &= \sum \vec{i} \left\{ \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \left( \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right\} \\
 &= \sum \vec{i} \left\{ \frac{\partial}{\partial x} \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) - \left( \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right\} \\
 &= \sum \vec{i} \left\{ \frac{\partial}{\partial x} (\nabla \cdot \vec{F}) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) F_1 \right\} \\
 &= \sum \vec{i} \left\{ \frac{\partial}{\partial x} (\nabla \cdot \vec{F}) - \nabla^2 F_1 \right\} \\
 &= \left( \sum \vec{i} \frac{\partial}{\partial x} \right) (\nabla \cdot \vec{F}) - \nabla^2 \left( \sum \vec{i} F_1 \right) \\
 \therefore \nabla \times (\nabla \times \vec{F}) &= \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F} \quad \blacksquare
 \end{aligned}$$

## WORKED EXAMPLES

### Example 1

Prove that  $\nabla \left( \frac{1}{r^n} \right) = -\frac{n}{r^{n+2}} \vec{r}$ .

### Solution

We have  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $r^2 = x^2 + y^2 + z^2$

$$\begin{aligned} \therefore \quad \frac{\partial r}{\partial x} &= \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \\ \therefore \quad \nabla \left( \frac{1}{r^n} \right) &= \vec{i} \frac{\partial}{\partial x} \left( \frac{1}{r^n} \right) + \vec{j} \frac{\partial}{\partial y} \left( \frac{1}{r^n} \right) + \vec{k} \frac{\partial}{\partial z} \left( \frac{1}{r^n} \right) \\ &= \vec{i} \left( \frac{-n}{r^{n+1}} \frac{\partial r}{\partial x} \right) + \vec{j} \left( \frac{-n}{r^{n+1}} \frac{\partial r}{\partial y} \right) + \vec{k} \left( \frac{-n}{r^{n+1}} \frac{\partial r}{\partial z} \right) \\ &= -\frac{n}{r^{n+1}} \left[ \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right] \\ &= -\frac{n}{r^{n+2}} (\vec{x} \vec{i} + \vec{y} \vec{j} + \vec{z} \vec{k}) = -\frac{n}{r^{n+2}} \vec{r} \end{aligned}$$

$\therefore \quad \nabla \left( \frac{1}{r^n} \right) = -\frac{n}{r^{n+2}} \vec{r}$

**Note** We have

$$\nabla \left( \frac{1}{r^n} \right) = -\frac{n}{r^{n+2}} \vec{r}$$

If  $n = 1, 2, 3, 4, \dots$

$$\begin{aligned} \nabla \left( \frac{1}{r} \right) &= -\frac{1}{r^3} \vec{r} \\ \nabla \left( \frac{1}{r^2} \right) &= -\frac{2}{r^4} \vec{r} \\ \nabla \left( \frac{1}{r^3} \right) &= -\frac{3}{r^5} \vec{r} \\ \nabla \left( \frac{1}{r^4} \right) &= -\frac{4}{r^6} \vec{r} \text{ and so on.} \end{aligned}$$

### Example 2

Prove that  $\nabla^2 \left( \frac{1}{r} \right) = 0.$

[AU 2009]

### Solution

We have from the above result  $\nabla \left( \frac{1}{r} \right) = -\frac{1}{r^3} \vec{r}.$

$$\begin{aligned}
\therefore \nabla^2 \left( \frac{1}{r} \right) &= \nabla \cdot \nabla \left( \frac{1}{r} \right) \\
&= \nabla \cdot \left( -\frac{1}{r^3} \vec{r} \right) \\
&= - \left[ \nabla \left( \frac{1}{r^3} \right) \cdot \vec{r} + \frac{1}{r^3} \nabla \cdot \vec{r} \right] \quad [\text{by identity 7}] \\
&= - \left[ -\frac{3}{r^5} (\vec{r} \cdot \vec{r}) + \frac{1}{r^3} 3 \right] \quad [\because \nabla \cdot \vec{r} = 3] \\
&= \frac{3}{r^5} r^2 - \frac{3}{r^3} \\
&= \frac{3}{r^3} - \frac{3}{r^3} = 0.
\end{aligned}$$

**Example 3**

**Prove that**  $\nabla^2(r^n) = n(n+1)r^{n-2}$ .

**Solution**

We have  $\vec{r} = \vec{x}i + \vec{y}j + \vec{z}k$  and  $r^2 = x^2 + y^2 + z^2$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned}
\therefore \nabla(r^n) &= \sum \vec{i} \frac{\partial}{\partial x}(r^n) \\
&= \sum \vec{i} nr^{n-1} \frac{\partial r}{\partial x} \\
&= \sum \vec{i} nr^{n-1} \frac{x}{r} \\
&= nr^{n-2} \sum \vec{x}i \\
&= nr^{n-2}(\vec{x}i + \vec{y}j + \vec{z}k) \\
&= nr^{n-2} \vec{r} \quad \text{if } n \geq 3
\end{aligned} \tag{1}$$

$$\begin{aligned}
\nabla^2(r^n) &= \nabla \cdot (\nabla r^n) \\
&= \nabla \cdot (nr^{n-2} \vec{r}) \\
&= n[\nabla r^{n-2} \cdot \vec{r} + r^{n-2}(\nabla \cdot \vec{r})]
\end{aligned}$$

$$\begin{aligned}
 &= n[(n-2)r^{n-4}\vec{r} \cdot \vec{r} + r^{n-2}3] \quad [\text{using (1)}] \\
 &= n[(n-2)r^{n-4}r^2 + 3r^{n-2}] \\
 &= nr^{n-2}[n-2+3] = n(n+1)r^{n-2}
 \end{aligned}$$

**Note** We have  $\nabla(r^n) = nr^{n-2}\vec{r}$

If  $n = 1, 2, 3, 4, \dots$

$$\begin{aligned}
 \nabla(r) &= \frac{1}{r}\vec{r} \\
 \nabla(r^2) &= 2 \cdot r^{2-2}\vec{r} = 2\vec{r} \\
 \nabla(r^3) &= 3r\vec{r} \\
 \nabla(r^4) &= 4r^2\vec{r} \\
 &\vdots \quad \vdots \\
 \nabla(r^{n-2}) &= (n-2)r^{n-4}\vec{r}, \text{ etc.}
 \end{aligned}$$

#### Example 4

Prove that  $\nabla \cdot \left( r \nabla \left( \frac{1}{r^3} \right) \right) = \frac{3}{r^4}$ .

#### Solution

We have  $\nabla \left( \frac{1}{r^3} \right) = -\frac{3}{r^5}\vec{r}$

$$\nabla \left( \frac{1}{r^4} \right) = \frac{-4}{r^6}\vec{r} \quad \text{and} \quad \nabla \cdot \vec{r} = 3$$

$$\begin{aligned}
 \therefore \nabla \cdot \left( r \nabla \left( \frac{1}{r^3} \right) \right) &= \nabla \cdot \left( r \frac{-3}{r^5}\vec{r} \right) \\
 &= \nabla \cdot \left( \frac{-3}{r^4}\vec{r} \right) = -3 \left[ \nabla \left( \frac{1}{r^4} \right) \cdot \vec{r} + \frac{1}{r^4} \nabla \cdot \vec{r} \right] \\
 &= -3 \left[ -\frac{4}{r^6}(\vec{r} \cdot \vec{r}) + \frac{3}{r^4} \right] = -3 \left[ -\frac{4}{r^6}r^2 + \frac{3}{r^4} \right] \\
 &= -3 \left[ \frac{-4}{r^4} + \frac{3}{r^4} \right] = \frac{3}{r^4}
 \end{aligned}$$

**Example 5**

If  $\phi$  and  $\psi$  satisfy Laplace equation, prove that the vector  $(\phi \nabla \psi - \psi \nabla \phi)$  is solenoidal.

**Solution**

Given  $\phi$  and  $\psi$  satisfy Laplace equation.

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (1)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (2)$$

To prove  $(\phi \nabla \psi - \psi \nabla \phi)$  is solenoidal, we have to prove

$$\operatorname{div}(\phi \nabla \psi - \psi \nabla \phi) = 0$$

$$\begin{aligned} \text{Now } \operatorname{div}(\phi \nabla \psi - \psi \nabla \phi) &= \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) \\ &= \nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi) \\ &= \nabla \phi \cdot \nabla \psi + \phi (\nabla \cdot \nabla \psi) - [\nabla \psi \cdot \nabla \phi + \psi (\nabla \cdot \nabla \phi)] \\ &= \phi \nabla^2 \psi - \psi \nabla^2 \phi \quad [\because \nabla \phi \cdot \nabla \psi = \nabla \psi \cdot \nabla \phi] \\ &= \phi 0 - \psi 0 = 0 \quad [\text{from (1) and (2)}] \end{aligned}$$

$\therefore (\phi \nabla \psi - \psi \nabla \phi)$  is solenoidal.

**Example 6**

Prove that  $\nabla \times (\nabla r^n) = \vec{0}$

**Solution**

$$\text{We have } \nabla r^n = n r^{n-2} \vec{r} \quad [\text{Ref. example 3}]$$

$$\begin{aligned} \therefore \nabla \times \nabla r^n &= \nabla \times (n r^{n-2} \vec{r}) = (\nabla n r^{n-2}) \times \vec{r} + n r^{n-2} (\nabla \times \vec{r}) \\ &= n (\nabla r^{n-2}) \times \vec{r} + \vec{0} \quad [\because \nabla \times \vec{r} = \vec{0}] \\ &= n(n-2) r^{n-4} \vec{r} \times \vec{r} = \vec{0} \quad [\because \vec{r} \times \vec{r} = \vec{0}] \end{aligned}$$

**Note** This shows that  $\nabla r^n$  is irrotational.

**Example 7**

Show that  $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$

**Solution**

We have,  $\nabla f(r) = f'(r) \frac{\vec{r}}{r}$

$$\begin{aligned}
 \therefore \quad \nabla^2 f(r) &= \nabla \cdot \nabla f(r) = \nabla \cdot f'(r) \frac{\vec{r}}{r} \\
 &= \left( \nabla \frac{f'(r)}{r} \right) \cdot \vec{r} + \frac{f'(r)}{r} (\nabla \cdot \vec{r}) \\
 &= \left( \nabla \frac{f'(r)}{r} \cdot \vec{r} \right) + \frac{3f'(r)}{r} \quad [\because \nabla \cdot \vec{r} = 3] \\
 &= \left( \frac{r \nabla f'(r) - f'(r) \nabla r}{r^2} \right) \cdot \vec{r} + \frac{3f'(r)}{r} \quad \left[ \because \nabla \left( \frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2} \right] \\
 &= \frac{\left( rf''(r) \frac{\vec{r}}{r} - f'(r) \frac{\vec{r}}{r} \right) \cdot \vec{r}}{r^2} + \frac{3f'(r)}{r} \quad \left[ \because \nabla f'(r) = f''(r) \frac{\vec{r}}{r} \right. \\
 &\quad \left. \text{and } \nabla \vec{r} = \frac{\vec{r}}{r} \right] \\
 &= \frac{[rf''(r) - f'(r)] \vec{r} \cdot \vec{r}}{r^3} + \frac{3f'(r)}{r} \\
 &= \frac{[rf''(r) - f'(r)]}{r^2} + \frac{3f'(r)}{r} \quad [\because \vec{r} \cdot \vec{r} = r^2] \\
 &= \frac{rf''(r) - f'(r)}{r} + \frac{3f'(r)}{r} \\
 &= f''(r) + \frac{2f'(r)}{r} \\
 &= \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}
 \end{aligned}$$

**Example 8**

Using  $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$ , prove that  $\nabla^4(e^r) = e^r + \frac{4}{r}e^r$ .

**Solution**

Let

$$f(r) = e^r.$$

$$f'(r) = e^r$$

$$f''(r) = e^r$$

Given

$$\nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r}$$

$$= e^r + \frac{2}{r}e^r$$

$$\nabla^4 f(r) = \nabla^2[\nabla^2 f(r)] = \nabla^2 \left[ e^r + \frac{2}{r}e^r \right]$$

$$\text{Let } F(r) = e^r + \frac{2}{r}e^r$$

$$F'(r) = e^r + 2 \left[ \frac{re^r - e^r \cdot 1}{r^2} \right] = e^r + 2 \frac{e^r}{r} - \frac{2}{r^2}e^r$$

$$F''(r) = e^r + 2 \left[ \frac{re^r - e^r \cdot 1}{r^2} \right] - 2 \left[ \frac{r^2e^r - 2r e^r}{r^4} \right]$$

$$\begin{aligned} \Rightarrow F''(r) &= e^r + 2 \left[ \frac{e^r}{r} - \frac{1}{r^2}e^r \right] - 2 \left[ \frac{e^r}{r^2} - \frac{2}{r^3}e^r \right] \\ &= e^r + 2 \frac{e^r}{r} - 4 \frac{e^r}{r^2} + 4 \frac{e^r}{r^3} \end{aligned}$$

$$\text{But } \nabla^2 F(r) = F''(r) + \frac{2}{r}F'(r)$$

$$\begin{aligned} \therefore \nabla^2 \left( e^r + \frac{2}{r}e^r \right) &= e^r + 2 \frac{e^r}{r} - 4 \frac{e^r}{r^2} + 4 \frac{e^r}{r^3} + \frac{2}{r} \left[ e^r + 2 \frac{e^r}{r} - \frac{2}{r^2}e^r \right] \\ &= e^r + \frac{4}{r}e^r \end{aligned}$$

$$\therefore \nabla^4(e^r) = \nabla^2 \left( e^r + \frac{2}{r}e^r \right) = e^r + \frac{4}{r}e^r.$$

## 1.8 INTEGRATION OF VECTOR FUNCTIONS

Let  $\vec{f}(t)$  and  $\vec{F}(t)$  be two vector functions of a scalar variable  $t$  such that

$\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$ . Then  $\vec{F}(t)$  is called an indefinite integral of  $\vec{f}(t)$  with respect to  $t$  and is written as  $\int \vec{f}(t) dt = \vec{F}(t) + \vec{c}$ , where  $\vec{c}$  is an arbitrary constant vector independent of  $t$  and is called the constant of integration.

The definite integral of  $\vec{f}(t)$  between the limits  $t = t_1$  and  $t = t_2$  is given by

$$\begin{aligned} \int_{t_1}^{t_2} \vec{f}(t) dt &= \left[ \vec{F}(t) \right]_{t_1}^{t_2} \\ &= \vec{F}(t_2) - \vec{F}(t_1) \end{aligned}$$

As in the case of differentiation of vectors, in order to integrate a vector function, we integrate its components.

If  $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ , then

$$\int \vec{f}(t) dt = \vec{i} \int f_1(t) dt + \vec{j} \int f_2(t) dt + \vec{k} \int f_3(t) dt$$

### 1.8.1 Line integral

An integral evaluated over a curve  $C$  is called a line integral. We call  $C$  as the path of integration. We assume every path of integration of a line integral to be piecewise smooth consisting of finitely many smooth curves.

#### Definition 1.11

A line integral of a vector point function  $\vec{F}(\vec{r})$  over a curve  $C$ , where  $\vec{r}$  is the position vector of any point on  $C$ , is defined by  $\int_C \vec{F} \cdot d\vec{r}$

If  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  and  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ , then

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \text{ and } \int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

Here  $F_1, F_2, F_3$  are functions of  $x, y, z$ , where  $x, y, z$  depend on a parameter  $t \in [a, b]$ , since  $\vec{r}(t)$  is the equation of the curve  $C$ .

Then we can write  $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$ .

If the path of integration  $C$  is a closed curve, we write  $\oint_C$  instead of  $\int_C$ .

**Note**

- Since  $\frac{d\vec{r}}{dt}$  is a tangent vector to the curve  $C$  the line integral  $\int_C \vec{F} \cdot d\vec{r}$  is also called the tangential line integral of  $\vec{F}$  over  $C$  and **line integral is a scalar**.
- Two other types of line integrals are also considered.  $\int_C \vec{F} \times d\vec{r}$  and  $\int_C \phi d\vec{r}$  are vectors.

**WORKED EXAMPLES****Example 1**

If  $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $C$  is the arc of the parabola  $y = 2x^2$  from  $(0, 0)$  to  $(1, 2)$ . [AU 2001]

**Solution**

Given  $\vec{F} = 3xy\vec{i} - y^2\vec{j}$

$\vec{r} = xi + yj$ , where  $\vec{r}$  is the position vector of any point  $(x, y)$  on  $y = 2x^2$ .

$$\therefore d\vec{r} = dx\vec{i} + dy\vec{j}$$

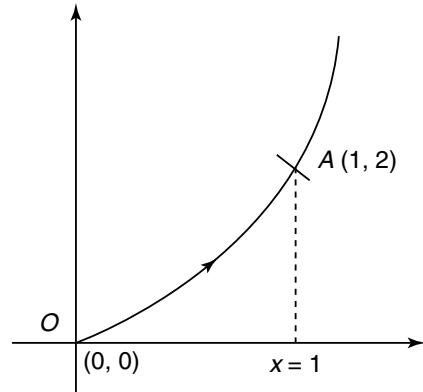
and  $\vec{F} \cdot d\vec{r} = (3xy\vec{i} - y^2\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$

$$= 3xydx - y^2dy$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C (3xydx - y^2dy)$$

Equation of  $C$  is  $y = 2x^2$   $\therefore dy = 4x dx$ .

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3x \cdot 2x^2 dx - 4x^4 \cdot 4x dx)$$



$$\begin{aligned}
 &= \int_0^1 (6x^3 - 16x^5) dx \\
 &= \left[ 6 \frac{x^4}{4} - 16 \frac{x^6}{6} \right]_0^1 \\
 &= \frac{3}{2} - \frac{8}{3} = \frac{9 - 16}{6} = -\frac{7}{6}
 \end{aligned}$$

**Example 2**

If  $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  from  $(0, 0, 0)$  to  $(1, 1, 1)$  along the curve  $C$  given by  $x = t, y = t^2, z = t^3$ .

**Solution**

Given  $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

and  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

and  $\vec{F} \cdot d\vec{r} = [(3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$

$$\begin{aligned}
 &= (3x^2 + 6y)dx - 14yzdy + 20xz^2dz
 \end{aligned}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$$

Given  $x = t, y = t^2, z = t^3$  is the curve.

$$\therefore dx = dt, \quad dy = 2t dt, \quad dz = 3t^2 dt$$

Limits  $t = 0, t = 1$

$$\begin{aligned}
 \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (3 \cdot t^2 + 6 \cdot t^2)dt - 14 \cdot t^5 \cdot 2t dt + 20t^7 \cdot 3t^2 dt \\
 &= \int_0^1 (9t^2 - 28t^6 + 60t^9) dt \\
 &= \left[ 9 \frac{t^3}{3} - 28 \frac{t^7}{7} + 60 \frac{t^{10}}{10} \right]_0^1 \\
 &= 3 - 4 + 6 = 5.
 \end{aligned}$$

**Example 3**

Evaluate the line integral  $\int_C (y^2 dx - x^2 dy)$  around the triangle whose vertices are  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$  in the positive sense.

**Solution**

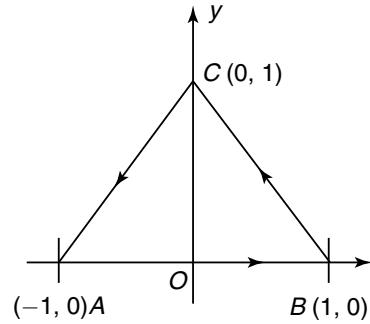
Given the path  $C$  consists of the sides of the  $\triangle ABC$ , where  $A(-1, 0)$ ,  $B(1, 0)$  and  $C(0, 1)$ .

Equation of  $AB$  is  $y = 0$

$$\text{Equation of } BC \text{ is } \frac{y-0}{0-1} = \frac{x-1}{1-0} \Rightarrow y = -x + 1$$

$$\text{Equation of } CA \text{ is } \frac{y-1}{1-0} = \frac{x-0}{0+1} \Rightarrow y = x + 1$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_{AB} (y^2 dx - x^2 dy) \\ &\quad + \int_{BC} (y^2 dx - x^2 dy) \\ &\quad + \int_{CA} (y^2 dx - x^2 dy) \end{aligned}$$



**On  $AB$ ,**  $y = 0$ ,  $\therefore dy = 0$ ,

$x$  varies from  $-1$  to  $1$

$$\therefore \int_{AB} (y^2 dx - x^2 dy) = \int_{-1}^1 0 \, dx = 0$$

**On  $BC$ ,**  $y = -x + 1$   $\therefore dy = -dx$

From  $B$  to  $C$ ,  $x$  varies from  $1$  to  $0$ .

$$\begin{aligned} \therefore \int_{BC} (y^2 dx - x^2 dy) &= \int_1^0 (-x+1)^2 dx - x^2(-dx) \\ &= \int_1^0 (x^2 - 2x + 1 + x^2) dx \\ &= \int_1^0 (2x^2 - 2x + 1) dx \\ &= \left[ 2\frac{x^3}{3} - 2\frac{x^2}{2} + x \right]_1^0 \\ &= 0 - \left( \frac{2}{3} - 1 + 1 \right) = -\frac{2}{3} \end{aligned}$$

**On CA,  $y = x + 1 \quad \therefore dy = dx$**

From C to A, x varies from 0 to -1

$$\begin{aligned}\therefore \int_{CA} (y^2 dx - x^2 dy) &= \int_0^{-1} (x+1)^2 dx - x^2 dx \\ &= \int_0^{-1} (x^2 + 2x + 1 - x^2) dx \\ &= \int_0^{-1} (2x + 1) dx \\ &= [x^2 + x]_0^{-1} = 1 - 1 - 0 = 0 \\ \therefore \int_C \vec{F} \cdot d\vec{r} &= 0 + \left(-\frac{2}{3}\right) + 0 = -\frac{2}{3}\end{aligned}$$

#### Example 4

If  $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where C is the part of the curve  $y = x^2$  between  $x = 1$  and  $x = 2$ . [AU 2003]

#### Solution

$$\text{Given } \vec{F} = 5xy\vec{i} + 2y\vec{j}$$

$$\text{and } \vec{r} = x\vec{i} + y\vec{j}$$

$$\therefore d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\begin{aligned}\therefore \vec{F} \cdot d\vec{r} &= (5xy\vec{i} + 2y\vec{j}) \cdot (dx\vec{i} + dy\vec{j}) \\ &= 5xy dx + 2y dy\end{aligned}$$

The curve C is  $y = x^2 \quad \therefore dy = 2xdx$

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{r} &= \int_C 5xy dx + 2y dy \\ &= \int_1^2 5x \cdot x^2 dx + 2x^2 \cdot 2xdx\end{aligned}$$

$$\begin{aligned}
 &= \int_1^2 (5x^3 + 4x^3) dx \\
 &= \int_1^2 9x^3 dx = 9 \left[ \frac{x^4}{4} \right]_1^2 = \frac{9}{4} [2^4 - 1] = \frac{135}{4}
 \end{aligned}$$

**Example 5**

If  $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the straight line joining  $(0, 0, 0)$  to  $(1, 1, 1)$ .

**Solution**

Given

$$\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \therefore d\vec{r} = d\vec{x} + d\vec{y} + d\vec{z}$$

$$\begin{aligned}
 \therefore \vec{F} \cdot d\vec{r} &= [(3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}] \cdot [d\vec{x} + d\vec{y} + d\vec{z}] \\
 &= (3x^2 + 6y)dx - 14yz dy + 20xz^2 dz
 \end{aligned}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C (3x^2 + 6y)dx - 14yz dy + 20xz^2 dz$$

Equation of the line joining  $(0, 0, 0)$  to  $(1, 1, 1)$  is

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} \Rightarrow x = y = z = t, \text{ say}$$

$$\therefore dx = dt, \quad dy = dt, \quad dz = dt$$

At the point  $(0, 0, 0)$ ,  $t = 0$  and at the point  $(1, 1, 1)$ ,  $t = 1$

$$\begin{aligned}
 \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (3t^2 + 6t)dt - 14t^2 dt + 20t^3 dt \\
 &= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3) dt \\
 &= \int_0^1 (20t^3 - 11t^2 + 6t) dt \\
 &= \left[ 20 \frac{t^4}{4} - 11 \frac{t^3}{3} + 6 \frac{t^2}{2} \right]_0^1 = 5 - \frac{11}{3} + 3 = \frac{13}{3}
 \end{aligned}$$

**Definition 1.12 Work done by a force**

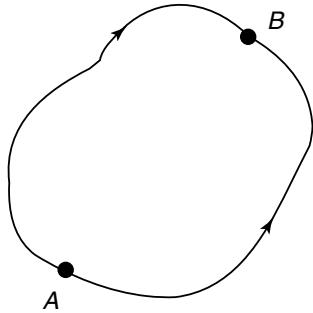
If  $\vec{F}(x, y, z)$  is a force acting on a particle which is moving along arc  $AB$  then  $\int_A^B \vec{F} \cdot d\vec{r}$  gives the total work done by the force  $\vec{F}$  in displacing the particle from  $A$  to  $B$ .

