DEPARTMENT OF MATHEMATICS

Introduction

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x}\, dx, \quad p > 0.$$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx, \qquad (p > 0, q > 0)$$

$$er\ f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

BETA FUNCTION

DEFINITION

The definite integral $\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$, for $p>0, q>0$

is called the **Beta function** and is dented by B(m,n)

(read as "Beta m, n").

$$\text{Thus, } \mathrm{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \quad p>0, q>0$$

Beta function is also called the ***Eulerian integral of the first kind***.

BETA FUNCTION

Symmetry Property: $\boxed{B(p, q) = B(q, p)}$

Proof: By definition $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$

Set $x=1-y \Rightarrow dx=-dy$ and $x=0 \Rightarrow y=1$ and $x=1 \Rightarrow y=0$

Therefore, $B(p,q)=\int_1^0 (1-y)^{p-1}y^{q-1}(-dy)$

$$=-\int_1^0 y^{q-1}(1-y)^{p-1}dy \quad =\int_0^1 y^{q-1}(1-y)^{p-1}dy = B(q,p)$$

BETA FUNCTION

SOME FACTS WORTH REMEMBERING

<u>I:</u> When n is a positive integer

$$B(m,n)=\frac{(n-1)!}{m(m+1)(m+2)\cdots\cdots(m+n-2)(m+n-1)}$$

<u>2:</u> When m is a positive integer

$$B(m,n)=\frac{(m-1)!}{n(n+1)(n+2)\cdots\cdots(n+m-2)(n+m-1)}$$

BETA FUNCTION

3: If both m and n are a positive integers

$$B\left(m,n\right)=\frac{\left(m-1\right)!\left(n-1\right)!}{\left(m+n-1\right)!}$$

BETA FUNCTION

Problem – 1: Express the following integral in terms of beta function.

$$\int_0^1 x^m\left(1-x^2\right)^n dx; m>-1, n>-1$$

SOLUTION:

We have

$$\int_0^1 x^m\left(1-x^2\right)^n dx \qquad = \int_0^1 x^{m-1}\left(1-x^2\right)^n x\,dx$$

Set $x^2 = y \Rightarrow 2xdx = dy$ or $xdx = \frac{1}{2}dy$.

Also $x = 0 \Rightarrow y = 0$ and $x = 1 \Rightarrow y = 1$

BETA FUNCTION

$$\therefore \int_0^1 x^m \left(1-x^2\right)^n dx = \frac{1}{2}\int_0^1 y^{\frac{(m-1)}{2}} (1-y)^n \, dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{(m+1)}{2}-1} (1-y)^{(n+1)-1} \, dy$$

$$= \frac{1}{2} \mathrm{B}\left(\frac{1}{2}(m+1), n+1\right)$$

BETA FUNCTION

Problem – 2: Show that if m, n are positive, then.

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\,\mathrm{B}(m,n).$$

SOLUTION:

$$\text{Let } I = \int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$$

Set $x = a + (b-a)y \Rightarrow dx = (b-a)dy.$

Also $x = a \Rightarrow y = 0$ and $x = b \Rightarrow y = 1$

BETA FUNCTION

$$\therefore I = \int_0^1 \left(a+\left(b-a\right)y-a\right)^{m-1}\left[b-\left(a+\left(b-a\right)y\right)\right]^{n-1}\left(b-a\right)dx$$

$$= \int_0^1 \left(b-a\right)^{m-1} y^{m-1}\left(b-a\right)^{n-1}\left(1-y\right)^{n-1}\left(b-a\right)dy$$

$$= \left(b\text{-}a\right)^{m-1+n-1+1}\int_0^1 y^{m-1}\left(1-y\right)^{n-1}dy$$

$$= \left(b\text{-}a\right)^{m+n-1}\int_0^1 y^{m-1}\left(1-y\right)^{n-1}dy$$

$$= \left(b\text{-}a\right)^{m+n-1}\mathrm{B}\left(m,n\right)$$

BETA FUNCTION

Problem – 3: Show that if *p >-1*, *q > -1*, then.

