Radius of Curvature

<div style="text-align:right">5</div>

5.1 INTRODUCTION

Let C be a smooth plane curve at each point on which a tangent can be drawn as in Fig. 5.1. Suppose that between P and Q on C, the curve bends continuously in one direction.

Let $\overset{\frown}{AP}= s$ and $\overset{\frown}{PQ}= \delta s$. Suppose PL and QM are tangents drawn to the curve at P and Q intersecting at R and cutting the X-axis at L and M, respectively.

5.1.1 Angle of Contingence

The angle QRT is called the angle of contingence of the arc PQ. Clearly it is the angle through which the tangent line turns as it touches each point between P and Q.

5.1.2 Measure of Curvature

The fraction

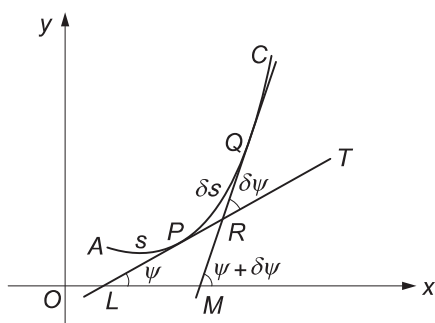$$\kappa_{av} = \frac{\text{Angle of contingence}}{\text{Length of arc}}$$

is the average bending or average curvature of the arc.

Curvature of a curve C at a point P, denoted by κ is given by

$$\kappa = \lim_{Q \to P} |\kappa_{av}|$$

$$= \left| \lim_{\delta s \to 0} \frac{\delta \psi}{\delta s} \right| = \left| \frac{d\psi}{ds} \right|$$

5.2 CURVATURE OF A CIRCLE

In the case of a circle (Fig. 5.2), its curvature is the same at every point. It is measured by the reciprocal of the radius. If r is the radius and O the centre then

$$\angle RTQ = \angle POQ = \frac{\overset{\frown}{PQ}}{r}$$

the angle being measured in circular measure (radians)

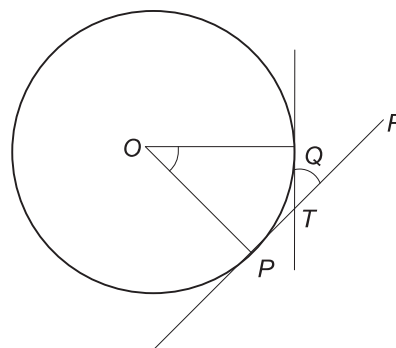$$\therefore \quad \frac{\text{Angle of contingence}}{\text{Length of arc}} = \frac{1}{r}$$

Figure 5.1 Curvature of a smooth plane curve
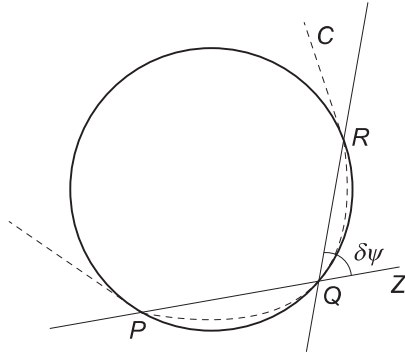

Figure 5.2 Curvature of a circle

Figure 5.3 Circle of curvature through P, Q, R on a curve C

5.2.1 Circle of Curvature

A unique circle can be drawn through three neighbouring points P, Q, R on any curve C (Fig. 5.3). As the points move closer together, PQ and QR ultimately become tangents both to the curve and to the circle. So, the curvature of a curve at a given point is defined with respect to the circle described at the point. We call the circle, the circle of curvature at that point. Its radius and centre are called the radius and centre of curvature, respectively. A chord of this circle drawn at the point of contact in any direction is referred to as the chord of curvature in that direction.

5.2.2 Formula for Radius of Curvature

Let PQ and QR be equal chords and the corresponding arcs $\overset{\frown}{PQ} = \overset{\frown}{QR} = \delta s$ and the angle of contingence $\angle RQZ = \delta \psi$.

