DEPARTMENT OF MATHEMATICS

MATHEMATICS - I

FOR BTECH 1st SEMESTER COURSE [COMMON TO ALL BRANCHES OF ENGINEERING]

TEXT BOOK: ADVANCED ENGINEERING MATHEMATICS BY ERWIN KREYSZIG

LECTURES – 27 & 28

Line Integrals
[Chapters – 9.1,9.2]

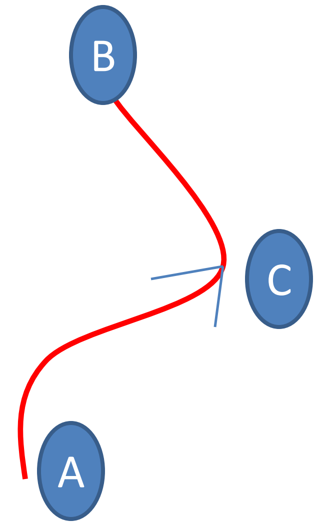
Content:

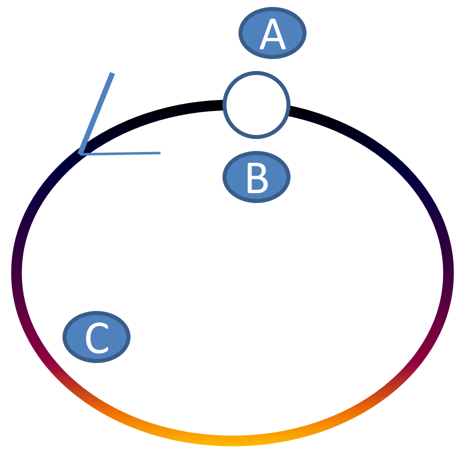
- Introduction to Line Integral
- Physical Application of Line Integrals
- Definition and Examples
- Properties
- Examples
- Test Knowledge
- Independent of Path
- Problem Solved
- Practice Problems

INTRODUCTION TO LINE INTEGRAL

Let C be a curve with a parametric representation

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$





The curve *C* is called <u>*path of integration*</u>, *A*: $\vec{r}(a)$ its initial point and *B*: $\vec{r}(b)$ its terminal point of the path of integration *C*. The direction from *A* to *B* in which t increases is called <u>*the positive direction*</u> on *C*. We can indicate the direction by arrow. If the point A and B coincide, then C is called a <u>*closed path*</u>.

INTRODUCTION TO LINE INTEGRAL

The curve *C* is called a ***smooth curve***, if *C* has a unique tangent at each of its points whose direction varies continuously as we move along *C*. Mathematically, *C* has representation

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Such that $\vec{r}(t)$ is differentiable and the derivative $\dfrac{d\vec{r}}{dt}$ is continuous and different from the zero vector at every point of *C*.

Physical Applications of Line Integrals

In physics, the line integrals are used, in particular, for computations of

- mass of a wire;
- center of mass and moments of inertia of a wire;
- work done by a force on an object moving in a vector field;
- magnetic field around a conductor (Ampere's Law);
- voltage generated in a loop (Faraday's Law of magnetic induction).

Definition and Evaluation of line Integral:

A line integral of a vector function $\vec{F}(\vec{r})$ over a curve *C* is defined by

$$\boxed{\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} dt --(1)}$$

Also interms of cartesian Coordinates with $d\vec{r} = [dx,\ dy,\ dz]$

we have from the above equation

$$\boxed{\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt} -(2)$$

Definition and Evaluation of line Integral:

SOME QUICK REMARKS

- If a path of integration C is a closed curve, then may also denote a line integral as $\oint_C$ $\left(\text{instead of} \int_C \right)$.

- Line integral arises naturally in mechanics in computation of work done by a fource in a displace along a given path.As such a line integral is also called a *work integral*.

Definition and Evaluation of line Integral:

- Line integral (2) is a definite integral of a function of t taken over the interval $a \leq t \leq b$ on the $t-axis$ in the *positive* direction. This definite integral exists if $\vec{F}$ is *continuous* and C is *piecewise smooth*, because this makes $\vec{F} \bullet \dfrac{d\vec{r}}{dt}$ *piecewise continous*.
- Line integral (2) with given $\vec{F}$ and C is *independent of of the choice of representation of* C.
- Line integral (2) with given $\vec{F}$ and given a and b is different along different paths joing a to b. That is the line integral (2) depends on C.

General Properties of Line Integral:

1. $\int_C k\vec{F}(\vec{r}).d\vec{r} = k\int_C \vec{F}(\vec{r})d\vec{r}$ where k is a constant

2. For vector functions $\vec{G}_1, \vec{G}_2, \vec{G}_3, \cdots, \vec{G}_n$ whose line integral of the form (1) exists and $a_1,\ a_2,\ a_3,\ \cdots a_n$ arbitrary constatns

$$\int_C \left(a_1\vec{G}_1(\vec{r}) + a_2\vec{G}_2(\vec{r}) + a_3\vec{G}_3(\vec{r}) + \cdots + a_n\vec{G}_n(\vec{r})\right) \bullet d\vec{r}$$

$$= a_1\int_C \vec{G}_1(\vec{r}) \bullet d\vec{r} + a_2\int_C \vec{G}_2(\vec{r}) \bullet d\vec{r} + a_3\int_C \vec{G}_3(\vec{r}) \bullet d\vec{r} + \cdots + a_n\int_C \vec{G}_n(\vec{r}) \bullet d\vec{r}$$

General Properties of Line Integral:

$$3.\int_{C} \vec{F}(\vec{r}).d\vec{r} = \int_{C_1} \vec{F}(\vec{r}).d\vec{r} + \int_{C2} \vec{F}(\vec{r}).d\vec{r} + \cdots + \int_{C_n} \vec{F}(\vec{r}).d\vec{r}$$

where C is the sum of the curve $C_1, C_2, C_3, \cdots C_n$.

