DEPARTMENT OF MATHEMATICs

MATHEMATICS - I

LECTURE

Functions of Several order partial derivatives and Taylor's series/theorem, Homogeneous functions and Euler's theorem.

Maximum and Minimum Values of a function: Necessary and Sufficient condition for maximum/minimum, Lagrange's Method of Multipliers for maximum/minimum.

TEXT BOOK: Differential Calculus – Shanti Narayan and P.K. Mittal

- **Higher order partial derivatives**
- **Taylor series theorem**

Question ?

- Find the value of 1+x-x^2 for x=2.
- Ans :1+2-$2^2 = 3 - 4 = -1$.

2) Find the value of $e^x, \sin x\,, \cos x, \ln x$ for x=2?

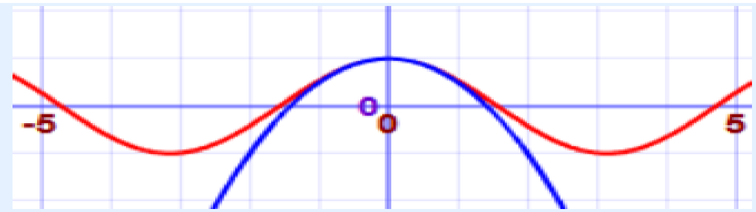
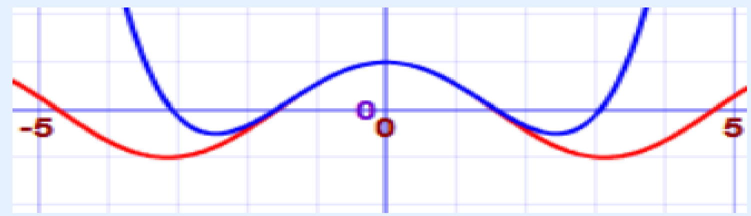
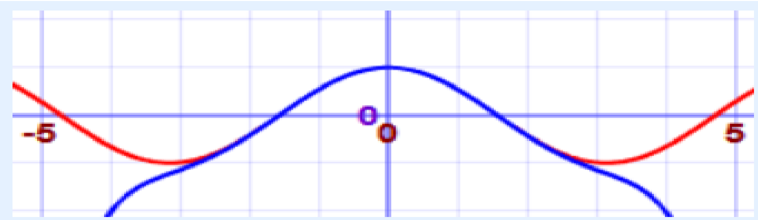
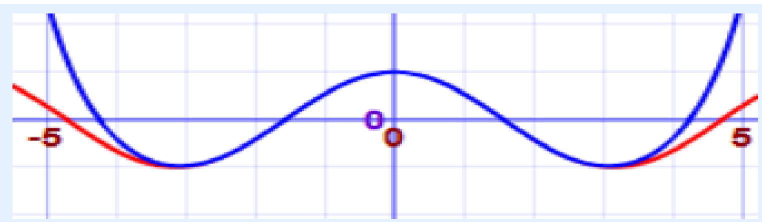
Taylor's series at 0

- f(x)=f(0)+$\frac{f'(0)}{1!}$x+$\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$ (1)
- For f(x)=e^x, f'(x)=f''(x)=f'''(x)=…=e^x
- f(0)= f'(0)=f''(0)=f'''(0)=…=$e^0=1$
- Hence by (1), we have
- e^x=1+x+$\frac{x^2}{2!}$+$\frac{x^3}{3!}+\cdots$
- So e^1 =1+1+$\frac{1}{2}+\frac{1}{6}+\cdots=$2.666
- Actual value of e=2.71

For cos x

- f(x)=cos x, f'(x)=-sin(x), f''(x)=-cos(x), f'''(x)=sin x
- f(0)=1, f'(0)=0, f''(0)=-1, f'''(0)=0....
- So cos x=1-$\frac{x^2}{2!} + \frac{x^4}{4!}$...
- So cos 0.25=1-$\frac{0.25^2}{2} + \frac{0.25^4}{24}$+...=1-0.031+0.0001
- =0.9691
- Actual value=0.9999.

Geometrical interpretation

$1 - x^2/2!$	
$1 - x^2/2! + x^4/4!$	
$1 - x^2/2! + x^4/4! - x^6/6!$	
$1 - x^2/2! + x^4/4! - x^6/6! + x^8/8!$	

Higher order partial derivatives:

- Let $z = f(x, y)$ be a function of two variables and let its first order partial derivatives exist at all point in the domain D of he function f. Then, the first order partial derivatives are also functions of x and y.
- Now, we define the second order partial derivatives if the following limits exists.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_x(x + \Delta x, y) - f_x(x, y)}{\Delta x}\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_x(x, y + \Delta y) - f_x(x, y)}{\Delta y}\right]$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_y(x + \Delta x, y) - f_y(x, y)}{\Delta x}\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_y(x, y + \Delta y) - f_y(x, y)}{\Delta y}\right]$$

- The derivatives f_{xy} and f_{yx} are called mixed derivatives. If f_{xy} and f_{yx} are continuous at a point (x, y), then $f_{xy} = f_{yx}$.