**Conservative force field**

A line integral  $\int_C \vec{F} \cdot d\vec{r}$  is independent of path in domain  $D$  if and only if  $\vec{F} = \nabla\phi$  for some scalar function  $\phi$  defined in  $D$ . Such a force field is called a conservative field.

In the conservative field the total work done by  $\vec{F}$  from  $A$  to  $B$  is

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C \nabla\phi \cdot d\vec{r} \\ &= \int_C \left( \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right) \\ &= \int_C \partial\phi = \int_A^B \partial\phi \\ \therefore \quad \int_C \vec{F} \cdot d\vec{r} &= [\phi]_A^B = \phi(B) - \phi(A)\end{aligned}$$



So in a conservative field the work done depends on the value of  $\phi$  at the end points  $A$  and  $B$  of the path, but not on the path.

**Note**

1.  $\phi$  is scalar potential.

2. If  $\vec{F}$  is conservative, then  $\vec{F} = \nabla\phi$

$$\Rightarrow \quad \nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0}$$

$\Rightarrow \quad \vec{F}$  is irrotational.

3. If  $C$  is a simple closed curve and  $\vec{F}$  is conservative, then  $\int_C \vec{F} \cdot d\vec{r} = 0$ .

## WORKED EXAMPLES

### Example 6

Show that  $\vec{F} = (e^x z - 2xy)\vec{i} - (x^2 - 1)\vec{j} + (e^x + z)\vec{k}$  is a conservative field. Hence evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where the end points of  $C$  are  $(0, 1, -1)$  and  $(2, 3, 0)$ .

### Solution

To prove that  $\vec{F}$  is conservative, we have to prove  $\nabla \times \vec{F} = \vec{0}$

$$\text{Now } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x z - 2xy & 1 - x^2 & e^x + z \end{vmatrix} = \vec{i}[0] - \vec{j}(e^x - e^x) + \vec{k}(-2x + 2x) = \vec{0}$$

Hence  $\vec{F}$  is conservative.  $\therefore \vec{F} = \nabla \phi$

$$\Rightarrow (e^x z - 2xy)\vec{i} + (1 - x^2)\vec{j} + (e^x + z)\vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = e^x z - 2xy \quad (1)$$

$$\frac{\partial \phi}{\partial y} = 1 - x^2 \quad (2)$$

$$\frac{\partial \phi}{\partial z} = e^x + z \quad (3)$$

Integrating (1) w. r. to  $x$ ,  $\phi = z e^x - x^2 y + f_1(y, z)$

Integrating (2) w. r. to  $y$ ,  $\phi = (1 - x^2)y + f_2(x, z)$

Integrating (3) w. r. to  $z$ ,  $\phi = e^x z + \frac{z^2}{2} + f_3(x, y)$

$$\therefore \phi = z e^x - x^2 y + y + \frac{z^2}{2} + C$$

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{r} &= [\phi]_{(0,1,-1)}^{(2,3,0)} \\ &= \left[ 0 - 2^2 \cdot 3 + 3 + C - \left( -1 - 0 + 1 + \frac{1}{2} + C \right) \right] \\ &= -12 + 3 - \frac{1}{2} = -\frac{19}{2}\end{aligned}$$

**Example 7**

If  $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$ , check whether the integral  $\int_C \vec{F} \cdot d\vec{r}$  is independent of the path  $C$ . [AU 2000]

**Solution**

Given  $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$

$$\begin{aligned}\text{Now } \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix} \\ &= \vec{i} \left\{ \frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2) \right\} \\ &\quad - \vec{j} \left\{ \frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy - 3x^2z^2) \right\} \\ &\quad + \vec{k} \left\{ \frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z^2) \right\} \\ &= \vec{i}\{0 - 0\} - \vec{j}\{-6x^2z + 6x^2z\} + \vec{k}\{4x - 4x\} = \vec{0}\end{aligned}$$

$\therefore \vec{F}$  is conservative.

Hence  $\int_C \vec{F} \cdot d\vec{r}$  is independent of the path  $C$ .

**Example 8**

Show that the vector field  $\vec{F}$ , where

$\vec{F} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}$ , is conservative and find its scalar potential.

**Solution**

Given  $\vec{F} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}$

$$\text{Now } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + y^2 + z^2 & x + z + 2xy & y + 2zx \end{vmatrix} = \vec{i}[1-1] - \vec{j}[2z-2z] + \vec{k}[1+2y-(1+2y)] = \vec{0}$$

$\therefore \vec{F}$  is conservative.

So, there exists scalar potential  $\phi$  such that  $\vec{F} = \nabla\phi$ .

$$\Rightarrow (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \frac{\partial\phi}{\partial x} = y + y^2 + z^2 \quad (1)$$

$$\frac{\partial\phi}{\partial y} = x + z + 2xy \quad (2)$$

$$\frac{\partial\phi}{\partial z} = y + 2zx \quad (3)$$

$$\text{Integrating (1) w. r. to } x, \quad \phi = (y + y^2 + z^2)x + f_1(y, z)$$

$$\text{Integrating (2) w. r. to } y, \quad \phi = (x + z)y + xy^2 + f_2(x, z)$$

$$\text{Integrating (3) w. r. to } z, \quad \phi = yz + z^2x + f_3(x, y)$$

$$\therefore \phi = xy + xy^2 + yz + xz^2 + C$$

**Example 9**

Show that  $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$  is a conservative field. Find the scalar potential and work done in moving an object in this field from  $(1, -2, 1)$  to  $(3, 1, 4)$ .

**Solution**

Given  $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

$$\begin{aligned} \text{Now } \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial}{\partial y} (3xz^2) - \frac{\partial}{\partial z} (x^2) \right] - \vec{j} \left[ \frac{\partial}{\partial x} (3xz^2) - \frac{\partial}{\partial z} (2xy + z^3) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (2xy + z^3) \right] \\ &= \vec{i}[0 - 0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2x - 2x] = \vec{0} \end{aligned}$$

$\therefore \vec{F}$  is conservative.

So there exists a scalar function  $\phi$  such that

$$\vec{F} = \nabla \phi.$$

$$\Rightarrow (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = 2xy + z^3 \quad (1)$$

$$\frac{\partial \phi}{\partial y} = x^2 \quad (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 \quad (3)$$

Integrating (1) partially w.r.to  $x$ ,  $\phi = x^2y + z^3x + f_1(y, z)$

Integrating (2) partially w.r.to  $y$ ,  $\phi = x^2y + f_2(x, z)$

Integrating (3) partially w.r.to  $z$ ,  $\phi = xz^3 + f_3(x, y)$

$$\therefore \phi = x^2y + xz^3 + C$$

Since  $\vec{F}$  is conservative, work done by the force  $\vec{F}$  from  $(1, -2, 1)$  to  $(3, 1, 4)$  is

$$\begin{aligned} &= [\phi]_{(1, -2, 1)}^{(3, 1, 4)} \\ &= [x^2y + xz^3 + C]_{(1, -2, 1)}^{(3, 1, 4)} \\ &= 3^2 \cdot 1 + 3 \cdot 4^3 + C - [(1^2(-2) + 1 \cdot 1^3) + C] \\ &= 9 + 192 + 1 = 202 \text{ units.} \end{aligned}$$

### EXERCISE 1.3

1. Prove that if  $\vec{F} = \phi \nabla \psi$ , then  $\vec{F} \cdot (\nabla \times \vec{F}) = 0$ .
2. Prove that  $\text{Curl}(\phi \text{ grad } \phi) = \vec{0}$ .
3. Show that  $\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$
4. Prove that  $\nabla \times (\phi \nabla \psi) = \nabla \phi \times \nabla \psi$
5. Prove that  $\nabla \times [f(r)\vec{r}] = \vec{0}$
6. Evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ , along the straight line joining the points  $(1, -2, 1)$  and  $(3, 2, 4)$  [AU 2007]
7. Find  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = (2y + 3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$  along the line joining the points  $(0, 0, 0)$  to  $(2, 1, 1)$ .
8. Find the work done in moving a particle in the force field  $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$  from  $t = 0$  to  $t = 1$  along the curve  $x = 2t^2$ ,  $y = t$ ,  $z = 4t^3$ .
9. Show that  $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$  is conservative. Find its scalar potential and find the work done in moving a particle from  $(1, -2, 1)$  to  $(3, 1, 2)$ .
10. Find the work done by the force  $\vec{F} = -xy\vec{i} + y^2\vec{j} + z\vec{k}$  in moving a particle over a circular path  $x^2 + y^2 = 4$ ,  $z = 0$  from  $(2, 0, 0)$  to  $(0, 2, 0)$ .
11. Find the work done when a force  $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$  moves a particle over in the  $xy$  plane from  $(0, 0)$  to  $(1, 1)$  along the curve

$y^2 = x$ . If the path is  $y = x$ , whether the work done is different or same. If it is same, state the reason.

12. Find the total work done in moving a particle in a force field given by  $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$  along the curve  $x = t^2 + 1$ ,  $y = 2t^2$ ,  $z = t^3$  from  $t = 1$  to  $t = 2$ .
13. For the vector function  $\vec{F} = 2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$  determine  $\int_C \vec{F} \cdot d\vec{r}$  around the unit circle with centres at the origin in the  $xy$  plane.
14. Evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = (x - 3y)\vec{i} + (x - 2y)\vec{j}$  and  $C$  is the closed curve in the  $xy$  plane.  $x = 2 \cos t$ ,  $y = 2 \sin t$  and  $t = 0$  to  $t = 2\pi$ .

### ANSWERS TO EXERCISE 1.3

6. 211 [Hint:  $\vec{F}$  is conservative]

7. 5            8.  $\frac{13}{6}$             9. 34            10.  $\frac{16}{3}$

11.  $\frac{-2}{3}, \frac{-2}{3}, \vec{F}$  is conservative.    12. 303    13. 0    14.  $24\pi$

## 1.9 GREEN'S THEOREM IN A PLANE

Green's theorem gives a relation between a double integral over a region  $R$  in the  $xy$  plane and the line integral over a closed curve  $C$  enclosing the region  $R$ . It helps to evaluate line integral easily.

### Statement of Green's theorem

If  $P(x, y)$  and  $Q(x, y)$  are continuous functions with continuous partial derivatives in a region  $R$  in the  $xy$  plane and on its boundary  $C$  which is a simple closed curve then

$$\oint_C (P dx + Q dy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where  $C$  is described in the anticlockwise sense (which is the positive sense).

**Corollary** Area of the region  $R$  bounded by  $C$  is  $= \iint_R dxdy$

$$= \frac{1}{2} \oint_C (xdy - ydx)$$

**Proof** Take  $P = -y$ ,  $Q = x$ , in Green's theorem.

$$\therefore \frac{\partial P}{\partial y} = -1 \text{ and } \frac{\partial Q}{\partial x} = 1, \text{ then}$$

$$\oint_C (-ydx + xdy) = \iint_R (1+1)dxdy = 2 \iint_R dxdy$$

$$\therefore \frac{1}{2} \oint_C (xdy - ydx) = \iint_R dxdy$$
■

## WORKED EXAMPLES

### Example 1

Find the area of a circle of radius  $a$  using Green's theorem.

[AU 2004, 2005]

### Solution

We know that area enclosed by a simple closed curve  $C$  is  $\frac{1}{2} \oint_C (xdy - ydx)$

Here the curve is a circle of radius  $a$  and it is  $x^2 + y^2 = a^2$

The parametric equations of the circle are

$$x = a \cos \theta, y = a \sin \theta, 0 \leq \theta \leq 2\pi$$

$$\begin{aligned}\therefore \text{area of the circle} &= \frac{1}{2} \int_0^{2\pi} \left( a \cos \theta \frac{dy}{d\theta} - a \sin \theta \frac{dx}{d\theta} \right) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [a \cos \theta \cdot a \cos \theta - a \sin \theta (-a \sin \theta)] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} a^2 (\cos^2 \theta + \sin^2 \theta) d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} d\theta = \frac{a^2}{2} [\theta]_0^{2\pi} = \pi a^2\end{aligned}$$

**Example 2**

Using Green's theorem evaluate  $\oint_C [(2xy - x^2)dx + (x^2 + y^2)dy]$  where  $C$  is the closed curve of the region bounded by  $y = x^2$  and  $y^2 = x$

**Solution**

Green's theorem is

$$\oint_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

The given line integral is  $\oint_C [(2xy - x^2)dx + (x^2 + y^2)dy]$

Here

$$P = 2xy - x^2, \quad Q = x^2 + y^2$$

$$\therefore \frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2x = 0$$

$$\therefore \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0$$

Hence  $\oint_C [(2xy - x^2)dx + (x^2 + y^2)dy] = 0$ .

**Example 3**

Using Green's theorem evaluate  $\iint_C [(x^2 - y^2)dx + 2xydy]$ , where  $C$  is the closed curve of the region bounded by  $y^2 = x$  and  $x^2 = y$ .

[AU 1996]

**Solution**

Green's theorem is

$$\iint_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

The given line integral is  $\iint_C [(x^2 - y^2)dx + 2xydy]$

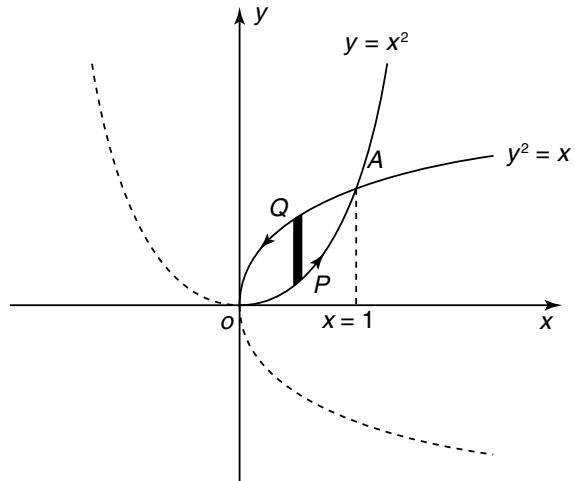
Here  $P = x^2 - y^2$ ,

$$\therefore \frac{\partial P}{\partial y} = -2y,$$

and  $Q = 2xy$

$$\therefore \frac{\partial Q}{\partial x} = 2y$$

$$\begin{aligned}\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= 2y + 2y \\ &= 4y\end{aligned}$$



$$\begin{aligned}\int_C (x^2 - y^2) dx + 2xy dy &= \iint_R 4y dxdy \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} 4y dy dx \\ &= 4 \int_0^1 \left[ \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx \\ &= 2 \int_0^1 (x - x^4) dx \\ &= 2 \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \\ &= 2 \left[ \frac{1}{2} - \frac{1}{5} \right] = 2 \left( \frac{5-2}{10} \right) = \frac{3}{5}\end{aligned}$$

#### Example 4

Evaluate  $\int_C [(\sin x - y)dx - \cos x dy]$ , where  $C$  is the triangle with vertices  $(0, 0)$ ,  $\left(\frac{\pi}{2}, 0\right)$  and  $\left(\frac{\pi}{2}, 1\right)$ . [AU 2003, 2006]

#### Solution

Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Given line integral is  $\int_C [(\sin x - y)dx - \cos x dy]$

Here  $P = \sin x - y$ ,  $Q = -\cos x$

$$\therefore \frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = \sin x$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin x + 1$$

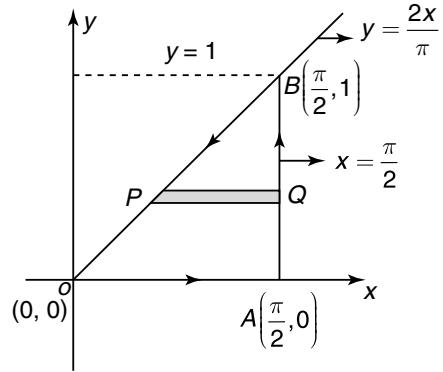
$$\int_C [(\sin x - y)dx - \cos x dy] = \iint_R (\sin x + 1) dx dy$$

$$\text{Equation of } OB \text{ is } \frac{y-0}{0-1} = \frac{x-0}{0-\frac{\pi}{2}}$$

$$\Rightarrow y = \frac{2}{\pi}x$$

$$\text{Equation of } AB \text{ is } x = \frac{\pi}{2}$$

In this region  $R$ ,  $x$  varies from  $\frac{\pi y}{2}$  to  $\frac{\pi}{2}$   
and  $y$  varies from 0 to 1.



$$\begin{aligned} \therefore \int_C [(\sin x - y)dx - \cos x dy] &= \int_0^1 \left[ \int_{\pi y/2}^{\pi/2} (\sin x + 1) dx \right] dy \\ &= \int_0^1 [-\cos x + x]_{\pi y/2}^{\pi/2} dy \\ &= \int_0^1 \left[ \left( -\cos \frac{\pi}{2} + \frac{\pi}{2} \right) - \left( -\cos \frac{\pi y}{2} + \frac{\pi y}{2} \right) \right] dy \\ &= \int_0^1 \left( \frac{\pi}{2} + \cos \frac{\pi y}{2} - \frac{\pi y}{2} \right) dy \\ &= \left[ \frac{\pi}{2}y + \frac{\sin \frac{\pi y}{2}}{\frac{\pi}{2}} - \frac{\pi y^2}{2} \right]_0^1 \\ &= \frac{\pi}{2} + \frac{2}{\pi} \sin \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2} + \frac{2}{\pi} - \frac{\pi}{4} = \frac{2}{\pi} + \frac{\pi}{4} \end{aligned}$$

**Example 5**

Evaluate by Green's theorem  $\int_C e^{-x}(\sin y dx + \cos y dy)$ ,  $C$  being the rectangle with vertices  $(0, 0)$ ,  $(\pi, 0)$ ,  $\left(\pi, \frac{\pi}{2}\right)$  and  $\left(0, \frac{\pi}{2}\right)$ . [AU 1995]

**Solution**

Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

The given line integral is

$$\int_C e^{-x}(\sin y dx + \cos y dy)$$

Here  $P = e^{-x} \sin y$

$$\therefore \frac{\partial P}{\partial y} = e^{-x} \cos y$$

and  $Q = e^{-x} \cos y$

$$\therefore \frac{\partial Q}{\partial x} = -e^{-x} \cos y$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^{-x} \cos y - e^{-x} \cos y = -2e^{-x} \cos y$$

$$\therefore \int_C e^{-x}(\sin y dx + \cos y dy) = \iint_R -2e^{-x} \cos y dxdy$$

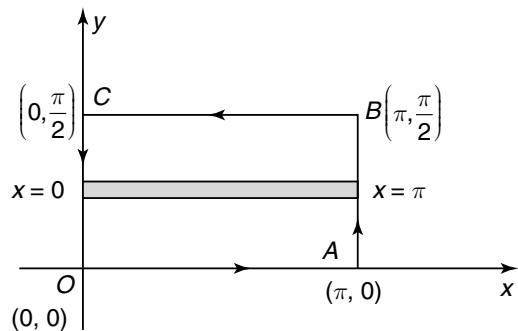
$$= -2 \int_0^{\frac{\pi}{2}} \int_0^{\pi} e^{-x} \cos y dxdy$$

$$= -2 \left[ \int_0^{\frac{\pi}{2}} \cos y dy \right] \left[ \int_0^{\pi} e^{-x} dx \right]$$

$$= -2 \left[ \sin y \right]_0^{\pi/2} \left[ \frac{e^{-x}}{-1} \right]_0^{\pi}$$

$$= 2 \left( \sin \frac{\pi}{2} \right) (e^{-\pi} - e^0)$$

$$= 2(e^{-\pi} - 1)$$



**Example 6**

**Find the area bounded between the curve  $y^2 = 4x$  and  $x^2 = 4y$  using Green's theorem.**

**Solution**

We know, by Green's theorem the area bounded by a simple closed

curve  $C$  is

$$\frac{1}{2} \oint_C (xdy - ydx)$$

Here  $C$  consists of the curves  $C_1$  and  $C_2$

$$\therefore \text{area} = \frac{1}{2} \left[ \int_{C_1} xdy - ydx + \int_{C_2} xdy - ydx \right] = \frac{1}{2} [I_1 + I_2]$$

**On  $C_1$ :**  $x^2 = 4y$

$$\therefore 2xdx = 4dy$$

$$\Rightarrow dy = \frac{1}{2} xdx$$

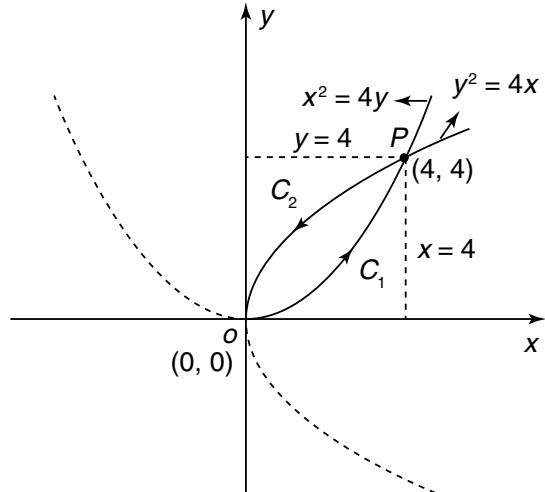
and  $x$  varies from 0 to 4.

$$\begin{aligned} \therefore I_1 &= \int_{C_1} xdy - ydx \\ &= \int_0^4 x \cdot \frac{1}{2} xdx - \frac{x^2}{4} dx \\ &= \int_0^4 \left( \frac{x^2}{2} - \frac{x^2}{4} \right) dx \\ &= \int_0^4 \frac{x^2}{4} dx = \frac{1}{4} \left[ \frac{x^3}{3} \right]_0^4 = \frac{1}{4} \frac{16}{3} = \frac{4}{3} \end{aligned}$$

**On  $C_2$ :**  $y^2 = 4x$

$$\therefore 2ydy = 4dx \Rightarrow dx = \frac{1}{2} ydy \text{ and } y \text{ varies from 4 to 0.}$$

$$\therefore I_2 = \int_{C_2} xdy - ydx$$



$$\begin{aligned}
 &= \int_4^0 \frac{y^2}{4} dy - y \cdot \frac{1}{2} y dy \\
 &= \int_4^0 \left( \frac{y^2}{4} - \frac{y^2}{2} \right) dy \\
 &= \int_4^0 -\frac{y^2}{4} dy = \frac{1}{4} \int_0^4 y^2 dy = \frac{1}{4} \left[ \frac{y^3}{3} \right]_0^4 = \frac{16}{3} \\
 \therefore \text{area} &= \frac{1}{2} \left[ \frac{16}{3} + \frac{16}{3} \right] = \frac{16}{3}
 \end{aligned}$$

**Example 7**

**Verify Green's theorem in a plane for the integral**  $\int_C (x - 2y)dx + xdy$  **taken around the circle**  $x^2 + y^2 = 4$ . [AU 2000]

**Solution**

Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Hence  $C$  is the circle  $x^2 + y^2 = 4$ .

Given line integral is  $\int_C (x - 2y)dx + xdy = \int_C Pdx + Qdy$

Here  $P = x - 2y$ ,  $Q = x$

$$\therefore \frac{\partial P}{\partial y} = -2, \quad \frac{\partial Q}{\partial x} = 1$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 + 2 = 3$$

$$\begin{aligned}
 \therefore \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy &= \iint_R 3 dxdy = 3 \iint_R dxdy \\
 &= 3 \text{ area of the circle} = 3\pi \cdot 4 = 12\pi.
 \end{aligned}$$

Now we shall compute the line integral directly.