$$4.\int_{a}^{b} \vec{F}(\vec{r}).d\vec{r} = -\int_{b}^{a} \vec{F}(\vec{r}).d\vec{r}$$

Other Forms of Line Integrals:

If we set $\vec{F} = F_1 \hat{i}$ and $F_1 = \dfrac{f}{\left(\dfrac{dx}{dt}\right)}$, so that $f = F_1\left(\dfrac{dx}{dt}\right)$,

(2) gives another form of line integral given by

$$\int_C f(\vec{r})dt = \int_a^b f\left(\vec{r}(t)\right) dt ------(3)$$

Evaluation of integral $\int_C f(\vec{r})ds$ with Arc length as parameter :

$$\int_C f(\vec{r})ds = \int_C f\{x(t), y(t), z(t)\}\left(\frac{ds}{dt}\right)dt$$

$$= \int_C f\left(\vec{r}(t)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}dt ---(4)$$

Where

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}$$

If limits of t along C are a and b, then

$$\int_C f(\vec{r})ds = \int_a^b f\{x(t), y(t), z(t)\}\left(\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\right)dt$$

Some Problems of Line Integral:

PROBLEM - 1

Calculate $\int_C \vec{F}(\vec{r}) \bullet d\vec{r}$, where $\vec{F} = \left[y^2, \quad -x^2 \right]$ and C is the straight line segment from $(0,0)$ to $(1,4)$.

SOLUTION:

Equation of the straight line segment from $(0,0)$ to $(1,4)$ is $y = 4x$.

$$\text{Let } x = t \Rightarrow y = 4t.$$

$$\text{Here } \vec{r}(t) = x\hat{i} + y\hat{j} = t\hat{i} + 4t\hat{j}$$

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t\hat{i} + 4t\hat{j}\right) = \hat{i} + 4\hat{j}$$

Some Problems of Line Integral:

At $(0,0)$, $t=0$ and at $(1,4)$, $t=1$.

Given that $\vec{F}=\left[y^2,\quad -x^2\right]$.

$$\Rightarrow \vec{F}\left(\vec{r}\left(t\right)\right)=\left[\left(4t\right)^2,\quad -t^2\right]=\left[16t^2,\quad -t^2\right].$$

$$\vec{F}\left(\vec{r}\left(t\right)\right)\bullet\frac{d\vec{r}}{dt}=\left(16t^2\hat{i}-t^2\hat{j}\right)\bullet\left(\hat{i}+4\hat{j}\right)$$

$$=16t^2-4t^2=12t^2$$

Some Problems of Line Integral:

Therefore, the required line integral is

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} dt$$

$$= \int_0^1 12t^2 dt = 12\int_0^1 t^2 dt$$

$$= 12\left[\frac{t^3}{3}\right]_0^1 = 12\left[\frac{1}{3} - 0\right] = 4.$$

Some Problems of Line Integral:

PROBLEM - 2

Calculate $\int_C \vec{F}(\vec{r}) \bullet d\vec{r}$, where $\vec{F} = \left[\cosh x, \quad \sinh y, \quad e^z\right]$

and $C : \vec{r} = \left[t, \quad t^2, \quad t^3\right]$ from $(0,0,0)$ to $(2,4,8)$.

SOLUTION: Here $\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t\hat{i} + t^2\hat{j} + t^3\hat{k}\right) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

At $(0,0,0)$, $\left(t, t^2, t^3\right) = (0,0,0) \Rightarrow t = 0$

At $(2,4,8)$, $\left(t, t^2, t^3\right) = (2,4,8) \Rightarrow t = 2.$

Some Problems of Line Integral:

Given that $\vec{F} = \left[\cosh x, \quad \sinh y, \quad e^{z}\right].$

$$\Rightarrow \vec{F}\left(\vec{r}\left(t\right)\right) = \left[\cosh t, \quad \sinh t^{2}, \quad e^{t^{3}}\right]$$

$$\vec{F}\left(\vec{r}\left(t\right)\right) \bullet \frac{d\vec{r}}{dt} = \left(\cosh t\hat{i} + \sinh t^{2}\hat{j} + e^{t^{3}}\hat{k}\right) \bullet \left(\hat{i} + 2t\hat{j} + 3t^{2}\hat{k}\right)$$

$$= \cosh t + 2t\sinh t^{2} + 3t^{2}e^{t^{3}}$$

Some Problems of Line Integral:

Therefore, the required line integral is

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_0^2 \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} dt$$

