Rolle's Theorem and Mean Value Theorems

3

3.1 INTRODUCTION

Continuous and differentiable functions have many interesting properties, some of which are studied through Rolle's theorem and other mean value theorems. Taylor's and Maclaurin's series, which are generalisations of the mean value theorems are useful in approximating transcendental functions. We briefly review the definitions and properties of continuity and derivability of functions.

Continuity

A function $f : [a, b] \to R$ is said to be continuous at a point $c \in (a, b)$ if $\lim_{x \to c} f(x) = f(c)$.

f is continuous on the open interval (a, b) if it is continuous at every point c of (a, b).

f is continuous on the closed interval $[a, b]$ if

(1) f is continuous on the open interval (a, b),

(2) f is continuous from the right at the left end point 'a'

$$i.e., \quad f(a+) = \lim_{x \to a+} f(x) = f(a) \text{ and}$$

(3) f is continuous from the left at the right end point 'b'

$$i.e., \quad f(b-) = \lim_{x \to b-} f(x) = f(b)$$

Properties of continuous functions

(1) If f is continuous in a closed interval $[a, b]$, then f is bounded in that interval and further it attains its bounds at least once in $[a, b]$ *i.e.*, there exist $c, d \in [a, b]$ such that $f(c) = \text{Sup} f = M$; $f(d) = \text{Inf} f = m$ (Fig. 3.1(a)).

(2) $f(x)$ attains every value between $f(a)$ and $f(b)$.

(3) If $f(a) f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$ (see Fig. 3.1(b)).

Differentiability

A function f: $[a, b] \to R$ is said to be differentiable (derivable) at a point $c \in (a, b)$ if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists and is finite. The limit is called the derivative of f at c and is denoted by $f'(c)$ or $\left.\frac{df}{dx}\right|_{x=c}$.

f is derivable in (a, b) if f is derivable at each point of (a, b). The function defined by these derived values is called the derivative $\frac{df}{dx}$ of f on (a, b).

The derivability of f at a point c implies continuity there. f is increasing or decreasing at c according as $f'(c) \gtrless 0$.

3.1.1 Rolle's Theorem

Theorem 3.1 If $f : [a, b] \to R$ is

(i) continuous in the closed interval $[a, b]$

(ii) derivable in the open interval (a, b) and

(iii) $f(a) = f(b)$ then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Proof: $f(x)$ is continuous in $[a, b]$

$\Rightarrow$ $f(x)$ is bounded and attains its bounds

$\Rightarrow$ There exist c, d in $[a, b]$ such that

$f(c) = \sup f = M$ and $f(d) = \text{Inf} f = m$ in $[a, b]$.

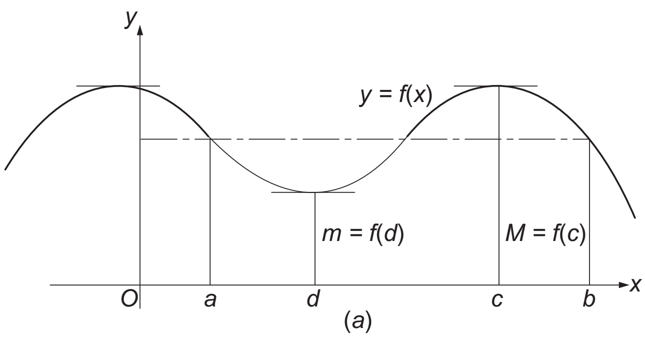

Figure 3.1a Continuous function attains its bounds: $f(c) = M, f(c) = m$

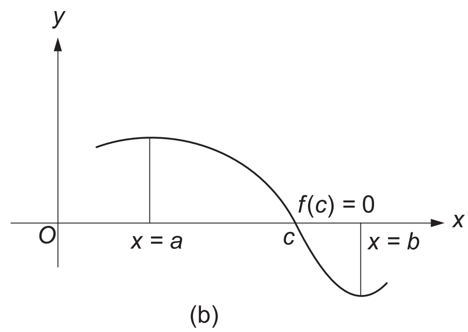

Figure 3.1b f is Continuous function; $f(a)f(b) < 0$. Then there exists c such that $f(c) = 0$

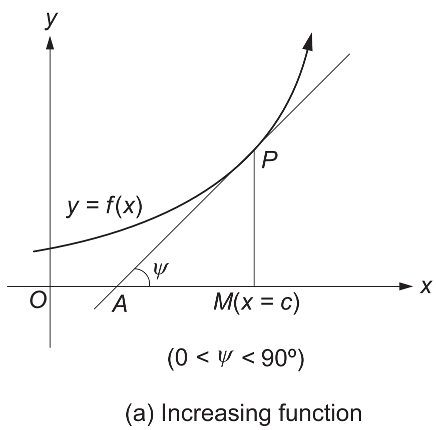

Figure 3.2a

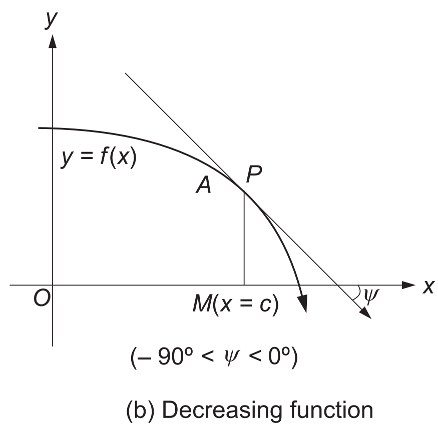

Figure 3.2b

Case (i) $M = m \Rightarrow f(x)$ is constant $\Rightarrow f'(x) = 0$ for all x.