Taylor series theorem: Let $z = f(x, y)$ be a function defined in some domain D in $\mathbb{R}^2$ which has continuous partial derivatives up to $(n+1)$th order in some neighbourhood of a point $P(x_0, y_0)$ in D. Then, for some point $(x_0 + h, y_0 + k)$ in this neighbourhood, we have the following expansion.

$$f(x_0+h, y_0+k)$$

$$= f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0)$$

$$+ R_n = T_n + R_n,$$

where T_n is called the n^{th} degree polynomial approximation to $f(x,y)$ and R_n is called the remainder term given by

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), 0 < \theta < 1.$$

Note-1: For linear approximation, the maximum absolute error in a rectangular region R: $|x - x_0| < \alpha, |y - y_0| < \beta$ is as follows.

$$E_l = \frac{1}{2!}\left[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}\right]$$

Now, let us assume $B_l = max\{|f_{xx}|, |f_{xy}|, |f_{yy}| \text{ for all } x, y \in R\}$. Then, we have

$$|E_l| \leq \frac{B_l}{2}[|x - x_0|^2 + 2|x - x_0||y - y_0| + |y - y_0|^2]$$

$$= \frac{B_l}{2}[|x - x_0| + |y - y_0|]^2 \leq \frac{B_l}{2}[\alpha + \beta]^2.$$

Thus, $|E_l| \leq \frac{B_l}{2}[\alpha + \beta]^2$ (1)

Note-2: For quadratic approximation, the maximum absolute error in a rectangular region R: $|x-x_0|<\alpha, |y-y_0|<\beta$ is as follows.

$$E_q = \frac{1}{3!}\left[(x-x_0)^3 f_{xxx} + 3(x-x_0)^2(y-y_0)f_{xxy} + 3(x-x_0)(y-y_0)^2 f_{xyy} + (y-y_0)^3 f_{yyy}\right]$$

Now, let us assume $B_q = max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}||f_{yyy}| \text{for all } x, y \in R\}$.

Then, we have

$$|E_q| \le \frac{B_q}{6}[|x-x_0|^3 + 3|x-x_0|^2|y-y_0| + 3|x-x_0||y-y_0|^2 + |y-y_0|^3]$$

$$= \frac{B_q}{6}[|x-x_0| + |y-y_0|]^3 \le \frac{B_q}{6}[\alpha+\beta]^3 \qquad (2)$$

Example-1: Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the following function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Also discuss about the continuity of f_{xy} and f_{yx} at $(0,0)$.

Solution: we have

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0,$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0.$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y^3 \Delta x}{\Delta x[\Delta x + y^2]} = y,$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x(\Delta y)^3}{\Delta y[x + \Delta y^2]} = 0.$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 00) - f_y(0,0)}{\Delta x} = 0,$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y} = 1.$$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$ and hence f_{xy} and f_{yx} are not continuous at (0,0).

Example-2: Expand $f(x,y) = 21 + x - 20y + 4x^2 + xy + 6y^2$ in Taylor's series of maximum order about the point $(-1,2)$.

Solution: Since the function f has degree 2 and hence all the terms of order greater than or equal to 3 are zero. Hence, the Taylor's expansion about the point $(-1,2)$ becomes

$$f(x,y) = f(-1,2) + \left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right] f(-1,2) + \frac{1}{2!}\left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right]^2 f(-1,2)$$

Since $x = -1 + h, y = 2 + k$.

we have $f(-1,2) = 6, f_x(-1,2) = -5,$

$f_y(-1,2) = 3, f_{xx}(-1,2) = 8, f_{xy}(-1,2) = 1,$

$f_{yy}(-1,2) = 12.$ Thus, we get

$$f(x,y) = 6 - 5(x+1) + 3(y-2) + 4(x+1)^2$$
$$+(x+1)(y-2) + 6(y-2)^2.$$

Taylor's series

- f(x,y)=f(a,b)+ $f_x(a,b)(x-a)+f_y(a,b)(y-b)$

$+\frac{f_{xx}(a,b)}{2}(x-a)^2+\frac{2f_{xy}(a,b)}{2}(x-a)(y-b)$

$+\frac{f_{yy}(a,b)}{2}(x-0)^2+\ldots$

Example-3: Find the linear and quadratic Taylor's series polynomial approximations to the function $f(x, y) = 2x^3 + 3y^3 - 4x^2y$ about the point $(1,2)$. Obtain the maximum absolute error in the region $R: |x - 1| < 0.01$ and $|y - 2| < 0.1$.

Solution: As like the above example, we can easily compute the following linear and quadratic approximations:

$$f_l(x, y) = 18 - 10(x - 1) + 32(y - 2),$$

$$f_q(x, y) = 18 - 10(x - 1) + 32(y - 2) \\ -2[(x - 1)^2 + 4(x - 1)(y - 2) - 9(y - 2)^2]$$

$$f_l(x,y) = 18 - 10(x-1) + 32(y-2),$$

$$f_q(x,y) = 18 - 10(x-1) + 32(y-2)$$
$$-2[(x-1)^2 + 4(x-1)(y-2) - 9(y-2)^2]$$

Maximum absolute error Type equation here. in the linear approximation $f_l(x,y)$ is as follows.

$$|E_l| \le \frac{B_l}{2}[|x-1| + |y-2|]^2 \le \frac{B_l}{2}[0.01 + 0.1]^2 = 0.00605B_l$$

where $B_l = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}| \text{ for all } x, y \in R\}$.