$$= \int_0^2 \left(\cosh t + 2t \sinh t^2 + 3t^2 e^{t^3} \right) dt$$

$$= \left[\sinh t + \cosh t^2 + e^{t^3} \right]_0^2$$

$$= \left[\left(\sinh 2 + \cosh 4 + e^8 \right) - (0 + 1 + 1) \right]$$

$$= \sinh 2 + \cosh 4 + e^8 - 2$$

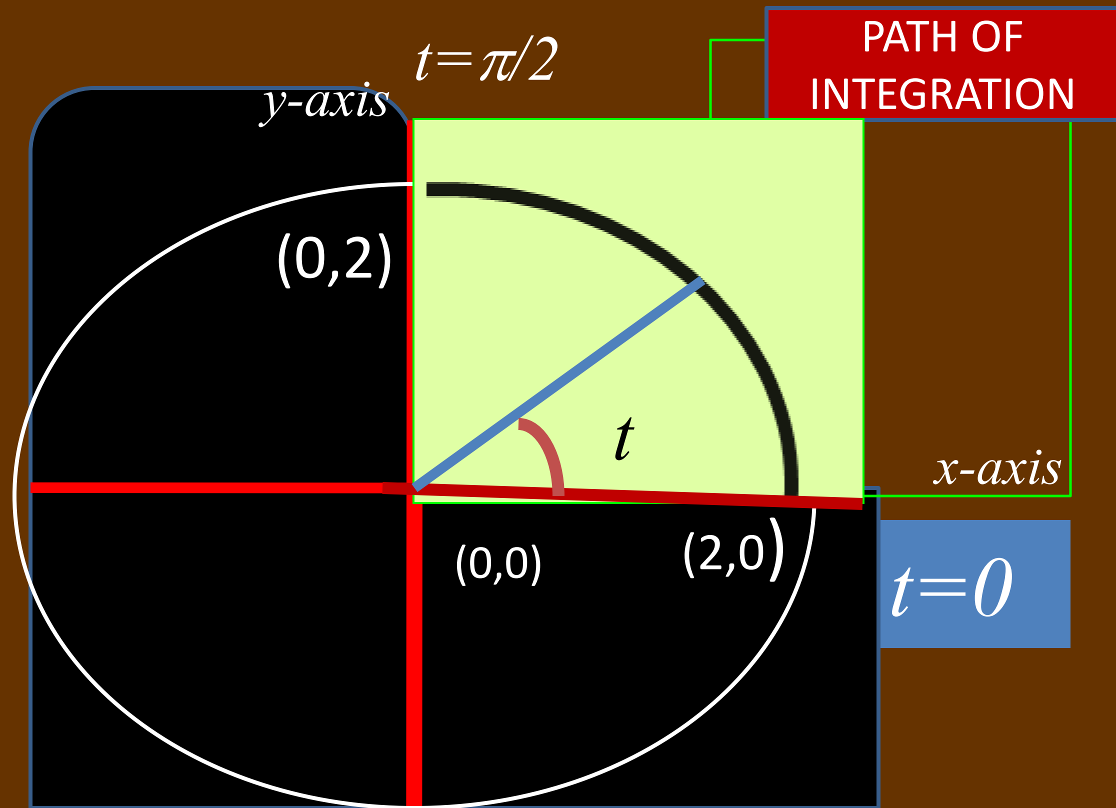
Some Problems of Line Integral:

PROBLEM - 3

Calculate $\int_C \vec{F}(\vec{r}) \bullet d\vec{r}$, where $\vec{F} = \left[xy, \quad x^2 y^2 \right]$ and C is the quarter - circle from $(2,0)$ to $(0,2)$ with center at $(0,0)$.

SOLUTION:

The radius of the circle with center at (0,0) with points *(2,0)* and *(0,2)* on it is 2 .



Some Problems of Line Integral:

Parametric representation of the quarter -circle from $(2,0)$ to $(0,2)$ with center at $(0,0)$ and radius 2 is

$$\vec{r}(t) = 2\cos t\hat{i} + 2\sin t\hat{j}, 0 \le t \le \frac{\pi}{2}.$$

$$\therefore \frac{\mathrm{d}\vec{r}}{dt} = \frac{d}{dt}\left(2\cos t\hat{i} + 2\sin t\hat{j}\right) = -2\sin t\hat{i} + 2\cos t\hat{j}$$

Given that $\vec{F} = \left[xy, \quad x^2y^2\right].$

$$\Rightarrow \vec{F}\left(\vec{r}(t)\right) = \left[4\sin t\cos t, \quad 16\sin^2 t\cos^2 t\right]$$

Some Problems of Line Integral:

$$\therefore \vec{F}\left(\vec{r}(t)\right)\bullet\frac{d\vec{r}}{dt}=\left(4\sin t\cos t\hat{i}+16\sin^2 t\cos^2 t\hat{j}\right)\bullet\left(-2\sin t\hat{i}+2\cos t\hat{j}\right)$$

$$=-8\sin^2 t\cos t+32\sin^2 t\cos^3 t$$

Therefore, the required line integral is

$$\int_C \vec{F}(\vec{r})\bullet d\vec{r}=\int_0^{\frac{\pi}{2}}\vec{F}\left(\vec{r}(t)\right)\bullet\frac{d\vec{r}}{dt}dt$$

$$=\int_0^{\frac{\pi}{2}}\left(-8\sin^2 t\cos t+32\sin^2 t\cos^3 t\right)dt$$

$$=-8\int_0^{\frac{\pi}{2}}\sin^2 t\cos t\,dt+32\int_0^{\frac{\pi}{2}}\sin^2 t\cos^3 t\,dt$$

Some Problems of Line Integral:

$$= -8\left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}} + 32\int_0^{\frac{\pi}{2}} \sin^2 t(1-\sin^2 t)\cos t\,dt$$

$$= -8\left[\frac{1}{3}-0\right] + 32\int_0^{\frac{\pi}{2}} \sin^2 t\cos t\,dt - 32\int_0^{\frac{\pi}{2}} \sin^4 t\cos t\,dt$$

$$= -\frac{8}{3} + 32\left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}} - 32\left[\frac{\sin^5 t}{5}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{8}{3} + 32\left[\frac{1}{3}-0\right] - 32\left[\frac{1}{5}-0\right]_0^{\frac{\pi}{2}} = -\frac{8}{3} + \frac{32}{3} - \frac{32}{5} = 8 - \frac{32}{5} = \frac{8}{5}$$

TEST YOUR KNOWLEDGE

Calculate $\int \vec{F}(\vec{r}).d\vec{r}$ $for\ the\ following\ d$ ate over a given curve C'

$$Q1.\vec{F}=[(x-y)^2,(y-x)^2]\,C:xy=1,1\leq x\leq 4$$

$$Q.2\,\vec{F}=[x-y,y-z.z-x],C:r(t)=[2\cos t,t,2\sin t]\ from\ (2,0,0)\ to\ (2,2\pi,0)$$

$$Q.3:\vec{F}=[e^{y^{\frac{2}{3}}},e^{x^{\frac{2}{3}}}],C:y=x^{\frac{3}{2}}$$

Some Problems of Line Integral:

PROBLEM - 4

Evaluate the integral $\int_C f(\vec{r})\, ds$, where $f = x^2 + y^2$

and $C: y = 3x$ from $(0,0)$ to $(2,6)$.

Solution:

Given path is the line $C: y = 3x$ from $(0,0)$ to $(2,6)$.