The parametric equations of the circle  $x^2 + y^2 = 4$  are  $x = 2 \cos \theta$ ,  $y = 2 \sin \theta$ ,  $0 \leq \theta \leq 2\pi$

$$\therefore dx = -2 \sin \theta d\theta, \quad dy = 2 \cos \theta d\theta$$

$$\begin{aligned}
 \therefore \int_C (x - 2y)dx + xdy &= \int_0^{2\pi} [(2\cos \theta - 4\sin \theta)(-2\sin \theta d\theta) \\
 &\quad + 2\cos \theta \cdot 2\cos \theta d\theta] \\
 &= \int_0^{2\pi} (-4\sin \theta \cos \theta + 8\sin^2 \theta + 4\cos^2 \theta) d\theta \\
 &= \int_0^{2\pi} [-2\sin 2\theta + 4\sin^2 \theta + 4(\cos^2 \theta + \sin^2 \theta)] d\theta \\
 &= \int_0^{2\pi} [-2\sin 2\theta + 2(1 - \cos 2\theta) + 4] d\theta \\
 &= \int_0^{2\pi} [-2\sin 2\theta - 2\cos 2\theta + 6] d\theta \\
 &= \left[ 2\frac{\cos 2\theta}{2} - 2\frac{\sin 2\theta}{2} + 6\theta \right]_0^{2\pi} \\
 &= \cos 4\pi - \sin 4\pi + 6 \times 2\pi - [\cos 0 - \sin 0 + 0] \\
 &= 1 + 12\pi - 1 = 12\pi
 \end{aligned} \tag{2}$$

$\therefore$  from (1) and (2)

$$\int_C [(x - 2y)dx + xdy] = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$\therefore$  Green's theorem is verified.

### Example 8

Verify Green's theorem in the plane for  $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ , where  $C$  is the boundary of the region bounded by  $x = 0, y = 0, x + y = 1$ .  
[AU 1999, 2009, 2010, 2011, 2012]

### Solution

Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

The given integral is

$$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

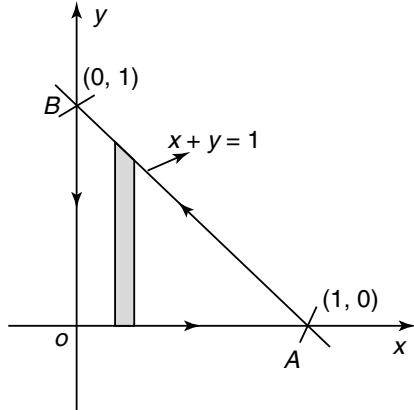
Here  $P = 3x^2 - 8y^2$ ,

$$\therefore \frac{\partial P}{\partial y} = -16y$$

and  $Q = 4y - 6xy$

$$\therefore \frac{\partial Q}{\partial x} = -6y$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6y + 16y = 10y$$



$$\begin{aligned} \therefore \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy &= \int_0^1 \int_0^{1-x} 10y \, dy \, dx \\ &= 10 \int_0^1 \left[ \frac{y^2}{2} \right]_0^{1-x} dx = 5 \int_0^1 (1-x)^2 \, dx \\ &= 5 \left[ \frac{(1-x)^3}{-3} \right]_0^1 = \frac{-5}{3}[0-1] = \frac{5}{3} \end{aligned}$$

We shall now compute the line integral  $\int_C Pdx + Qdy$

Now  $\int_C Pdx + Qdy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$$\begin{aligned} &= \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy \\ &\quad + \int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy \\ &\quad + \int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy \\ &= I_1 + I_2 + I_3 \end{aligned}$$

**On OA:**  $y = 0 \quad \therefore dy = 0$  and  $x$  varies from 0 to 1.

$$\therefore I_1 = \int_0^1 3x^2 dx = 3 \left[ \frac{x^3}{3} \right]_0^1 = 1$$

**On AB:**  $x + y = 1 \Rightarrow y = 1 - x \therefore dy = -dx$

and  $x$  varies 1 to 0.

$$\begin{aligned}\therefore I_2 &= \int_1^0 (3x^2 - 8(1-x)^2)dx + [4(1-x) - 6x(1-x)](-dx) \\ &= \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6(x-x^2)]dx \\ &= \left[ x^3 - \frac{8(1-x)^3}{-3} - 4 \frac{(1-x)^2}{-2} + 6 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \right]_1^0 \\ &= \left[ 0 + \frac{8}{3} + 2 + 0 - (1+6) \left( \frac{1}{2} - \frac{1}{3} \right) \right] \\ &= \frac{8}{3} + 2 - 1 - 1 = \frac{8}{3}\end{aligned}$$

**On BO:**  $x = 0 \therefore dx = 0$  and  $y$  varies from 1 to 0

$$\therefore I_3 = \int_1^0 4ydy = 2 \left[ y^2 \right]_1^0 = -2$$

$$\therefore \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad (2)$$

(1) and (2) give the same value.

Hence Green's theorem is verified.

### Example 9

Verify Green's theorem for  $\int_C (xy + y^2)dx + x^2dy$  where  $C$  is the boundary of the area between  $y = x^2$  and  $y = x$ . [AU 2002, 2009]

### Solution

Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

The given line integral is

$$\int_C (xy + y^2) dx + x^2 dy$$

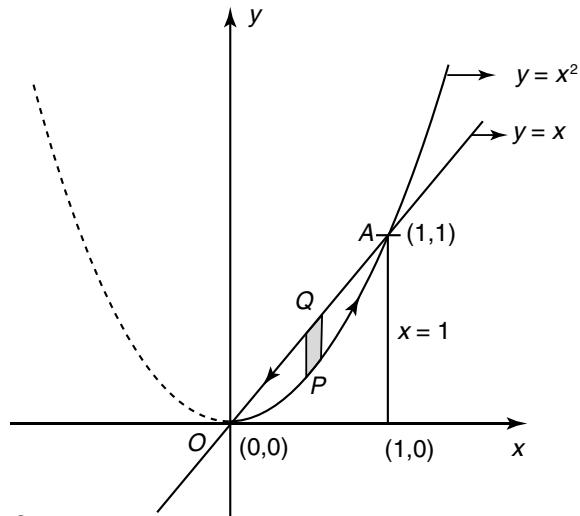
Here  $P = xy + y^2$

$$\therefore \frac{\partial P}{\partial y} = x + 2y$$

and  $Q = x^2$

$$\frac{\partial Q}{\partial x} = 2x$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x - 2y = x - 2y$$



$$\begin{aligned} \therefore \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy &= \int_0^1 \int_{x^2}^x (x - 2y) dy dx \\ &= \int_0^1 \left[ xy - \frac{2y^2}{2} \right]_{x^2}^x dx \\ &= \int_0^1 [x^2 - x^2 - (x^3 - x^4)] dx \\ &= \int_0^1 [(x^4 - x^3)] dx \\ &= \left[ \frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20} \end{aligned} \quad (1)$$

We shall now compute the line integral  $\int_C Pdx + Qdy$

$$\begin{aligned} \text{Now } \int_C Pdx + Qdy &= \int_C (xy + y^2) dx + x^2 dy \\ &= \int_{C_1} (xy + y^2) dx + x^2 dy + \int_{C_2} (xy + y^2) dx + x^2 dy \\ &= I_1 + I_2 \end{aligned}$$

**On  $C_1$ :**  $y = x^2$ ,  $\therefore dy = 2x dx$  and  $x$  varies from 0 to 1.

$$I_1 = \int_0^1 (x \cdot x^2 + x^4) dx + x^2 \cdot 2x dx$$

$$\begin{aligned}
 &= \int_0^1 (x^3 + x^4 + 2x^3) dx \\
 &= \int_0^1 (3x^3 + x^4) dx \\
 &= \left[ 3\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}
 \end{aligned}$$

**On C<sub>2</sub>:**  $y = x$ ,  $\therefore dy = dx$  and  $x$  varies from 1 to 0.

$$\begin{aligned}
 \therefore I_2 &= \int_1^0 (x \cdot x + x^2) dx + x^2 dx \\
 &= \int_0^1 3x^2 dx = 3 \left[ \frac{x^3}{3} \right]_1^0 = -1 \\
 \therefore \int_C Pdx + Qdy &= \frac{19}{20} - 1 = -\frac{1}{20} \quad (2)
 \end{aligned}$$

(1) and (2) give the same value.

Hence Green's theorem is verified.

### Example 10

Verify Green's theorem in the plane for  $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$

where  $C$  is the square with vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(2, 2)$ ,  $(0, 2)$ .

### Solution

Green's theorem is

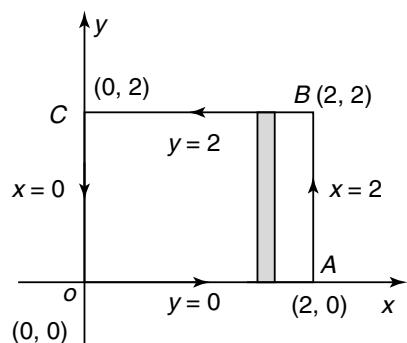
$$\int_C (Pdx + Qdy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

The given integral is

$$\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$$

$$\text{Here } P = x^2 - xy^3$$

$$\therefore \frac{\partial P}{\partial y} = -3xy^2$$



and

$$Q = y^2 - 2xy$$

$$\therefore \frac{\partial Q}{\partial x} = -2y$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2y + 3xy^2$$

$$\begin{aligned}\therefore \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy &= \int_0^2 \int_0^2 (-2y + 3xy^2) dx dy \\&= \int_0^2 \left[ -2yx + \frac{3x^2}{2}y^2 \right]_0^2 dy \\&= \int_0^2 \left[ -4y + \frac{3}{2} \cdot 2^2 y^2 \right] dy \\&= \int_0^2 [-4y + 6y^2] dy \\&= \left[ -4 \frac{y^2}{2} + 6 \frac{y^3}{3} \right]_0^2 \\&= -2 \cdot 2^2 + 2 \cdot 2^3 = 8\end{aligned}\tag{1}$$

We now compute the line integral  $\int_C P dx + Q dy$ .

Now

$$\begin{aligned}\int_C P dx + Q dy &= \int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy \\&= \int_{OA} (x^2 - xy^3) dx + (y^2 - 2xy) dy \\&\quad + \int_{AB} (x^2 - xy^3) dx + (y^2 - 2xy) dy \\&\quad + \int_{BC} (x^2 - xy^3) dx + (y^2 - 2xy) dy \\&\quad + \int_{CO} (x^2 - xy^3) dx + (y^2 - 2xy) dy \\&= I_1 + I_2 + I_3 + I_4\end{aligned}$$

**On OA:**  $y = 0, \therefore dy = 0$  and  $x$  varies from 0 to 2

$$\therefore I_1 = \int_0^2 x^2 dx = \left[ \frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

**On AB:**  $x = 2, \therefore dx = 0$  and  $y$  varies from 0 to 2

$$\begin{aligned} \therefore I_2 &= \int_0^2 (y^2 - 4y) dy \\ &= \left[ \frac{y^3}{3} - 4 \cdot \frac{y^2}{2} \right]_0^2 = \frac{8}{3} - 2 \cdot 2^2 = -\frac{16}{3} \end{aligned}$$

**On BC:**  $y = 2, \therefore dy = 0$  and  $x$  varies from 2 to 0

$$\begin{aligned} \therefore I_3 &= \int_2^0 (x^2 - x \cdot 2^3) dx \\ &= \left[ \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} \right]_2^0 = -\left( \frac{8}{3} - 4 \cdot 4 \right) = \frac{40}{3} \end{aligned}$$

**On CO:**  $x = 0, \therefore dx = 0$  and  $y$  varies from 2 to 0

$$\begin{aligned} \therefore I_4 &= \int_2^0 y^2 dy = \left[ \frac{y^3}{3} \right]_2^0 = -\frac{8}{3} \\ \therefore \int_C Pdx + Qdy &= \frac{8}{3} + \left( -\frac{16}{3} \right) + \frac{40}{3} + \left( -\frac{8}{3} \right) = \frac{24}{3} = 8 \quad (2) \end{aligned}$$

(1) and (2) give the same value.

Hence Green's theorem is verified.

## 1.10 SURFACE INTEGRALS

Suppose a surface is bounded by a simple closed curve  $C$ , then we can regard the surface as having two sides separated by  $C$ . One of which is arbitrarily chosen as the positive side and the other is the negative side. If the surface is a closed surface, then the outer side is taken as the positive side and the inner side is the negative side. A unit normal at any point of the positive side of the surface is denoted by  $\vec{n}$  and is called the outward drawn normal and its direction is considered positive.

Any integral which is evaluated over a surface is called a surface integral.

### Definition 1.11 Surface Integral

Let  $S$  be a surface of finite area which is smooth or piecewise smooth (e.g. a sphere is a smooth surface and a cube is a piecewise smooth surface). Let  $\vec{F}(x, y, z)$  be a vector point function defined at each point of  $S$ . Let  $P$  be any point on the surface and let  $\vec{n}$  be the outward unit normal at  $P$ . Then the surface integral of  $\vec{F}$  over  $S$  is defined as  $\iint_S \vec{F} \cdot \vec{n} dS$

If we associate a vector  $d\vec{S}$  (called vector area) with the differential of surface area  $dS$  such that  $|d\vec{S}| = dS$  and direction of  $d\vec{S}$  is  $\vec{n}$ , then

$$d\vec{S} = \vec{n} dS$$

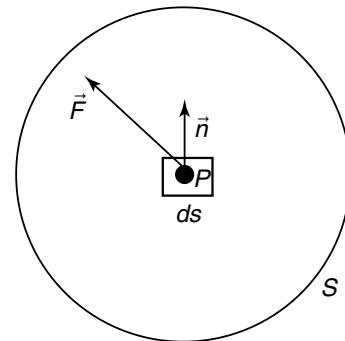
$\therefore \iint_S \vec{F} \cdot \vec{n} dS$  can also be written as  $\iint_S \vec{F} \cdot d\vec{S}$

#### Note

1. In physical application the integral

$\iint_S \vec{F} \cdot d\vec{S}$  is called the normal flux of

$\vec{F}$  through the surface  $S$ , because this integral is a measure of the volume emerging from  $S$  per unit time.



#### 1.10.1 Evaluation of surface integral

To evaluate a surface integral over a surface it is usually expressed as a double integral over the orthogonal projection of  $S$  on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface  $S$  in not more than one point.

Let  $R$  be the orthogonal projection of  $S$  on the  $xy$  plane.

Then the element surface  $dS$  is projected to an element area  $dx dy$  in the  $xy$  plane as in fig.

$\therefore dx dy = dS \cos \theta$ , where  $\theta$  is the angle between the planes of  $dS$  and  $xy$ -plane.

Let  $\vec{n}$  be the unit normal to  $dS$  and  $\vec{k}$  is the unit normal to the  $xy$ -plane.

Since angle between the planes is equal to the angle between the normals,

$\theta$  is the angle between the normals  $\vec{n}$  and  $\vec{k}$ .

$$\begin{aligned}\therefore \cos \theta &= \frac{\vec{n} \cdot \vec{k}}{|\vec{n}| |\vec{k}|} \\ &= \vec{n} \cdot \vec{k} \quad [\text{Since } |\vec{n}| = 1, |\vec{k}| = 1]\end{aligned}$$

We take the acute angle between the normals and so we take  $|\vec{n} \cdot \vec{k}|$

$$\therefore dx dy = dS |\vec{n} \cdot \vec{k}|$$

$$\Rightarrow dS = \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

Hence

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

Similarly, taking the projection on the  $yz$  and  $zx$  planes, we get

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dy dz}{|\vec{n} \cdot \vec{i}|}$$

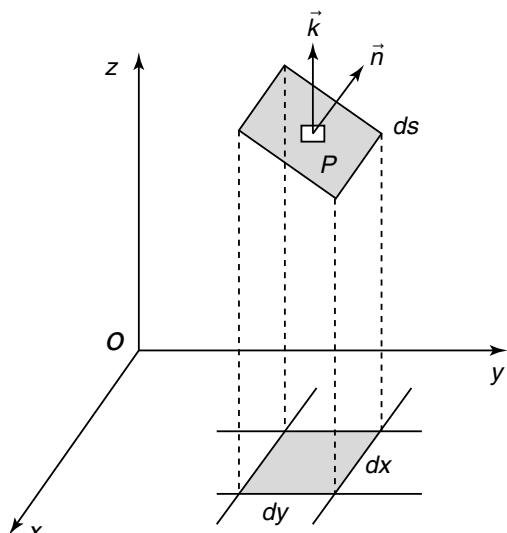
and

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dz dx}{|\vec{n} \cdot \vec{j}|}$$

### Corollary

The surface area

$$\begin{aligned}\iint_S dS &= \iint_R \frac{dx dy}{|\vec{n} \cdot \vec{k}|} = \iint_{R_1} \frac{dy dz}{|\vec{n} \cdot \vec{i}|} \\ &= \iint_{R_2} \frac{dz dx}{|\vec{n} \cdot \vec{j}|}\end{aligned}$$



## 1.11 VOLUME INTEGRAL

Any integral which is evaluated over a volume bounded by a surface is called a volume integral.

If  $V$  is the volume bounded by a surface  $S$ , then

$$\iiint_V \phi(x, y, z) dV \text{ and } \iiint_V \vec{F} dV \text{ are called volume integrals.}$$

If we divide  $V$  into rectangular blocks by drawing planes parallel to the coordinate planes, then  $dV = dx dy dz$ .

$$\therefore \iiint_V \phi dV = \iiint_V \phi(x, y, z) dx dy dz$$

If  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ , then

$$\iiint_V \vec{F} dV = \vec{i} \iiint_V F_1 dx dy dz + \vec{j} \iiint_V F_2 dx dy dz + \vec{k} \iiint_V F_3 dx dy dz$$

### WORKED EXAMPLES

#### Example 1

Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$  if  $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$  and  $S$  is the surface of the plane  $3x + 2y + 6z = 6$  contained in the first octant.

#### Solution

Given  $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$  and the surface  $3x + 2y + 6z = 6$ .

Let  $\phi = 3x + 2y + 6z$

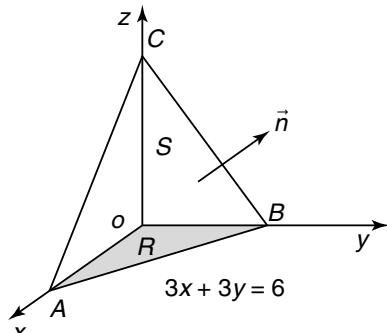
Let  $R$  be the projection of  $S$  in the  $xy$  plane.

$\therefore R$  is the  $\triangle AOB$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

where  $\vec{n}$  is unit normal to  $S$  and  $\vec{k}$  is the unit normal to  $xy$ -plane.

$$\begin{aligned} \text{Normal to the surface is } \nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= 3\vec{i} + 2\vec{j} + 6\vec{k} \end{aligned}$$



$$\therefore \text{unit normal is } \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{9+4+36}} = \frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k})$$

$$\begin{aligned}\therefore \vec{F} \cdot \vec{n} &= (4y\vec{i} + 18z\vec{j} - x\vec{k}) \cdot \frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k}) \\ &= \frac{1}{7}(12y + 36z - 6x) = \frac{6}{7}(2y + 6z - x) \\ \vec{n} \cdot \vec{k} &= \frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k}) \cdot \vec{k} = \frac{6}{7}\end{aligned}$$

$$\begin{aligned}\therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iint_R \frac{6}{7} (2y + 6z - x) \frac{dx dy}{\frac{6}{7}} \\ &= \iint_R (2y + 6z - x) dx dy\end{aligned}$$

We have  $3x + 2y + 6z = 6$

$$\Rightarrow 6z = 6 - 3x - 2y$$

$$\therefore 2y + 6z - x = 2y + 6 - 3x - 2y - x = 6 - 4x$$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \iint_R (6 - 4x) dx dy$$

The plane  $3x + 2y + 6z = 6$  meets  $xy$ -plane in line  $AB$ .

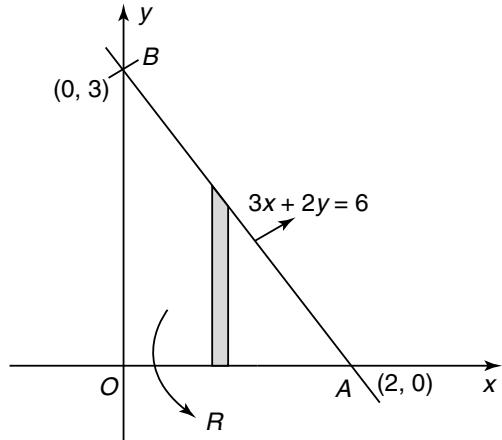
$\therefore$  the equation of  $AB$  is  $3x + 2y = 6$

$\therefore$  Point A is  $(2, 0)$  and B is  $(0, 3)$  and  $3x + 2y = 6$

$$\Rightarrow y = \frac{6 - 3x}{2}$$

$\therefore$  In  $R$ ,  $x$  varies from 0 to 2 and

$$y \text{ varies from 0 to } \frac{6 - 3x}{2}$$



$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \int_0^2 \int_0^{\frac{6-3x}{2}} (6 - 4x) dy dx$$

$$\begin{aligned}
&= 2 \int_0^2 \int_0^{\frac{6-3x}{2}} (3-2x) dy dx \\
&= 2 \int_0^2 [(3-2x)y]_{0}^{\frac{6-3x}{2}} dx \\
&= 2 \int_0^2 (3-2x) \frac{(6-3x)}{2} dx \\
&= 3 \int_0^2 (3-2x)(2-x) dx \\
&= 3 \int_0^2 (6-7x+2x^2) dx \\
&= 3 \left[ 6x - \frac{7x^2}{2} + 2 \frac{x^3}{3} \right]_0^2 \\
&= 3 \left[ 6 \times 2 - 7 \times \frac{4}{2} + 2 \times \frac{8}{3} \right] \\
&= 3 \left[ 12 - 14 + \frac{16}{3} \right] = -6 + 16 = 10
\end{aligned}$$

**Example 2**

Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$  if  $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$  and  $S$  is part of the surface  $x^2 + y^2 + z^2 = 1$ , which lies in the first octant.

**Solution**

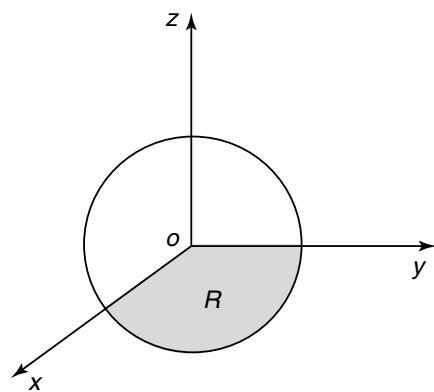
Given  $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

and the surface  $x^2 + y^2 + z^2 = 1$

Let  $\phi = x^2 + y^2 + z^2$

The normal to the surface

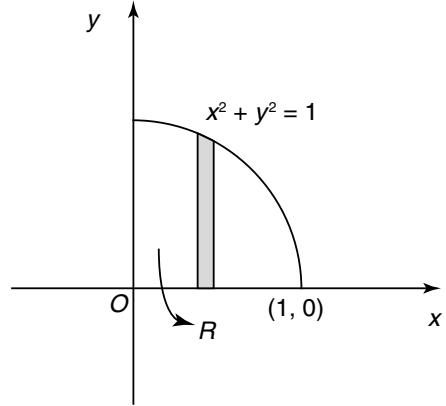
$$\begin{aligned}
\text{is } \nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\
&= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}
\end{aligned}$$



$$\begin{aligned}
 \therefore \text{unit normal } \vec{n} &= \frac{\nabla\phi}{|\nabla\phi|} \\
 &= \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\
 &= \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} \\
 &= x\vec{i} + y\vec{j} + z\vec{k} \quad [ \because x^2 + y^2 + z^2 = 1 ] \\
 \therefore \vec{F} \cdot \vec{n} &= (yz\vec{i} + zx\vec{j} + xy\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) \\
 &= xyz + xyz + xyz = 3xyz
 \end{aligned}$$

The projection of the surface of the sphere in the first octant in the  $xy$  plane is  $R$ , which is the quadrant of the circle  $x^2 + y^2 = 1$ ,  $z = 0$ ,  $x \geq 0$ ,  $y \geq 0$  and  $\vec{k}$  is the unit normal to  $R$ .

$$\begin{aligned}
 \therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iint_R \vec{F} \cdot \vec{n} \frac{dx dy}{|\vec{n} \cdot \vec{k}|} \\
 &= \iint_R 3xyz \frac{dx dy}{|\vec{n} \cdot \vec{k}|} \\
 \text{But } \vec{n} \cdot \vec{k} &= (x\vec{i} + y\vec{j} + z\vec{k}) \cdot \vec{k} = z \\
 \therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iint_R 3xyz \frac{1}{z} dx dy \\
 &= \iint_R 3xy dx dy \\
 &= \int_0^1 \int_0^{\sqrt{1-x^2}} 3xy dx dy \\
 &= 3 \int_0^1 \left[ \frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} x dx = \frac{3}{2} \int_0^1 x(1-x^2) dx \\
 &= \frac{3}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{2} \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}
 \end{aligned}$$



**Example 3**

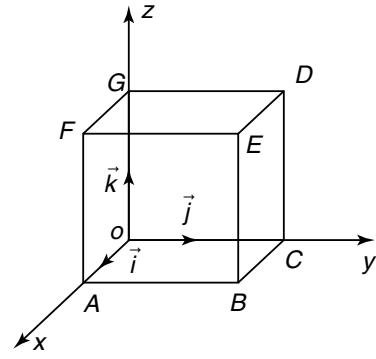
Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$  and  $S$  is the surface of the cube bounded by the planes  $x = 0, x = 1, y = 1, z = 0, z = 1$ .