Now, $\max|f_{xx}| = \max|12x - 8y|$

$$= \max|12(x-1) - 8(y-2) - 4| \le 12 \times 0.01 + 8 \times 0.1 + 4 = 4.92,$$

Similarly, we can compute $\max|f_{xy}| \le 8.08$ and $\max|f_{yy}| \le 37.8$.

Hence, $B_l = 37.8$ and therefore, $|E_l| \le 0.00605 \times 37.8 = 0.23$.

Similarly, using equation (2) in note-2, we have

$$\left|E_q\right| \leq \frac{B_q}{6}\left[|x-1|+|y-2|\right]^3 \leq \frac{B_q}{6}[0.01+0.1]^3$$

$$= \frac{B_q}{6} \times 0.001331,$$

where $B_q = \max\left\{|f_{xxx}|, \left|f_{xxy}\right|, \left|f_{xyy}\right|\left|f_{yyy}\right| \text{for all } x, y \in R\right\} =$

$\max\{12, 8, 0, 18\} = 18.$

Hence, we obtain $\left|E_q\right| \leq \frac{18}{6} \times 0.001331 = 0.004$

Example-4: Find an approximate value of $f(1.1,0.8)$ using Taylor's series quadratic approximation polynomial for the function $f(x,y)=\tan^{-1}(xy)$.

Solution: let $(x_0,y_0)=(1.0,1.0), h=0.1, k=-0.2$. Then, by using the Taylor series quadratic approximation, we get

$$f(1.0+0.1,1.0-0.2) \sim f(1,1)+\left(hf_x+kf_y\right)_{(1,1)}$$

$$+\frac{1}{2}\left(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy}\right)_{(1,1)}=0.7229.$$

- **Homogeneous Functions**
- **Euler's Theorem**

Definition (Homogeneous function):- A function $f(x, y)$ is said to be homogeneous of degree n in x and y if it can be expressed in any one of the following forms:

i. $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

ii. $f(x, y) = x^n g(y/x)$

iii. $f(x, y) = y^n g(x/y)$

Similarly, the above definition can be extended for a function of n variables.

Examples: 1) $x^2 + xy$ 2) $xyz/\left(x^4 + y^4 + z^4\right)$ 3) $\sqrt{x}/\sqrt{x^2 + y^2 + z^2}$

4) $\left(x^2 + y\right)/\left(x + y^2\right)$

The above examples 1, 2 [redacted] are homogeneous of degree 2, [redacted] -1 but the example is not homogeneous.

Theorem (Euler's Theorem):- If $f(x,y)$ is a homogeneous function of degree n in x and y and has continuous first and second order partial derivatives, then

a) $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$ and b) $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f.$

Proof (a): Since $f(x,y)$ is a homogeneous function of degree n in x and y and hence we can write $f(x,y)=x^n g(y/x)$.

Differentiating partially w. r. t. x and y, we get

$$\frac{\partial f}{\partial x}=nx^{n-1}g(y/x)-yx^{n-2}g'(y/x),\qquad \frac{\partial f}{\partial y}=x^{n-1}g'(y/x).$$

Hence, we obtain $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nx^n g(y/x)-yx^{n-1}g'(y/x)+yx^{n-1}g'(y/x)=nf.$

Proof of (b): differentiating the result (a) w. r. t. x and y, we get the followings:

$$x\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}+y\frac{\partial^2 f}{\partial x\partial y}=n\frac{\partial f}{\partial x},\qquad x\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}=n\frac{\partial f}{\partial y}.$$

Then,

$$x\left\{x\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}+y\frac{\partial^2 f}{\partial x\partial y}\right\}+y\left\{x\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}\right\}=n\left\{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right\}$$

$$\Longrightarrow x^2\frac{\partial^2 f}{\partial x^2}+\left\{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right\}+xy\left\{\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial x}\right\}+y^2\frac{\partial^2 f}{\partial y^2}=n\left\{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right\}$$

which clearly shows the result (b) since $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$ and $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$.

Example-1: If $u(x,y) = cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right), 0 < x < 1,$ then show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -\frac{1}{2}\cot u.$

Solution: For all $x,\ y,\ 0 < x,\ y < 1, (x+y)/\left(\sqrt{x}+\sqrt{y}\right) < 1$, so that $u(x,y)$ is defined. The given function can be written as

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}\left(1+\sqrt{y/x}\right)} = \sqrt{x}\frac{1+y/x}{1+\sqrt{y/x}}.$$

Therefore, $\cos u$ is a homogeneous function of degree $1/2$. Now, using Euler's theorem for homogeneous function for the function $f = \cos u$ and $n = 1/2$, we get

$$x\frac{\partial \cos u}{\partial x} + y\frac{\partial \cos u}{\partial y} = \frac{1}{2}\cos u$$

$$-x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u\,.$$

Example-2:

If $u(x,y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, $x, y > 0$, then evaluate $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$.

Solution: Here we observed that $u(\lambda x, \lambda y) = \lambda^2 u(x,y)$. Therefore, $u(x,y)$ is a homogeneous function of degree 2. Using the second result of Euler's theorem, we obtain

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u.$$

Example-3: If $u(x,y) = \frac{x^3+y^3}{x+y}, (x,y) \neq (0,0)$, then evaluate $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}$.