Now radius of circumcircle of

$$\triangle PQR = \frac{PR}{2 \sin \widehat{PQR}}$$

$$\therefore \quad \rho = \lim \frac{PR}{2 \sin \widehat{PQR}} = \lim \frac{2\delta s}{2 \sin \delta \psi}$$

$$= \lim \frac{\delta s}{\delta \psi} \cdot \lim \frac{\delta \psi}{\sin \delta \psi} = \frac{ds}{d\psi} \quad (5.1)$$

Also, the perpendicular bisectors of chords PQ and QR intersect at the circumcentre of PQR. Thus, in the limit, the centre of curvature at any point on a curve C is the point of intersection of the normal at that point with the normal at a neighbouring point.

5.2.3 Intrinsic Equation

The equation of a curve may be written in various forms. A relation between s and ψ is called the intrinsic equation of the curve: $s = f(\psi)$. The formula for ρ in this case is

$$\rho = \frac{ds}{d\psi} \qquad \text{[From Eq. (5.1)]}$$

For example, the intrinsic equation of a circle of radius 'a' is $s = a\psi$.

$$\therefore \quad \rho = \frac{ds}{d\psi} = a \text{ and the curvature } \kappa = \frac{1}{\rho} = \frac{1}{a}.$$

5.2.4 Transformations

From Fig. 5.4, we have $\cos \psi = \dfrac{dx}{ds}, \sin \psi = \dfrac{dy}{ds}$

Differentiating w.r.t 's'

$$-\sin \psi \frac{d\psi}{ds} = \frac{d^2x}{ds^2}, \qquad \cos \psi \frac{d\psi}{ds} = \frac{d^2y}{ds^2}$$

$$\frac{1}{\rho} = \frac{-\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}} \qquad (5.2)$$

Squaring and adding,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right) \qquad (5.3)$$

since $\quad \delta s^2 = (\text{Chord } PQ)^2 = (\delta x)^2 + (\delta y)^2$

or $\quad 1 = \left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2$
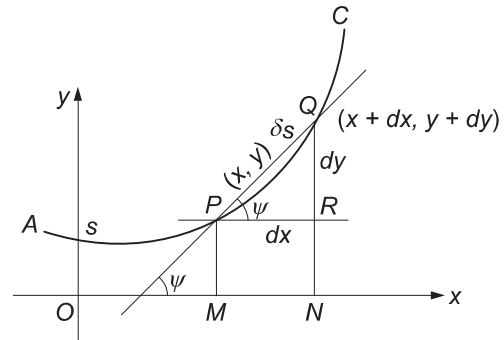

Figure 5.4 Elemental right triangle

Formulas (5.2) and (5.3) are only suitable for the case where x and y are known functions of s.

Example 5.1
Find the radius of curvature at any point of the catenary (chainette) $s = c \tan \psi$.

Solution Equation of the curve is
$$s = c \tan \psi \qquad (1)$$

Differentiating Eq. (1) w.r.t ψ, we have
$$\rho = \frac{ds}{d\psi} = c \ \sec^2 \psi$$
$$= c\left(1 + \frac{s^2}{c^2}\right) = \frac{c^2 + s^2}{c}.$$

Example 5.2
Find the radius of curvature at any point of the cycloid $s = \sqrt{8ay}$ having the intrinsic equation $s = 4a \sin \psi$.

Solution Equation of the curve is
$$s = 4a \sin \psi \qquad (1)$$

Differentiating Eq. (1) w.r.t ψ, we have
$$\rho = \frac{ds}{d\psi} = 4a \ \cos \psi$$
$$= 4a\sqrt{1 - \frac{s^2}{16a^2}}$$
$$= \sqrt{16a^2 - s^2} = 4a\sqrt{1 - \frac{y}{2a}}.$$

Example 5.3
Find the radius of curvature at any point of the cardioid $s = 4a \sin \dfrac{\psi}{3}$.

Solution Equation of the curve is
$$s = 4a \sin \frac{\psi}{3} \qquad (1)$$

Differentiating Eq. (1) w.r.t ψ, we have
$$\rho = \frac{ds}{d\psi} = \frac{4a}{3} \cos \frac{\psi}{3}.$$

Example 5.4
Find the radius of curvature at any point of the parabola
$$s = a \log(\tan \psi + \sec \psi) + a \tan \psi \sec \psi.$$

Solution Equation of the parabola is
$$s = a \log(\tan \psi + \sec \psi) + a \tan \psi \sec \psi \qquad (1)$$