Let $x = t$. So, $y = 3x = 3t$.

Therefore, the parametric representation of the given straight line is $\vec{r}(t) = x\hat{i} + y\hat{j} = t\hat{i} + 3t\hat{j}$.

Some Problems of Line Integral:

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t\hat{i} + 3t\hat{j}\right) = \hat{i} + 3\hat{j}$$

$$\therefore \quad \frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt} = \left(\hat{i} + 3\hat{j}\right) \bullet \left(\hat{i} + 3\hat{j}\right) = 1 + 9 = 10$$

At $(0,0)$, $t = 0$ and at $(2,6)$, $t = 2$.

Given that $f = x^2 + y^2$.

$$\Rightarrow f\left(\vec{r}(t)\right) = (t)^2 + (3t)^2 = 10t^2.$$

Some Problems of Line Integral:

Therefore, the required line integral is

$$\int_C f(\vec{r})\,ds = \int_C f\left(\vec{r}(t)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\,dt = \int_0^2 f\left(\vec{r}(t)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\,dt$$

$$= \int_0^2 \left(10t^2\right)\sqrt{10}\,dt \qquad = 10\sqrt{10}\int_0^2 t^2\,dt \qquad = 10\sqrt{10}\left[\frac{t^3}{3}\right]_0^2$$

$$= 10\sqrt{10}\left[\frac{(2)^3}{3} - 0\right] = 10\sqrt{10}\left(\frac{8}{3}\right) = \frac{80\sqrt{10}}{3}.$$

Some Problems of Line Integral:

PROBLEM - 5

Evaluate the integral $\int_C f(\vec{r})\,ds$, where $f = x^2 + y^2 + z^2$ and $C: \vec{r} = [\cos t, \quad \sin t, \quad 2t], \quad 0 \le t \le 4\pi.$

Solution: The parametric representation of the given path C is $\vec{r}(t) = [\cos t, \quad \sin t, \quad 2t].$

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\cos t\hat{i} + \sin t\hat{j} + 2t\hat{k}\right) = -\sin t\hat{i} + \cos t\hat{j} + 2\hat{k}.$$

Some Problems of Line Integral:

$$\therefore \quad \frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt} = \left(-\sin t\hat{i} + \cos t\hat{j} + 2\hat{k}\right) \bullet \left(-\sin t\hat{i} + \cos t\hat{j} + 2\hat{k}\right)$$

$$= \sin^2 t + \cos^2 t + 4 = 5$$

Given that $f = x^2 + y^2 + z^2$.

$$\Rightarrow f\left(\vec{r}\left(t\right)\right) = \left(\cos t\right)^2 + \left(\sin t\right)^2 + \left(2t\right)^2$$

$$= \cos^2 t + \sin^2 t + 4t^2 = 4t^2 + 1.$$

Some Problems of Line Integral:

Therefore, the required line integral is

$$\int_C f(\vec{r})\,ds = \int_C f\left(\vec{r}\left(t\right)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\,dt = \int_0^{4\pi} f\left(\vec{r}\left(t\right)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\,dt$$

$$= \int_0^{4\pi}\left(4t^2+1\right)\sqrt{5}\,dt = \sqrt{5}\int_0^{4\pi}\left(4t^2+1\right)dt \quad = \sqrt{5}\left[\frac{4}{3}t^3+t\right]_0^{4\pi}$$

$$= \sqrt{5}\left[\frac{4}{3}\left(4\pi\right)^3+4\pi-0\right] \quad = \frac{256}{3}\sqrt{5}\pi^3+4\sqrt{5}\pi$$

Some Problems of Line Integral:

PROBLEM - 6

Evaluate the integral $\int_C f(\vec{r})ds$, where $f = x^2 + (xy)^{\frac{1}{3}}$ and C is the hypocycloid $\vec{r} = \left[\cos^3 t, \quad \sin^3 t\right], 0 \le t \le \pi.$

Solution:

The parametric representation of the given path C is $\vec{r} = \left[\cos^3 t, \quad \sin^3 t\right].$

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\cos^3 t\hat{i} + \sin^3 t\hat{j}\right) = -3\cos^2 t\sin t\hat{i} + 3\sin^2 t\cos t\hat{j}.$$

Some Problems of Line Integral:

$$\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}$$

$$=\left(-3\cos^2 t\sin t\hat{i}+3\sin^2 t\cos t\hat{j}\right)\bullet\left(-3\cos^2 t\sin t\hat{i}+3\sin^2 t\cos t\hat{j}\right)$$

$$=\left(-3\cos^2 t\sin t\right)^2+\left(3\sin^2 t\cos t\right)^2 \;=9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t$$

$$=9\cos^2 t\sin^2 t$$

Given that $f=x^2+(xy)^{\frac{1}{3}}$.

$$\Rightarrow f\left(\vec{r}(t)\right)=\left(\cos^3 t\right)^2+\left(\cos^3 t\sin^3 t\right)^{\frac{1}{3}}$$

Some Problems of Line Integral:

$$= \cos^6 t + \cos t \sin t$$

Therefore, the required line integral is

$$\int_C f(\vec{r})\,ds = \int_C f\left(\vec{r}\left(t\right)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\,dt = \int_0^{\pi} f\left(\vec{r}\left(t\right)\right)\sqrt{\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}}\,dt$$

$$= \int_0^{\pi}\left(\cos^6 t + \cos t \sin t\right) 3\cos t \sin t\,dt$$

$$= 3\int_0^{\pi}\cos^7 t \sin t\,dt + 3\int_0^{\pi}\cos^2 t \sin^2 t\,dt$$

Some Problems of Line Integral:

$$= 3\left[-\frac{\cos^8 t}{8}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\left(2\cos t\sin t\right)^2 dt$$

$$= 3\left[\left(-\frac{1}{8}\right)-\left(-\frac{1}{8}\right)\right] + \frac{3}{4}\int_0^{\pi}\sin^2 2t dt$$

$$= 0 + \frac{3}{4}\int_0^{\pi}\left(\frac{1+\cos 4t}{2}\right)dt \qquad = \frac{3}{8}\left[t + \frac{1}{4}\sin 4t\right]_0^{\pi}$$

$$= \frac{3}{8}\left[\left(\pi + \frac{1}{4}(0)\right)-\left(0 + \frac{1}{4}(0)\right)\right] \qquad = \frac{3}{8}\pi$$