**Solution**

$$\text{Given } \vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$S$  is the surface of the cube, which is piecewise smooth surface consisting of six smooth surfaces.

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} dS &= \iint_{ABEF} \vec{F} \cdot \vec{n} dS + \iint_{OCDG} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{BCDE} \vec{F} \cdot \vec{n} dS + \iint_{OAFG} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{OABC} \vec{F} \cdot \vec{n} dS + \iint_{DEFG} \vec{F} \cdot \vec{n} dS\end{aligned}$$



**On the face ABEF:**  $x = 1, \vec{n} = \vec{i}$

$$\therefore \vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{i} = 4xz = 4z$$

$$dS = \frac{dy dz}{|\vec{n} \cdot \vec{i}|} = \frac{dy dz}{|\vec{i} \cdot \vec{i}|} = dy dz$$

$$\therefore \iint_{ABEF} \vec{F} \cdot \vec{n} dS = \int_0^1 \int_0^1 4z dz dy$$

$$= 4 \cdot [y]_0^1 \left[ \frac{z^2}{2} \right]_0^1 = 4 \cdot 1 \cdot \frac{1}{2} = 2$$

**On the face OCDG:**  $x = 0, \vec{n} = -\vec{i}$

$$\vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i}) = -4xz = 0$$

$$\therefore \iint_{OCDG} \vec{F} \cdot \vec{n} dS = 0$$

**On the face BCDE:**  $y = 1, \vec{n} = \vec{j}$

$$dS = \frac{dx dz}{|\vec{n} \cdot \vec{j}|} = \frac{dx dz}{|\vec{j} \cdot \vec{j}|} = dx dz$$

$$\vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j} = -y^2 = -1$$

$$\begin{aligned}\therefore \iint_{BCDE} \vec{F} \cdot \vec{n} dS &= \int_0^1 \int_0^1 (-1) dx dz \\ &= -[x]_0^1 [z]_0^1 = -1\end{aligned}$$

**On the face OAFG:**  $y = 0, \vec{n} = -\vec{j}$

$$\vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{j}) = y^2 = 0$$

$$\therefore \iint_{OAFG} \vec{F} \cdot \vec{n} dS = 0$$

**On the face DEFG:**  $z = 1, \vec{n} = \vec{k}$

$$\vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{k} = yz = y$$

$$dS = \frac{dx dy}{|\vec{n} \cdot \vec{k}|} = \frac{dx dy}{|\vec{k} \cdot \vec{k}|} = dx dy$$

$$\begin{aligned}\therefore \iint_{DEFG} \vec{F} \cdot \vec{n} dS &= \int_0^1 \int_0^1 y dx dy \\ &= [x]_0^1 \left[ \frac{y^2}{2} \right]_0^1 = 1 \times \frac{1}{2} = \frac{1}{2}\end{aligned}$$

**On the face OABC:**  $z = 0, \vec{n} = -\vec{k}$

$$\vec{F} \cdot \vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{k}) = -yz = 0$$

$$\therefore \iint_{OABC} \vec{F} \cdot \vec{n} dS = 0$$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = 2 + (-1) + \frac{1}{2} = \frac{3}{2}$$

**Example 4**

Evaluate  $\iiint_V \nabla \cdot \vec{F} dV$  if  $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$  and  $V$  is the volume of the region enclosed by the cube  $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ .

**Solution**

Given

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\begin{aligned}\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 2x + 2y + 2z = 2(x + y + z)\end{aligned}$$

$$\begin{aligned}\therefore \iiint_V \nabla \cdot \vec{F} dV &= \int_0^1 \int_0^1 \int_0^1 2(x + y + z) dx dy dz \\ &= 2 \int_0^1 \int_0^1 \left[ \frac{x^2}{2} + (y + z)x \right]_0^1 dy dz \\ &= 2 \int_0^1 \int_0^1 \left[ \frac{1}{2} + y + z \right] dy dz \\ &= 2 \int_0^1 \left[ \left( \frac{1}{2} + z \right) y + \frac{y^2}{2} \right]_0^1 dz \\ &= 2 \int_0^1 \left\{ \left[ \frac{1}{2} + z \right] + \frac{1}{2} \right\} dz \\ &= 2 \int_0^1 (1 + z) dz = 2 \left[ z + \frac{z^2}{2} \right]_0^1 = 2 \left[ 1 + \frac{1}{2} \right] = 3.\end{aligned}$$

**1.12 GAUSS DIVERGENCE THEOREM**

The divergence theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

**Statement of Gauss divergence theorem**

Let  $V$  be the volume bounded by a closed surface  $S$ . If a vector function  $\vec{F}$  is continuous and has continuous partial derivatives inside and on  $S$ , then the surface integral of  $\vec{F}$  over  $S$  is equal to the volume integral of divergence of  $\vec{F}$  taken throughout  $V$ .

i.e.  $\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{F} dV$

If  $\vec{n}$  is the outward normal to the surface  $d\vec{S} = \vec{n} dS$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

### Gauss divergence theorem in cartesian coordinates

If  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ , then

$$\vec{F} \cdot \vec{n} = F_1(\vec{i} \cdot \vec{n}) + F_2(\vec{j} \cdot \vec{n}) + F_3(\vec{k} \cdot \vec{n})$$

$$\begin{aligned} \vec{F} \cdot \vec{n} dS &= F_1(\vec{i} \cdot \vec{n}) dS + F_2(\vec{j} \cdot \vec{n}) dS + F_3(\vec{k} \cdot \vec{n}) dS \\ &= F_1 dy dz + F_2 dz dx + F_3 dy dx \end{aligned}$$

and

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Hence, Gauss theorem becomes

$$\iint_S F_1 dy dz + F_2 dz dx + F_3 dx dy = \iiint_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx dy dz$$

### WORKED EXAMPLES

#### Example 5

If  $S$  is any closed surface enclosing volume  $V$  and  $\vec{F} = ax \vec{i} + by \vec{j} + cz \vec{k}$  prove that  $\iint_S \vec{F} \cdot \vec{n} dS = (a+b+c)V$  [AU 2004]

#### Solution

Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Given

$$\vec{F} = ax \vec{i} + by \vec{j} + cz \vec{k}$$

$$\therefore \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a + b + c$$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \iiint_V (a+b+c) dV \\ = (a+b+c) \iiint_V dV = (a+b+c)V.$$

**Example 6**

Using Gauss divergence theorem, evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$  where  $\vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$  and  $S$  is the sphere  $x^2 + y^2 + z^2 = a^2$  [AU 2000]

**Solution**

Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Given  $\vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$

$$\begin{aligned} \therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) \\ &= 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) \end{aligned}$$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \iiint_V 3(x^2 + y^2 + z^2) dx dy dz$$

We shall evaluate this triple integral by using spherical polar coordinates.

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

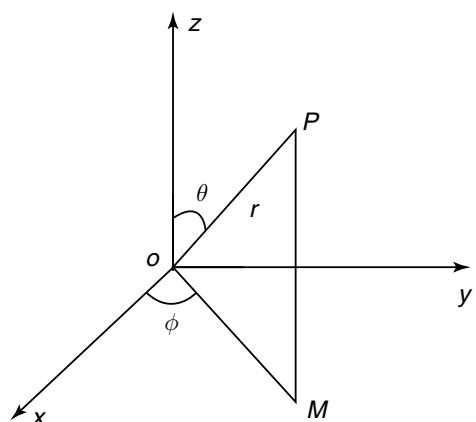
$$\begin{aligned} \text{then } dx dy dz &= \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right| dr d\theta d\phi \\ &= r^2 \sin \theta dr d\theta d\phi \end{aligned}$$

$$\text{and } x^2 + y^2 + z^2 = r^2$$

Here  $r$  varies from 0 to  $a$ ,

$\theta$  varies from 0 to  $\pi$  and

$\phi$  varies from 0 to  $2\pi$ .



$$\begin{aligned}
\therefore \iint_S \vec{F} \cdot \vec{n} dS &= \int_0^{2\pi} \int_0^{\pi} \int_0^a 3r^4 \sin \theta dr d\theta d\phi \\
&= 3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^a r^4 dr \\
&= 3 \left[ \phi \right]_0^{2\pi} \left[ -\cos \theta \right]_0^{\pi} \left[ \frac{r^5}{5} \right]_0^a \\
&= 3 \cdot 2\pi (-\cos \pi + \cos 0) \cdot \frac{a^5}{5} \\
&= 6\pi \cdot 2 \cdot \frac{a^5}{5} = \frac{12\pi}{5} a^5
\end{aligned}$$

**Note** We have  $\operatorname{div} \vec{F} = 3(x^2 + y^2 + z^2)$ . Since the equation of the surface  $x^2 + y^2 + z^2 = a^2$ , we cannot take  $\operatorname{div} \vec{F} = 3a^2$  because  $\vec{F}$  is defined in the volume inside  $S$  and on  $S$ .  $x^2 + y^2 + z^2 = a^2$  is true only for points on  $S$ .

### Example 7

Verify Gauss divergence theorem for  $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$  over the cube bounded by  $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ .

[AU 2003, 2004, 2006, 2010]

### Solution

Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Given  $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\begin{aligned}
\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) \\
&= 4z - 2y + y = 4z - y
\end{aligned}$$

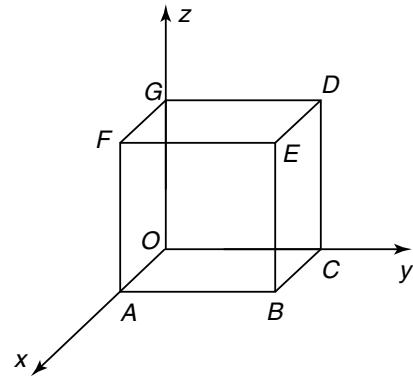
$$\therefore \iiint_V \nabla \cdot \vec{F} dV = \int_0^1 \int_0^1 \int_0^1 (4z - y) dx dy dz \quad [\because dV = dx dy dz]$$

$$\begin{aligned}
 &= \int_0^1 \int_0^1 [(4z-y)x]_0^1 dy dz \\
 &= \int_0^1 \int_0^1 [4z-y] dy dz \\
 &= \int_0^1 \left[ 4zy - \frac{y^2}{2} \right]_0^1 dz \\
 &= \int_0^1 \left[ 4z - \frac{1}{2} \right] dz = \left[ 4\frac{z^2}{2} - \frac{1}{2}z \right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}
 \end{aligned} \tag{1}$$

We shall now evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$

Here the surface  $S$  consists of the six faces of the cube.

$$\begin{aligned}
 \therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iint_{S_1} \vec{F} \cdot \vec{n} dS + \iint_{S_2} \vec{F} \cdot \vec{n} dS \\
 &\quad + \iint_{S_3} \vec{F} \cdot \vec{n} dS + \iint_{S_4} \vec{F} \cdot \vec{n} dS \\
 &\quad + \iint_{S_5} \vec{F} \cdot \vec{n} dS + \iint_{S_6} \vec{F} \cdot \vec{n} dS
 \end{aligned}$$



We shall simplify the computation and put it in the form of a table.

Face	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	$dS$
$S_1 = ABEF$	$x = 1$	$\vec{i}$	$4xz = 4z$	$dy dz$
$S_2 = OCDG$	$x = 0$	$-\vec{i}$	$-4xz = 0$	$dy dz$
$S_3 = BCDE$	$y = 1$	$\vec{j}$	$-y^2 = -1$	$dx dz$
$S_4 = OAFG$	$y = 0$	$-\vec{j}$	$y^2 = 0$	$dx dz$
$S_5 = DEFG$	$z = 1$	$\vec{k}$	$yz = y$	$dx dy$
$S_6 = OABC$	$z = 0$	$-\vec{k}$	$-yz = 0$	$dx dy$

$$\begin{aligned}
 \therefore \iint_S \vec{F} \cdot \vec{n} dS &= \int_0^1 \int_0^1 4z \, dy \, dz = 4 \left[ y \right]_0^1 \left[ \frac{z^2}{2} \right]_0^1 = 4 \cdot 1 \cdot \frac{1}{2} = 2 \\
 \iint_{S_2} \vec{F} \cdot \vec{n} dS &= \iint_{S_2} 0 \, dy \, dz = 0 \\
 \iint_{S_3} \vec{F} \cdot \vec{n} dS &= \int_0^1 \int_0^1 -1 \, dx \, dz = - \left[ x \right]_0^1 \left[ z \right]_0^1 = -1 \\
 \iint_{S_4} \vec{F} \cdot \vec{n} dS &= \iint_{S_4} 0 \, dx \, dz = 0 \\
 \iint_{S_5} \vec{F} \cdot \vec{n} dS &= \int_0^1 \int_0^1 y \, dx \, dy = \left[ x \right]_0^1 \left[ \frac{y^2}{2} \right]_0^1 = \frac{1}{2} \\
 \iint_{S_6} \vec{F} \cdot \vec{n} dS &= \iint_{S_6} 0 \, dx \, dy = 0 \\
 \therefore \iint_S \vec{F} \cdot \vec{n} dS &= 2 + 0 + (-1) + 0 + \frac{1}{2} + 0 = \frac{3}{2} \tag{2}
 \end{aligned}$$

From (1) and (2),  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$

Hence Gauss's divergence theorem is verified.

### **Example 8**

**Verify Gauss divergence theorem for**

$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$  taken over the rectangular parallelepiped bounded by  $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$ .

[AU 1994, 2009]

### **Solution**

Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Given

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

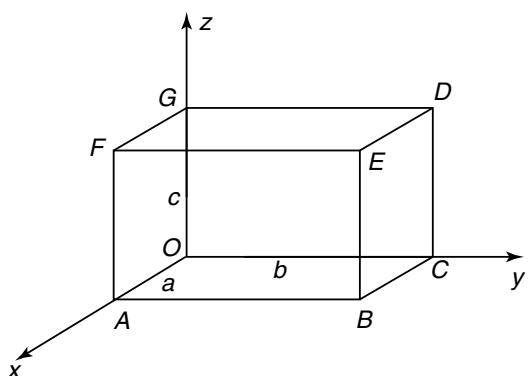
$$\begin{aligned}
 \therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) \\
 &= 2x + 2y + 2z = 2(x + y + z)
 \end{aligned}$$

$$\begin{aligned}
\therefore \iiint_V \nabla \cdot \vec{F} dV &= \int_0^c \int_0^b \int_0^a 2(x + y + z) dx dy dz \\
&= 2 \int_0^c \int_0^b \left[ \frac{x^2}{2} + (y+z)x \right]_0^a dy dz \\
&= 2 \int_0^c \int_0^b \left[ \frac{a^2}{2} + (y+z)a \right] dy dz \\
&= 2 \int_0^c \left[ \frac{a^2}{2} y + a \left( \frac{y^2}{2} + zy \right) \right]_0^b dz \\
&= 2 \int_0^c \left[ \frac{a^2}{2} b + a \left( \frac{b^2}{2} + zb \right) \right] dz \\
&= 2 \int_0^c \left[ \frac{a^2 b + ab^2}{2} + abz \right] dz \\
&= 2 \left[ \frac{a^2 b + ab^2}{2} z + ab \frac{z^2}{2} \right]_0^c \\
&= 2 \left[ ab \frac{(a+b)}{2} c + ab \frac{c^2}{2} \right] \\
&= 2 \frac{abc}{2} (a+b+c) = abc(a+b+c)
\end{aligned} \tag{1}$$

We shall now evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$

Here the surface  $S$  consists of the six faces of the parallelopiped.

$$\begin{aligned}
\therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iint_{S_1} \vec{F} \cdot \vec{n} dS \\
&\quad + \iint_{S_2} \vec{F} \cdot \vec{n} dS \\
&\quad + \iint_{S_3} \vec{F} \cdot \vec{n} dS \\
&\quad + \iint_{S_4} \vec{F} \cdot \vec{n} dS + \iint_{S_5} \vec{F} \cdot \vec{n} dS + \iint_{S_6} \vec{F} \cdot \vec{n} dS
\end{aligned}$$



We shall simplify the computation and put it in the form of a table.

Face	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	$dS$
$S_1 = \text{ABEF}$	$x = a$	$\vec{i}$	$x^2 - yz = a^2 - yz$	$dy dz$
$S_2 = \text{OCDG}$	$x = 0$	$-\vec{i}$	$-(x^2 - yz) = yz$	$dy dz$
$S_3 = \text{BCDE}$	$y = b$	$\vec{j}$	$y^2 - zx = b^2 - zx$	$dz dx$
$S_4 = \text{OAFG}$	$y = 0$	$-\vec{j}$	$-(y^2 - zx) = zx$	$dz dx$
$S_5 = \text{DEFG}$	$z = c$	$\vec{k}$	$z^2 - xy = c^2 - xy$	$dx dy$
$S_6 = \text{OABC}$	$z = 0$	$-\vec{k}$	$-(z^2 - xy) = xy$	$dx dy$

$$\therefore \iint_{S_1} \vec{F} \cdot \vec{n} dS = \int_0^c \int_0^b (a^2 - yz) dy dz$$

$$\begin{aligned}
 &= \int_0^c \left[ a^2 y - \frac{y^2}{2} z \right]_0^b dz \\
 &= \int_0^c \left[ a^2 b - \frac{b^2}{2} z \right] dz \\
 &= \left[ a^2 bz - \frac{b^2}{2} \frac{z^2}{2} \right]_0^c \\
 &= a^2 bc - \frac{b^2 c^2}{4}
 \end{aligned}$$

$$\begin{aligned}
 \iint_{S_2} \vec{F} \cdot \vec{n} dS &= \int_0^c \int_0^b yz dy dz \\
 &= \left[ \frac{y^2}{2} \right]_0^b \left[ \frac{z^2}{2} \right]_0^c \\
 &= \frac{b^2}{2} \cdot \frac{c^2}{2} = \frac{b^2 c^2}{4}
 \end{aligned}$$

$$\begin{aligned}
\iint_{S_3} \vec{F} \cdot \vec{n} dS &= \int_0^c \int_0^a (b^2 - zx) dz dx \\
&= \int_0^a \left[ b^2 z - \frac{z^2}{2} x \right]_0^c dx \\
&= \int_0^a \left[ b^2 c - \frac{c^2}{2} x \right] dx \\
&= \left[ b^2 c x - \frac{c^2}{2} \frac{x^2}{2} \right]_0^a = b^2 c a - \frac{c^2 a^2}{4} \\
\iint_{S_4} \vec{F} \cdot \vec{n} dS &= \int_0^c \int_0^a zx dz dx \\
&= \left[ \frac{z^2}{2} \right]_0^c \left[ \frac{x^2}{2} \right]_0^a = \frac{c^2}{2} \cdot \frac{a^2}{2} = \frac{c^2 a^2}{4} \\
\iint_{S_5} \vec{F} \cdot \vec{n} dS &= \int_0^b \int_0^a (c^2 - xy) dx dy \\
&= \int_0^b \left[ c^2 x - \frac{x^2}{2} y \right]_0^a dy \\
&= \int_0^b \left[ ac^2 - \frac{a^2}{2} y \right] dy \\
&= \left[ ac^2 y - \frac{a^2}{2} \frac{y^2}{2} \right]_0^b = ac^2 b - \frac{a^2 b^2}{4} \\
\iint_{S_6} \vec{F} \cdot \vec{n} dS &= \int_0^b \int_0^a xy dx dy = \left[ \frac{x^2}{2} \right]_0^a \left[ \frac{y^2}{2} \right]_0^b = \frac{a^2 b^2}{4} \\
\iint_S \vec{F} \cdot \vec{n} dS &= a^2 bc - \frac{b^2 c^2}{4} + \frac{b^2 c^2}{4} + b^2 ca - \frac{c^2 a^2}{4} + \frac{c^2 a^2}{4} \\
&\quad + ac^2 b - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4} \\
&= a^2 bc + b^2 ca + ac^2 b = abc(a + b + c) \tag{2}
\end{aligned}$$

From (1) and (2),  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$

Hence the theorem is verified.

**Example 9**

Verify divergence theorem for  $\vec{F} = x^2 \vec{i} + z \vec{j} + yz \vec{k}$  over the cube formed by the planes  $x = \pm 1, y = \pm 1, z = \pm 1$ . [AU 2006, 2013]

**Solution**

Gauss divergence theorem is  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$

$$\text{Given } \vec{F} = x^2 \vec{i} + z \vec{j} + yz \vec{k}$$

$$\begin{aligned} \therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz) \\ &= 2x + 0 + y = 2x + y \end{aligned}$$

$$\begin{aligned} \therefore \iiint_V \nabla \cdot \vec{F} dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + y) dx dy dz \\ &= \int_{-1}^1 \int_{-1}^1 \left[ x^2 + yx \right]_{-1}^1 dy dz \\ &= \int_{-1}^1 \int_{-1}^1 [1 + y - (1 - y)] dy dz \\ &= \int_{-1}^1 \int_{-1}^1 2y dy dz = 0 \quad (1) \end{aligned}$$

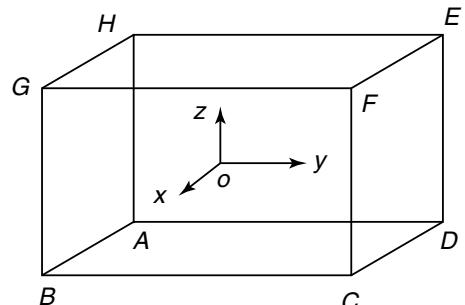
$\left[ \int_{-a}^a f(x) dx = 0 \text{ if } f(x) \text{ is odd function} \right]$

Here  $y$  is odd function

We shall now compute  $\iint_S \vec{F} \cdot \vec{n} dS$

$S$  is the surface consisting of the six faces of the cube.

$$\begin{aligned} \therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iint_{S_1} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{S_2} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{S_3} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{S_4} \vec{F} \cdot \vec{n} dS + \iint_{S_5} \vec{F} \cdot \vec{n} dS + \iint_{S_6} \vec{F} \cdot \vec{n} dS \end{aligned}$$



We shall simplify the computations and put it in the form of a table.

Faces	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	$dS$
$S_1 = \text{BCFG}$	$x = 1$	$\vec{i}$	$x^2 = 1$	$dy dz$
$S_2 = \text{ADEH}$	$x = -1$	$-\vec{i}$	$-x^2 = -1$	$dy dz$
$S_3 = \text{CDEF}$	$y = 1$	$\vec{j}$	$z$	$dz dx$
$S_4 = \text{ABGH}$	$y = -1$	$-\vec{j}$	$-z$	$dz dx$
$S_5 = \text{EFGH}$	$z = 1$	$\vec{k}$	$yz = y$	$dx dy$
$S_6 = \text{ABCD}$	$z = -1$	$-\vec{k}$	$-yz = y$	$dx dy$

$$\therefore \iint_{S_1} \vec{F} \cdot \vec{n} dS = \int_{-1}^1 \int_{-1}^1 dy dz = [y]_{-1}^1 [z]_{-1}^1 = (1+1)(1+1) = 4$$

$$\iint_{S_2} \vec{F} \cdot \vec{n} dS = \int_{-1}^1 \int_{-1}^1 -1 dy dz = -[y]_{-1}^1 [z]_{-1}^1 = -[1+1][1+1] = -4$$

$$\iint_{S_3} \vec{F} \cdot \vec{n} dS = \int_{-1}^1 \int_{-1}^1 z dz dx = 0 \quad [\because z \text{ is odd function}]$$

$$\iint_{S_4} \vec{F} \cdot \vec{n} dS = \int_{-1}^1 \int_{-1}^1 -z dz dx = -\int_{-1}^1 \int_{-1}^1 z dz dx = 0$$

$$\iint_{S_5} \vec{F} \cdot \vec{n} dS = \int_{-1}^1 \int_{-1}^1 y dx dy = 0 \quad [\because y \text{ is odd function}]$$

$$\iint_{S_6} \vec{F} \cdot \vec{n} dS = \int_{-1}^1 \int_{-1}^1 y dx dy = 0$$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = 4 - 4 + 0 + 0 + 0 + 0 = 0 \quad (2)$$

$$\text{From (1) and (2), } \iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Hence Gauss's divergence theorem is verified.