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

SOLUTION:

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{p-1} \theta \cos^{q-1} \theta \sin\theta \cos\theta d\theta$$

BETA FUNCTION

$$= \int_0^{\frac{\pi}{2}} \sin^{p-1}\theta \left(1-\sin^2\theta\right)^{\frac{q-1}{2}} \sin\theta \cos\theta d\theta$$

Set $sin^2\theta = x \Rightarrow dx = 2sin\theta\, cos\theta\, d\theta$

$\Rightarrow sin\theta\, cos\theta\, d\theta = ½dx$

Also $\theta = 0 \Rightarrow x = 0$ and $\theta = \pi/2 \Rightarrow x = 1$

$$\therefore I = \int_0^1 x^{\frac{p-1}{2}} (1-x)^{\frac{q-1}{2}} \frac{1}{2} dx = \frac{1}{2}\int_0^1 x^{\frac{p-1}{2}} (1-x)^{\frac{q-1}{2}} dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1} (1-x)^{\frac{q+1}{2}-1} dx$$

BETA FUNCTION

$$= \frac{1}{2} \mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2} \right)$$

4. Prove that $\dfrac{\mathrm{B}(m+1,n)}{\mathrm{B}(m,n)} = \dfrac{m}{m+n}$

SOLUTION:

We have $\mathrm{B}(m+1,n) = \mathrm{B}(n,m+1)$ (By symmetry)

$$= \int_0^1 x^{n-1} (1-x)^{(m+1)-1}\, dx \qquad = \int_0^1 (1-x)^m x^{n-1} dx$$

BETA FUNCTION

$$=\left[(1-x)^m \int x^{n-1}dx\right]_0^1 - \int_0^1 \left(\frac{d}{dx}(1-x)^m\right)\left(\int x^{n-1}dx\right)dx$$

$$=\left[\frac{(1-x)^m x^n}{n}\right]_0^1 - \int_0^1 \left(m(1-x)^{m-1}\right)\left(\frac{x^n}{n}\right)dx$$

$$=[0-0]+\frac{m}{n}\int_0^1 (1-x)^{m-1} x^n dx \quad =\frac{m}{n}\int_0^1 (1-x)^{m-1} x^n dx$$

$$=\frac{m}{n}\int_0^1 (1-x)^{m-1} x^{n-1} x dx$$

$$=\frac{m}{n}\int_0^1 (1-x)^{m-1} x^{n-1}\left(1-(1-x)\right)dx$$

BETA FUNCTION

$$= \frac{m}{n}\left(\int_0^1 (1-x)^{m-1} x^{n-1} dx - \int_0^1 (1-x)^{m} x^{n-1} dx \right)$$

$$= \frac{m}{n}\left(\int_0^1 x^{n-1} (1-x)^{m-1} dx - \int_0^1 x^{n-1} (1-x)^{m} dx \right)$$

$$= \frac{m}{n}\left(\mathrm{B}(n,m) - \mathrm{B}(n,m+1) \right)$$

$$= \frac{m}{n}\left(\mathrm{B}(m,n) - \mathrm{B}(m+1,n) \right) \qquad \text{(By symmetry)}$$

BETA FUNCTION

Thus, we get $\mathrm{B}(m+1,n)=\dfrac{m}{n}\left(\mathrm{B}(m,n)-\mathrm{B}(m+1,n)\right)$

$$\Rightarrow\ \mathrm{B}(m+1,n)=\frac{m}{n}\mathrm{B}(m,n)-\frac{m}{n}\mathrm{B}(m+1,n)$$

$$\Rightarrow\ \left(1+\frac{m}{n}\right)\mathrm{B}(m+1,n)=\frac{m}{n}\mathrm{B}(m,n)$$

$$\Rightarrow\ \frac{m+n}{n}\mathrm{B}(m+1,n)=\frac{m}{n}\mathrm{B}(m,n)$$

$$\Rightarrow \frac{\mathrm{B}(m+1,n)}{\mathrm{B}(m,n)}=\frac{m}{m+n}.$$

BETA FUNCTION

5. Prove that $B(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx, \quad m>0, n>0.$

SOLUTION:

By definition $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Set $x=\frac{1}{1+y} \Rightarrow dx=-\frac{1}{(1+y)^2}dy$

Also $x\to 0 \Rightarrow y\to\infty$

and $x\to 1 \Rightarrow y\to 0$

BETA FUNCTION

$$\therefore \mathrm{B}(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left(1-\frac{1}{1+y}\right)^{n-1} \left(-\frac{1}{(1+y)^2}\right) dy$$

$$= -\int_{\infty}^{0} \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}} dy = \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_{0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

MATHEMATICS - I

Unit-II

Gamma Function

Gamma Function

Definition:

Gamma function of a real number $p > 0$, denoted by $\Gamma(p)$ is dependent on the parameter p, and is defined by the improper integral as

$$\boxed{\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx, \quad p > 0}$$

Gamma function is also **Eulerian integral of the second kind**.