Solution: Here we observed that $u(x,y) = x^2\left[1 + (y/x)^3\right]/[1 + (y/x)]$. Therefore, $u(x,y)$ is a homogeneous function of degree 2. Using Euler's theorem, we obtain

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

Differentiating partially w. r. t. x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x} \Longrightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} = 0.$$

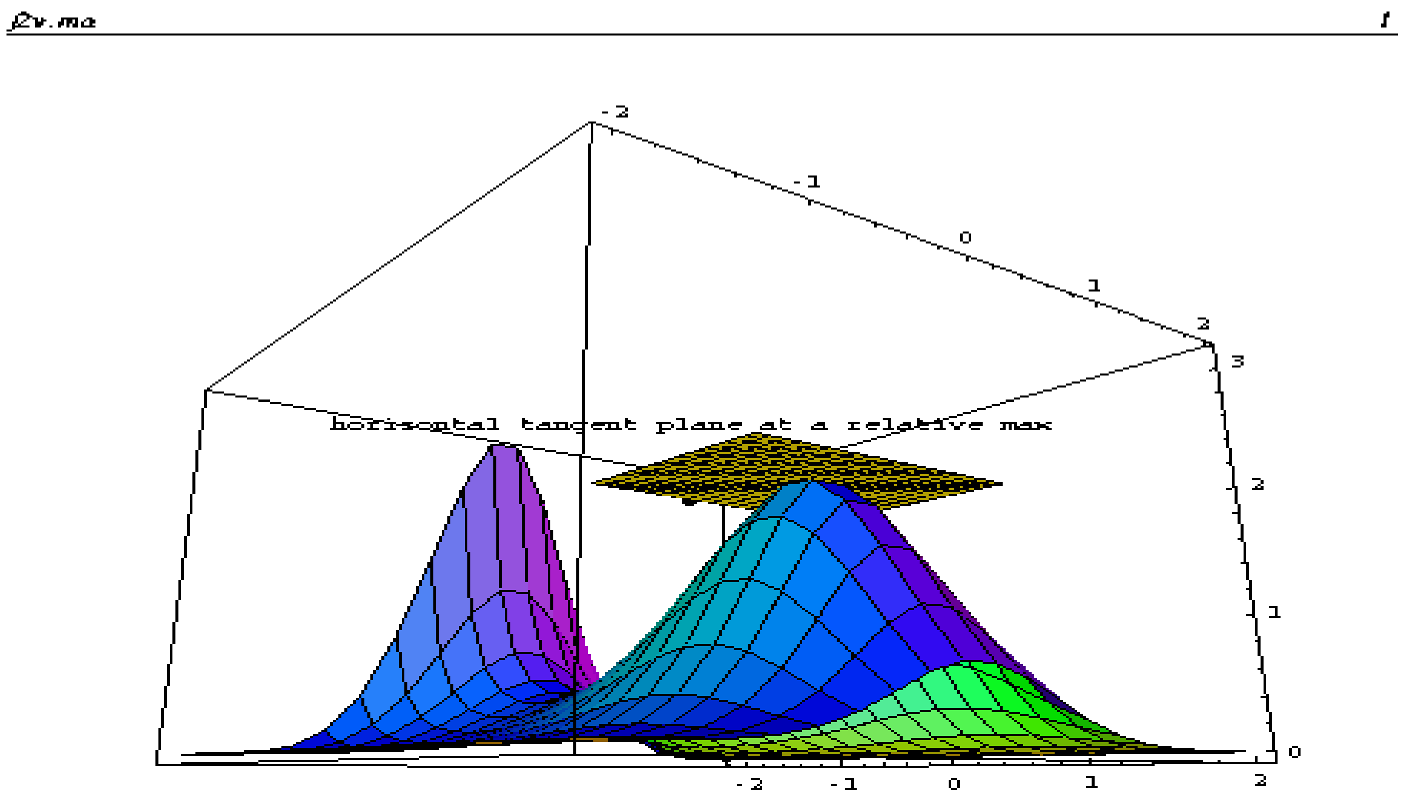
- **Maximum and minimum values of a function**
- **Langrange’s Method of Multiplier**

Definition (Local maximum and Local minimum):- Let a function $f(x, y)$ be defined in some closed and bounded domain R. Let (a, b) be an interior point of R and $(a + h, b + k)$ be a point in its neighborhood and lies inside in R. Then, we define the following definitions.

The point (a, b) is called a point of local (or relative) minimum, if $f(a + h, b + k) \geq f(a, b)$ for all h, k. Then, $f(a, b)$ is called a local (or relative) minimum.

The point (a, b) is called a point of local (or relative) maximum, if $f(a + h, b + k) \leq f(a, b)$ for all h, k. Then, $f(a, b)$ is called a local (or relative) maximum.

Geometrical interpretation



Note: 1) The smallest and the largest values attained by a function over the entire region including the boundary are called absolute (or global) minimum and maximum values respectively.

2) The points at which minimum/maximum values of the function occur are also called points of extrema and the minimum and maximum values taken together are called the extrema values of the function.

