MATHEMATICS-I

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TEXT BOOK: DIFFERENTIAL CALCULUS BY SHANTI NARAYAN & P.K.MITTAL

LECTURE - 1

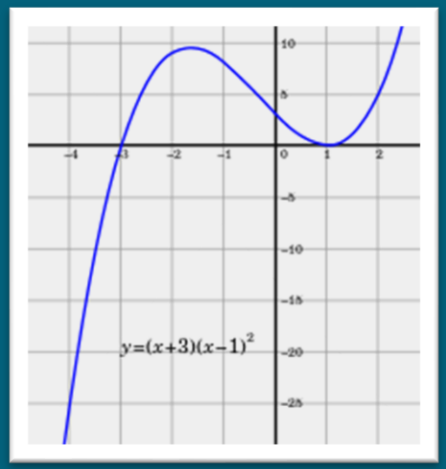
mean value theorem [Rolle's Theorem , Lagrange's and and Cauchy MVT(Chapter-8.1 & 8.2)

DEPARTMENT OF MATHEMATICS, CVRCE

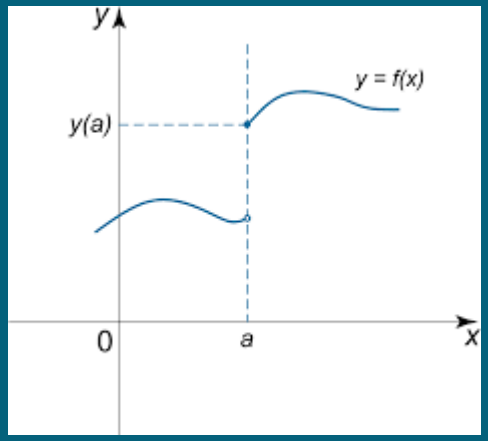
Outline

- Preliminary
- Rolle's theorem and geometrical interpretation
- Examples and counterexamples
- Lagrange’s Mean Value Theorem and its interpretation
- Cauchy mean value theorem and its interpretation
- Implications
- Applications

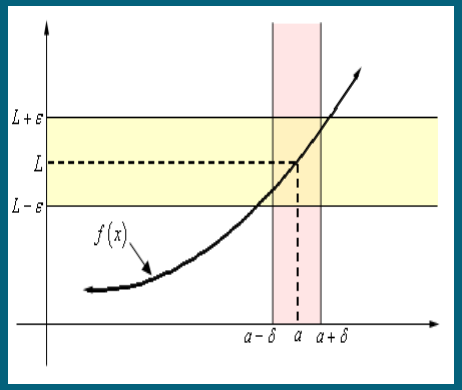
Definition: A function is continuous at point ‘a’ if for every small ε>0 there exist δ>0 such that for all x
|x- a|< δ implies |f(x)-f(a)|< ε.