6. Evaluate the integral $\int_C f(\vec{r})ds$,

where $f = x^3 y$, $C: \vec{r} = [2cost, 2sint], 0 \le t \le \frac{\pi}{2}$

Solution : Given that the parametric equation of the path C is

$\vec{r}(t) = 2cost\hat{\imath} + 2sint\hat{\jmath}$, $0 \le t \le \frac{\pi}{2}$

$\Rightarrow \vec{r}'(t) = -2sint\hat{\imath} + 2cost\hat{\jmath}$

$\Rightarrow |\vec{r}'(t)| = \sqrt{(-2sint)^2 + (2cost)^2} = 2 = \frac{ds}{dt}$

Substituting x=2cost, y=2sint in a given f=$x^3y, we\ obtain\ f(\vec{r}) =$ $16cos^3tsint$A

So $\int f(\vec{r})ds = \int f(\vec{r})(\frac{ds}{dt})dt = 32\int_0^{\frac{\pi}{2}} cos^3tsintdt$

$$-32\left[\frac{\cos^4 t}{4}\right]_0^{\frac{\pi}{2}} = -8(0-1) = 8$$

ANS

TEST YOUR KNOWLEDGE

$\int f(\vec{r})ds$ *with Arc length as parameter*

$$Q1.\ f = x^3 y,\ C: \vec{r} = [2\cos t, 2\sin t].\ 0 \le t \le \frac{\pi}{2}$$

$$Q\,2.\ f = \sqrt{16x^2 + 81y^2},\ C: \vec{r}(t) = [3\cos t, 2\sin t],\ 0 \le t \le \pi$$

$$Q.3\ f = 1 - \sinh^2 x,\ C: \vec{r} = [t, \cosh t],\ 0 \le t \le 2$$

LINE INTEGRAL INDEPENDENT OF PATH

A line integral of the type

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_A^B \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

is said to be ***independent of path*** in a domain *D* in space if for every pair of end points *A* and *B* in *D*, in the above integral has **the same value** for all paths in D the *begin at A* and *end at B*.

LINE INTEGRAL INDEPENDENT OF PATH

THEOREM - 1 A line integral of the type

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right)$$

with continuous F_1, F_2, F_3 in a domain D in space is independent of path in D if and only if $\vec{F} = [F_1, F_2, F_3]$ is the gradient of some function f in D. Moreover,

$$\boxed{\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right) = f(B) - f(A)}$$

LINE INTEGRAL INDEPENDENT OF PATH

EXAMPLE – 1:

Show that the integral $\int_C \vec{F} \bullet d\vec{r} = \int_C \left(3x^2 dx + 2yzdy + y^2 dz\right)$ is independent of path in any domain in space and find its value if C has the initial point $A:\ (0,1,2)$ and the terminal point $B:(1,-1,7)$.

SOLUTION:

$$\text{Given that } \vec{F} = 3x^2\hat{i} + 2yz\hat{j} + y^2\hat{k}$$

Let us see whether there exists a scalar function f such that

$$\vec{F} = grad\ f$$

LINE INTEGRAL INDEPENDENT OF PATH

$$\Rightarrow 3x^2\hat{i} + 2yz\hat{j} + y^2\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\frac{\partial f}{\partial x} = 3x^2 ------(1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2yz ------(2)$$

$$\frac{\partial f}{\partial z} = y^2 -------(3)$$

Integrating (1) w.r.t. x, we get $f(x,y,z) = x^3 + k(y,z) ---(4)$

$k(y,z)$ is a function of y and z.

LINE INTEGRAL INDEPENDENT OF PATH

Differentiating (4) w.r.t. y, we get $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(x^3 + k(y,z)\right)$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial k}{\partial y} --(5)$$

From (2) and (5), we have $2yz = \dfrac{\partial k}{\partial y} ---(6)$

Integrating (6) w.r.t. y, we get $k(y,z) = y^2 z + \phi(z) ---(7)$

where $\phi(z)$ is a function of z.

So from (4) we have $f(x,y,z) = x^3 + y^2 z + \phi(z) ---(8)$

LINE INTEGRAL INDEPENDENT OF PATH

Differentiating (8) w.r.t. z, we get $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left(x^3 + y^2 z + \phi(z)\right)$

$$\Rightarrow \frac{\partial f}{\partial z} = y^2 + \frac{d\phi}{dz} - - - (9)$$

From (3) and (9), we have $y^2 = y^2 + \dfrac{d\phi}{dz} \Rightarrow \dfrac{d\phi}{dz} = 0$

On integration w.r.t. z, we get $\phi(z) = A\ constant = c(\text{say})$

So from (8) we have $f(x, y, z) = x^3 + y^2 z + c - - - (10)$

LINE INTEGRAL INDEPENDENT OF PATH

So we find that there exists a function

$f(x,y,z) = x^3 + y^2 z + c$ such that $\vec{F} = grad f$.

Therefore, the integral $\int_C \vec{F} \bullet d\vec{r} = \int_C \left(3x^2 dx + 2yzdy + y^2 dz\right)$ is

independent of path and its value is given by

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \left(3x^2 dx + 2yzdy + y^2 dz\right) = f(1,-1,7) - f(0,1,2).$$

$$= \left[x^3 + y^2 z + c\right]_{\text{at } (1,-1,7)} - \left[x^3 + y^2 z + c\right]_{\text{at } (0,1,2)}$$

$$= \left[(1)^3 + (-1)^2 (7) + c\right] - \left[(0)^3 + (1)^2 (2) + c\right] = 8 - 2 = 6.$$

LINE INTEGRAL INDEPENDENT OF PATH

THEOREM - 2

A line integral of the type

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right)$$

is independent of path in a domain *D* if and only if its value around every closed path in D is zero.