Theorem 2.6: (Necessary conditions for a function to have an extremum) Let the function $f(x, y)$ be continuous and possess first order partial derivatives at a point $P(a, b)$. Then, the necessary conditions for the existence of an extreme value of f at the point P are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Condition for maxima and minima

- f(a+h,b+k)=f(a,b)+$f_x(a,b)h + f_y(a,b)k + \frac{1}{2}\left(f_{xx}h^2 + f_{yy}k^2 + 2hkf_{xy}\right) + \cdots$
- f(a+h,b+k)-f(a,b)= $\frac{1}{2}\left(f_{xx}h^2 + f_{yy}k^2 + 2hkf_{xy}\right)|_c$

Let f_{xx}=A, f_{xy}=B, f_{yy}=C,

A $h^2 + Ck^2 + 2hkB$=A(h^2+$\frac{2B}{A}hk$) +Ck^2

= A$\left[\left(\text{h}+\frac{B}{A}k\right)^2 - \frac{B^2}{A^2}k^2\right] + Ck^2 =$ A$\left[\left(\text{h}+\frac{B}{A}k\right)^2\right]$- $\frac{B^2}{A}k^2 + \frac{CA}{A}k^2$

A$\left[\left(\text{h}+\frac{B}{A}k\right)^2\right]$+$\left(\frac{CA-B^2}{A}\right)k^2 = A\left[\left(\text{h}+\frac{B}{A}k\right)^2\right]$+$\left(\frac{CA-B^2}{A}\right)k^2$

Theorem 2.7: (Sufficient conditions for a function to have an extremum) Let the function $f(x, y)$ be continuous and possess first and second order partial derivatives at a point $P(a, b)$. If $P(a, b)$ is a critical point, then the point P is appoint of

relative minimum if $rt - s^2 > 0$ and $r > 0$,

relative maximum if $rt - s^2 > 0$ and $r < 0$,

where $r = f_{xx}(a, b), s = f_{xy}(a, b)$ and $t = f_{yy}(a, b)$.

No conclusion about an extremum can be drawn if $rt - s^2 = 0$ or $r = t = s = 0$ and further investigation is required by taking the terms involving higher order derivatives. If $\boldsymbol{rt - s^2 < 0}$, then the function f has no minimum or maximum at this point. In this case, the point is called ***saddle point***.

Example-1: Find the absolute maximum and minimum values of $f(x,y) = 4x^2 + 9y^2 - 8x - 12y + 4$ over the rectangle in the first quadrant bounded by the lines $x = 2, y = 3$ and the coordinate axes.

Solution: We have $f_x = 8x - 8 = 0, f_y = 18y - 12 = 0$. The critical point is $(x,y) = (1, 2/3)$. Now, $r = f_{xx} = 8,\ s = f_{xy} = 0,\ t = f_{yy} = 18,\ rt - s^2 = 144$. Since $rt - s^2 > 0$ and $r > 0$, the point $(1, 2/3)$ is a point of relative minimum and the minimum value is $f(1, 2/3) = -4$.

Now, to find the absolute maximum/minimum values, it is required to analyze on the boundary of the region (i.e. the rectangle OABC, where the coordinates of O, A, B and C are (0,0), (2,0), (2,3) and (0,3) respectively).

On the line OA: Here, $y = 0 \Longrightarrow f(x, y) = g(x) = 4x^2 - 8x + 4$.

Now, setting $g'(x) = 0 \Longrightarrow 8x - 8 = 0 \Longrightarrow x = 1$. $g''(x) = 8 > 0$. Hence, at $x = 1$, the function has a minimum value which is $g(1) = 0$ and at the corners $(0,0)$, $(2,0)$, we have $f(0,0) = g(0) = 4, f(2,0) = g(2) = 4$.

Similarly, along the other boundary lines, we have the following results:

On the line $\boldsymbol{x = 2}$: Here, $g(y) = 9y^2 - 12y + 4$.

Now, setting $g^{'}(y) = 0 \Longrightarrow y = 2/3$. $g^{''}(y) = 18 > 0$. Hence, at $y = 2/3$, the function has a minimum value which is $f(2, 2/3) = 0$ and at the corner $(2, 3)$, we have $f(2,3) = 49$.

On the line $\boldsymbol{x = 3}$: Here, $g(x) = 4x^2 - 8x + 49$.

Now, setting $g^{'}(x) = 0 \Longrightarrow x = 1$. $g^{''}(y) = 8 > 0$. Hence, at $x = 1$, the function has a minimum value which is $f(1, 3) = 0$ and at the corner $(0, 3)$, we have $f(0,3) = 49$.

On the line $\boldsymbol{x = 0}$: Here, $g(y) = 9y^2 - 12y + 4$ which is a same case for $x = 2$.

Therefore, the absolute minimum value is -4 which occurs at the point $(1, 2/3)$ and the absolute maximum value is 49 at the points $(2, 3)$ and $(0, 3)$.

Example-2: Find the absolute extrema of $f(x, y) = 2x^2 + y^2 - 2x - 2y - 4$ over the region bounded by the lines $x = 0, y = 0$ and $2x + y = 1$.

Solution: We have $f_x = 4x - 2 = 0, f_y = 2y - 2 = 0$. The critical point is $(x, y) = (1/2, 1)$ which is not inside the region. So, the extrema may exist on the boundary of the region. Now, to find the absolute maximum/minimum values, it is required to analyze on the boundary of the region (i.e. the triangle OAB, where the coordinates of O, A and B are (0,0), (1/2,0) and (0,1) respectively).