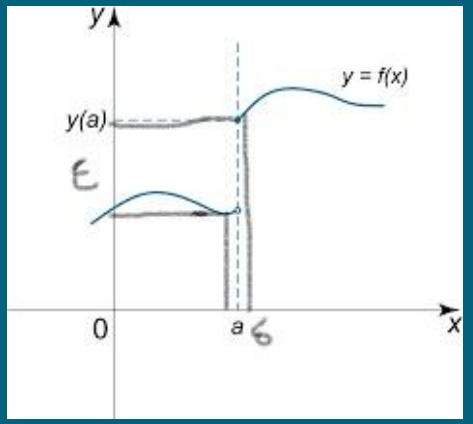


continuous function

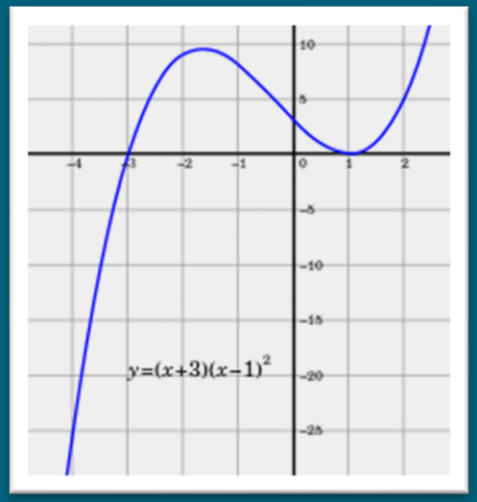


non-continuous function

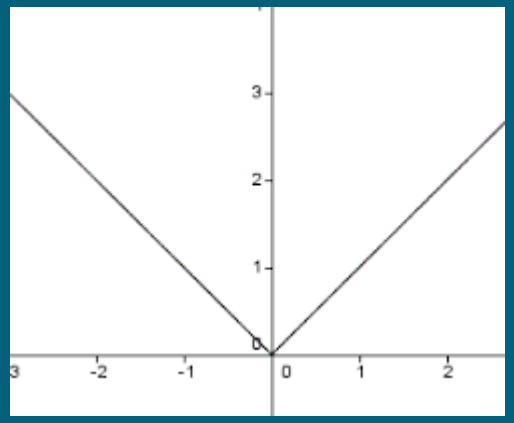




A function is differentiable at a point ‘a’ if the derivative $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.