Differentiating Eq. (1) w.r.t ψ, we have
$$\rho = \frac{ds}{d\psi}$$
$$= \frac{a(\sec^2 \psi + \sec \psi \tan \psi)}{\tan \psi + \sec \psi}$$
$$\quad + a(\sec^3 \psi + \tan^2 \psi \sec \psi)$$
$$= a \sec \psi + a \sec \psi(\sec^2 \psi + \sec^2 \psi - 1)$$
$$= 2a \sec^3 \psi.$$

EXERCISE 5.1

1. Find the radius of curvature at any point of the tractrix $s = c \log \sec \psi$.
   Ans: $\rho = c \tan \psi$

2. Prove that for the curve $s = m(\sec^3 \psi - 1)$, the radius of curvature is $\rho = 3m \tan \psi \sec^3 \psi$. Hence show that $3my_1 y_2 = 1$ and this differential equation is satisfied by the semicubical parabola $27my^2 = 8x^3$.

3. Prove that for the curve
   $$s = a \log \cot \left(\frac{\pi}{4} - \frac{\psi}{2}\right) + a \frac{\sin \psi}{\cos^2 \psi},$$
   the radius of curvature is $\rho = 2a \sec^3 \psi$.

4. Determine the radius of curvature of the curve
   $$x = a \cos \frac{s}{a}, y = a \sin \frac{s}{a}.$$
   [**Hint:** Use Equations (5.2) and (5.3)]
   Ans: $\rho = a$

5.3 RADIUS OF CURVATURE— CARTESIAN FORMULA

[Equation of the curve is of the form $y = f(x)$]
We know that the slope of the tangent at any point P on a curve C is
$$\frac{dy}{dx} = \tan \psi$$

Differentiating w.r.t 'x'

$$\frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx} = \sec^2\psi \frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$= \sec^3\psi \frac{d\psi}{ds} \qquad \left(\because \frac{dx}{ds} = \cos\psi\right)$$

or $\qquad \rho = \dfrac{ds}{d\psi} = \dfrac{\sec^3\psi}{\frac{d^2y}{dx^2}}$ $\qquad$ (5.4)

$$= \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+y_1^2)^{3/2}}{y_2} \quad (5.5)$$

where $\quad y_1 = \dfrac{dy}{dx}, \quad y_2 = \dfrac{d^2y}{dx^2}.$

Note

If the equation is of the form $x = \phi(y)$ then

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}. \qquad (5.6)$$

Example 5.5
Find the radius of curvature at $P = (\sqrt{2}, \sqrt{2})$ on the curve $x^2 + y^2 = 4$.

Solution   The equation of the given curve, which is a circle, is

$$x^2 + y^2 = 4 \qquad (1)$$

Differentiating Eq. (1) w.r.t 'x', we have

$$2x + 2y\frac{dy}{dx} = 0 \qquad (2)$$

$$\Rightarrow \quad y_1|_p = -\left.\frac{x}{y}\right|_p = -\frac{\sqrt{2}}{\sqrt{2}} = -1$$

Differentiating Eq. (2) again w.r.t 'x', we get

$$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0 \qquad (3)$$

$$\Rightarrow \quad y_2|_p = -\left.\frac{1+y_1^2}{y}\right|_p$$

$$= -\frac{1+(-1)^2}{\sqrt{2}} = -\sqrt{2}$$

$$\therefore \quad \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{[1+(-1)^2]^{3/2}}{\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{\sqrt{2}} = 2 = \text{radius of the circle.}$$

Example 5.6
Find the radius of curvature at $P = \left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

Solution   The equation of the curve is
$$x^3 + y^3 = 3axy \qquad (1)$$

Differentiating Eq. (1) w.r.t 'x', we have

$$3x^2 + 3y^2 y_1 = 3a(y + xy_1) \quad \text{where} \quad y_1 = \frac{dy}{dx}$$

$$\Rightarrow \quad y_1 = \frac{ay - x^2}{y^2 - ax} \qquad (2)$$

Differentiating Eq. (2) w.r.t 'x', we have

$$y_2 = \frac{\left(y^2 - ax\right)\left(ay_1 - 2x\right) - \left(ay - x^2\right)\left(2yy_1 - a\right)}{\left(y^2 - ax\right)^2} \qquad (3)$$

$$y_1|_p = \frac{a\cdot\dfrac{3a}{2} - \dfrac{9a^2}{4}}{\dfrac{9a^2}{4} - a\cdot\dfrac{3a}{2}} = \frac{\dfrac{6a^2}{4} - \dfrac{9a^2}{4}}{\dfrac{9a^2}{4} - \dfrac{6a^2}{4}} = -1$$