On the line OA: Here, $y = 0 \Longrightarrow f(x,y) = g(x) = 2x^2 - 2x - 4$.

Now, setting $g'(x) = 0 \Longrightarrow 4x - 2 = 0 \Longrightarrow x = 1/2$. $g''(x) = 4 > 0$. Hence, at $x = 1/2$, the function has a minimum value which is $f(1/2, 0) = g(1/2) = -9/2$ and at the corner $(0,0)$, we have $f(0,0) = -4$.

Similarly, along the other boundary lines, we have the following results:

On the line $2x + y = 1$ i.e. $y = 1 - 2x$: Here, $g(x) = 6x^2 - 2x - 5$.

Now, setting $g'(x) = 0 \Longrightarrow x = 1/6$. $g''(x) = 12 > 0$. Hence, at $x = 1/6$, the function has a minimum value which is $f(1/6, 2/3) = -31/6$.

On the line $x = 0$: Here, $g(y) = y^2 - 2y - 4 \Longrightarrow g^{'}(y) = 2y - 2 = 0 \Longrightarrow y = 1$. $g^{''}(y) = 2 > 0$. Hence, at $y = 1$, the function has a minimum value which is $f(0,1) = -4$.

Therefore, the absolute minimum value is $-31/6$ which occurs at the point $(1/6, 2/3)$.

Example-3: Find the absolute maximum and minimum values of $f(x, y) = 3x^2 + y^2 - x$ over the region $2x^2 + y^2 \leq 1$.

Solution: We have $f_x = 6x - 1 = 0, f_y = 2y = 0$. The critical point is $(x, y) = (1/6, 0)$. Now, $r = f_{xx} = 6$, $s = f_{xy} = 0$, $t = f_{yy} = 2$, $rt - s^2 = 12$. Since $rt - s^2 > 0$ and $r > 0$, the point $(1/6, 0)$ is a point of minimum and the minimum value at this point is $f(1/6, 0) = -1/6$.

On the boundary of the region, we have $y^2 = 1 - 2x^2$; $-1/\sqrt{2} \leq x \leq 1/\sqrt{2}$. Substituting in $f(x, y)$, we obtain $f(x, y) = 3x^2 + \left(1 - 2x^2\right) - x = 1 - x + x^2 = g(x)$ which is a function of one variable. Setting $dg/dx = 0$, we get

$$\frac{dg}{dx} = 2x - 1 \Rightarrow x = \frac{1}{2} \text{ and also } \frac{d^2g}{dx^2} = 2 > 0.$$

For $x = \frac{1}{2}$, we get $y^2 = 1 - 2x^2 = 1/2$ i.e. $y = \pm 1/\sqrt{2}$. Hence, the points $\left(1/2\,, 1/\sqrt{2}\right)$ are the points of minimum and the minimum value is $f\left(1/2\,, 1/\sqrt{2}\right) = 3/4$. At the vertices, we have $f\left(1/\sqrt{2}\,, 0\right) = \left(3 - \sqrt{2}\right)/4$, $f\left(-1/\sqrt{2}\,, 0\right) = \left(3 + \sqrt{2}\right)/4$ and $f(0, \pm 1) = 1$. Hence the given function has absolute minimum value $-1/12$ at $(1/6\,, 0)$ and absolute maximum value $\left(3 + \sqrt{2}\right)/4$ at $\left(-1/\sqrt{2}\,, 0\right)$.

Example-4: Find the relative maximum/minimum values of the function $f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$.

Solution: We have $f_x = 4x^3 - 4yz = 0, f_y = 4y^3 - 4xz = 0, f_z = 4z^3 - 4xy = 0$. Therefore, $x^3 = yz, y^3 = xz, z^3 = xy \Longrightarrow x^3y^3z^3 = x^2y^2z^2 \Longrightarrow x^2y^2z^2(xyz - 1) = 0$. Therefore, all points which satisfy $xyz = 0$ or $xyz = 1$ are critical points an dthe solutions of these equations are $(0,0,0)$, $(1,1,1)$, $(\pm1,\pm1,1)$, $(1,\pm1,\pm1)$, $(\pm1,1,\pm1)$ with the same sign taken for the two coordinates. Now $f_{xx} = 12x^2$, $f_{xy} = -4z$, $f_{xz} = -4y$, $f_{yz} = -4x$, $f_{yy} = 12y^2$, $f_{zz} = 12z^2$. At $(0,0,0)$, all the second order partial derivatives are zero. Therefore, no conclusion can be drawn.

Now, we have $A = \begin{bmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{bmatrix}$

Depending on whether A or $B = -A$ is positive definite, we can decide the points of minimum or maximum. The leading minors are

$$M_1 = 12x^2, M_2 = 16\left(9x^2y^2 - z^2\right), M_3$$
$$= 192\left(9x^2y^2z^2 - x^4 - y^4 - z^4\right) - 12xyz.$$

At all the above mentioned points, we find that $M_1, M_2, M_3 > 0$. Hence, A is a positive definite matrix and the given function has relative minimum at all these points, since $f_{xx},\ f_{yy},\ f_{zz} > 0$. The relative minimum values at all these points is same and is given by $f(1,1,1) = -1$.