Differentiable function



Non differentiable function

Rolle's Theorem

Statement

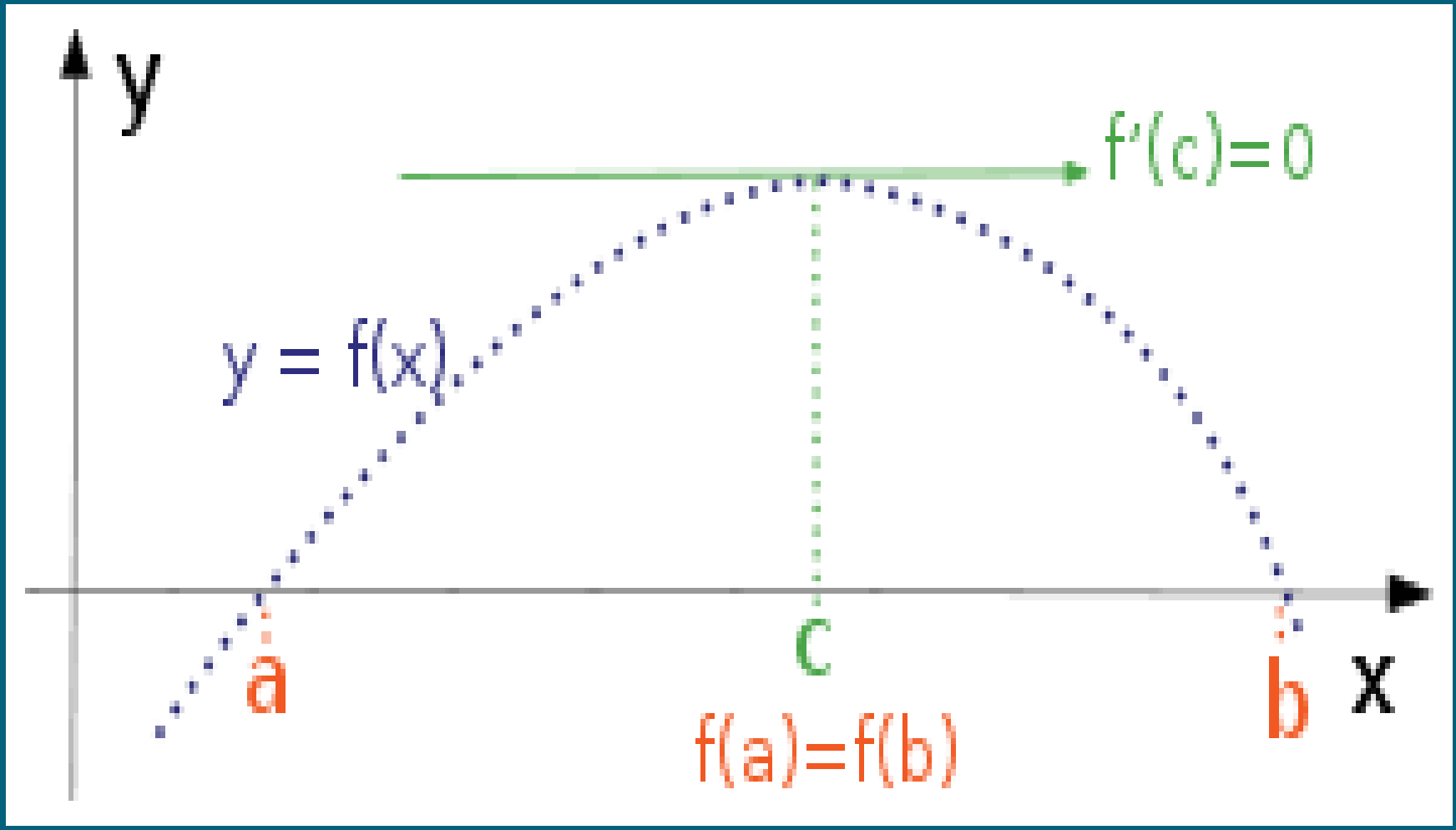
If a real valued function f(x) is

(i) continuous in a closed interval [a, b],

(ii) derivable in an interval (a, b), and

(iii) f(a) = f(b),

then there exist at least one value $c \in (a,b)$ such that f ' (c) = o.


y
f′(c)=0
y = f(x)
a
c
b
x
f(a)=f(b)

Solved Problems

Ex-1: Verify Rolle's theorem for

$$f(x)=x(x+3)e^{-x/2} \quad \text{in } [-3,0].$$

Solution

Let $f(x)=x(x+3)e^{-x/2}$, $x \in [-3,0]$.

so that f(-3) = 0 = f(0).

Also the function is derivable in the interval [-3, 0].

We have $f'(x)=(2x+3)e^{-x/2}+x(x+3)e^{-x/2}\left(-\frac{1}{2}\right)$

$$=\frac{(-x^2+x+6)}{2}e^{-x/2}$$

Now $f'(x) = 0 \Leftrightarrow -x^2 + x + 6 = 0.$

The equation $-x^2 + x + 6 = 0$ is satisfied by x = -2, 3.

Of these two values of x for which f' (x) is zero, -2 belongs to the open interval (-3, 0) under consideration. Hence the verification.

Ex-2: Discuss applicability of Rolle's Theorem to the function f(x) = |x| in [-1,1].

The function f(x) is continuous but not differentiable in the entire open interval (-1, 1) and so Rolle's Theorem is not applicable to the given function f(x) in [-1, 1]