$$y_2|_p = \frac{\dfrac{3a^2}{4}\left(-a - \dfrac{6a}{2}\right) - \left(\dfrac{6a^2}{4} - \dfrac{9a^2}{4}\right)(-3a - a)}{\left(\dfrac{9a^4}{16}\right)}$$

$$= \frac{32}{3a}$$

The radius of curvature at P

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{|y_2|} = \frac{\left[1+(-1)^2\right]^{3/2}}{\frac{32}{3a}} = \frac{3a}{8\sqrt{2}}.$$

Example 5.7
Find the radius of curvature at (3, 3) on the curve $x^3 + xy^2 - 6y^2 = 0$.

Solution   The equation of the curve, in implicit form, is

$$\phi(x, y) = x^3 + xy^2 - 6y^2 = 0 \qquad (1)$$

Differentiating Eq. (1) w.r.t 'x', we have

$$3x^2 + y^2 + 2xyy_1 - 12yy_1 = 0 \qquad (2)$$

At $P = (3, 3)$

$$3 \cdot 3^2 + 3^2 + 2 \cdot 3 \cdot 3y_1 \big|_p - 12 \cdot 3 \cdot y_1 \big|_p = 0$$

$$\Rightarrow y_1 \big|_p = 2 \qquad (3)$$

Differentiating Eq. (2) w.r.t 'x', we have

$$6x + 2yy_1 + 2yy_1 + 2xy_1^2 + 2xyy_2$$
$$-12y_1^2 - 12yy_2 = 0$$

Putting $(x, y) = (3, 3)$ and $y_1 \big|_p = 2$

we get $18 + 12 + 12 + 24 + 18y_2 \big|_p$

$-48 - 36y_2 \big|_p = 0 \Rightarrow y_2 \big|_p = 1$

The radius of curvature at $P = (3, 3)$ on Eq. (1)

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{|y_2|} = \frac{\left(1 + 2^2\right)^{3/2}}{1} = 5^{3/2} = 5\sqrt{5}.$$

Example 5.8
Show that the radius of curvature at any point of the catenary $y = c \cosh(x/c)$ is equal to the length of the portion of the normal intercepted between the curve and the x-axis.

Solution   Equation of the catenary
$$y = c \cosh \frac{x}{c} \qquad (1)$$

Differentiating Eq. (1) w.r.t 'x', we have

$$y_1 = \frac{dy}{dx} = c \sinh \frac{x}{c} \cdot \frac{1}{c} \Rightarrow y_1 \big|_p = \sinh \frac{x}{c} \qquad (2)$$

$$y_2 = \frac{d^2y}{dx^2} = \cosh \frac{x}{c} \cdot \frac{1}{c} \Rightarrow y_2 \big|_p = \frac{1}{c} \cosh \frac{x}{c} \qquad (3)$$

∴ Radius of curvature

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{\left(1 + \sinh^2 \frac{x}{c}\right)^{3/2}}{\frac{1}{c} \cosh \frac{x}{c}}$$

$$= \frac{\cosh^3 \frac{x}{e}}{\frac{1}{c} \cosh \frac{x}{c}} = c \cosh^2 \frac{x}{c} = \frac{y^2}{c} \qquad (4)$$

Length of the normal at $P(x, y)$ is

$$PN = y\sqrt{1 + y_1^2} = y \cosh \frac{x}{c} = \frac{y^2}{c} \qquad \text{[by Eq. (2)]}$$

∴   Radius of curvature at any point P on the catenary = Length of the normal.

5.4   RADIUS OF CURVATURE— PARAMETRIC FORM

[Equations of the curve are of the form: $x = x(t)$, $y = y(t)$]

Let $x = x(t), y = y(t)$ be the parametric equations of the curve $y = f(x)$.

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}$$

where $'$ denotes differentiation w.r.t parameter 't'.