Gamma Function

Recursion Relation

$\Gamma(p+1) = p\,\Gamma(p)$

Proof:

By definition, $\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx$

$$= \left[x^p \int e^{-x} dx - \int \left[\left(\frac{d}{dx} \left(x^p \right) \right) \left(\int e^{-x} dx \right) \right] dx \right]_0^\infty$$

$$= \left[-x^p e^{-x} \right]_0^\infty + p \int_0^\infty e^{-x} x^{p-1} \, dx \quad = 0 + p\Gamma(p)$$

$$= p\Gamma(p)$$

Gamma Function

Problem – 6: For a positive integer *n* prove that

$\Gamma(n+1) = n!$

Proof:

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\,\Gamma(n-2)$$
$$\dots$$
$$= n(n-1)(n-2)\dots 3.2.1\Gamma(1)$$
$$= n(n-1)(n-2)\dots 3.2.1 = n!$$

Gamma Function

Therefore, ***Γ(1)= 1***

2. Gamma function for negative values of p i.e. $p < 0$ is undefined as

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \qquad \text{(from recursion relation)}$$

As $$p \to 0,\ \Gamma(0) = \lim_{p\to 0} \frac{\Gamma(1)}{p} = \lim_{p\to 0} \frac{1}{p} \to \infty$$

Thus Γ(0) is undefined and it follows that Γ(-1), Γ(-2), Γ(-3), etc. are all undefined.

Gamma Function

<u>Problem – 7:</u> For positive real numbers m and n prove that

$$\mathrm{B}(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

<u>Solution:</u> By definition we have,

$$\Gamma(m)=\int_0^\infty e^{-x}x^{m-1}dx \text{ and } \Gamma(n)=\int_0^\infty e^{-y}y^{m-1}dy$$

Setting $x=u^2$ and $y=v^2$ we have

Gamma Function

$$\Gamma(m)=\int_0^\infty e^{-u^2}\left(u^2\right)^{m-1}2u\,du \text{ and } \Gamma(n)=\int_0^\infty e^{-v^2}\left(v^2\right)^{n-1}2v\,dv$$

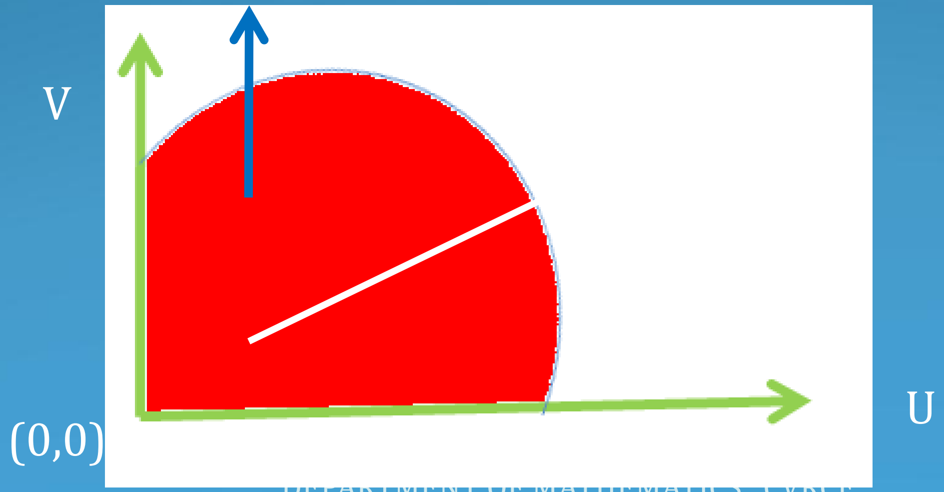
$$\Rightarrow \Gamma(m)=2\int_0^\infty e^{-u^2}u^{2m-1}du \text{ and } \Gamma(n)=2\int_0^\infty e^{-v^2}v^{2n-1}dv$$

$$\therefore \Gamma(m)\Gamma(n)=\left(2\int_0^\infty e^{-u^2}u^{2m-1}du\right)\left(2\int_0^\infty e^{-v^2}v^{2n-1}dv\right)$$

$$=4\int_0^\infty\int_0^\infty e^{-\left(u^2+v^2\right)}u^{2m-1}v^{2n-1}du\,dv$$

Gamma Function

The region of double integration R is R: $0 < u < \infty$, $0 < v < \infty$, i.e. the first quadrant as shown in the figure given below. By changing the variables u,v to the new variables r, θ, by the polar substitution (transformation) $u = r\cos\theta$, $v = r\sin\theta$, the transformed region of integration R^* is $R^*: 0 < r < \infty$, $0 < \theta < \pi/2$.