LINE INTEGRAL INDEPENDENT OF PATH

Let $F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)$ be continuous functions having continuous first partial derivatives in a domain D in space.

$F_1 dx + F_2 dy + F_3 dz$ is called $\boxed{\text{exact}}$ in D if it is the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

of a differentiable function $f\left(x, y, z\right)$ everywhere in D, i.e., if we have

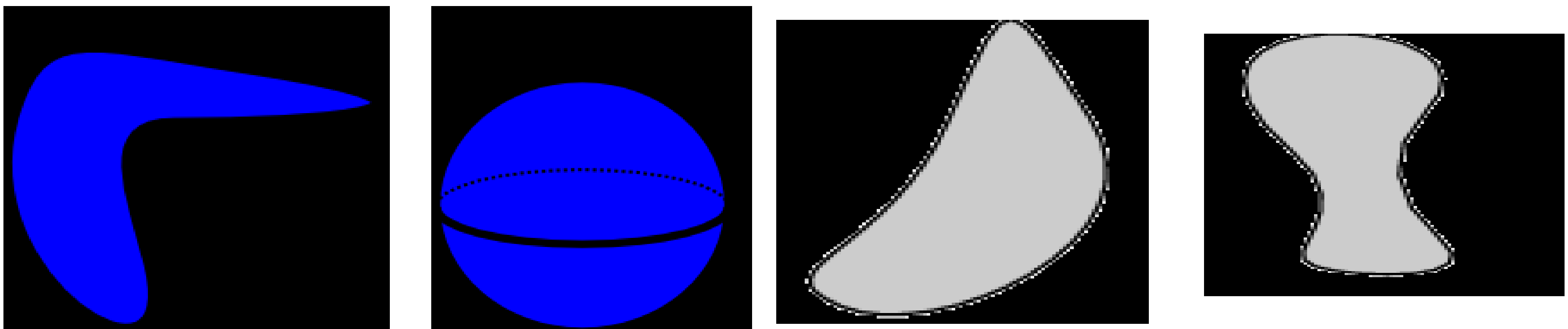
$F_1 dx + F_2 dy + F_3 dz = df$.

From $F_1 dx + F_2 dy + F_3 dz \; = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

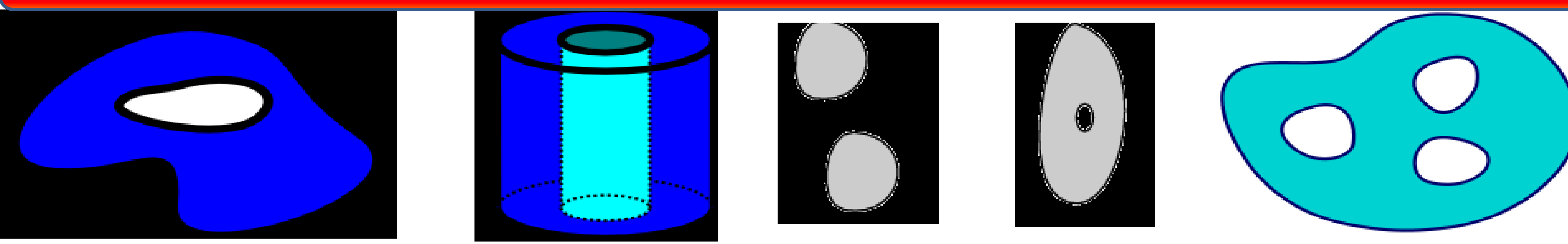
we have $F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z}, i.e.\ \vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} = grad f$.

LINE INTEGRAL INDEPENDENT OF PATH

A domain D is called a ***simply connected domain*** if every closed curve in D can be continuously shrunk to any point in D without leaving D.



SIMPLY CONNECTED DOMAINS



NOT SIMPLY CONNECTED DOMAINS

LINE INTEGRAL INDEPENDENT OF PATH

THEOREM – 3:

Let $F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)$ be continuous function having continuous first partial derivatives in a domain D in space. If the line integral $\int_C \left(F_1dx + F_2dy + F_3dz\right)$ is independent of path in D,

Then $F_1dx + F_2dy + F_3dz$ is exact then in D

$$curl\ \vec{F} = curl\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) = \vec{0}$$

i.e. $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

LINE INTEGRAL INDEPENDENT OF PATH

THEOREM – 4:

Let $F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)$ be continuous function havingcontinuous first partial derivatives in a domain D in space. If $curl\ \vec{F} = curl\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) = \vec{0}$

and D is simply connected then the integral

$\int_C \left(F_1 dx + F_2 dy + F_3 dz\right)$ is independent of path in D.

LINE INTEGRAL INDEPENDENT OF PATH

Example – 1:

Show that the form under integral sign is exact in the plane and evaluate the integral.

$$\int_{(0,\pi)}^{\left(2,\frac{3\pi}{2}\right)} e^{x}(\cos y\,dx - \sin y\,dy)$$

Solution:

We know that a differential form $F_1dx + F_2dy + F_3dz$ is exact if $Curl\vec{F} = Curl\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) = \vec{0}.$

Set $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = e^x \cos y\hat{i} - e^x \sin y\hat{j} + 0\hat{k}$

LINE INTEGRAL INDEPENDENT OF PATH

$$\therefore curl\left(\vec{F}\right)$$

$$= curl\left(e^x \cos y\hat{i} - e^x \sin y\hat{j} + 0\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \cos y & -e^x \sin y & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(-e^x \sin y\right)\right)\hat{i} + \left(\frac{\partial}{\partial z}\left(e^x \cos y\right) - \frac{\partial}{\partial z}(0)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}\left(-e^x \sin y\right) - \frac{\partial}{\partial z}\left(e^x \cos y\right)\right)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