**Example 10**

**Verify divergence theorem for the function  $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$  taken over the surface of the region, bounded by the cylinder  $x^2 + y^2 = 4$  and  $z = 0, z = 3$ .** [AU 2003, 2006, 2007, 2011]

**Solution**

Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \operatorname{div} \vec{F} dV$$

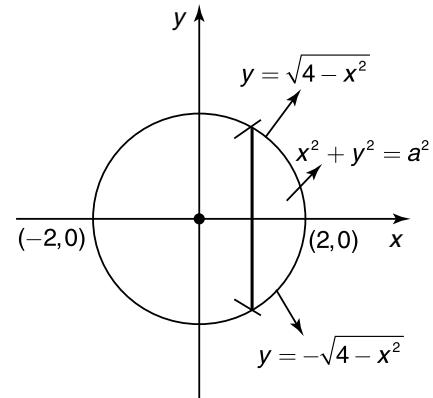
Given  $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$$\begin{aligned}\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) \\ &= 4 - 4y + 2z\end{aligned}$$

Also given  $x^2 + y^2 = 4$

$$\Rightarrow y^2 = 4 - x^2 \Rightarrow y = \pm\sqrt{4 - x^2}$$

$$\text{and } y = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$



$$\begin{aligned}\therefore \iiint_V \nabla \cdot \vec{F} dV &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^3 (4 - 4y + 2z) dz dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[ (4 - 4y)z + 2 \frac{z^2}{2} \right]_0^3 dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4 - 4y) \cdot 3 + 9] dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [21 - 12y] dy dx \\ &= \int_{-2}^2 \left[ 21y - 12 \frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^2 [21(\sqrt{4-x^2} + \sqrt{4-x^2}) - 6(4-x^2 - (4-x^2))] dx\end{aligned}$$

$$\begin{aligned}
&= \int_{-2}^2 42 \sqrt{4-x^2} dx \\
&= 84 \int_0^2 \sqrt{4-x^2} dx \quad [\because \sqrt{4-x^2} \text{ is even function}] \\
&= 84 \left[ \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2 \\
&= 84 [0 + 2 \sin^{-1} 1 - 0] = 84 \cdot 2 \frac{\pi}{2} = 84\pi
\end{aligned} \tag{2}$$

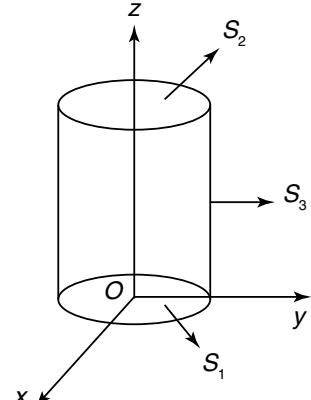
We shall now compute the surface integral  $\iint_S \vec{F} \cdot \vec{n} dS$ .

$S$  consists of the bottom surface  $S_1$ , top surface  $S_2$  and the curved surface  $S_3$  of the cylinder.

**On  $S_1$ :** Equation is  $z = 0$ ,  $\vec{n} = -\vec{k}$

$$\therefore \vec{F} \cdot \vec{n} = -z^2 = 0$$

$$\therefore \iint_{S_1} \vec{F} \cdot \vec{n} dS = 0$$



**On  $S_2$ :** Equation is  $z = 3$ ,  $\vec{n} = \vec{k}$

$$\therefore \vec{F} \cdot \vec{n} = z^2 = 9, \quad dS = \frac{dx dy}{|\vec{n} \cdot \vec{k}|} = \frac{dx dy}{|\vec{k} \cdot \vec{k}|} = dx dy$$

$$\begin{aligned}
\therefore \iint_{S_2} \vec{F} \cdot \vec{n} dS &= \iint_{S_2} 9 dx dy = 9 \iint_{S_2} dx dy \\
&= 9 \text{ area of the circle } S_2 = 9 \cdot \pi \cdot 2^2 = 36\pi.
\end{aligned}$$

**On  $S_3$ :** Equation is  $x^2 + y^2 = 4$

Let  $\phi = x^2 + y^2$

$$\begin{aligned}
\therefore \nabla \phi &= \vec{i} \frac{\partial}{\partial x} \phi + \vec{j} \frac{\partial}{\partial y} \phi + \vec{k} \frac{\partial \phi}{\partial z} \\
&= \vec{i} 2x + \vec{j} 2y + \vec{k} 0 = 2(x\vec{i} + y\vec{j})
\end{aligned}$$

$$\therefore \text{the normal } \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2 + y^2}} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{4}} = \frac{1}{2}(x\vec{i} + y\vec{j})$$

$$\therefore \vec{F} \cdot \vec{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}) \cdot \frac{1}{2}(x\vec{i} + y\vec{j}) = 2x^2 - y^3$$

Since  $S_3$  is the surface of a cylinder  $x^2 + y^2 = 4$ , we use cylindrical polar coordinates to evaluate

$$\iint_{S_3} \vec{F} \cdot \vec{n} dS$$

$$\therefore x = 2 \cos \theta, \quad y = 2 \sin \theta, \quad z = z$$

$$dS = 2 d\theta dz \quad \theta \text{ varies from } 0 \text{ to } 2\pi$$

$$\text{and} \quad z \text{ varies from } 0 \text{ to } 3$$

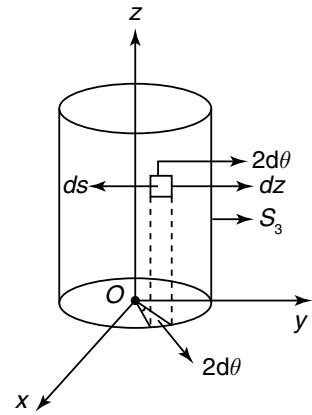
$$\begin{aligned} \therefore \iint_{S_3} \vec{F} \cdot \vec{n} dS &= \int_0^3 \int_0^{2\pi} (2 \cdot 4 \cos^2 \theta - 8 \sin^3 \theta) 2d\theta dz \\ &= 16 \int_0^3 \int_0^{2\pi} (\cos^2 \theta - \sin^3 \theta) d\theta dz \\ &= 16 \int_0^3 \int_0^{2\pi} \left[ \frac{1 + \cos 2\theta}{2} - \frac{1}{4} (3 \sin \theta - \sin 3\theta) \right] d\theta dz \\ &= 16 \int_0^3 \left[ \frac{1}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) - \frac{1}{4} \left( -3 \cos \theta + \frac{\cos 3\theta}{3} \right) \right]_0^{2\pi} dz \\ &= 16 \int_0^3 \left\{ \frac{1}{2} \left[ 2\pi + \frac{\sin 4\pi}{2} - 0 \right] - \frac{1}{4} \left[ -3 \cos 2\pi + \frac{\cos 6\pi}{3} - \left( -3 \cos 0 + \frac{\cos 0}{3} \right) \right] \right\} dz \\ &= 16 \int_0^3 \left( \pi + \frac{3}{4} - \frac{1}{12} - \frac{3}{4} + \frac{1}{12} \right) dz \end{aligned}$$

$$= 16 \pi \int_0^3 dz = 16 \pi [z]_0^3 = 16 \pi \times 3 = 48\pi$$

$$\iint_S \vec{F} \cdot \vec{n} dS = 36\pi + 48\pi = 84\pi \quad (2)$$

$$\text{From (1) and (2), } \iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Hence Gauss's divergence theorem is verified.



**Example 11**

Verify Gauss divergence theorem for  $\vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$  over the region bounded by the upper hemisphere  $x^2 + y^2 + z^2 = a^2$  and the plane  $z = 0$ .

**Solution**

Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Given  $\vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$

$$\begin{aligned}\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(a(x+y)) + \frac{\partial}{\partial y}(a(y-x)) + \frac{\partial}{\partial z}(z^2) \\ &= a + a + 2z = 2(a+z)\end{aligned}$$

$$\begin{aligned}\therefore \iiint_V \nabla \cdot \vec{F} dV &= \iiint_V 2(a+z) dV \\ &= 2a \iiint_V dV + 2 \iiint_V z dV \\ &= 2aV + 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z dz dy dx \\ &= 2aV + 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[ \frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dy dx \\ &= 2aV + 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left( \frac{a^2 - x^2 - y^2}{2} \right) dy dx \quad \left[ \because V = \frac{2}{3}\pi a^3 \right] \\ &= 2a \frac{2\pi}{3} a^3 + \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2) dy dx \\ &= \frac{4\pi a^4}{3} + \int_{-a}^a 2 \int_0^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2) dy dx \quad [\because a^2 - x^2 - y^2 \text{ is even in } y] \\ &= \frac{4\pi a^4}{3} + 2 \int_{-a}^a \left[ (a^2 - x^2)y - \frac{y^3}{3} \right]_0^{\sqrt{a^2-x^2}} dx\end{aligned}$$

$$\begin{aligned}
&= \frac{4\pi a^4}{3} + 2 \int_{-a}^a \left[ (a^2 - x^2) \sqrt{a^2 - x^2} - \frac{(a^2 - x^2)^{3/2}}{3} \right] dx \\
&= \frac{4\pi a^4}{3} + 2 \int_{-a}^a \left[ (a^2 - x^2)^{3/2} - \frac{(a^2 - x^2)^{3/2}}{3} \right] dx \\
&= \frac{4\pi a^4}{3} + 2 \cdot \frac{2}{3} \int_{-a}^a (a^2 - x^2)^{3/2} dx \\
&= \frac{4\pi a^4}{3} + \frac{4}{3} \times 2 \int_0^a (a^2 - x^2)^{3/2} dx \quad [\because (a^2 - x^2)^{3/2} \text{ is even}] \\
&= \frac{4\pi a^4}{3} + \frac{8}{3} I
\end{aligned}$$

where  $I = \int_0^a (a^2 - x^2)^{3/2} dx$

Put  $x = a \sin \theta \therefore dx = a \cos \theta d\theta$

When  $x = 0, \sin \theta = 0 \Rightarrow \theta = 0$  and when  $x = a, \sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\begin{aligned}
\therefore I &= \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)^{3/2} a \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} a^3 \cos^3 \theta \cdot a \cos \theta d\theta \\
&= a^4 \int_0^{\pi/2} \cos^4 \theta d\theta \\
&= a^4 \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2} = a^4 \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi a^4}{16} \\
\iiint_V \nabla \cdot \vec{F} &= \frac{4\pi a^4}{3} + \frac{8}{3} \cdot \frac{3\pi a^4}{16} = \frac{(8+3)}{6} \pi a^4 = \frac{11}{6} \pi a^4 \tag{1}
\end{aligned}$$

Now we shall compute the double integral  $\iint_S \vec{F} \cdot \vec{n} dS$

$S$  consists of  $S_1$  and  $S_2$

$$\therefore \iint_S \vec{F} \cdot \vec{n} dS = \iint_{S_1} \vec{F} \cdot \vec{n} dS_1 + \iint_{S_2} \vec{F} \cdot \vec{n} dS_2$$

**On  $S_1$ :**  $z = 0, \vec{n} = -\vec{k}$

$$\begin{aligned}\therefore \vec{F} \cdot \vec{n} &= (a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}) \cdot (-\vec{k}) \\ &= -z^2 = 0\end{aligned}$$

$$\therefore \iint_{S_1} \vec{F} \cdot \vec{n} dS = 0$$

**On  $S_2$ :**  $x^2 + y^2 + z^2 = a^2$

$$\text{Let } \phi = x^2 + y^2 + z^2$$

$$\begin{aligned}\therefore \nabla \phi &= 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \\ &= 2(x\vec{i} + y\vec{j} + z\vec{k})\end{aligned}$$

$$\therefore \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$\begin{aligned}\vec{F} \cdot \vec{n} &= [a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}] \cdot \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a} \\ &= (x+y)x + (y-x)y + \frac{z^3}{a} = x^2 + y^2 + \frac{z^3}{a}\end{aligned}$$

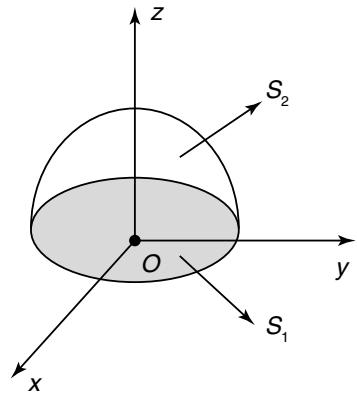
$$\iint_{S_2} \vec{F} \cdot \vec{n} dS = \iint_R \vec{F} \cdot \vec{n} \frac{dx dy}{|\vec{n} \cdot \vec{k}|}, \text{ where } R \text{ is the projection of } S_2 \text{ on the } xy\text{-plane.}$$

$$\begin{aligned}\therefore \iint_{S_2} \vec{F} \cdot \vec{n} dS &= \iint_R \left( x^2 + y^2 + \frac{z^3}{a} \right) \frac{dx dy}{\frac{z}{a}} \\ &= \iint_R \left( \frac{a(x^2 + y^2)}{z} + z^2 \right) dx dy \\ &= \iint_R \left( \frac{a(x^2 + y^2)}{z} + [a^2 - x^2 - y^2] \right) dx dy\end{aligned}$$

Changing to polar coordinate, we have

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2 \quad \text{and} \quad dx dy = r dr d\theta$$

$$\therefore \iint_{S_2} \vec{F} \cdot \vec{n} dS = \iint_0^{a/2} \left\{ \frac{ar^2}{\sqrt{a^2 - r^2}} + (a^2 - r^2) \right\} r dr d\theta$$



$$\begin{aligned}
&= \int_0^a \int_0^{2\pi} \left\{ \frac{-a(a^2 - r^2) + a^3}{\sqrt{a^2 - r^2}} + (a^2 - r^2) \right\} r dr d\theta \\
&= \int_0^a \int_0^{2\pi} \left\{ -a\sqrt{a^2 - r^2} + \frac{a^3}{\sqrt{a^2 - r^2}} + (a^2 - r^2) \right\} r dr d\theta \\
&= \int_0^{2\pi} d\theta \int_0^a \left\{ -a\sqrt{a^2 - r^2} + \frac{a^3}{\sqrt{a^2 - r^2}} + (a^2 - r^2) \right\} r dr \\
&= [\theta]_0^{2\pi} \int_0^a (-a\sqrt{a^2 - r^2})r + a^3(a^2 - r^2)^{-1/2}r + (a^2 - r^2)r dr \\
&= 2\pi \left\{ \int_0^a \frac{a}{2}(a^2 - r^2)(-2r)dr - \frac{a^3}{2} \int_0^a (a^2 - r^2)^{-1/2}(-2r)dr \right. \\
&\quad \left. + \int_0^a (a^2r - r^3)dr \right\} \\
&= 2\pi \left\{ \frac{a}{2} \left[ \frac{(a^2 - r^2)^{3/2}}{\frac{3}{2}} \right]_0^a - \frac{a^3}{2} \left[ \frac{(a^2 - r^2)^{1/2}}{\frac{1}{2}} \right]_0^a + \left[ a^2 \frac{r^2}{2} - \frac{r^4}{4} \right]_0^a \right\} \\
&= 2\pi \left[ \frac{a}{3}(0 - a^3) - a^3(0 - a) + \frac{a^4}{2} - \frac{a^4}{4} \right] \\
&= 2\pi \left[ -\frac{a^4}{3} + a^4 + \frac{a^4}{4} \right] \\
\therefore \int \int \overrightarrow{F} \cdot \vec{n} dS &= 2\pi \times \frac{11a^4}{12} = \frac{11\pi a^4}{6} \\
\therefore \int \int \overrightarrow{F} \cdot \vec{n} dS &= 0 + \frac{11\pi a^4}{6} = \frac{11\pi a^4}{6} \tag{2}
\end{aligned}$$

From (1) and (2),  $\int \int \overrightarrow{F} \cdot \vec{n} dS = \int \int \int \nabla \cdot \vec{F} dV$

Hence Gauss's divergence theorem is verified.

**Example 12**

Evaluate  $\iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy$  over the surface  $z = 0, z = h$ ,  $x^2 + y^2 = a^2$ . [AU 2005]

**Solution**

We know Gauss divergence theorem in cartesian form is

$$\iint_S F_1 dy dz + F_2 dz dx + F_3 dx dy = \iiint_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx dy dz$$

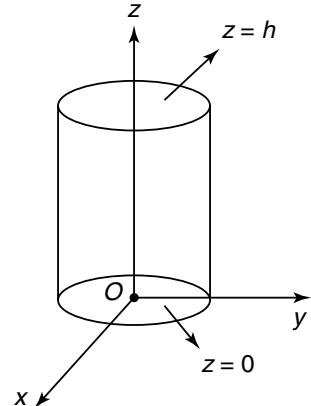
Given surface integral is

$$\iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy$$

$$\text{Here } F_1 = x^3, \quad F_2 = x^2 y, \quad F_3 = x^2 z$$

$$\therefore \frac{\partial F_1}{\partial x} = 3x^2, \quad \frac{\partial F_2}{\partial y} = x^2, \quad \frac{\partial F_3}{\partial z} = x^2$$

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$



$$\iint_S F_1 dy dz + F_2 dz dx + F_3 dx dy = \iiint_V 5x^2 dx dy dz$$

$$= 5 \int_{z=0}^h \int_{y=-a}^a \int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x^2 dx dy dz$$

$$= 5 \int_0^h \int_{y=-a}^a \left[ 2 \int_0^{\sqrt{a^2-y^2}} x^2 dx \right] dy dz \quad [\because x^2 \text{ is even}]$$

$$= 10 \int_{z=0}^h \int_{y=-a}^a \left[ \frac{x^3}{3} \right]_0^{\sqrt{a^2-y^2}} dy dz$$

$$= \frac{10}{3} \int_{z=0}^h \int_{y=-a}^a (a^2 - y^2)^{3/2} dy dz$$

$$\begin{aligned}
&= \frac{10}{3} \int_0^h dz \left[ 2 \int_0^a (a^2 - y^2)^{3/2} dy \right] \\
&= \frac{20}{3} \left[ z \right]_0^h \int_0^a (a^2 - y^2)^{3/2} dy \\
&= \frac{20}{3} h \int_0^a (a^2 - y^2)^{3/2} dy = \frac{20}{3} h I
\end{aligned}$$

where  $I = \int_0^a (a^2 - y^2)^{3/2} dy$

Put  $y = a \sin \theta$ ;  $dy = a \cos \theta d\theta$

$$\begin{aligned}
\therefore I &= \int_0^{\pi/2} (a^2 - a^2 \sin^2 \theta)^{3/2} a \cos \theta d\theta \\
&= a^4 \int_0^{\pi/2} \cos^3 \theta \cos \theta d\theta \\
&= a^4 \int_0^{\pi/2} \cos^4 \theta d\theta = a^4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi a^4}{16}
\end{aligned}$$

$y$	$\theta$
0	0
$a$	$\pi/2$

$$\therefore \iint_S F_1 dy dz + F_2 dz dx + F_3 dx dy = \frac{20}{3} h \times \frac{3\pi a^4}{16} = \frac{5}{4} \pi a^4 h$$

### Example 13

Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = (2x + 3z)\vec{i} - (xz + y)\vec{j} + (y^2 + 2z)\vec{k}$  and

$S$  is the surface of the sphere having centre  $(3, -1, 2)$  and radius 3.

### Solution

Gauss divergence theorem is  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$

Given  $\vec{F} = (2x + 3z)\vec{i} - (xz + y)\vec{j} + (y^2 + 2z)\vec{k}$

$$\begin{aligned}
\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}(-(xz + y)) + \frac{\partial}{\partial z}(y^2 + 2z) \\
&= 2 - 1 + 2 = 3
\end{aligned}$$

$S$  is the sphere with centre at  $(3, -1, 2)$  and radius  $r = 3$ .

$$\therefore \text{volume of the sphere } V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 3^3 = 36\pi$$

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V 3 dV = 3V = 3 \times 36\pi = 108\pi$$

## 1.13 STOKE'S THEOREM

Stoke's theorem gives a relation between **line integral** and **surface integral**.

**Theorem 1.1** If  $S$  is an open surface bounded by a simple closed curve  $C$  and if  $\vec{F}$  is continuous having continuous partial derivatives in  $S$  and on  $C$ , then  $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$ , where  $C$  is traversed in the positive direction.

### Note

1. Green's theorem in the plane is a particular case of Stoke's theorem.
2. If  $S$  is the region  $R$  in the  $xy$ -plane, bounded by the simple closed curve  $C$ , then  $\vec{n} = \vec{k}$  is the outward unit normal.

$$\therefore \text{Stoke's theorem in the plane is } \oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{Curl } \vec{F} \cdot \vec{k} dR$$

### Cartesian form of Stoke's theorem

If  $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  then

$$\begin{aligned} \text{Curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \vec{i} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \vec{j} \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \vec{k} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

and  $\vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$

$\therefore$  the cartesian form of Stoke's theorem is  $\oint_C (F_1 dx + F_2 dy + F_3 dz)$

$$= \iint_S \left[ \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dy dz + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) dz dx + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \right]$$

**Note**

$$\text{If } \vec{F} = P\vec{i} + Q\vec{j}, \quad \vec{r} = x\vec{i} + y\vec{j} \quad \therefore d\vec{r} = dx\vec{i} + dy\vec{j}$$

then  $\vec{F} \cdot d\vec{r} = P dx + Q dy$

$$\begin{aligned}\text{Curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\ &= \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}\end{aligned}$$

$$\therefore \text{Curl } \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\therefore$  Stokes theorem in the plane is  $\oint_C (P dx + Q dy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$   
which is Green's theorem.

**WORKED EXAMPLES**
**Example 1**

Prove that  $\oint_C \vec{r} \cdot d\vec{r} = 0$ , where  $C$  is the simple closed curve.

**Solution**

Let  $\vec{r}$  be the position vector of any point  $P(x, y, z)$  on  $C$ .

$$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Stokes theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

Here  $\vec{F} = \vec{r}$ .

$$\begin{aligned} \therefore \operatorname{Curl} \vec{F} = \operatorname{Curl} \vec{r} &= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{array} \right| \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) + (0-0) = \vec{0} \\ \therefore \oint_C \vec{r} \cdot d\vec{r} &= 0 \end{aligned}$$

**Example 2**

If  $S$  is the surface of the sphere  $x^2 + y^2 + z^2 = a^2$ , then show that

$$\iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS = 0.$$

**Solution**

Suppose the sphere is cut by a plane into two parts  $S_1$  and  $S_2$  and let  $C$  be the curve binding these two parts.