- **Langrange's Method of Multiplier**

inadequate of substitution method

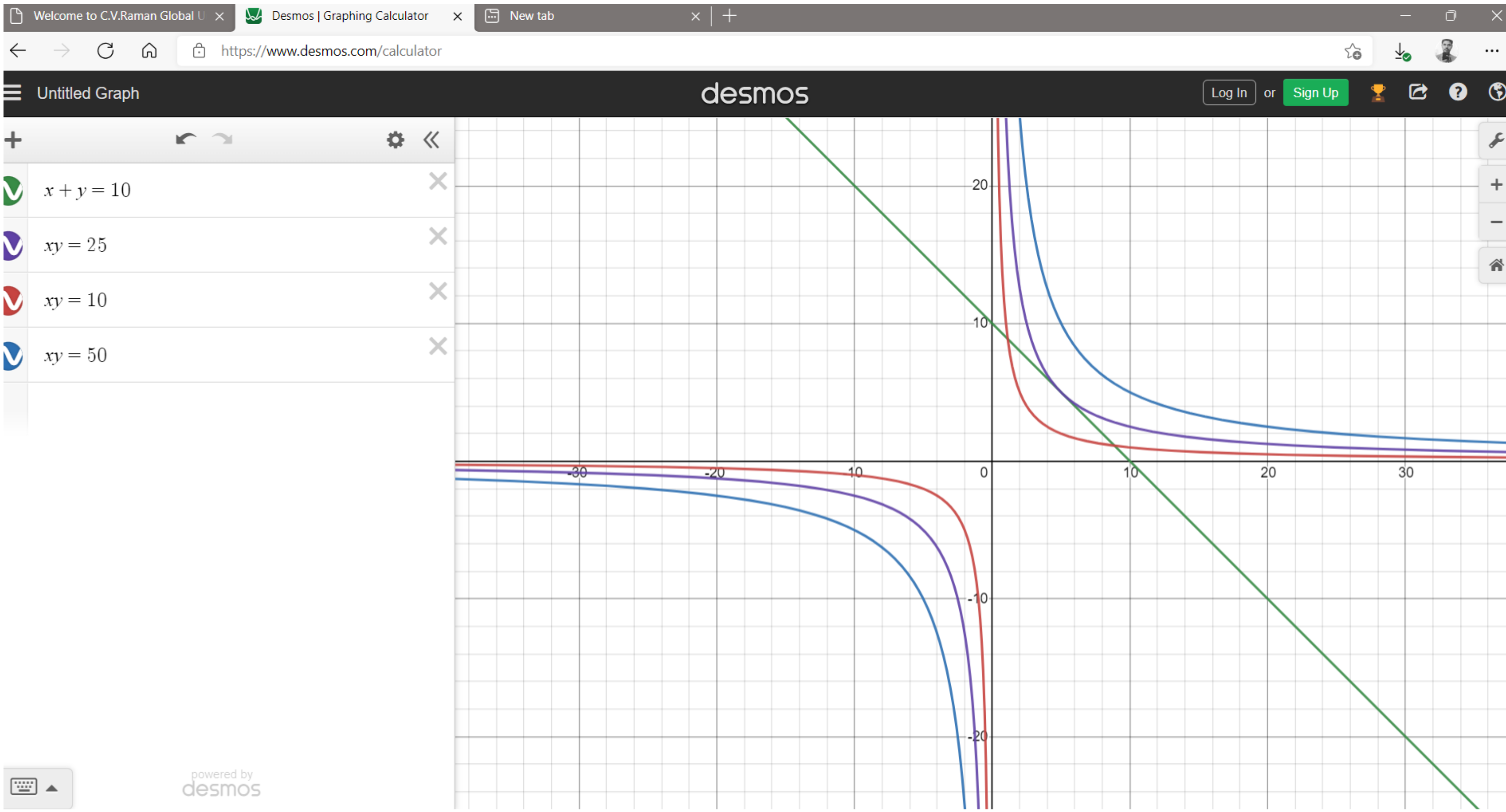
Assume the objective is to maximize

$$f(x, y) = x.y$$

subject to $x + y = 10$.

the constraint implies $y = 10 - x$, which can be substituted into the objective function to create $p(x) = x(10 - x) = 10x - x^2$. The first order necessary condition gives $\frac{\partial p}{\partial x} = 10 - 2x = 0$. which can be solved for x=5 and consequently y=10-5=5.

Geometrical interpretation



Langrange's Method of Multiplier

In many practical problems, we need to find the minimum/maximum value of a function $f(x_1, x_2, \dots, x_n)$ when the variables are not independent but are interconnected by one or more constraints of the form

$$\phi_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, k \qquad (1)$$

where generally $n > k$. That is why Lagrange's method of multipliers is necessary.

Method: To find the extremum of the function $f(x_1, x_2, \dots, x_n)$ under the conditions

$$\phi_i(x_1, x_2, \dots, x_n) = 0, i = 1, 2, \dots, k.$$

For this, we construct an auxiliary function of the form

$$F(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_k)$$

$$= f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{k} \lambda_i \phi_i(x_1, x_2, \dots, x_n),$$

where λ_i's are undetermined parameters and are known as Lagrange multipliers.

Then, to determine the stationary points, we consider the following conditions

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \cdots = \frac{\partial F}{\partial x_n}$$

which gives the following set of equations:

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0, \qquad j = 1, 2, \dots, n. \tag{2}$$

Form equations (1) and (2), we obtain $(n+k)$ equations in $(n+k)$ unknowns $x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_k$. Solving these equations we get the required stationary points $(x_1, x_2, \dots, x_n)$ at which the function is extremum.