LINE INTEGRAL INDEPENDENT OF PATH

So the given differential form

$$F_1dx + F_2dy + F_3dz = e^x\left(\cos y dx - \sin y dy\right) \text{ is exact.}$$

So there exists function, say $f\left(x, y, z\right)$ such that

$$df = F_1dx + F_2dy + F_3dz = e^x\left(\cos y dx - \sin y dy\right).$$

$$i.e.\ \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = e^x\left(\cos y dx - \sin y dy\right)$$

LINE INTEGRAL INDEPENDENT OF PATH

On comparing the coefficients of dx, dy and dz we get

$$\frac{\partial f}{\partial x} = e^{x} \cos y ---(1)$$

$$\frac{\partial f}{\partial y} = -e^{x} \sin y --(2)$$

$$\frac{\partial f}{\partial z} = 0 ------(3)$$

LINE INTEGRAL INDEPENDENT OF PATH

On integrating (1) partially w. r. t. x we get

$$f = \int e^x \cos y dx + \varphi(y,z) = e^x \cos y + k(y,z) --(4)$$

$k(y,z)$ is a function of y and z.

Differentiating (4) partially w.r.t. y we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^x \cos y + k(y,z)\right) = -e^x \sin y + \frac{\partial k}{\partial y} --(5)$$

From (2) and (5), we get

$$-e^x \sin y = -e^x \sin y \Rightarrow \frac{\partial k}{\partial y} = 0$$

$i.e.\ k$ is a function of z alone, say $k(y,z) = c(z)$

LINE INTEGRAL INDEPENDENT OF PATH

From (4) we have

$$f = e^x \cos y + c(z) - -(6).$$

Differentiating (6) partially w. r. t. z we get

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(e^x \cos y + c(z)\right) = \frac{dc}{dz} - -(7)$$

From (3) and (7), we get

$$\frac{dc}{dz} = 0 \Rightarrow c(z) = \text{ A constant} = C \ (\text{say})$$

From (6) we have

$$f = e^x \cos y + C - - - (8)$$

LINE INTEGRAL INDEPENDENT OF PATH

Hence the exact differential is

$$e^x \cos y dx - e^x \sin y dy = df = d\left(e^x \cos y + c\right)$$

$$\therefore \int_{(0,\pi)}^{\left(2,\frac{3\pi}{2}\right)} e^x\left(\cos y dx - \sin y dy\right) = \int_{(0,\pi)}^{\left(2,\frac{3\pi}{2}\right)} df = \int_{(0,\pi)}^{\left(2,\frac{3\pi}{2}\right)} d\left(e^x \cos y + c\right)$$

LINE INTEGRAL INDEPENDENT OF PATH

$$= \left[e^{x} \cos y + c \right]_{(0,\pi)}^{(2,\frac{3\pi}{2})}$$

$$= \left(e^{2} \cos \frac{3\pi}{2} + c \right) - \left(e^{0} \cos \pi + c \right)$$

$$= (0 + c) - (1 + c) = 1$$

ANS

EXAMPLE – 2:

Show that the from under the integral sign is exact in the plane or in the space and evaluate the integral.

$$\int_{0,0,0}^{4,1,2} (3ydx + 3xdy + 2zdz)$$

We know that the differential form $F_1dx + F_2dy + F_3dz$ is exact if Curl$\vec{F}=\vec{0}$

Here $\vec{\mathbf{F}} = 3y\hat{\mathbf{i}} + 3x\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$

$$\therefore curl\left(\vec{F}\right) = curl\left(3y\hat{i} + 3x\hat{j} + 2z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 3x & 2z \end{vmatrix}$$

LINE INTEGRAL INDEPENDENT OF PATH

$$= \left[\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(3x)\right]\hat{i} + \left[\frac{\partial}{\partial z}(3y) - \frac{\partial}{\partial x}(2z)\right]\hat{j} + \left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(3y)\right]\hat{k}$$

$$= [0-0]\hat{i} + [0-0]\hat{j} + [3-3]\hat{k} = \vec{0}$$

$$\Rightarrow Curl\ \vec{F} = \vec{0}$$

So the given differential 3ydx+3xdy+2zdz is exact

To find the exact differential

Suppose that there a function f(x,y,z) in space such that *3ydx+3xdy+2zdz=df.*

LINE INTEGRAL INDEPENDENT OF PATH

$$\Rightarrow 3ydx+3xdy+2zdz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

On comparing the coefficient of dx ,dy and dz on both sides

$$\frac{\partial f}{\partial x}=3y---(1)$$

$$\frac{\partial f}{\partial y}=3x---(2)$$

$$\frac{\partial f}{\partial z}=2z---(3)$$

LINE INTEGRAL INDEPENDENT OF PATH

Integrating (1) w.r.t. x, we get $f(x,y,z)=3xy+k(y,z)---(4)$

$k(y,z)$ is a function of y and z.

Differentiating (4) w.r.t. y, we get $\dfrac{\partial f}{\partial y}=\dfrac{\partial}{\partial y}\left(3xy+k(y,z)\right)$

$$\Rightarrow \frac{\partial f}{\partial y}=3x+\frac{\partial k}{\partial y}--(5)$$

From (2) and (5), we have $3x \;=3x+\dfrac{\partial k}{\partial y}\Rightarrow\dfrac{\partial k}{\partial y}=0$

Since $k(y,z)=0$ and $\dfrac{\partial \mathrm{k}}{\partial \mathrm{y}}$=0, integrating w.r.t. y,

we get $k(y,z)=A$ function of z-alone $=\varphi\left(\mathrm{z}\right)\left(\mathrm{say}\right)$

So from (4) we have $f(x,y,z)=3xy+\varphi\left(\mathrm{z}\right)---(6)$

LINE INTEGRAL INDEPENDENT OF PATH

Differentiating (6) w.r.t. z, we get $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left(3xy + \varphi(z)\right)$.