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(y'/x'\right)$$

$$= \frac{x'y'' - x''y'}{(x')^2}\frac{1}{x'} = \frac{x'y'' - x''y'}{(x')^3}$$

$$\rho = \frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2} = \left[1 + \left(y'/x'\right)^2\right]^{\frac{3}{2}} \cdot \frac{(x')^3}{x'y'' - x''y'}$$

$$= \frac{\left[\left(x^1\right)^2 + \left(y^1\right)^2\right]^{\frac{3}{2}}}{(x')^3} \cdot \frac{(x')^3}{x'y'' - x''y'}$$

$$= \frac{\left[(x')^2 + \left(y^1\right)^2\right]^{\frac{3}{2}}}{x'y'' - x''y'}$$

∴ Radius of curvature, $\rho = \dfrac{\left[(x')^2 + \left(y'\right)^2\right]^{\frac{3}{2}}}{[x'y'' - x''y']}$ (5.6)

Curvature, $\dfrac{1}{\rho} = \dfrac{[x'y'' - x''y']}{\left[(x')^2 + \left(y'\right)^2\right]^{\frac{3}{2}}}$

Example 5.9
Show that the radius of curvature at any point of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is $4a \cos(t/2)$.

Solution   The parametric equations of the cycloid are

$$x = a(t + \sin t), \quad y = a(1 - \cos t) \qquad (1)$$

Differentiating w.r.t 't' we get

$$x' = \frac{dx}{dt} = a\,(1 + \cos t),$$

$$y' = \frac{dy}{dt} = a\,(0 + \sin t) \qquad (2)$$

$$x'' = \frac{d^2x}{dt^2} = a\,(0 - \sin t),$$

$$y'' = \frac{d^2y}{dt^2} = a\cos t \qquad (3)$$

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{3/2}}{x'y'' - x''y'}$$

$$= \frac{a^3\left[(1 + \cos t)^2 + (\sin t)^2\right]^{3/2}}{a^2\left[(1 + \cos t)\cos t + \sin^2 t\right]} \qquad (4)$$

$$= \frac{a\left[2\,(1 + \cos t)\right]^{3/2}}{(1 + \cos t)}$$

$$= a \cdot 2\sqrt{2}\,(1 + \cos t)^{1/2}$$

$$= a \cdot 2\sqrt{2} \cdot \left(2\cos^2\frac{t}{2}\right)^{1/2} = 4a\cos\frac{t}{2}$$

Example 5.10
Find the radius of curvature at any point 't' of the curve

$$x = a\,(\cos t + t \sin t), \quad y = a\,(\sin t - t \cos t). \quad (1)$$

Solution   Differentiating Eq. (1) w.r.t 't', we have

$$x' = \frac{dx}{dt} = a\,(-\sin t + \sin t + t \cos t) = at\cos t$$

$$y' = \frac{dy}{dt} = a\,(\cos t - \cos t + t \sin t) = at\sin t$$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t\,\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} = \frac{1}{at}\sec^3 t$$

$$\rho = \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{\left[1 + (\tan t)^2\right]^{3/2}}{\frac{1}{at}\sec^3 t}$$

$$= \frac{\sec^3 t}{\frac{1}{at}\sec^3 t} = at$$

Example 5.11
Find the radius of curvature at any point of the parabola $y^2 = 4ax$.

Solution   The equation of the curve is

$$y^2 = 4ax \qquad (1)$$

Differentiating Eq. (1) w.r.t 'x', we get

$$2yy_1 = 4a \;\Rightarrow\; y_1 = \frac{2}{a}y \qquad (2)$$

Differentiating again

$$2y_1^2 + 2yy_2 = 0$$

or $\qquad y_2 = -\dfrac{y_1^2}{y} = -\dfrac{4a^2}{y^3} \qquad (3)$

The radius of curvature is

$$\rho = \pm\frac{\left(1 + y_1^2\right)^{3/2}}{y_2}$$

$$= \pm\frac{\left(1 + \frac{4a^2}{y^2}\right)^{3/2}}{-\frac{4a^2}{y^3}} = \frac{\left(y^2 + 4a^2\right)^{3/2}}{\frac{4a^2}{y^3}}\cdot\frac{1}{y^3}$$

$$= \frac{\left(4ax + 4a^2\right)^{3/2}}{4a^2} = \frac{4a \cdot 2\sqrt{a}\,(x + a)^{3/2}}{4a^2}$$

$$= \frac{2}{\sqrt{a}}\,(x + a)^{3/2}.$$

Example 5.12
Find the radius of curvature at any point of the curve $y = c\log\sec(x/c)$.