Gamma Function

$$\therefore \frac{\partial u}{\partial r}=\frac{\partial}{\partial r}\left(r\cos\theta\right)=\cos\theta,\ \frac{\partial u}{\partial \theta}=\frac{\partial}{\partial \theta}\left(r\cos\theta\right)=-r\sin\theta,$$

$$\frac{\partial v}{\partial r}=\frac{\partial}{\partial r}\left(r\sin\theta\right)=\sin\theta,\ \frac{\partial v}{\partial \theta}=\frac{\partial}{\partial \theta}\left(r\sin\theta\right)=r\cos\theta.$$

Therefore, the Jacobian of transformation is given by

$$J=\frac{\partial(u,v)}{\partial(r,\theta)}=\begin{vmatrix}\dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta}\\[2ex] \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r$$

Gamma Function

$$\therefore |J| = |r| = r$$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}(r\sin\theta)^{2m-1}(r\cos\theta)^{2n-1}|J|\,drd\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}(r\sin\theta)^{2m-1}(r\cos\theta)^{2n-1}\,rdrd\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2m-1+2n-1+1}e^{-r^2}\sin^{2m-1}\theta\cos^{2n-1}\theta\;drd\theta$$

Gamma Function

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2m+2n-1} e^{-r^2} \sin^{2m-1}\theta \cos^{2n-1}\theta \; dr d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(\int_0^{\infty} r^{2m+2n-1} e^{-r^2} \; dr\right)\sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$

$$= 4\left(\int_0^{\infty} r^{2m+2n-1} e^{-r^2} dr\right)\left(\int_0^{\frac{\pi}{2}} s\,\text{in}^{2m-1}\theta \cos^{2n-1} d\theta\right)$$

$$= 2\left(\int_0^{\infty} r^{2m+2n-2} e^{-r^2} \, 2rdr\right)\left(\int_0^{\frac{\pi}{2}} s\,\text{in}^{2m-1}\theta \cos^{2n-1} d\theta\right)$$

Gamma Function

$$= 2\left(\int_0^\infty t^{m+n-1} e^{-t}\,dt\right)\left(\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta\right) \qquad \left(\text{Setting } t = r^2\right)$$

$$= 2\Gamma(m+n)\left(\frac{1}{2}\mathrm{B}\left(\frac{2m-1+1}{2}, \frac{2n-1+1}{2}\right)\right)$$

$$\left(\because \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right)$$

$$\text{Thus, } \Gamma(m)\Gamma(n) = \Gamma(m+n)\mathrm{B}(m+1, n+1)$$

$$\therefore \mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Gamma Function

8. Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Solution:
(Method – I)

We know that $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Setting $m = n = \frac{1}{2}$, we get

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

Gamma Function

$$\Rightarrow\left(\Gamma\left(\frac{1}{2}\right)\right)^2=B\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$=\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$$

Set $x=\sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

Also $x=0 \Rightarrow \theta=0$ and $x=1 \Rightarrow \theta=\pi/2$

$$\Rightarrow\left(\Gamma\left(\frac{1}{2}\right)\right)^2=\int_0^{\frac{\pi}{2}}\left(\sin^2\theta\right)^{-\frac{1}{2}}\left(1-\sin^2\theta\right)^{-\frac{1}{2}}2\sin\theta\cos\theta\,d\theta$$

Gamma Function

$$\Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\cos\theta} 2\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Gamma Function

Solution:
(Method – II)

By definition we have,

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-x}x^{\frac{1}{2}-1}\,dx=\int_0^\infty e^{-x}x^{-\frac{1}{2}}\,dx$$