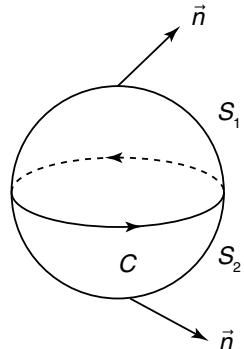
$$\begin{aligned} \text{Then } \iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS &= \iint_{S_1} \operatorname{Curl} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{S_2} \operatorname{Curl} \vec{F} \cdot \vec{n} dS \end{aligned}$$

$$\text{By Stoke's theorem, } \iint_{S_1} \operatorname{Curl} \vec{F} \cdot \vec{n} dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$\text{and } \iint_{S_2} \operatorname{Curl} \vec{F} \cdot \vec{n} dS = -\oint_C \vec{F} \cdot d\vec{r}, \text{ because for } S_2 \text{ the}$$

positive sense of the curve  $C$  is the opposite direction of  $C$  in  $S_1$

$$\therefore \iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS = \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} = 0$$

**Example 3**

Evaluate by Stoke's theorem  $\int_C e^x dx + 2y dy - dz$ , where  $C$  is the curve  $x^2 + y^2 = 4, z = 2$ . [AU 2001]

**Solution**

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS$$

Given

$$\int_C (e^x dx + 2y dy - dz)$$

Here

$$\vec{F} \cdot d\vec{r} = e^x dx + 2y dy - dz$$

 $\therefore$ 

$$\vec{F} = e^x \vec{i} + 2y \vec{j} - \vec{k}$$

 $\therefore$ 

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

$$\Rightarrow \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS = 0$$

Hence

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

$$\oint_C e^x dx + 2y dy - dz = 0$$

### Example 4

Evaluate  $\int_C (xy dx + xy^2 dy)$  by Stoke's theorem, where  $C$  is the square in the  $xy$ -plane with vertices  $(1, 0), (-1, 0), (0, 1), (0, -1)$ . [AU 2002]

### Solution

$$\text{Stoke's theorem is } \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

Given

$$\int_C (xy dx + xy^2 dy) \quad \text{and} \quad \vec{r} = x \vec{i} + y \vec{j}$$

$$d\vec{r} = dx \vec{i} + dy \vec{j}.$$

Here

$$\vec{F} \cdot d\vec{r} = xy dx + xy^2 dy$$

 $\therefore$ 

$$\vec{F} = xy \vec{i} + xy^2 \vec{j}$$

 $\therefore$ 

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$\begin{aligned}
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} \\
 &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(y^2 - x)
 \end{aligned}$$

$$\Rightarrow \text{Curl } \vec{F} = (y^2 - x)\vec{k}$$

Also given  $C$  is the square in the  $xy$  plane with vertices  $(1, 0)$ ,  $(-1, 0)$ ,  $(0, 1)$ ,  $(0, -1)$ .

$$\therefore \vec{n} = \vec{k} \text{ and } dS = dx dy$$

$$\therefore \text{Curl } \vec{F} \cdot \vec{n} = (y^2 - x)\vec{k} \cdot \vec{k} = y^2 - x$$

$$\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS = \iint_R (y^2 - x) dx dy$$

where  $R$  is the region inside the square.

$$\text{That is } \iint_C xy dx + xy^2 dy = \iint_R (y^2 - x) dx dy$$

We shall now evaluate this double integral.

$$\text{Equation of } AB \text{ in intercept form is } \frac{x}{1} + \frac{y}{1} = 1$$

$$\Rightarrow x + y = 1$$

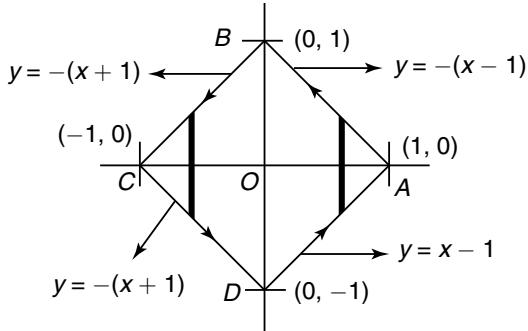
$$\begin{aligned}
 \Rightarrow y &= -x + 1 \\
 &= -(x - 1)
 \end{aligned}$$

$$\text{Equation of } BC \text{ is } \frac{x}{-1} + \frac{y}{1} = 1$$

$$\Rightarrow y - x = 1 \Rightarrow y = x + 1$$

$$\text{Equation of } CD \text{ is } \frac{x}{-1} + \frac{y}{-1} = 1$$

$$\Rightarrow x + y = -1 \Rightarrow y = -(x + 1)$$



Equation of  $AD$  is  $\frac{x}{1} + \frac{y}{-1} = 1$

$$\Rightarrow y - x = -1 \Rightarrow y = x - 1$$

$$\begin{aligned}\therefore \int_C (xy \, dx + xy^2 \, dy) &= \int_{-1}^0 \int_{-(x+1)}^{x+1} (y^2 - x) \, dy \, dx + \int_0^1 \int_{x-1}^{-(x-1)} (y^2 - x) \, dy \, dx \\ &= \int_{-1}^0 \left[ \frac{y^3}{3} - xy \right]_{-(x+1)}^{x+1} \, dx + \int_0^1 \left[ \frac{y^3}{3} - xy \right]_{x-1}^{-(x-1)} \, dx \\ &= \int_{-1}^0 \frac{1}{3} \left\{ [(x+1)^3 - (-(x+1))^3] - x[x+1 - (-(x+1))] \right\} \, dx \\ &\quad + \int_0^1 \left\{ \frac{1}{3} [(-(x-1))^3 - (x-1)^3] - x[-(x-1) - (x-1)] \right\} \, dx \\ &= \int_{-1}^0 \left\{ \frac{1}{3} [(x+1)^3 + (x+1)^3] - x[(x+1) + (x+1)] \right\} \, dx \\ &\quad + \int_0^1 \left\{ -\frac{1}{3} [(x-1)^3 + (x-1)^3] + x[x-1 + x-1] \right\} \, dx \\ &= \int_{-1}^0 \left[ \frac{2}{3}(x+1)^3 - 2x(x+1) \right] \, dx + \int_0^1 \left[ -\frac{2}{3}(x-1)^3 + 2x(x-1) \right] \, dx \\ &= \left[ \frac{2}{3} \frac{(x+1)^4}{4} - 2 \left( \frac{x^3}{3} + \frac{x^2}{2} \right) \right]_{-1}^0 + \left[ -\frac{2}{3} \frac{(x-1)^4}{4} + 2 \left( \frac{x^3}{3} - \frac{x^2}{2} \right) \right]_0^1 \\ &= \frac{2}{3} \left( \frac{1}{4} \right) - 2 \left\{ 0 - \left[ \frac{1}{3}(-1)^3 + \frac{(-1)^2}{2} \right] \right\} - \frac{2}{3} \left[ 0 - \left( \frac{1}{4} \right) \right] + 2 \left[ \frac{1}{3} - \frac{1}{2} \right] \\ &= \frac{1}{6} - \frac{2}{3} + 1 + \frac{1}{6} + \frac{2}{3} - 1 = \frac{2}{6} = \frac{1}{3}\end{aligned}$$

### Example 5

Verify Stoke's theorem for  $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ , where  $S$  is the surface of the cube  $x = 0, x = 2, y = 0, y = 2, z = 0$  and  $z = 2$  above the  $xy$ -plane. [AU 2005]

### Solution

Given  $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ .

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS$$

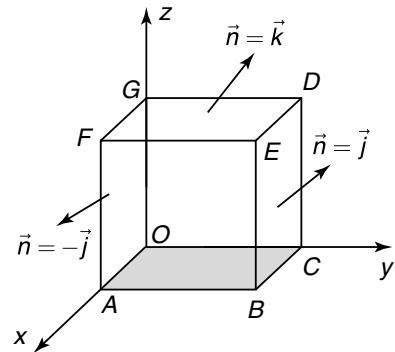
Now  $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$$\begin{aligned} &= \vec{i} \left[ \frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4) \right] \\ &\quad - \vec{j} \left[ \frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y-z+2) \right] + \vec{k} \left[ \frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2) \right] \\ &= \vec{i}[(0-y)] - \vec{j}[-z-(-1)] + \vec{k}(0-1) = -y\vec{i} + (z-1)\vec{j} - \vec{k} \end{aligned}$$

We shall compute  $\iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS$ .

Given  $S$  is the open surface consisting of 5 faces of the cube except the face  $OABC$ .

$$\begin{aligned} \iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS &= \iint_{S_1} \operatorname{Curl} \vec{F} \cdot \vec{n} dS + \iint_{S_2} \operatorname{Curl} \vec{F} \cdot \vec{n} dS + \iint_{S_3} \operatorname{Curl} \vec{F} \cdot \vec{n} dS \\ &\quad + \iint_{S_4} \operatorname{Curl} \vec{F} \cdot \vec{n} dS + \iint_{S_5} \operatorname{Curl} \vec{F} \cdot \vec{n} dS \end{aligned}$$



Face	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	$dS$
$S_1 = ABEF$	$x = 2$	$\vec{i}$	$-y$	$dy dz$
$S_2 = OCDG$	$x = 0$	$-\vec{i}$	$y$	$dy dz$
$S_3 = BCDE$	$y = 2$	$\vec{j}$	$z-1$	$dx dz$
$S_4 = OAFG$	$y = 0$	$-\vec{j}$	$-(z-1)$	$dx dz$
$S_5 = DEFG$	$z = 2$	$\vec{k}$	$-1$	$dx dy$

$$\begin{aligned}
\therefore \iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_0^2 \int_0^2 -y \, dy \, dz = \int_0^2 dz \cdot \int_0^2 (-y) \, dy \\
&= [z]_0^2 \left[ \frac{-y^2}{2} \right]_0^2 = 2(-2) = -4 \\
\iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_0^2 \int_0^2 y \, dy \, dz = \int_0^2 dz \int_0^2 y \, dy \\
&= [z]_0^2 \left[ \frac{y^2}{2} \right]_0^2 = 2 \cdot 2 = 4 \\
\iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_0^2 \int_0^2 (z-1) \, dz \, dx = \int_0^2 dx \cdot \int_0^2 (z-1) \, dz \\
&= [x]_0^2 \cdot \left[ \frac{(z-1)^2}{2} \right]_0^2 \\
&= 2 \cdot \frac{1}{2} \{ (2-1)^2 - (-1)^2 \} = 1 - 1 = 0 \\
\iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_0^2 \int_0^2 -(z-1) \, dz \, dx = 0 \quad [\text{as above}] \\
\iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_0^2 \int_0^2 -1 \, dx \, dy = -[x]_0^2 [y]_0^2 = -4 \\
\iint_S \text{Curl } \vec{F} \cdot \vec{n} \, dS &= -4 + 4 + 0 + 0 - 4 = -4 \tag{1}
\end{aligned}$$

We shall now compute the line integral over the simple closed curve  $C$  bounding the surface consisting of the edges  $OA$ ,  $AB$ ,  $BC$  and  $CO$  in  $z = 0$  plane

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

$$\begin{aligned}
\text{Now } \vec{F} \cdot d\vec{r} &= [(y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}] \\
&= (y-z+2)dx + (yz+4)dy - xzdz
\end{aligned}$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = (y+2)dx + 4dy \quad [\because z = 0]$$

**On OA:**  $y = 0 \quad \therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = 2dx$   
 $x$  varies from 0 to 2

$$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2dx = 2[x]_0^2 = 4$$

**On AB:**  $x = 2 \quad \therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = 4dy;$   
 $y$  varies from 0 to 2

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4dy = 4[y]_0^2 = 8$$

**On BC:**  $y = 2 \quad \therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = 4dx;$   
 $x$  varies from 2 to 0

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 4dx = 4[x]_2^0 = 4(-2) = -8$$

**On CO:**  $x = 0 \quad \therefore dx = 0, \vec{F} \cdot d\vec{r} = 4dy;$   
 $y$  varies from 2 to 0

$$\therefore \int_{CO} \vec{F} \cdot d\vec{r} = \int_2^0 4dy = 4[y]_2^0 = -8$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = 4 + 8 - 8 - 8 = -4 \quad (2)$$

From (1) and (2),  $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$

Hence Stoke's theorem is verified.

### Example 6

Verify Stoke's theorem for  $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$  in the rectangular region in the  $xy$  plane bounded by the lines  $x = 0, x = a, y = 0, y = b$ .

[AU 2004, 2005, 2010]

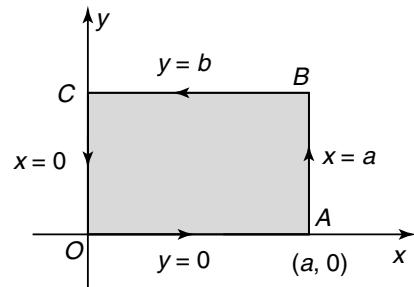
### Solution

Given  $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

Stoke's theorem is

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$$

$$\begin{aligned}\operatorname{Curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) \\ &= 4y\vec{k}\end{aligned}$$



Since the surface is a rectangle in the  $xy$ -plane, normal  $\vec{n} = \vec{k}$

$$\therefore \operatorname{Curl} \vec{F} \cdot \vec{n} = 4y\vec{k} \cdot \vec{k} = 4y$$

$$\begin{aligned}\iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS &= \int_0^a \int_0^b 4y dx dy \\ &= \int_0^a dx \int_0^b 4y dy = [x]_0^a 4 \left[ \frac{y^2}{2} \right]_0^b = 2ab^2\end{aligned}\quad (1)$$

We shall now compute the line integral.

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

Now

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xydy$$

$$\text{On } OA: \quad y = 0 \quad \therefore dy = 0 \text{ and } \vec{F} \cdot d\vec{r} = x^2dx$$

$x$  varies from 0 to  $a$

$$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \left[ \frac{x^3}{3} \right]_0^a = \frac{a^3}{3}$$

$$\text{On } AB: \quad x = a \quad \therefore dx = 0 \text{ and } \vec{F} \cdot d\vec{r} = 2aydy$$

$y$  varies from 0 to  $b$

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2ay dy = 2a \left[ \frac{y^2}{2} \right]_0^b = ab^2$$

**On BC:**  $y = b \quad \therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = (x^2 - b^2)dx$

$x$  varies from  $a$  to 0

$$\begin{aligned}\therefore \int_{BC} \vec{F} \cdot d\vec{r} &= \int_a^0 (x^2 - b^2)dx = \left[ \frac{x^3}{3} - b^2 x \right]_a^0 \\ &= 0 - \left( \frac{a^3}{3} - b^2 a \right) = ab^2 - \frac{a^3}{3}\end{aligned}$$

**On CO:**  $x = 0 \quad \therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = 0$

$$\therefore \int_{CO} \vec{F} \cdot d\vec{r} = 0$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2 \quad (2)$$

From (1) and (2),  $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$

Hence Stoke's theorem is verified.

**Note** Stoke's theorem in the plane is Green's theorem. This is indeed Green's theorem verification.

### Example 7

Verify Stoke's theorem for  $\vec{F} = y^2 z \vec{i} + z^2 x \vec{j} + x^2 y \vec{k}$  where  $S$  is the open surface of the cube formed by the planes  $x = -a, x = a, y = -a, y = a, z = -a, z = a$  in which  $z = -a$  is cut open. [AU 2009]

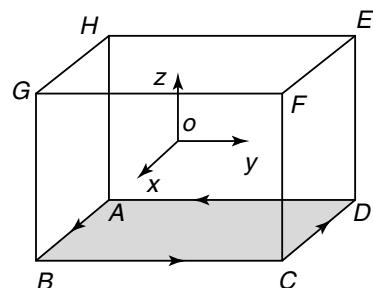
### Solution

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$$

Given  $\vec{F} = y^2 z \vec{i} + z^2 x \vec{j} + x^2 y \vec{k}$

$$\text{Now } \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$



$$\begin{aligned}
&= \vec{i} \left[ \frac{\partial}{\partial y} (x^2 y) - \frac{\partial}{\partial z} (z^2 x) \right] - \vec{j} \left[ \frac{\partial}{\partial x} (x^2 y) - \frac{\partial}{\partial z} (y^2 z) \right] \\
&\quad + \vec{k} \left[ \frac{\partial}{\partial x} (z^2 x) - \frac{\partial}{\partial y} (y^2 z) \right] \\
&= (x^2 - 2zx)\vec{i} + (y^2 - 2xy)\vec{j} + (z^2 - 2yz)\vec{k}
\end{aligned}$$

We shall compute  $\iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$

Given  $S$  is the open surface consisting of the 5 faces of the cube except face  $ABCD$ .

$$\begin{aligned}
\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS &= \iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n} dS + \iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n} dS \\
&\quad + \iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n} dS + \iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n} dS + \iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n} dS
\end{aligned}$$

Face	Equation	Outward normal $\vec{n}$	$\text{Curl } \vec{F} \cdot \vec{n}$	$dS$
$S_1 = \text{BCFG}$	$x = a$	$\vec{i}$	$a^2 - 2az$	$dy dz$
$S_2 = \text{ADEH}$	$x = -a$	$-\vec{i}$	$-(a^2 + 2az)$	$dy dz$
$S_3 = \text{CDEF}$	$y = a$	$\vec{j}$	$a^2 - 2ax$	$dz dx$
$S_4 = \text{ABGH}$	$y = -a$	$-\vec{j}$	$-(a^2 + 2ax)$	$dz dx$
$S_5 = \text{EFGH}$	$z = a$	$\vec{k}$	$a^2 - 2ay$	$dx dy$

$$\begin{aligned}
\iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n} dS &= \int_{-a}^a \int_{-a}^a (-a^2 - 2az) dy dz \\
&= \int_{-a}^a dy \int_{-a}^a (a^2 - 2az) dz \\
&= [y]_{-a}^a \left[ a^2 z - 2a \frac{z^2}{2} \right]_{-a}^a \\
&= [a + a][a^2(a + a) - a(a^2 - a^2)] \\
&= 2a \cdot 2a^3 = 4a^4
\end{aligned}$$

$$\begin{aligned}
\iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_{-a}^a \int_{-a}^a -(a^2 + 2az) \, dy \, dz \\
&= - \int_{-a}^a dy \cdot \int_{-a}^a (a^2 + 2az) \, dz \\
&= - \left[ y \right]_{-a}^a \cdot \left[ a^2 z + 2a \frac{z^2}{2} \right]_{-a}^a \\
&= -2a \cdot [a^2(a+a) + a(a^2 - a^2)] \\
&= -2a \cdot 2a^3 = -4a^4
\end{aligned}$$

$$\begin{aligned}
\iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_{-a}^a \int_{-a}^a (a^2 - 2ax) \, dz \, dx \\
&= - \int_{-a}^a dz \cdot \int_{-a}^a (a^2 - 2ax) \, dx \\
&= \left[ z \right]_{-a}^a \cdot \left[ a^2 x - 2a \frac{x^2}{2} \right]_{-a}^a \\
&= 2a \cdot [a^2 \cdot 2a - a(a^2 - a^2)] = 2a \cdot 2a^3 = 4a^4
\end{aligned}$$

$$\begin{aligned}
\iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_{-a}^a \int_{-a}^a -(a^2 + 2ax) \, dz \, dx \\
&= - \int_{-a}^a dz \cdot \int_{-a}^a (a^2 + 2ax) \, dx \\
&= - \left[ z \right]_{-a}^a \cdot \left[ a^2 x + 2a \frac{x^2}{2} \right]_{-a}^a \\
&= -2a \cdot a^2(2a) + a(a^2 - a^2) = -2a \cdot 2a^3 = -4a^4
\end{aligned}$$

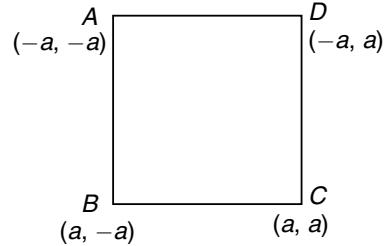
$$\begin{aligned}
\iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n} \, dS &= \int_{-a}^a \int_{-a}^a (a^2 - 2ay) \, dxdy \\
&= \int_{-a}^a dx \cdot \int_{-a}^a (a^2 - 2ay) \, dy \\
&= \left[ x \right]_{-a}^a \cdot \left[ a^2 y + 2a \frac{y^2}{2} \right]_{-a}^a \\
&= 2a \cdot [a^2 \cdot 2a - a(a^2 - a^2)] = 2a \cdot 2a^3 = 4a^4
\end{aligned}$$

$$\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS = 4a^4 - 4a^4 + 4a^4 - 4a^4 + 4a^4 = 4a^4 \quad (1)$$

We shall now compute the line integral over the simple closed curve  $C$  consisting of the edges  $AB, BC, CD, DA$ .

Here  $z = -a, dz = 0$

$$\begin{aligned} \therefore \vec{F} \cdot d\vec{r} &= y^2 z dx + z^2 x dy + x^2 y dz \\ &= -ay^2 dx + a^2 x dy \end{aligned}$$



**On  $AB$ :**  $y = -a \quad \therefore dy = 0$

$$\text{and } \vec{F} \cdot d\vec{r} = -a^3 dx$$

and  $x$  varies from  $-a$  to  $a$

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a}^a -a^3 dx = -a^3 [x]_{-a}^a = -a^3 \cdot 2a = -2a^4$$

**On  $BC$ :**  $x = a \quad \therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = a^3 dy$

and  $y$  varies from  $-a$  to  $a$

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_{-a}^a a^3 dy = a^3 [y]_{-a}^a = a^3 \cdot 2a = 2a^4$$

**On  $CD$ :**  $y = a \quad \therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = -a^3 dx$

and  $x$  varies from  $a$  to  $-a$

$$\therefore \int_{CD} \vec{F} \cdot d\vec{r} = \int_a^{-a} -a^3 dx = -a^3 [x]_a^{-a} = -a^3 (-2a) = 2a^4$$

**On  $DA$ :**  $x = -a \quad \therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = -a^3 dy$

and  $y$  varies from  $a$  to  $-a$

$$\therefore \int_{DA} \vec{F} \cdot d\vec{r} = \int_a^{-a} -a^3 dy = -a^3 [y]_a^{-a} = -a^3 (-2a) = 2a^4$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = -2a^4 + 2a^4 + 2a^4 + 2a^4 = 4a^4 \quad (2)$$

From (1) and (2),  $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$

Hence Stoke's theorem is verified.

### Example 8

**Verify Stoke's theorem for the vector field  $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$  over the upper half surface  $x^2 + y^2 + z^2 = 1$ , bounded by its projections on the  $xy$ -plane.** [AU 2003, 2004, 2006, 2013]

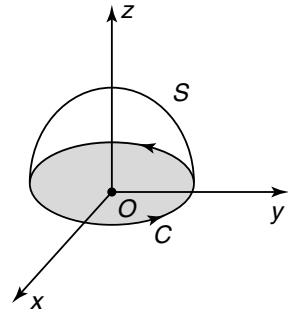
### Solution

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

Given  $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

$$\begin{aligned} \therefore \text{Curl } \vec{F} &= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{array} \right| \\ &= \vec{i} \left[ \frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2) \right] \\ &\quad - \vec{j} \left[ \frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x - y) \right] + \vec{k} \left[ \frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x - y) \right] \\ &= \vec{i}[-2yz + 2yz] - \vec{j}[0 - 0] + \vec{k}[0 - (-1)] = \vec{k} \end{aligned}$$



$$\therefore \text{Curl } \vec{F} \cdot \vec{n} = \vec{k} \cdot \vec{n}$$

The surface is the upper hemisphere  $x^2 + y^2 + z^2 = 1$

$$\begin{aligned} \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS &= \iint_S \vec{k} \cdot \vec{n} dS \\ &= \iint_R \vec{k} \cdot \vec{n} \frac{dxdy}{|\vec{k} \cdot \vec{n}|}, \end{aligned}$$

where  $R$  is the projection of  $S$  on the  $xy$ -plane.

$\therefore R$  is the circle  $x^2 + y^2 = 1$  in the  $xy$ -plane.

$$\begin{aligned}\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS &= \iint_R dx dy \\ &= \text{area of the circle} = \pi \cdot 1^2 = \pi\end{aligned}\quad (1)$$

Now  $C$  is the circle  $x^2 + y^2 = 1$  in the  $z = 0$  plane.

Parametric equations are  $x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq 2\pi$

$$\begin{aligned}\therefore \oint_C \vec{F} \cdot d\vec{r} &= \oint_C [(2x - y)dx - yz^2 dy - y^2 z dz] \\ &= \oint_C (2x - y)dx \quad [\because z = 0] \\ x = \cos \theta \Rightarrow dx &= -\sin \theta d\theta \\ \therefore \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (2\cos \theta - \sin \theta)(-\sin \theta) d\theta \\ &= \int_0^{2\pi} (-2\sin \theta \cos \theta + \sin^2 \theta) d\theta \\ &= \int_0^{2\pi} \left[ -\sin 2\theta + \frac{1 - \cos 2\theta}{2} \right] d\theta \\ &= \left[ \frac{\cos 2\theta}{2} + \frac{1}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) \right]_0^{2\pi} \\ &= \frac{1}{2} \left[ (\cos 4\pi - \cos 0) + 2\pi - \frac{\sin 4\pi}{2} - 0 \right] \\ &= \frac{1}{2} [1 - 1 + 2\pi] = \pi\end{aligned}\quad (2)$$

From (1) and (2),  $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$

Hence Stoke's theorem is verified.