Solution   The equation of the curve is
$$y = c\log\sec(x/c) \qquad (1)$$

Differentiating Eq. (1) w.r.t 'x', we get

$$y_1 = c\cos\frac{x}{c}\cdot\sec\frac{x}{c}\cdot\tan\frac{x}{c}\cdot\frac{1}{c} = \tan\frac{x}{c} \qquad (2)$$

$$y_2 = \sec^2\frac{x}{c}\cdot\frac{1}{c} \qquad (3)$$

The radius of curvature is

$$\rho = \pm\frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \frac{\left(1 + \tan^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\sec^2\frac{x}{c}}$$

$$= \frac{\sec^3\frac{x}{c}}{\frac{1}{c}\sec^2\frac{x}{c}} = c\sec\frac{x}{c}. \qquad (4)$$

Example 5.13
Find the radius of curvature at (–2, 0) on the curve $y^2 = x^3 + 8$.

Solution   Here we take y as the independent variable and use the formula

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^2}{\frac{d^2x}{dy^2}}$$

Differentiating the given equation w.r.t to 'y', we get

$$3x^2 \frac{dx}{dy} = 2y$$

$$\left(\frac{dx}{dy}\right)_p = \frac{2y}{3x^2}\bigg|_{(-2,0)} = 0$$

Differentiating again w.r.t to 'y', we get

$$3x^2 \frac{d^2x}{dy^2} + 6x \left(\frac{dx}{dy}\right)^2 = 2$$

$$\left(\frac{d^2x}{dy^2}\right)_p = \frac{2 - 6x\left(\frac{dx}{dy}\right)^2}{3x^2}\bigg|_p = \frac{2}{3(-2)^2} = \frac{1}{6}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{(1+0)^{3/2}}{\frac{1}{6}} = 6.$$

Example 5.14
Find the radius of curvature at any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$. Show that it is $2\frac{(SP)^{3/2}}{\sqrt{a}}$ where S is the focus of the parabola.

Solution   The equations of the parabola are

$$x = at^2, y = 2at \tag{1}$$

Differentiating Eq. (1) w.r.t to 't', we get

$$x' = \frac{dx}{dt} = 2at, \quad y' = \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t} \tag{2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{dt}{dx} \tag{3}$$

$$= -\frac{1}{t^2} \cdot \frac{t}{2at} = -\frac{1}{2at^3} \tag{4}$$

$$\rho = \pm \frac{\left(1 + y_1^2\right)^{3/2}}{y_2} = \pm \frac{\left(1 + \frac{1}{t^2}\right)^{3/2}}{-\frac{1}{2at^3}}$$

$$= \frac{\left(1 + t^2\right)^{3/2}}{t^3} \cdot \frac{2at^3}{1} = 2a\left(1 + t^2\right)^{3/2} \tag{5}$$

$$S = (a, 0), \quad P = \left(at^2, 2at\right)$$

$$(SP)^2 = \left(at^2 - a^2\right) + (2at - 0)^2$$

$$= a^2\left[\left(t^2 - 1\right) + 4t^2\right] = a^2\left(t^2 + 1\right)^2$$

$$SP = a\left(t^2 + 1\right) \tag{6}$$

From Eqs. (4) and (5)

$$\rho = 2a\left(1 + t^2\right)^{3/2} = \frac{2\left[a\left(1 + t^2\right)\right]^{3/2}}{\sqrt{a}} \tag{7}$$

$$= \frac{2}{\sqrt{a}}(SP)^3.$$

5.5 RADIUS OF CURVATURE—POLAR FORM

[Equation of the curve is of the form $r = f(\theta)$]

Let O be the pole and OX be the initial line as shown in Fig. 5.5. Let p be the length of the perpendicular ON on the tangent PT at P on C.