Example-1: Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Solution: Consider the following auxiliary function

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda\left(xyz - a^3\right).$$

We obtain the necessary conditions for extremum as

$$\frac{\partial F}{\partial x} = 2x + \lambda yz = 0, \qquad \frac{\partial F}{\partial y} = 2y + \lambda xz = 0,$$

$$\frac{\partial F}{\partial z} = 2z + \lambda xy = 0.$$

From these equations, we obtain $\lambda xyz = -2x^2$, $\lambda xyz = -2y^2$, $\lambda xyz = -2z^2$.

Therefore, $x^2 = y^2 = z^2$. Now, using the condition $xyz = a^3$, we obtain the solutions as (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$, $(-a, -a, a)$. At each of these points, the value of the given function is $x^2 + y^2 + z^2 = 3a^2$.

Now, the arithmetic mean and geometric mean of $x^2,\ y^2, z^2$ are $\left(x^2 + y^2 + z^2\right)/3$ and $\left(x^2 y^2 z^2\right)^{1/3} = a^2$. Since $A.M \geq G.M.$, we obtain $x^2 + y^2 + z^2 \geq 3a^2$. Hence, all the points are the points of constrained minimum and the minimum value of $x^2 + y^2 + z^2$ is $3a^2$.

Example-2: Find the extreme values of $f(x,y,z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.

Solution: Consider the auxiliary function

$$F(x,y,z,\lambda_1,\lambda_2) = 2x + 3y + z + \lambda_1\left(x^2 + y^2 - 5\right) + \lambda_2(x + z - 1).$$

For the extremum, we have the following necessary conditions

$$\frac{\partial F}{\partial x} = 2 + 2\lambda_1 x + \lambda_2 = 0, \frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0, \frac{\partial F}{\partial z} = 1 + \lambda_2 = 0.$$

$$\Longrightarrow \lambda_2 = -1, \qquad 3 + 2\lambda_1 y = 0, \qquad 1 + 2\lambda_1 x = 0$$

$$\Longrightarrow x = -1/(2\lambda_1), \qquad y = -3/(2\lambda_1).$$

Substituting in the constraint $x^2 + y^2 = 5$, we get $\lambda_1 = \pm 1/\sqrt{2}$.

For $\lambda_1 = 1/\sqrt{2}$:

we have $x = -\sqrt{2}/2$, $y = -3\sqrt{2}/2$, $z = 1 - x = (2 + \sqrt{2})/2$

and $f(x, y, z) = 1 - 5\sqrt{2}$.

Now, for $\lambda_1 = -1/\sqrt{2}$:

we have $x = \sqrt{2}/2$, $y = 3\sqrt{2}/2$, $z = 1 - x = (2 - \sqrt{2})/2$

and $f(x, y, z) = 1 + 5\sqrt{2}$.

Example-2: Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution: Let (x, y) be a point on the ellipse and (u, v) be a point on the line. Then, the shortest distance between the line and the ellipse is the square root of the minimum value of

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2$$

subject to the constraints

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \text{ and } \phi_2(u, v) = 2u - v - 10 = 0.$$

Now, we define the auxiliary function as

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right)$$
$$+\lambda_2(2u - v - 10).$$

For extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2(x-u) + \lambda_1 \frac{x}{2} = 0 \Longrightarrow \lambda_1 x = 4(u-x),$$

$$\frac{\partial F}{\partial y} = 2(y-v) + \lambda_1 \frac{2y}{9} = 0 \Longrightarrow \lambda_1 y = 4(v-y),$$

$$\frac{\partial F}{\partial u} = -2(x-u) + 2\lambda_2 = 0 \Longrightarrow \lambda_2 = x-u,$$

$$\frac{\partial F}{\partial v} = -2(y-v) + \lambda_2 = 0 \Longrightarrow \lambda_2 = 2(y-v).$$

Eliminating λ_1 and λ_2 from the above equations, we get

$$4(u-x)y = 9(v-y)x \text{ and } x-u = 2(y-v).$$

Dividing the two equations, we obtain $8y = 9x$. Substituting in equation of the ellipse, we get

$$\frac{x^2}{4} + \frac{9x^2}{64} = 1 \Longrightarrow x^2 = \frac{64}{25} \Longrightarrow x = \pm 8/5 \ and \ y = \pm 9/5.$$

Corresponding to $x = 8/5$ and $y = 9/5$, we get

$$\frac{8}{5} - u = 2\left(\frac{9}{5} - v\right) \Longrightarrow u = 2v - 2$$

Substituting in the equation of the line $2u + v - 10 = 0$, we get $u = 18/5 \ and \ v = 14/5$. Hence, an extremum is obtained when $(x, y) = (8/5, 9/5)$ and $(u, v) = (18/5, 14/5)$. So, the distance between the two points is $\sqrt{5}$.

Similarly, another extremum can be found out when $(x, y) = (-8/5, -9/5)$ and $(u, v) = (22/5, 6/5)$. So, the distance between the two points is $3\sqrt{5}$.

Therefore, the shortest distance between the line and ellipse is $\sqrt{5}$.