Lagrange's Mean Value Theorem

If a function f is

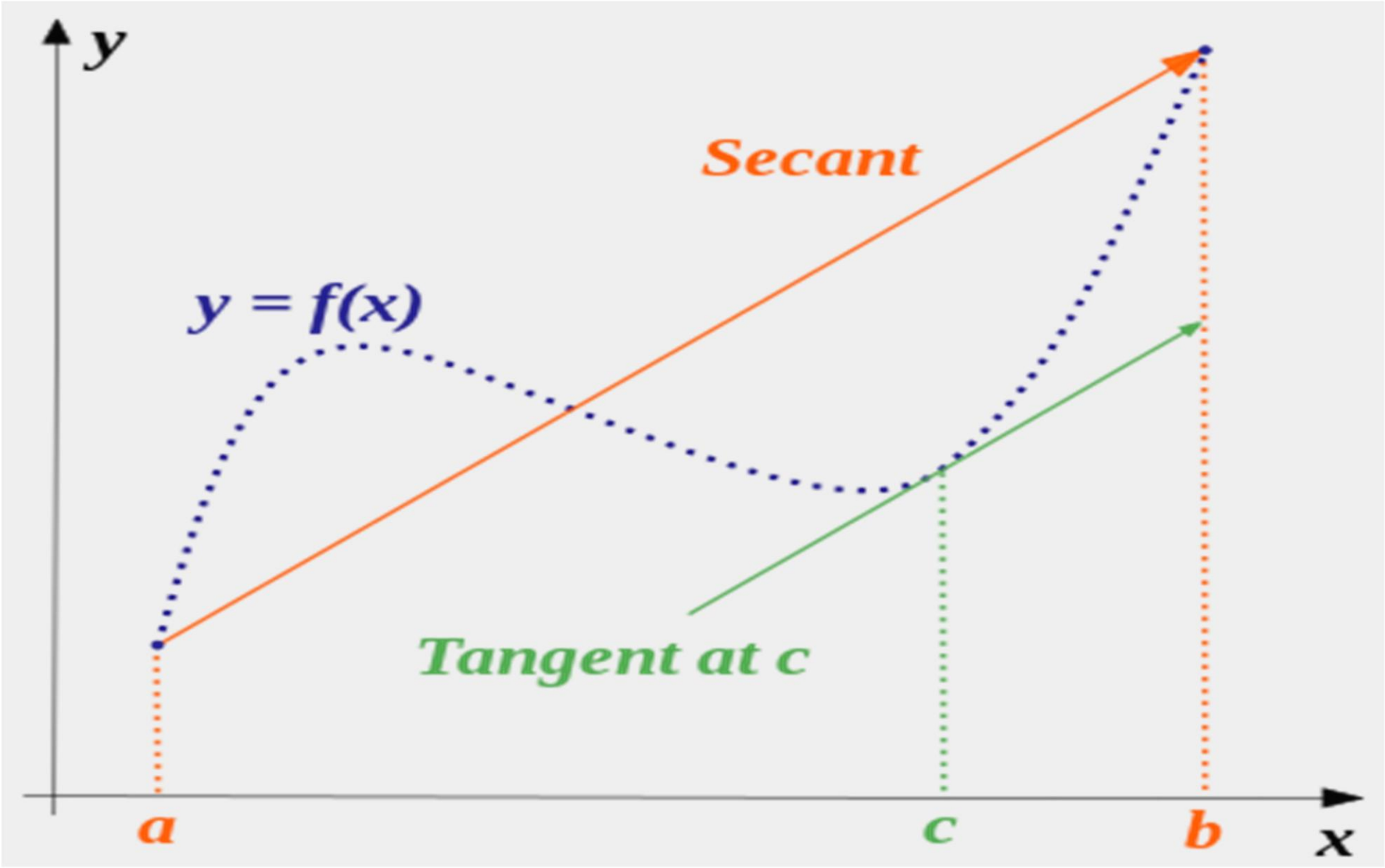
(i) Continuous in a closed interval [a,b] and

(ii) derivable in the open interval (a, b),

then there exists at least one value c ϵ(a, b) such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

Geometrical interpretation



Ex-1: Find 'c' of the mean value theorem, if

$f(x) = x(x-1)(x-2); \qquad x \in [0, ½]$

Solution $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x.$

f is a polynomial function. So, it is continuous and derivable for all x.

$a = 0, \quad b = \frac{1}{2};$

$f(a) = f(0) = 0, \quad f(b) = f\left(\frac{1}{2}\right) = \frac{3}{8}$

The conditions of Lagrange's mean value theorem are satisfied.

Also, $f'(x) = 3x^2 - 6x + 2$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{1/2} = \frac{3}{4}$$

$$12c^2 - 24c + 5 = 0$$

$$\Rightarrow c = \frac{6 \pm \sqrt{21}}{6}$$

Since $\frac{6 + \sqrt{21}}{6} \notin (0, 1/2)$

the required $c = \frac{6 - \sqrt{21}}{6}$.