### **Example 9**

Verify Stoke's theorem for  $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$  taken around the rectangle bounded by the lines  $x = \pm a, y = 0$  and  $y = b$ .

[AU 2001, 2009]

### **Solution**

Stoke's theorem is

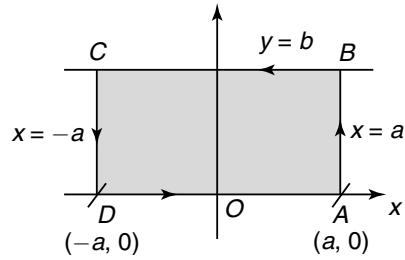
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$$

Given  $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

$$\begin{aligned}\text{Curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} \\ &= \vec{i}[0-0] - \vec{j}(0-0) + \vec{k}(-2y-2y) \\ &= -4y\vec{k}\end{aligned}$$

Since  $S$  is the rectangular surface,  $\vec{n} = \vec{k}$

$$\begin{aligned}\iint_S \text{Curl } \vec{F} \cdot \vec{n} dS &= \iint_S -4y\vec{k} \cdot \vec{k} dx dy \\ &= -4 \int_0^b \int_{-a}^a y dx dy \\ &= -4 \left[ \frac{y^2}{2} \right]_0^b [x]_{-a}^a \\ &= -2b^2 \cdot 2a = -4ab^2\end{aligned}\tag{1}$$



We shall now compute the line integral  $\oint_C \vec{F} \cdot d\vec{r}$ .

$$\begin{aligned}\text{Now } \vec{F} \cdot d\vec{r} &= [(x^2 + y^2)\vec{i} - 2xy\vec{j}] \cdot [dx\vec{i} + dy\vec{j}] \\ &= (x^2 + y^2)dx - 2xydy\end{aligned}$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r}$$

**On  $AB$ :**  $x = a$   $\therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = -2aydy$

and  $y$  varies from 0 to  $b$

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b (-2a)y dy = -2a \left[ \frac{y^2}{2} \right]_0^b = -ab^2$$

**On  $BC$ :**  $y = b$   $\therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = (x^2 + b^2)dx$

and  $x$  varies from  $a$  to  $-a$

$$\begin{aligned}\therefore \int_{BC} \vec{F} \cdot d\vec{r} &= \int_a^{-a} (x^2 + b^2) dx = \left[ \frac{x^3}{3} + b^2 x \right]_a^{-a} \\ &= \frac{1}{3}(-a^3 - a^3) + b^2(-a - a) = \frac{-2}{3}a^3 - 2ab^2\end{aligned}$$

**On CD:**  $x = -a$      $\therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = 2aydy$

and  $y$  varies from  $b$  to  $0$

$$\therefore \int_{CD} \vec{F} \cdot d\vec{r} = \int_b^0 2ay dy = 2a \left[ \frac{y^2}{2} \right]_b^0 = a(0 - b^2) = -ab^2$$

**On DA:**  $y = 0$      $\therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = x^2 dx$

and  $x$  varies from  $-a$  to  $a$

$$\therefore \int_{DA} \vec{F} \cdot d\vec{r} = \int_{-a}^a x^2 dx = 2 \int_0^a x^2 dx = 2 \left[ \frac{x^3}{3} \right]_0^a = \frac{2}{3}a^3$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = -ab^2 - \frac{2}{3}a^3 - 2ab^2 - ab^2 + \frac{2}{3}a^3 = -4ab^2$$

From (1) and (2),  $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$

Hence Stoke's theorem is verified.

### Example 10

Evaluate  $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$  where  $C$  is the boundary of the triangle with the vertices  $(2, 0, 0)$ ,  $(0, 3, 0)$  and  $(0, 0, 6)$ , using Stoke's theorem. [AU 2009]

### Solution

Stoke's theorem is

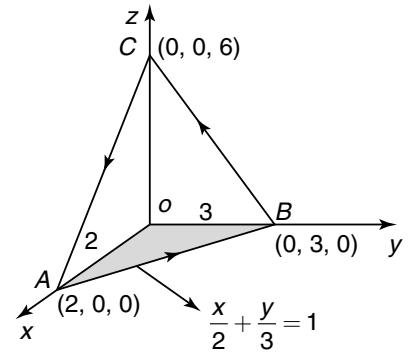
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS,$$

where  $S$  is the surface of the triangle  $ABC$  bounded by the curve  $C$ , consisting of the sides of the triangle in the figure.

Given  $\vec{F} \cdot d\vec{r} = (x+y)dx + (2x-z)dy + (y+z)dz$

Here  $\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$$\begin{aligned}\therefore \text{Curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y) \right] \\ &= \vec{i}[1 - (-1)] - \vec{j}[0 - 0] + \vec{k}(2 - 1) = 2\vec{i} + \vec{k}\end{aligned}$$



Equation of the plane  $ABC$  is  $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$  [intercept form]

$$\therefore \phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6}, \quad \frac{\partial \phi}{\partial x} = \frac{1}{2}, \quad \frac{\partial \phi}{\partial y} = \frac{1}{3}, \quad \frac{\partial \phi}{\partial z} = \frac{1}{6}$$

$$\begin{aligned}\therefore \nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= \frac{1}{2}\vec{i} + \frac{1}{3}\vec{j} + \frac{1}{6}\vec{k} = \frac{1}{6}(3\vec{i} + 2\vec{j} + \vec{k})\end{aligned}$$

$$\therefore \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\frac{1}{6}(3\vec{i} + 2\vec{j} + \vec{k})}{\frac{1}{6}\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k})$$

$$\therefore \text{Curl } \vec{F} \cdot \vec{n} = (2\vec{i} + \vec{k}) \cdot \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k}) = \frac{1}{\sqrt{14}}(6 + 1) = \frac{7}{\sqrt{14}}$$

$$\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS = \iint_S \frac{7}{\sqrt{14}} dS = \frac{7}{\sqrt{14}} \iint_R \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

where  $R$  is the orthogonal projection of  $S$  on the  $xy$ -plane.

But

$$\vec{n} \cdot \vec{k} = \frac{1}{\sqrt{14}} (3\vec{i} + 2\vec{j} + \vec{k}) \cdot \vec{k} = \frac{1}{\sqrt{14}}$$

$$\begin{aligned}\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS &= \frac{7}{\sqrt{14}} \iint_R \frac{dx dy}{\frac{1}{\sqrt{14}}} \\ &= 7 \iint_R dx dy \\ &= 7 \times \text{Area of } \triangle OAB = 7 \cdot \frac{1}{2} \cdot 2 \cdot 3 = 21 \\ \therefore \oint_C [(x+y)dx + (2x-z)dy + (y+z)dz] &= 21\end{aligned}$$

**Example 11**

Evaluate  $\iint_S \text{Curl } \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$  and  $S$  is the open surface bounded by the planes  $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$  above the  $xy$  plane. [AU 2010]

**Solution**

Since the surface is an open surface, we use Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS,$$

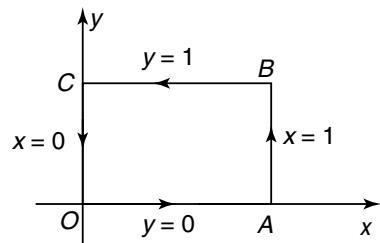
where  $S$  is the open surface bounded by the curve  $C$ .

Here  $C$  is the sides of the square in the  $xy$ -plane,  $z = 0$  bounded by  $x = 0, x = 1, y = 0, y = 1$ .

$$\begin{aligned}\therefore \iint_S \text{Curl } \vec{F} \cdot \vec{n} dS &= \int_{OA} \vec{F} \cdot d\vec{r} \\ &\quad + \int_{AB} \vec{F} \cdot d\vec{r} \\ &\quad + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}\end{aligned}$$

Given  $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$



$$\therefore \vec{F} \cdot d\vec{r} = (y - z)dx + yzdy - xzdz$$

Since  $C$  lies on the  $xy$ -plane,  $z = 0$ ,  $\therefore dz = 0$

$$\therefore \vec{F} \cdot d\vec{r} = ydx$$

**On  $OA$ :**  $y = 0$   $\therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = 0$

$$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = 0$$

**On  $AB$ :**  $x = 1$   $\therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = 0$

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = 0$$

**On  $BC$ :**  $y = 1$   $\therefore dy = 0$  and  $\vec{F} \cdot d\vec{r} = dx$

and  $x$  varies from 1 to 0

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_1^0 dx = [x]_1^0 = -1$$

**On  $CO$ :**  $x = 0$   $\therefore dx = 0$  and  $\vec{F} \cdot d\vec{r} = 0$

$$\therefore \int_{CO} \vec{F} \cdot d\vec{r} = 0$$

Hence  $\int_C \vec{F} \cdot d\vec{r} = 0 + 0 + (-1) + 0 = -1$

$$\iint_S \text{Curl } \vec{F} \cdot \vec{n} dS = -1$$

### Example 12

Using Stoke's theorem, evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$

and  $C$  is the boundary of the triangle with vertices at  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(1, 1, 0)$ . [AU 2012]

### Solution

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

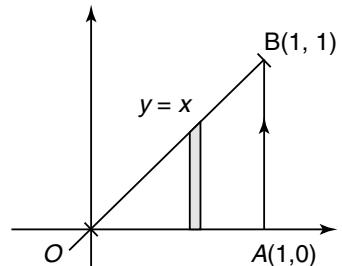
$$\begin{aligned}
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix} \\
&= \left[ \frac{\partial}{\partial y}(-x-z) - \frac{\partial}{\partial z}(x^2) \right] \vec{i} - \left[ \frac{\partial}{\partial x}(-x-z) - \frac{\partial}{\partial z}(y^2) \right] \vec{j} \\
&\quad + \left[ \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2) \right] \vec{k} \\
&= (0) \vec{i} - [-1] \vec{j} + [2x - 2y] \vec{k} \\
&= \vec{j} + 2(x-y) \vec{k}.
\end{aligned}$$

Given  $C$  is the boundary of the triangle formed by the points  $(0, 0, 0)$ ,  $(1, 0, 0)$  and  $(1, 1, 0)$  which lie in the  $xy$ -plane.  $\therefore \vec{n} = \vec{k}$

$$\therefore \operatorname{curl} \vec{F} \cdot \vec{n} = 2(x-y)$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_S 2(x-y) dx dy$$

Equation of  $OB$  is  $y = x$



$$\begin{aligned}
\therefore \oint_C \vec{F} \cdot d\vec{r} &= 2 \iint_0^1 \int_0^x (x-y) dy dx \\
&= 2 \int_0^1 \left[ xy - \frac{y^2}{2} \right]_0^x dx \\
&= 2 \int_0^1 \left[ x^2 - \frac{x^2}{2} - 0 \right] dx \\
&= 2 \int_0^1 \frac{x^2}{2} dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}.
\end{aligned}$$

## EXERCISE 1.4

1. Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = 12x^2y\vec{i} - 3yz\vec{j} + 2z\vec{k}$  and  $S$  is the portion of the plane  $x + y + z = 1$  included in the first octant. [AU 2007]
2. Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = (2x^2 - 3z)\vec{i} + 2y\vec{j} - 4xz\vec{k}$ , where  $S$  is the surface of the solid bounded by the planes  $x = 0, y = 0, z = 0$  and  $2x + 2y + z = 4$ .
3. Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = z\vec{i} + x\vec{j} - y^2z\vec{k}$  and  $S$  is the curved surface of the cylinder  $x^2 + y^2 = 1$  included in the first octant between the planes  $z = 0$  and  $z = 2$ .
4. If  $\vec{F} = xy^2\vec{i} - yz^2\vec{j} + zx^2\vec{k}$ , find  $\iint_S \vec{F} \cdot \vec{n} dS$  over the sphere  $x^2 + y^2 + z^2 = 1$ .
5. Evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{F} = 4xzi - y^2\vec{j} + yz\vec{k}$  and  $S$  is the surface of the cube bounded by  $x = 0, x = 1, y = 0, y = 1, z = 0$  and  $z = 1$ .
6. Evaluate  $\oint_C (x^2 + xy)dx + (x^2 + y^2)dy$ , where  $C$  is the square formed by the lines  $y = \pm 1, x = \pm 1$ , by Green's theorem.
7. Using Green's theorem evaluate  $\oint_C (x^2 + y)dx - xy^2 dy$  taken around the square whose vertices are  $(0, 0), (1, 0), (1, 1), (0, 1)$
8. Using Green's theorem find the value of  $\int_C (xy - x^2)dx + x^2y dy$  along the closed curve  $C$  formed by  $y = 0, x = 1$  and  $y = x$ .
9. Verify Green theorem for  $\int_C (15x^2 - 4y^2)dx + (2y - 3x)dy$ , where  $C$  is the curve enclosing the area bounded by  $y = x^2, x = y^2$
10. Verify Green theorem in the plane for  $\int_C (3x^2 - 8y^3)dx + (4y - 6xy)dy$ , where  $C$  is the boundary of the region defined by  $x = 0, y = 0, x + y = 1$ . [AU 1999, 2009, 2010, 2011]
11. Using Green's theorem find the area of  $x^{2/3} + y^{2/3} = a^{2/3}$ .

**[Hint:** Area =  $\frac{1}{2} \int_C (xdy - ydx)$ ,  $C$  is the boundary of the curve]

12. Using Green's theorem in  $xy$  plane find the area of the region in the  $xy$  plane bounded by  $y^3 = x^2$  and  $y = x$ .
13. Using Green's theorem evaluate  $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ , where  $C$  is the boundary of the area in the  $xy$  plane bounded by  $x$ -axis and the semi circle  $x^2 + y^2 = 1$  in the upper half of the plane.
14. Verify Gauss divergence theorem for  $\vec{F} = x^2\vec{i} + z\vec{i} + yz\vec{k}$  taken over the cube bounded by  $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ .
15. Verify Gauss divergence theorem for  $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$  over the parallelepiped bounded by the planes  $x = 0, x = 1, y = 0, y = 2, y = 2, z = 0, z = 3$ .
16. Verify Gauss divergence theorem for  $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$  over a unit cube.
17. Verify Gauss divergence theorem for  $\vec{F} = (x^3 - yz)\vec{i} - zx^2y\vec{j} + 2\vec{k}$  over the cube  $x = 0, x = a, y = 0, y = a, z = 0, z = a$  [AU 2010]
18. Verify the divergence theorem for  $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ , where  $S$  is the rectangular parallelepiped bounded by  $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$ . [AU 2011]
19. Using divergence theorem show that  $\iint_S x^2 dy + y^2 dz dy + 2z(xy - x - y) dx dy = \frac{1}{2}$ , where  $S$  is the surface of the cube  $x = y = z = 0, y = z = 1$ .
20. Use divergence theorem to evaluate  $\iint_S (2xy\vec{i} + yz^2\vec{j} + xz\vec{k}) \cdot d\vec{S}$ , where  $S$  is the surface of the region bounded by  $x = y = z = 0, y = 3, x + 2z = 6$ .
21. Prove that  $\iint_S [x(y-z)\vec{i} + y(z-x)\vec{j} + z(x-y)\vec{k}] \cdot dS = 0$ , where  $S$  is any closed surface.
22. Verify Stoke's theorem for  $\vec{F} = 2z\vec{i} + x\vec{j} + y^2\vec{k}$ , where  $S$  is the surface of the paraboloid  $z = 4 - x^2 - y^2$  and  $C$  is the simple closed curve in the  $xy$  plane.
23. Verify Stoke's theorem for  $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ , where  $S$  is the upper half surface of the sphere  $x^2 + y^2 + z^2 = 1$  and  $C$  its boundary. [AU 2005]

24. Verify Stoke's theorem for  $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j} + xyz\vec{k}$  over the surface of the box bounded by the planes  $x = 0, y = 0, x = a, y = b, z = c$  above the  $xy$  plane.
25. Verify Stoke's theorem for  $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$  in the rectangular region in the  $xy$  plane bounded by  $x = 0, x = a, y = 0, y = b$ .
26. Verify Stoke's theorem for  $\vec{F} = -y^3\vec{i} + x^3\vec{j}$  and the closed curve  $C$  is the boundary of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .
27. If  $\phi$  is scalar point function, use Stoke's theorem to prove  $\text{curl}(\text{grad } \phi) = 0$ .
28. Evaluate  $\iint_S \nabla \times \vec{F} \cdot \vec{n} dS$ , where  $S$  is the surface  $x^2 + y^2 + z^2 = a^2$  above the  $xy$ -plane and  $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$ .
29. Evaluate  $\int_C yzdx + zx dy + xy dz$ , where  $C$  is the curve  $x^2 + y^2 = 1, z = y^2$ .
30. Evaluate  $\iint_S \nabla \times \vec{F} \cdot \vec{n} dS$  for  $\vec{F} = (2x - y + z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$  over the surface of the cylinder  $x^2 + y^2 = 4$ , bounded by the plane  $z = 9$  and open at the end  $z = 0$ .

### ANSWERS TO EXERCISE 1.4

- |                     |                     |                    |                          |                    |
|---------------------|---------------------|--------------------|--------------------------|--------------------|
| 1. $\frac{49}{120}$ | 2. $\frac{16}{3}$   | 3. 3               | 4. $\frac{4}{3}\pi$      | 5. $\frac{3}{2}$   |
| 6. 0                | 7. $-\frac{4}{3}$   | 8. $-\frac{1}{12}$ | 11. $\frac{3}{8}\pi a^3$ | 12. $\frac{1}{10}$ |
| 13. $\frac{4}{3}$   | 20. $\frac{351}{2}$ | 28. 0              | 29. 0                    | 30. $8\pi$         |

## PART A – QUESTIONS AND ANSWERS

1. If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $|\vec{r}| = r$ , then find  $\nabla r$ .

**Ans:** Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore r^2 = x^2 + y^2 + z^2, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \vec{i} \frac{\partial r}{\partial x} + \vec{j} \frac{\partial r}{\partial y} + \vec{k} \frac{\partial r}{\partial z}$$

$$\therefore \nabla r = \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} = \frac{1}{r} (x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{r}}{r}$$

2. Find  $\nabla \left( \frac{1}{r} \right)$ .

**Ans:** Refer worked Example 5(iii) Page 1.14.

3. Prove that  $\nabla(r^n) = nr^{n-2}\vec{r}$ .

[AU 2000, 2006]

**Ans:** Refer worked Example 5(ii) Page 1.14.

4. Find grad  $\phi$  at the point  $(1, -2, -1)$ , where  $\phi = 3x^2y - y^3z^2$ .

$$\text{Ans: } \text{grad } \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Given

$$\phi = 3x^2y - y^3z^2$$

$$\therefore \frac{\partial \phi}{\partial x} = 6xy, \quad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2z^2, \quad \frac{\partial \phi}{\partial z} = -2y^3z$$

$$\text{At the point } (1, -2, -1), \quad \frac{\partial \phi}{\partial x} = 6 \cdot 1 \cdot (-2) = -12$$

$$\frac{\partial \phi}{\partial y} = 3 - 3 \cdot (-2)^2 (-1)^2 = 3 - 12 = -9$$

$$\frac{\partial \phi}{\partial z} = -2 \cdot (-2)^3 \cdot (-1) = -16$$

$$\therefore \text{at the point } (1, -2, -1), \quad \nabla \phi = -12\vec{i} - 9\vec{j} - 16\vec{k}$$

- 5. What is the greatest rate of increase of  $\phi = xyz^2$  at the point (1, 0, 3)? [AU 2005, 2007]**

**Ans:** We know that the greatest increase in  $\phi$  is  $= |\nabla\phi|$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Given

$$\phi = xyz^2, \quad \frac{\partial\phi}{\partial x} = yz^2, \quad \frac{\partial\phi}{\partial y} = xz^2, \quad \frac{\partial\phi}{\partial z} = 2xyz$$

∴

$$\nabla\phi = \vec{i}yz^2 + \vec{j}xz^2 + \vec{k}2xyz$$

At the point (1, 0, 3),  $\nabla\phi = 0\vec{i} + 9\vec{j} + 0\vec{k} = 9\vec{j}$

∴

$$|\nabla\phi| = |9| = 9$$

- 6. If  $\phi = x^2 + y^2 + z^2 - 8$ , then find grad  $\phi$  at the point (2, 0, 2). [AU 2007]**

**Ans:**  $\text{grad } \phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Given

$$\phi = x^2 + y^2 + z^2 - 8$$

∴

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = 2z$$

At the point (2, 0, 2),  $\frac{\partial\phi}{\partial x} = 4, \quad \frac{\partial\phi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial z} = 4$

∴ at the point (2, 0, 2),  $\nabla\phi = 4\vec{i} + 4\vec{k}$ .

- 7. Find the unit normal vector to the surface  $x^2 + xy + z^2 = 4$  at the point (1, -1, 2). [AU 2005]**

**Ans:** Given surface is  $x^2 + xy + z^2 = 4$

∴

$$\phi = x^2 + xy + z^2$$

∴

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2x + y) + \vec{j}x + \vec{k}2z$$

At the point (1, -1, 2),  $\nabla\phi = (2-1)\vec{i} + \vec{j} + 4\vec{k} = \vec{i} + \vec{j} + 4\vec{k}$

∴ unit normal is

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{1+1+16}} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$$

- 8. Find the directional derivative of  $\phi = xyz$  at  $(1, 1, 1)$  in the direction of  $\vec{i} + \vec{j} + \vec{k}$**

**Ans:** Given

$$\phi = xyz$$

Now

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}yz + \vec{j}zx + \vec{k}xy\end{aligned}$$

At the point  $(1, 1, 1)$ ,  $\nabla\phi = \vec{i} + \vec{j} + \vec{k}$

Given direction is  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\therefore$  the directional derivative in the direction of  $\vec{a}$  is

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (\vec{i} + \vec{j} + \vec{k}) \cdot \frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}} = \frac{1+1+1}{\sqrt{3}} = \sqrt{3}.$$

- 9. The temperature at a point  $(x, y, z)$  in space is given by  $T(x, y, z) = x^2 + y^2 - z$ . A mosquito located at the point  $(4, 4, 2)$  desires to fly in such a direction that it gets cooled faster. Find the direction in which it should fly. [AU 2004]**

**Ans:** Given  $T = x^2 + y^2 - z$

Direction of maximum temperature is

$$\begin{aligned}\nabla T &= \vec{i}\frac{\partial T}{\partial x} + \vec{j}\frac{\partial T}{\partial y} + \vec{k}\frac{\partial T}{\partial z} \\ &= \vec{i}2x + \vec{j}2y + \vec{k}(-1)\end{aligned}$$

At the point  $(4, 4, 2)$ ,  $\nabla T = 8\vec{i} + 8\vec{j} - \vec{k}$

$\therefore$  direction that gets cooled faster is  $-\nabla T = -8\vec{i} - 8\vec{j} + \vec{k}$ , which is the direction in which the mosquito should fly.

- 10. Find the normal derivative of  $\phi = x^3 - y^3 + z$  at the point  $(1, 1, 1)$ .**

**Ans:** Normal derivative is the derivative in the direction of the normal and it is  $\nabla\phi$ .

Given  $\phi = x^3 - y^3 + z^3$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}3x^2 + \vec{j}(-3y^2) + \vec{k} \cdot 1$$

At the point  $(1, -1, 1)$   $\nabla\phi = \vec{i} - 3\vec{j} + \vec{k}$

$$\text{Normal derivative } = |\nabla\phi| = \sqrt{9+9+1} = \sqrt{19}$$

- 11. Find the angle between the surfaces  $x^2 + y^2 + z^2 = 9$  and  $z = x^2 + y^2 - 3$  at the point  $(2, -1, 2)$ .**

**Ans:** Given

$$\phi_1 = x^2 + y^2 + z^2 \text{ and } \phi_2 = x^2 + y^2 - z$$

$$\begin{aligned}\therefore \nabla\phi_1 &= \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z} \\ &= \vec{i}2x + \vec{j}2y + \vec{k}2z\end{aligned}$$

$$\text{At the point } (2, -1, 2), \quad \nabla\phi_1 = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\text{and } \nabla\phi_2 = \vec{i}2x + \vec{j}2y + \vec{k}(-1)$$

$$\text{At the point } (2, -1, 2), \quad \nabla\phi_2 = 4\vec{i} - 2\vec{j} - \vec{k}$$

If  $\theta$  is the angle between the surfaces, then

$$\begin{aligned}\cos\theta &= \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+16} \cdot \sqrt{16+4+1}} \\ &= \frac{16+4-4}{\sqrt{36} \cdot \sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \\ \therefore \theta &= \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)\end{aligned}$$

- 12. Find the equation of the tangent plane to the surface  $x^2 + y^2 - z = 0$  at the point  $(2, -1, 5)$ .**

**Ans:** Given

$$\phi = x^2 + y^2 - z \text{ and the point is } (2, -1, 5)$$

$$\begin{aligned}\nabla\phi &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \\ &= \vec{i}2x + \vec{j}2y + \vec{k}(-1)\end{aligned}$$

At the point  $(2, -1, 5)$ ,  $\nabla\phi = 4\vec{i} - 2\vec{j} - \vec{k}$ , which is the normal to the surface.