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{d\varphi}{dz} \text{---}(7)$$

From (3) and (7), we have $2z = \dfrac{d\varphi}{dz}$

$$\Rightarrow \frac{d\varphi}{dz} = 2z$$

On integration w.r.t. z, we get $\varphi(z) = z^2 + c$, where c is an arbitrary constant.

LINE INTEGRAL INDEPENDENT OF PATH

So from (6) we have $f(x,y,z)=3xy+z^2+c.$

$$\therefore \int_{(0,0,0)}^{(4,1,2)} [3ydx+3xdy+2zdz] = \int_{(0,0,0)}^{(4,1,2)} df = \int_{(0,0,0)}^{(4,1,2)} d\left(3xy+z^2+c\right)$$

$$=\left[3xy+z^2+c\right]_{(0,0,0)}^{(4,1,2)}$$

$$=\left[3xy+z^2+c\right]_{At\ (4,1,2)}-\left[3xy+z^2+c\right]_{At(0,0,0)}$$

$$=\left[\left(3(4)(1)+(2)^2+c\right)-\left(3(0)(0)+(0)^2+c\right)\right]$$

= 16

ANS

EXAMPLE – 3:

Check for path –independence . In case of independence, integrate form (0,0,0) to (a, b, c).

$$2xy^2dx + 2x^2ydy + dz$$

We know that the differential form $F_1dx + F_2dy + F_3dz$ is exact if Curl$\vec{F}=\vec{0}$

Here $\vec{F} = 2xy^2\hat{i} + 2x^2y\hat{j} + \hat{k}$

$$\text{So } curl\left(\vec{F}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 & 2x^2ydy & 1 \end{vmatrix}$$

LINE INTEGRAL INDEPENDENT OF PATH

$$=\left[\frac{\partial}{\partial y}(1)-\frac{\partial}{\partial z}\left(2x^2y\right)\right]\hat{i}+\left[\frac{\partial}{\partial z}\left(2xy^2\right)-\frac{\partial}{\partial x}(1)\right]\hat{j}$$

$$+\left[\frac{\partial}{\partial x}\left(2x^2y\right)-\frac{\partial}{\partial y}\left(2xy^2\right)\right]\hat{k}$$

$$=[0-0]\hat{i}+[0-0]\hat{j}+[4xy-4xy]\hat{k}=0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}$$

So the given differential form s exact.(path independent)

To find the exact differential
Suppose there a function f(x, y, z) in space such that
$2xy^2dx+2x^2ydy+dz=df$

LINE INTEGRAL INDEPENDENT OF PATH

$$\Rightarrow 2xy^2\,dx + 2x^2y\,dy + dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

On comparing the coefficient of dx ,dy and dz on both sides

$$\frac{\partial f}{\partial x} = 2xy^2 \quad ---(1)$$

$$\frac{\partial f}{\partial y} = 2x^2y \quad --\ (2)$$

$$\frac{\partial f}{\partial z} = 1 \quad ---(3)$$

LINE INTEGRAL INDEPENDENT OF PATH

Integrating (1) w.r.t. x, we get $f(x,y,z) = x^2y^2 + k(y,z) - - - (4)$

$k(y,z)$ is a function of y and z.

Differentiating (4) w.r.t. y, we get $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^2 + k(y,z)\right)$

$$\Rightarrow \frac{\partial f}{\partial y} = 2x^2y + \frac{\partial k}{\partial y} - - (5)$$

From (2) and (5), we have $2x^2y = 2x^2y + \frac{\partial k}{\partial y} \Rightarrow \frac{\partial k}{\partial y} = 0$

Since $k(y,z) = 0$ and $\frac{\partial \mathrm{k}}{\partial \mathrm{y}} = 0$, integrating w.r.t. y,

we get $k(y,z) = A$ function of z-alone $= \varphi(\mathrm{z})(\text{say})$

So from (4) we have $f(x,y,z) = x^2y^2 + \varphi(\mathrm{z}) - - - (6)$

LINE INTEGRAL INDEPENDENT OF PATH

Differentiating (6) w.r.t. z, we get $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2 y^2 + \varphi(z)\right)$.

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{d\varphi}{dz} - - - (7)$$

From (3) and (7), we have $1 = \frac{d\varphi}{dz}$

$$\Rightarrow \frac{d\varphi}{dz} = 1$$

On integration w.r.t. z, we get $\varphi(z) = z + K$, where K is an arbitrary constant.

LINE INTEGRAL INDEPENDENT OF PATH

So from (6) we have $f(x,y,z)=x^2y^2+z+K.$

$$\therefore \int_{(0,0,0)}^{(a,b,c)} [3ydx+3xdy+2zdz] = \int_{(0,0,0)}^{(a,b,c)} df = \int_{(0,0,0)}^{(a,b,c)} d\left(x^2y^2+z+K\right)$$

$$=\left[x^2y^2+z+K\right]_{(0,0,0)}^{(a,b,c)}$$

$$=\left[\left((a)^2(b)^2+(c)+K\right)-\left((0)^2(0)^2+(0)+K\right)\right]$$

$$=a^2b^2+c$$

ANS

EXAMPLE – 4:

Check for path –independence . In case of independence, integrate form (0,0,0) to (a, b, c).

$ydx - zxdy + zdz$

We know that the differential form $F_1dx + F_2dy + F_3dz$ is exact if CurlF=0

Here $\vec{\mathbf{F}} = y\hat{\mathbf{i}} - zx\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$$curl(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

LINE INTEGRAL INDEPENDENT OF PATH

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -zx & z \end{vmatrix} = \left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(-zx)\right]\hat{i}+\left[\frac{\partial}{\partial z}(y)-\frac{\partial}{\partial x}(z)\right]\hat{j}$$

$$+\left[\frac{\partial}{\partial x}(-zx)-\frac{\partial}{\partial y}(y)\right]\hat{k}$$

$$=(0+x)\hat{i}+(0-0)\hat{j}+(-z-1)\hat{k} \neq \vec{O}$$

So the given differential form s exact.(path dependent)

ANS