Setting $x = u^2$ in the above equation we get,

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-u^2}\,du$$

We can have another representation of $\Gamma\left(\frac{1}{2}\right)$ by setting $u = v$,

$$\text{i.e. } \Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-v^2}\,dv$$

Gamma Function

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2=\left(2\int_0^\infty e^{-u^2}\,du\right)\left(2\int_0^\infty e^{-v^2}\,dv\right)$$

$$\Rightarrow\left(\Gamma\left(\frac{1}{2}\right)\right)^2=4\int_0^\infty\int_0^\infty e^{-\left(u^2+v^2\right)}\,du\,dv$$

$$\therefore\frac{\partial u}{\partial r}=\frac{\partial}{\partial r}\left(r\cos\theta\right)=\cos\theta,\frac{\partial u}{\partial\theta}=\frac{\partial}{\partial\theta}\left(r\cos\theta\right)=-r\sin\theta,$$

$$\frac{\partial v}{\partial r}=\frac{\partial}{\partial r}\left(r\sin\theta\right)=\sin\theta,\frac{\partial v}{\partial\theta}=\frac{\partial}{\partial\theta}\left(r\sin\theta\right)=r\cos\theta.$$

Gamma Function

Therefore, the Jacobian of transformation is given by

$$J = \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r \qquad \therefore |J| = |r| = r$$

The region of double integration R is R: $0 < u < \infty$, $0 < v < \infty$, i.e. the first quadrant. By changing the variables u,v to the new variables r, θ, by the polar substitution (transformation) $u = r\cos\theta$, $v = r\sin\theta$, the transformed region of integration R^* is $R^*: 0 < r < \infty$, $0 < \theta < \pi/2$.

Gamma Function

$$\therefore \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}}\left(\int_0^{\infty} e^{-r^2}\, r\,dr\right)d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}\left(-e^{-r^2}\right)\right]_0^{\infty} d\theta = 4\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}\left(-e^{-r^2}\right)\right]_0^{\infty} d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$$

$$\therefore \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \qquad \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Gamma Function

Worth Remembering

$$1.\int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta d\theta = \frac{1}{2}\mathrm{B}\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$2.\mathrm{B}(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$3.\ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1.$$

Gamma Function

Problem – 9: For a positive real number m prove that

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

Solution: We have $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Setting $n = m$ in the above equation we get.

$$B(m,m)=\frac{\left(\Gamma(m)\right)^2}{\Gamma(2m)}=\int_0^1 x^{m-1}(1-x)^{m-1}\,dx$$

Gamma Function

Set $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

Also $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \pi/2$.

Setting $x = \sin^2\theta$ in the above equation we get

$$\frac{\left(\Gamma(m)\right)^2}{\Gamma(2m)} = \int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right)^{m-1} \left(1-\sin^2\theta\right)^{m-1} 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right)^{m-1} \left(\cos^2\theta\right)^{m-1} \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2m-1}\theta\, d\theta$$

Gamma Function

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}\, d\theta \quad = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$\Rightarrow \frac{(\Gamma(m))^2}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta$$

Set $\varphi = 2\theta \Rightarrow d\varphi = 2d\theta \Rightarrow d\theta = ½d\varphi$

Also $\theta = \pi/2 \Rightarrow \varphi = \pi$ and $\theta = 0 \Rightarrow \varphi = 0.$

Setting $\varphi = 2\theta$ in the above equation we get

$$\Rightarrow \frac{(\Gamma(m))^2}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \frac{1}{2} \int_0^{\pi} \sin^{2m-1} \varphi d\varphi$$

Gamma Function

$$= \frac{1}{2^{2m-1}} 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\varphi d\varphi \qquad = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1}\varphi \cos^0\varphi d\varphi$$

$$= \frac{2}{2^{2m-1}}\left(\frac{1}{2}\mathrm{B}\left(m,\frac{1}{2}\right)\right) \qquad = \frac{1}{2^{2m-1}}\left(\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}\right)$$

Thus we have $\dfrac{(\Gamma(m))^2}{\Gamma(2m)} = \dfrac{1}{2^{2m-1}}\left(\dfrac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}\right)$

Gamma Function

$$\Rightarrow \frac{\left(\Gamma(m)\right)^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\left(\frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}\right)$$

$$\Rightarrow \Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

Gamma Function

10. Prove that $\int_0^2 \left(8 - x^3\right)^{-\frac{1}{3}} dx = \frac{2\pi}{3\sqrt{3}}$

SOLUTION:

Let $I = \int_0^2 \left(8 - x^3\right)^{-\frac{1}{3}} dx$

Set $x^3 = 8y \Rightarrow 3x^2 dx = 8dy$

$$\Rightarrow dx = \frac{8}{3x^2} dy = \frac{8}{3} x^{-2} dy = \frac{8}{3} \left(8y\right)^{-\frac{2}{3}} dy$$

Also $x = 0 \Rightarrow y = 0$ and $x = 2 \Rightarrow y = 1.$

Setting $x^3 = 8y$ in the above equation we get

Gamma Function

$$\therefore I = \int_0^1 (8-8y)^{-\frac{1}{3}} \left(\frac{8}{3}(8y)^{-\frac{2}{3}} dy \right)$$

$$= (8)^{-\frac{1}{3}} \left(\frac{8}{3} \right) (8)^{-\frac{2}{3}} \int_0^1 (1-y)^{-\frac{1}{3}} y^{-\frac{2}{3}} dy$$

$$= \frac{1}{3} \int_0^1 y^{-\frac{2}{3}} (1-y)^{-\frac{1}{3}} dy \qquad = \frac{1}{3} \int_0^1 y^{\frac{1}{3}-1} (1-y)^{\frac{2}{3}-1} dy$$

$$= \frac{1}{3} \mathrm{B}\left(\frac{1}{3}, \frac{2}{3} \right)$$

Gamma Function

$$= \frac{1}{3}\left(\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}\right) = \frac{1}{3}\left(\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}\right)$$

$$= \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{3}\left(\frac{\pi}{\sin\frac{\pi}{3}}\right) = \frac{1}{3}\left(\frac{\pi}{\left(\frac{\sqrt{3}}{2}\right)}\right)$$

$$\left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi},\ \text{where } 0 < n < 1.\right]$$

$$= \frac{2\pi}{3\sqrt{3}}$$

Gamma Function

11.Compute $\Gamma(4.5)$

Solution

Using $\Gamma(p+1) = p\,\Gamma(p)$

$$
\begin{aligned}
\Gamma(4.5) = \Gamma(3.5+1) &= 3.5\,\Gamma(3.5) \\
&= (3.5)\,(2.5)\,\Gamma(2.5) \\
&= (3.5)\,(2.5)\,(1.5)\,\Gamma(1.5) \\
&= (3.5)\,(2.5)\,(1.5)\,(0.5)\,\Gamma(0.5) \\
&= 6.5625\,\sqrt{\pi} = 11.62875
\end{aligned}
$$

Gamma Function

12. Evaluate $I=\int_0^\infty x^4\, e^{-x^4}\, dx.$

Solution:

Put $x^4=t \Rightarrow 4x^3\, dx=dt,$

$$dx=\frac{1}{4}t^{-\frac{3}{4}}\,dt$$

$$\therefore I=\int_0^\infty t\cdot e^{-t}\cdot\frac{t^{-\frac{3}{4}}}{4}\,dt=\frac{1}{4}\int_0^\infty e^{-t}\cdot t^{\frac{1}{4}}dt$$

$$=\frac{1}{4}\Gamma\left(1+\frac{1}{4}\right)=\frac{1}{4}\Gamma\left(\frac{5}{4}\right).$$

Gamma Function

$$\int_0^{\frac{\pi}{2}} \cos^p \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^p \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}\left(\frac{\sqrt{\pi}}{2}\right)$$

$If \;\; p = 2r,$

$$I = \frac{\Gamma\left(r+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma\left(r+1\right)}$$

$$= \frac{\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{2\cdot r!}$$

$$= \frac{(2r-1)(2r-3)\cdots 3\cdot 1}{2r\cdot(2r-2)\cdot(2r-4)\cdots 2}\cdot\frac{\pi}{2}$$

$If \;\; p = 2r+1,$

$$I = \frac{\Gamma(r+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(r+\frac{3}{2}\right)}$$

$$= \frac{r!\sqrt{\pi}}{2\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{2^r\cdot r!}{(2r+1)(2r-1)\cdots 5\cdot 3\cdot 1}$$

$$= \frac{2\cdot 4\cdot 6\cdots(2r-2)\cdot 2r}{1\cdot 3\cdot 5\cdots(2r-1)(2r+1)}$$

Gamma Function

13. $Evaluate \int_0^{\frac{\pi}{2}} \sin^{10}\theta\, d\theta$

Solution

$$\int_0^{\frac{\pi}{2}} \sin^{10}\theta\, d\theta = \frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{\pi}{2} = \frac{63}{256}\pi$$