So, the direction ratios of the normal to the plane are  $4, -2, -1$

$\therefore$  the equation of the tangent plane at the point  $(2, -1, 5)$  is

$$4(x-2) + (-2)(y+1) + (-1)(z-5) = 0 \\ \Rightarrow 4x - 2y - z = 5.$$

**13. Find  $\operatorname{div} \vec{r}$  if  $\vec{r} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$**

**Ans:** Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \operatorname{div} \vec{r} = \nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1=3$$

**14. Prove that  $\operatorname{div} \left( \frac{\vec{r}}{r} \right) = \frac{2}{r}$ .**

[AU 2003]

**Ans:** We have  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $r = |\vec{r}|$

$$\begin{aligned} \operatorname{div} \left( \frac{\vec{r}}{r} \right) &= \nabla \cdot \frac{\vec{r}}{r} = \nabla \cdot \left( \frac{1}{r} \vec{r} \right) \\ &= \nabla \left( \frac{1}{r} \right) \cdot \vec{r} + \frac{1}{r} \nabla \cdot \vec{r} \\ &= -\frac{1}{r^3} (\vec{r} \cdot \vec{r}) + \frac{3}{r} \quad [\because \nabla \left( \frac{1}{r} \right) = -\frac{\vec{r}}{r^3}] \\ &= -\frac{1}{r^3} r^2 + \frac{3}{r} = -\frac{1}{r} + \frac{3}{r} = \frac{2}{r} \end{aligned}$$

**15. If  $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ , find  $\operatorname{div} (\operatorname{curl} \vec{F})$ .**

[AU 1996]

**Ans:** Given  $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$$\begin{aligned} \therefore \operatorname{Curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix} \\ &= \vec{i} \left( \frac{\partial}{\partial y} z^3 - \frac{\partial}{\partial z} x^3 \right) - \vec{j} \left( \frac{\partial}{\partial x} z^3 - \frac{\partial}{\partial z} x^3 \right) + \vec{k} \left( \frac{\partial}{\partial x} y^3 - \frac{\partial}{\partial z} x^3 \right) \\ &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0} \end{aligned}$$

$$\therefore \operatorname{div} (\operatorname{curl} \vec{F}) = 0$$

**Aliter:** We know that  $\nabla \cdot (\nabla \times \vec{F}) = 0$  for all  $\vec{F}$ .

- 16. Prove that  $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} - (6xy + 2x^2y^2)\vec{k}$  is solenoidal.**

**Ans:** Given  $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} - (6xy + 2x^2y^2)\vec{k}$

$$\therefore \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$\begin{aligned} &= \frac{\partial}{\partial x}(2x^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(-6xyz - 2x^2y^2) \\ &= 4xy + 2xy - 6xy = 0 \end{aligned}$$

$\therefore \vec{F}$  is solenoidal.

- 17. Find  $a$  such that  $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$  is solenoial.** [AU 2004, 2007]

**Ans:** Let  $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$

Given  $\vec{F}$  is solenoidal.

$$\therefore \nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$\Rightarrow 3 + a + 2 = 0 \Rightarrow a = -5.$$

- 18. Calculate curl  $\vec{F}$  if  $\vec{F} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$  at the point  $(1, -1, 1)$ .**

[AU 1996]

**Ans:** Given  $\vec{F} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$

$$\operatorname{Curl} \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\begin{aligned} &= \vec{i} \left[ \frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz) \right] \end{aligned}$$

$$\begin{aligned}
 &= \vec{i}[0 - 2yz - 0] - \vec{j}[z^2 - xy] + \vec{k}[6xy - xz] \\
 &= -2y\vec{i} - (z^2 - xy)\vec{j} + (6xy - xz)\vec{k}
 \end{aligned}$$

At the point  $(1, -1, 1)$ ,  $\operatorname{curl} \vec{F} = 2\vec{i} - (1+1)\vec{j} + (-6-1)\vec{k} = 2\vec{i} - 2\vec{j} - 7\vec{k}$

- 19. Find  $\operatorname{curl} \vec{F}$ , where  $\vec{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$ .** [AU 2003]

**Ans:** Given  $\vec{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz) = \nabla\phi$ .

where  $\phi = x^3 + y^3 + z^3 - 3xyz$

We know that  $\operatorname{Curl grad} \phi = 0$  always.

$$\therefore \nabla \times \vec{F} = \vec{0}$$

- 20. If  $\phi$  is a scalar point function, prove that  $\nabla\phi$  is solenoidal and irrotational if  $\phi$  is a solution of Laplace equation.** [AU 2006]

**Ans:** Given  $\phi$  satisfies Laplace equation.

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \Rightarrow \nabla^2 \phi = 0 \quad (1)$$

Now  $\operatorname{div} \nabla\phi = \nabla \cdot \nabla\phi = \nabla^2\phi = 0$  [by (1)]

$\therefore \nabla\phi$  is solenoidal.

$\operatorname{Curl} \nabla\phi = \nabla \times \nabla\phi = 0$  always.

$\therefore \nabla\phi$  is irrotational.

- 21. If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ , then find  $\nabla \times \vec{r}$ .** [AU 2005]

**Ans:** Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\begin{aligned}
 \operatorname{Curl} \vec{r} &= \nabla \times \vec{r} = \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{array} \right| \\
 &= \vec{i}[0 - 0] - \vec{j}[0 - 0] + \vec{k}(0 - 0) = \vec{0}
 \end{aligned}$$

- 22. If  $u = x^2 - y^2$ , then prove that  $\nabla^2 u = 0$ .**

**Ans:** We know that  $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ , since  $u$  is a function of  $x$  and  $y$ .

$$\text{Given } u = x^2 - y^2$$

$$\therefore \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2$$

and  $\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2,$

$$\therefore \nabla^2 u = 2 - 2 = 0$$

**23. Find the values of  $a, b, c$  if**

$\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$  is irrotational.

**Ans:** Given  $\vec{F}$  is irrotational.

$$\therefore \nabla \times \vec{F} = \vec{0} \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i} \left[ \frac{\partial}{\partial y} (4x + cy + 2z) - \frac{\partial}{\partial z} (bx - 3y - z) \right] - \vec{j} \left[ \frac{\partial}{\partial x} (4x + cy + 2z) - \frac{\partial}{\partial z} (x + 2y + az) \right] + \vec{k} \left[ \frac{\partial}{\partial x} (bx - 3y - z) - \frac{\partial}{\partial y} (x + 2y + az) \right] = \vec{0}$$

$$\Rightarrow \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = \vec{0}$$

$$\therefore c+1=0, 4-a=0, b-2=0$$

$$\Rightarrow a=4, b=2, c=-1$$

**24. If  $\vec{A}$  and  $\vec{B}$  are irrotational, prove that  $\vec{A} \times \vec{B}$  is solenoidal.**

[AU2013]

**Ans:** Given  $\vec{A}$  and  $\vec{B}$  are irrotational.

$$\therefore \nabla \times \vec{A} = \vec{0}, \quad \nabla \times \vec{B} = \vec{0}$$

$$\begin{aligned} \text{We know that } \nabla \cdot (\vec{A} \times \vec{B}) &= \vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}) \\ &= \vec{A} \cdot \vec{0} - \vec{B} \cdot \vec{0} = 0 \end{aligned}$$

$\therefore \vec{A} \times \vec{B}$  is solenoidal.

25. If  $\vec{F} = x^2 \vec{i} + xy^2 \vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  from (0, 0) to (1, 1) along the path  $y = x$ . [AU 2005, 2006]

**Ans:** Given  $\vec{F} = x^2 \vec{i} + xy^2 \vec{j}$

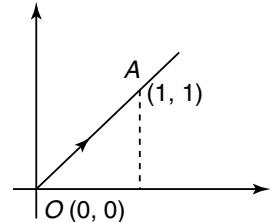
and  $\vec{r} = x\vec{i} + y\vec{j}$  be position vector of any point on  $y = x$

$$\therefore d\vec{r} = dx\vec{i} + dy\vec{j} \text{ and } dy = dx$$

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= x^2 dx + xy^2 dy \\ &= x^2 dx + x \cdot x^2 dx = (x^2 + x^3)dx\end{aligned}$$

and  $x$  varies from 0 to 1.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (x^2 + x^3)dx = \left[ \frac{x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$



26. If  $\vec{F} = 3xy\vec{i} - y^2\vec{j}$  evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the curve  $y = 2x^2$  in the  $xy$  plane from (0, 0) to (1, 2) [AU 2001]

**Ans:** Given  $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ ,  $d\vec{r} = dx\vec{i} + dy\vec{j}$

$$\therefore \vec{F} \cdot d\vec{r} = 3xydx - y^2dy$$

$$y = 2x^2 \Rightarrow dy = 4xdx$$

$$\therefore \vec{F} \cdot d\vec{r} = 3x \cdot 2x^2 dx - 4x^4 \cdot 4xdx$$

$$= (6x^3 - 16x^5)dx;$$

$C$  is the curve in the  $xy$  plane from the point (0, 0) to the point (1, 2)

$\therefore x$ -varies from 0 to 1.

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (6x^3 - 16x^5)dx \\ &= \left[ \frac{6x^4}{4} - \frac{16x^6}{6} \right]_0^1 = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = -\frac{7}{6}.\end{aligned}$$

27. If  $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$ , then check if the integral  $\int_C \vec{F} \cdot d\vec{r}$  is independent of the path  $C$ . [AU 2000]

**Ans:** The integral  $\int_C \vec{F} \cdot d\vec{r}$  is independent of the path if  $\nabla \times \vec{F} = \vec{0}$ .

$$\text{Given } \vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$$

$$\begin{aligned} \text{Now } \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix} \\ &= \vec{i} \left[ \frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2) \right] - \vec{j} \left[ \frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy - 3x^2z^2) \right] \\ &\quad + \vec{k} \left[ \frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z^2) \right] \end{aligned}$$

$$\Rightarrow \vec{i}[0 - 0] - \vec{j}[-6x^2z - (-6x^2z)] + \vec{k}[4x - 4x] = \vec{0}$$

$\therefore \int_C \vec{F} \cdot d\vec{r}$  is independent of the path.

28. If  $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the part of the curve  $y = x^3$  between  $x = 1$  and  $x = 2$ . [AU 2003]

**Ans:** Given  $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ ,  $\vec{r} = x\vec{i} + y\vec{j}$  is any point on  $y = x^3$

$$\begin{aligned} d\vec{r} &= dx\vec{i} + dy\vec{j} \text{ and } dy = 3x^2 dx \\ \therefore \vec{F} \cdot d\vec{r} &= 5xydx + 2ydy \end{aligned}$$

$$\begin{aligned} &= 5x \cdot x^3 dx + 2 \cdot x^3 \cdot 3x^2 dx \\ &= (5x^4 + 6x^5)dx \end{aligned}$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_1^2 (5x^4 + 6x^5)dx \\ &= \left[ 5 \frac{x^5}{5} + 6 \frac{x^6}{6} \right]^2 = 2^5 + 2^6 - (1+1) = 32 + 64 - 2 = 94. \end{aligned}$$

**29. Show that  $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$  is a conservative vector field.**

[AU 2001]

**Ans:** Given  $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\text{Now } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0] = \vec{0}$$

$\therefore \vec{F}$  is a conservative vector field.

**30. Find the work done, when a force  $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2x + y)\vec{j}$  moves a particle from the origin to the point (1, 1) along  $y^2 = x$ .**

[AU 1996]

**Ans:** Work done along  $C$  is  $= \int_C \vec{F} \cdot d\vec{r}$

where  $C$  is the curve  $y^2 = x$  from (0, 0) to (1, 1)

Given  $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2x + y)\vec{j}$  and  $r = x\vec{i} + y\vec{j}$

$$\therefore \vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)dx - (2x + y)dy$$

Also  $y^2 = x \quad \therefore 2ydy = dx$  and  $y$  varies from 0 to 1

$$\therefore \vec{F} \cdot d\vec{r} = (y^4 - y^2 + y^2)2ydy - (2y^2 \cdot y + y)dy$$

$$= (2y^5 - 2y^3 - y)dy$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2y^5 - 2y^3 - y)dy$$

$$= \left[ 2 \frac{y^6}{6} - 2 \cdot \frac{y^4}{4} - \frac{y^2}{2} \right]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = -\frac{2}{3}.$$

**31. Using Green's theorem in a plane evaluate  $\oint_C (2x - y)dx + (x + y)dy$ , where  $C$  is the circle  $x^2 + y^2 = 4$  in the  $xy$  plane.**

**Ans:** Green's theorem is

$$\int_C Pdx + Qdy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

To find  $\int_C (2x - y)dx + (x + y)dy$

Here  $P = 2x - y, Q = x + y$

$$\therefore \frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

$$\begin{aligned}\therefore \oint_C (2x - y)dx + (x + y)dy &= \iint_R 2dxdy \\ &= 2 \times \text{area of the region } R, \\ &= 2 (\text{the area of the circle } x^2 + y^2 = 4) \\ &= 2\pi \cdot 2^2 = 8\pi\end{aligned}$$

**32. Find the area of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  by using Green's theorem.**

[AU 2010]

**Ans:** Area of the region by Green's theorem is

$$A = \frac{1}{2} \int_C (xdy - ydx), \text{ where } C \text{ is the ellipse.}$$

Parametric equations of the ellipse are

$$\begin{aligned}x &= a \cos \theta, & y &= b \sin \theta, & 0 \leq \theta \leq 2\pi \\ dx &= -a \sin \theta d\theta, & dy &= b \cos \theta d\theta\end{aligned}$$

$$\begin{aligned}\therefore A &= \frac{1}{2} \int_0^{2\pi} [a \cos \theta \cdot b \cos \theta - b \sin \theta (-a \sin \theta)] d\theta \\ &= \frac{ab}{2} \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) d\theta \\ &= \frac{ab}{2} \int_0^{2\pi} d\theta = \frac{ab}{2} [\theta]_0^{2\pi} = \frac{ab}{2} \cdot 2\pi = \pi ab\end{aligned}$$

**33. Evaluate  $\int_C (x^2 - xy)dx + (x^2 + y^2)dy$ , where  $C$  is the square formed**

**by the lines  $y = \pm 1, x = \pm 1$  using Green's theorem.**

**Ans:** To find  $\int_C (x^2 - xy)dx + (x^2 + y^2)dy$

Here  $P = x^2 + xy$ ,  $Q = x^2 + y^2$

$$\frac{\partial P}{\partial y} = x, \quad \frac{\partial Q}{\partial x} = 2x$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x = x$$

$$\text{By Green's theorem, } \int_C P dx + Q dy = \iint_R x dxdy$$

$$= \iint_{[-1,1]}^{1,1} x dxdy$$

$$= \left[ \int_{-1}^1 dy \right] \left[ \int_{-1}^1 x dx \right]$$

$$= [y]_{-1}^1 \cdot \left[ \frac{x^2}{2} \right]_{-1}^1 = 2 \cdot 0 = 0$$

**34. Using Green's theorem evaluate  $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ , where**

**C is the boundary of the square enclosed by the lines  $x = 0, y = 0, x = 2, y = 2$ .**

**Ans:** To find  $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$

Here  $P = 2x^2 - y^2$ ,  $Q = x^2 + y^2$

$$\frac{\partial P}{\partial y} = -2y, \quad \frac{\partial Q}{\partial x} = 2x$$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x + 2y = 2(x + y)$$

By Green's theorem,

$$\therefore \int_C P dx + Q dy = \iint_R 2(x + y) dxdy$$

$$= 2 \int_0^2 \left[ \int_0^2 (x + y) dx \right] dy$$

$$= 2 \int_0^2 \left[ \frac{x^2}{2} + xy \right]_0^2 dy = 2 \int_0^2 \left[ \frac{4}{2} + 2y \right] dy$$

$$= 2[2y + y^2]_0^2 = 2(2 \cdot 2 + 4) = 16$$

**35. Evaluate  $\iint_S (xdydz + ydzdx + zdxdy)$ , where  $S$  is the surface of the sphere of radius  $a$ .**

**Ans:** Gauss theorem is  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \operatorname{div} \vec{F} dV$

$$\text{Here } \vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\therefore \operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1=3$$

$$\begin{aligned} \therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iiint_V 3 dV = 3 \iiint_V dV \\ &= 3V, \text{ where } V \text{ is the volume of the sphere.} \end{aligned}$$

$$= 3 \cdot \frac{4}{3} \pi a^3 = 4\pi a^3.$$

**36. Using Stoke's theorem prove that  $\operatorname{curl}(\operatorname{grad} \phi) = 0$ . [AU 2011]**

**Ans:** Stoke's theorem is  $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \operatorname{Curl} \vec{F} \cdot \vec{n} dS$ , where  $S$  is any open surface bounded by simple closed curve  $C$ .

$$\text{Put } \vec{F} = \nabla \phi$$

$$\begin{aligned} \text{then } \iint_S (\nabla \times \nabla \phi) \cdot \vec{n} dS &= \oint_C \nabla \phi \cdot d\vec{r} \\ &= \oint_C \left( \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) \\ &= \oint_C d\phi \end{aligned}$$

Since  $C$  is a simple closed curve both the ends are same, hence

$$\oint_C d\phi = 0$$

$$\therefore \iint_S (\nabla \times \nabla \phi) \cdot \vec{n} dS = 0$$

This is true for any surface  $S$ . So  $\nabla \times \nabla \phi = 0$

$$\Rightarrow \operatorname{Curl}(\operatorname{grad} \phi) = 0$$

37. If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $\vec{a}$  is a constant vector prove that  $\operatorname{div}(\vec{a} \times \vec{r}) = 0$ .

**Ans:** Given  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\operatorname{div}(\vec{a} \times \vec{r}) = \nabla \cdot (\vec{a} \times \vec{r})$$

$$\text{Let } \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\begin{aligned}\therefore \vec{a} \times \vec{r} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \\ &= \vec{i}(a_2z - a_3y) - \vec{j}(a_1z - a_3x) + \vec{k}(a_1y - a_2x)\end{aligned}$$

$$\begin{aligned}\therefore \operatorname{div}(\vec{a} \times \vec{r}) &= \frac{\partial}{\partial x}(a_2z - a_3y) - \frac{\partial}{\partial y}(a_1z - a_3x) + \frac{\partial}{\partial z}(a_1y - a_2x) \\ &= 0 + 0 + 0 = 0\end{aligned}$$

38. If  $S$  is the surface of the rectangular parallelepiped formed by the planes  $x = 0, x = a, y = 0, y = b, z = 0, z = c$ , then find  $\iint_S \vec{r} \cdot \vec{n} dS$ .

**Ans:** Gauss divergence theorem is  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \operatorname{div} \vec{F} dV$

$$\text{To find } \iint_S \vec{r} \cdot \vec{n} dS.$$

$$\text{Here } \vec{F} = \vec{r} \quad \therefore \operatorname{div} \vec{F} = \operatorname{div} \vec{r}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

$$\begin{aligned}\therefore \iint_S \vec{F} \cdot \vec{n} dS &= \iiint_V 3 dV = 3V \\ &= 3 \times \text{Volume of the rectangular parallelepiped.} \\ &= 3abc.\end{aligned}$$

39. If  $S$  any closed surface show that  $\iint_S \operatorname{Curl} \vec{F} \cdot \vec{n} dS = 0$ .

**Ans:** Gauss divergence theorem is  $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \operatorname{div} \vec{F} dV$

Replace  $\vec{F}$  by  $\text{Curl } \vec{F}$ , then  $\text{div Curl } \vec{F} = \nabla \cdot (\nabla \times \vec{F}) = 0$

$$\begin{aligned} \iiint_V \text{div Curl } \vec{F} dV &= 0 \\ \therefore \iint_S \text{curl } \vec{F} \cdot \vec{n} dS &= 0. \end{aligned}$$

- 40. Find the unit normal to the surface  $x^2y + 2xz^2 = 8$  at the point  $(1, 0, 2)$ .** [AU 2008]

**Ans:** Given  $\phi = x^2y + 2xz^2 \quad \therefore \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$

$$= \vec{i}(2xy + 2z^2) + \vec{j}(x^2) + \vec{k}(4xz)$$

At the point  $(1, 0, 2)$ ,  $\nabla \phi = \vec{i}(0+8) + \vec{j} \cdot 1 + \vec{k}(8) = 8\vec{i} + \vec{j} + 8\vec{k}$

$$\therefore \text{unit normal} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{64+1+64}} = \frac{1}{\sqrt{129}}(8\vec{i} + \vec{j} + 8\vec{k})$$

- 41. If the directional derivative of the function  $\phi = x^2 + y^2 + z^2$  at the point  $(1, 2, 3)$  in the direction of  $\alpha\vec{i} + \vec{j} + \vec{k}$  is 2, find  $\alpha$ .** [AU 2008, 2009]

**Ans:** Given

$$\phi = x^2 + y^2 + z^2$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = 2[x\vec{i} + y\vec{j} + z\vec{k}]$$

At the point  $(1, 2, 3)$ ,  $\nabla \phi = 2[\vec{i} + 2\vec{j} + 3\vec{k}]$

Given  $\vec{a} = \alpha\vec{i} + \vec{j} + \vec{k}$  and the directional derivative in the direction of  $\vec{a}$  is 2.

But the directional derivative in the direction of  $\vec{a}$  is  $\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$\therefore \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = 2$$

$$\Rightarrow 2[\vec{i} + 2\vec{j} + 3\vec{k}] \cdot \frac{[\alpha\vec{i} + \vec{j} + \vec{k}]}{\sqrt{\alpha^2 + 1 + 1}} = 2$$

$$\begin{aligned} \Rightarrow \frac{\alpha+2+3}{\sqrt{\alpha^2+2}} &= 1 \\ \Rightarrow (\alpha+5)^2 &= \alpha^2 + 2 \Rightarrow \alpha^2 + 10\alpha + 25 = \alpha^2 + 2 \\ \Rightarrow 10\alpha &= -23 \Rightarrow \alpha = -\frac{23}{10} \end{aligned}$$

**42. Prove that  $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$  is irrotational.** [AU 2011]

**Ans:** Given  $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}[x-x] - \vec{j}[y-y] + \vec{k}[z-z] = \vec{0}$$

$\therefore \vec{F}$  is irrotational.

**43. Find  $\lambda$  such that  $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2z)\vec{k}$  is solenoidal.** [AU 2012]

**Ans:** Given  $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2z)\vec{k}$

$\vec{F}$  is solenoidal if  $\nabla \cdot \vec{F} = 0$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial x}(3x - y + z) + \frac{\partial}{\partial y}(4x + \lambda y - z) + \frac{\partial}{\partial z}(x - y + 2z) &= 0 \\ \Rightarrow & 3 + \lambda + 2 = 0 \\ \Rightarrow & \lambda = -5 \end{aligned}$$

**44. State Gauss divergence theorem.** [AU 2012]

**Ans:** Refer page 1-109

**45. Find the directional derivative of  $\phi = xyz$  at  $(1, 1, 1)$  in the direction  $\vec{i} + \vec{j} + \vec{k}$ .** [AU 2013]

**Ans:** Given  $\phi = xyz$

$$\begin{aligned} \text{Now } \nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= \vec{i}yz + \vec{j}zx + \vec{k}xy \end{aligned}$$

At the point  $(1, 1, 1)$ ,  $\nabla \phi = \vec{i} + \vec{j} + \vec{k}$

Given direction  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\therefore$  directional derivative in the direction

$$\text{of } \vec{a} \text{ is } = \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i} + \vec{j} + \vec{k}) \cdot \frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}} = \frac{1+1+1}{\sqrt{3}} = \sqrt{3}.$$