Cauchy Mean Value Theorem

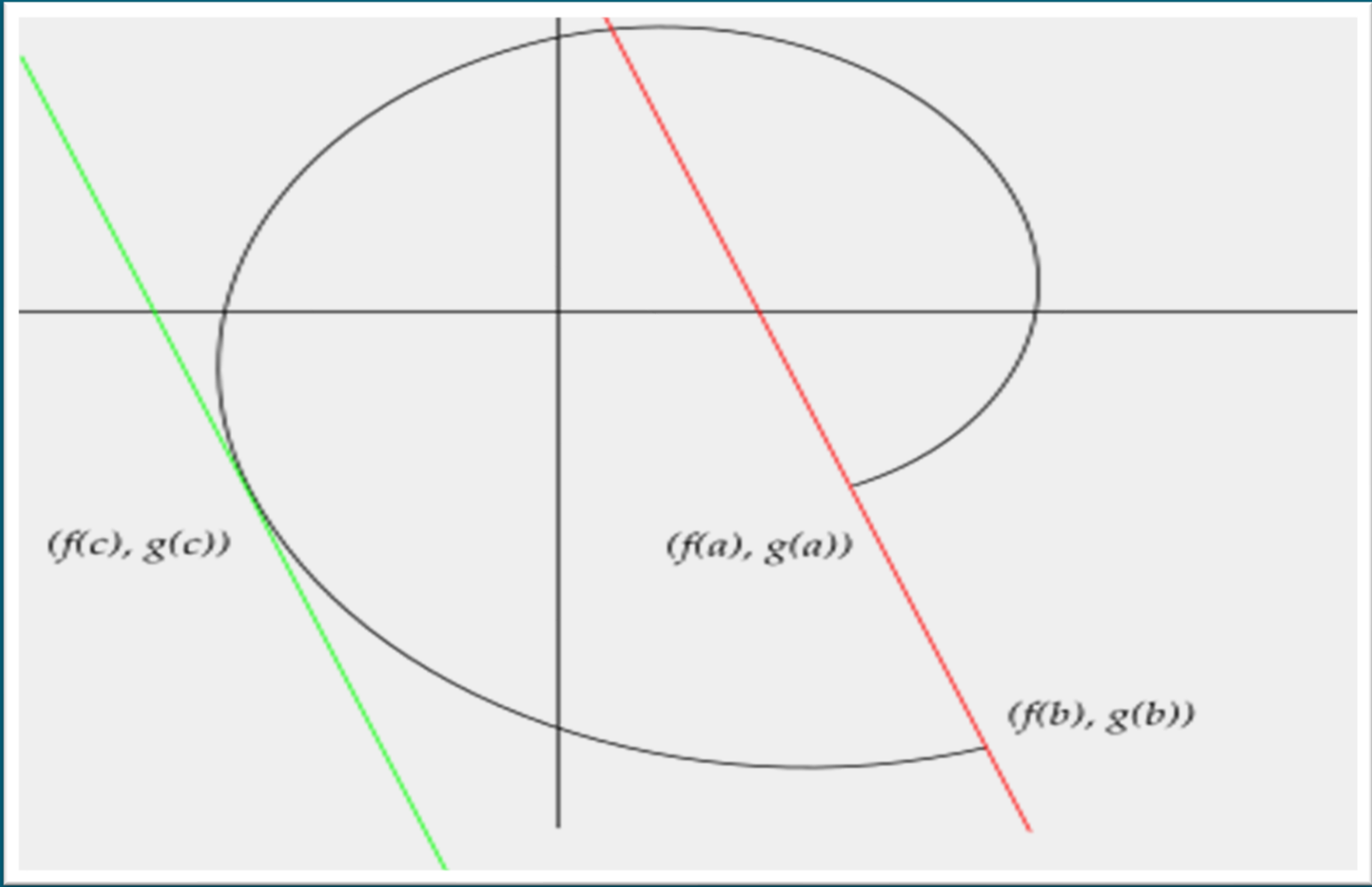
If two functions $f(x)$ and $g(x)$ are

(i) continuous in a closed interval [a, b],
(ii) derivable in an interval (a, b), and
(iii) $g'(x) \neq 0$ for all $x \in (a, b)$,

Then there exists at least one c $\in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Cauchy Mean Value Theorem



Implications

- F ‘(x)=o for all x implies the function F is constant.

Proof:$F'(x) = 0 = \frac{F(b)-f(a)}{b-a} \Longrightarrow F(a) = F(b)$⟹F is constant.

- If F’=G’ for all x implies F=G+c, where c is a constant.

Proof: Let H=F-G then H’=F’-G’=o⟹F=G+c.

- F’(x)>o for all x implies the function is strictly increasing.
- F’(x)<o for all x implies the function is strictly decreasing.

- Proof: let us assume that there exists a and b such that $a<b$ but $f(a)>f(b) \Longrightarrow f(b)-f(a)<0$ but $b-a>0$ so $f'(c)<0$. Hence a contradiction. Similar proof follows for second case.

Application

- If the distance between place A and place B is 50 kms. In between the speed limit is 30 kms/hour. If a driver covers the distance in an hour then there is at least a point where the driver's speed was more then the speed limit.

Thank you