14. $Evaluate \int_0^{\frac{\pi}{2}} \cos^{9}\theta\, d\theta$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^{9}\theta\, d\theta = \frac{2.4.6.8}{1.3.5.7.9} = \frac{384}{945}$$

Gamma Function

15. $$\text{Evaluate } \int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta \cdot \cos^7\theta\, d\theta$$

Solution

$$\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta \cdot \cos^7\theta\, d\theta = \frac{1}{2}\beta\left(\frac{6+1}{2}, \frac{7+1}{2}\right) = \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{\Gamma\left(\frac{15}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\left(\frac{7}{2}-3\right).3!}{\left(\frac{15}{2}-1\right)\left(\frac{15}{2}-2\right)\left(\frac{15}{2}-3\right)\left(\frac{15}{2}-4\right)\left(\frac{15}{2}-5\right)\left(\frac{15}{2}-6\right)\left(\frac{15}{2}-7\right)}$$

$$= \frac{2^4}{3.7.11.13}$$

MATHEMATICS - I

Unit-II

Error Function

ERROR FUNCTION

Error function of *x* denoted by *erf x*, is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Error function is also called ***the error integral*** or ***the probability integral***.

PROPERTIES ERROR FUNCTION

1. $erf(0) = 0$

2. $erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$

3. It is defined for all x, -∞ < x < ∞, monotonically increasing in the interval (0, ∞); passes through origin.

4. It is an odd function since

$$erf(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-v^2}(-dv)$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^{x} e^{-v^2}dv = -erf(x)$$

PROPERTIES ERROR FUNCTION

5. $$erf(-\infty) = -erf(\infty) = -1$$

6. complementary error function of x, denoted by

$$erf_c(x) = 1 - erf(x) = erf(\infty) - erf(x)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

7. $$erf_c(x) + erf_c(-x) = [1 - erf(x)] + [1 - erf(-x)]$$

$$= 2 - erf(x) + erf(x) \quad = 2$$

PROBLEMS INVOLVING ERROR FUNCTION

16. Prove that $\frac{d}{dx}\left[erf\left(-ax\right)\right] = -\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$

SOLUTION:

By definition, we have $erf(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}\,dt$

$$= \frac{2}{\sqrt{\pi}}\int_0^y \left(\sum_{n=0}^{\infty}\frac{\left(-t^2\right)^n}{n!}\right)dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^y \left(1-\frac{t^2}{1!}+\frac{t^4}{2!}-\frac{t^6}{3!}+\frac{t^8}{4!}-+\cdots\right)dt$$

PROBLEMS INVOLVING ERROR FUNCTION

$$= \frac{2}{\sqrt{\pi}}\left[t - \frac{t^3}{3} + \frac{t^5}{5(2!)} - \frac{t^7}{7(3!)} + \frac{t^9}{9(4!)} - + \cdots\right]_0^y$$

$$\therefore erf(y) = \frac{2}{\sqrt{\pi}}\left[y - \frac{y^3}{3} + \frac{y^5}{5(2!)} - \frac{y^7}{7(3!)} + \frac{y^9}{9(4!)} - + \cdots\right]$$

Setting $y = -ax$ in the above equation we get

$$\therefore erf(ax) = \frac{2}{\sqrt{\pi}}\left[-ax + \frac{a^3x^3}{3} - \frac{a^5x^5}{5(2!)} + \frac{a^7x^7}{7(3!)} - \frac{a^9x^9}{9(4!)} + - \cdots\right]$$

PROBLEMS INVOLVING ERROR FUNCTION

$$\therefore \frac{d}{dx}\left[erf\left(ax\right)\right] = -\frac{2}{\sqrt{\pi}}\left[a - a^3x^2 + \frac{a^5x^4}{2!} - \frac{a^7x^6}{3!} + \frac{a^9x^8}{4!} - + \cdots\right].$$

$$= -\frac{2a}{\sqrt{\pi}}\left[1 - a^2x^2 + \frac{\left(a^2x^2\right)^2}{2!} - \frac{\left(a^2x^2\right)^3}{3!} + \frac{\left(a^2x^2\right)^4}{4!} - + \cdots\right].$$

$$= -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$

$$\therefore \frac{d}{dx}\left[erf\left(ax\right)\right] = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}.$$