Let $r = f(\theta)$ be the equation of the curve C. Suppose that the tangent PT at $P(r, \theta)$ on the curve C make an angle ψ with the initial line OX and the angle between the radius vector OP and the tangent PT be ϕ.
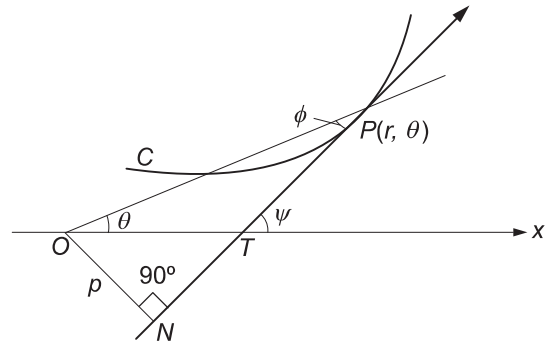
Then

$$\psi = \theta + \phi$$

Figure 5.5 Tangent at $P(r, \theta)$ on a polar curve

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad (5.7)$$

Since

$$\tan\phi = \frac{rd\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{r}{r_1} \qquad (5.8)$$

where $r_1 = \dfrac{dr}{d\theta}$

Differentiating Eq. (5.8) w.r.t θ, we get

$$\sec^2\phi \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)}{\left(\frac{dr}{d\theta}\right)^2} = \frac{r_1^2 - rr_2}{r_1^2}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2} \cdot \frac{1}{1 + \left(\frac{r}{r_1}\right)^2} = \frac{r_1^2 - rr_2}{r^2 + r_1^2} \quad (5.9)$$

where $\qquad r_1 = \dfrac{dr}{d\theta}, r_2 = \dfrac{d^2r}{d\theta^2}$

Also, $\qquad \dfrac{ds}{d\theta} = \sqrt{r^2 + r_1^2} \qquad (5.10)$

Substituting in Eq. (5.7) we get

$$\frac{1}{\rho} = \frac{1}{\left(r^2 + r_1^2\right)^{1/2}}\left(1 + \frac{r_1^2 - rr_2}{r^2 + r_1^2}\right)$$

$$= \frac{r^2 + 2r_1^2 - rr_2}{\left(r^2 + r_1^2\right)^{3/2}}$$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2} \qquad (5.11)$$

Example 5.15
Find the radius of curvature at any point (r, θ) of the cardioid $r = a(1 - \cos\theta)$.

Solution   Equation of the curve is
$$r = a(1 - \cos\theta) \qquad (1)$$

Differentiating w.r.t 'θ'
$$r_1 = \frac{dr}{d\theta} = a\sin\theta, \quad r_2 = \frac{d^2r}{d\theta^2} = a\cos\theta \qquad (2)$$

$$\left(r^2 + r_1^2\right)^{3/2} = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}$$

$$= \left[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}$$

$$= a^3\left[\left(2\sin^2\frac{\theta}{2}\right)^2 + \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2\right]^{3/2}$$

$$= a^3 \cdot \left(2\sin\frac{\theta}{2}\right)^{2 \cdot \frac{3}{2}} \times \left[\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right]^{3/2}$$

$$= 8a^3\sin^3\frac{\theta}{2} \qquad (3)$$

$$r^2 + 2r_1 - rr_2 = r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)$$

$$= a^2(1 - \cos\theta)^2 + 2a^2\sin^2\theta$$

$$\quad - a^2\cos\theta(1 - \cos\theta)$$

$$= a^2\left[1 + \cos^2\theta - 2\cos\theta\right.$$

$$\left. + 2\sin^2\theta - \cos\theta + \cos^2\theta\right]$$

$$= 3a^2(1 - \cos\theta) = 6a^2\sin^2\frac{\theta}{2}$$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1 - rr_2} = \frac{8a^3\sin^3\frac{\theta}{2}}{6a^2\sin^2\frac{\theta}{2}}$$

$$= \frac{4}{3}a\sin\frac{\theta}{2} = \frac{2}{3}\sqrt{2ar}.$$

Example 5.16
Show that the radius of curvature of the curve $r^n = a^n\cos n\theta$ is $a^n r^{-n+1} / (n + 1)$ for $n \neq -1$.

Solution   The equation of the given curve is
$$r^n = a^n\cos n\theta \qquad (1)$$

By logarithmic differentiation w.r.t 'θ' we get

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n\sin n\theta}{\cos n\theta} \Rightarrow r_1 = \frac{dr}{d\theta} = -r\tan n\theta \quad (2)$$

Differentiating again w.r.t θ

$$r_2 = \frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta}\tan n\theta - nr\sec^2 n\theta$$

$$= r\tan^2 n\theta - nr\sec^2 n\theta \qquad (3)$$

$$\therefore \ \rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1 - rr_2}$$

$$= \frac{\left(r^2 + r^2\tan^2 n\theta\right)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{(n+1)r^2\sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta}$$