In particular, $f'(c) = 0 \quad (a < c < b)$

Case (ii) $M \neq m$ At least one of these numbers is different from the equal values $f(a)$ and $f(b)$.

Suppose $M = f(c),\ M \neq f(a),$
$M \neq f(b) \Rightarrow c \neq a, c \neq b$

Since $f(c) = M = \sup f$ in $[a, b]$

$$\therefore \quad f(x) \leq f(c) \quad \text{for all } x \in [a, b] \tag{3.1}$$

In particular, if $h > 0$ such that $(c - h) \in [a, b]$, we have

$$f(c-h) \leq f(c) \Rightarrow \frac{f(c-h) - f(c)}{-h} \geq 0 \tag{3.2}$$

$$\Rightarrow Lf'(c) \geq 0 \tag{3.3}$$

Similarly, if $h > 0$ such that $(c+h) \in [a, b]$, we have

$$f(c+h) \leq f(c)$$

$$\Rightarrow \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\Rightarrow Rf'(c) \leq 0 \tag{3.4}$$

Since $f'(c)$ exists, we have

$$f'(c) = 0 \qquad \text{by Eqs. (3.3) and (3.4)}$$

Note
The conditions of Rolle's theorem are sufficient but are not necessary. That is to say, we may have the conclusion of Rolle's theorem, namely, the existence of c in $[a, b]$ such that $f'(c) = 0$ even if any of the conditions are not satisfied. Illustrated below are counter-examples.

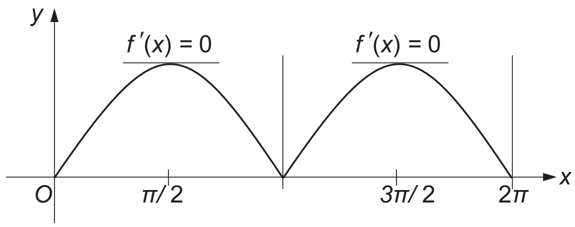

Figure 3.3 Graph of $f(x) = |\sin x|$

Example 3.1

$f(x) = \sin\frac{1}{x}$ is discontinuous at '0' in $[-1, 1]$

(Condition (1) fails to hold.)

But $f'(x) = \frac{-1}{x^2}\cos\frac{1}{x} = 0$

at $x = x_n = \frac{2}{(2n+1)\pi} \in (-1, 1)$.

Example 3.2

$f(x) = |\sin x|$ is not differentiable at $x = \pi$ in $(0, 2\pi)$

$$\begin{aligned} f(x) &= \sin x \quad \forall\ x \in [0, \pi] \\ &= -\sin x \quad \forall\ x \in [\pi, 2\pi] \end{aligned}$$

(Condition (2) fails to hold.)

But $f'(x) = \cos x$ in $(0, \pi)$ and

$f'(x) = -\cos x$ in $(\pi, 2\pi)$

which vanish at $x = \pi/2$, in $(0, \pi)$ and at $x = 3\pi/2$ in $(\pi, 2\pi)$, respectively (Fig. 3.3).

Example 3.3

$f(x) = \sin x$ in $[0, 3\pi/4]$

f is continuous in $[0, 3\pi/4]$ and derivable in $(0, 3\pi/4)$. But $f(0) = 0$ while $f(3\pi/4) = \sin 3\pi/4 = \sin\pi/4 = 1/\sqrt{2}$ (Condition (3) is not satisfied.)

But f is derivable and

$$f'(x) = \cos x = 0 \text{ at } x = \pi/2 \in (0, 3\pi/4).$$

3.1.2 Geometrical Interpretation

Rolle's theorem states that the graph of f under the assumption has a horizontal tangent for at least one point c where $c \in (a, b)$.

Example 3.4

Verify Rolle's theorem for $f(x) = x^2 - 1$ in $[-1, 1]$.

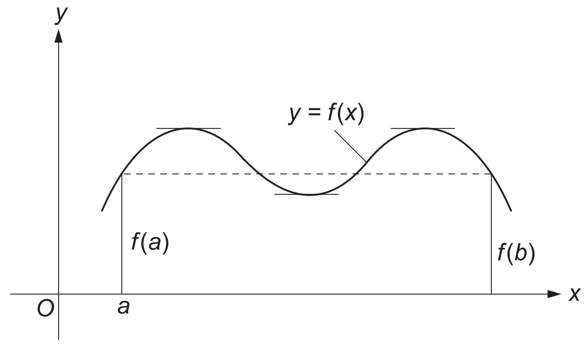

Figure 3.4 Geometrical illustration of Rolle's theorem

Solution $f(x) = x^2 - 1$ is a polynomial function which is continuous and derivable for all x.

In particular, f is continuous in $[-1, 1]$ and derivable in $(-1, 1)$. Also, $f(-1) = 0 = f(1)$

Thus, f satisfies all the conditions of Rolle's theorem and hence there exists at least one point x in $(-1, 1)$ such that

$$f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$$\therefore \quad c = \frac{1}{2}$$

Hence Rolle's theorem is verified.

Example 3.5

Examine if Rolle's theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.

Solution

$$f(x) = |x| = \begin{cases} x, & 0 \le x \le 1 \\ -x, & -1 \le x < 0 \end{cases}$$

$$\left.\begin{array}{l} f \text{ is continuous and derivable at } (0,1) \\ \text{and } (-1, 0) \text{ and continuous at } x = \pm 1 \end{array}\right\} \quad (1)$$

Continuity at $x = 0$

$$\begin{aligned} f(0+) &= \lim_{x\to 0+} f(x) = \lim_{x\to 0+} x = 0 \\ f(0-) &= \lim_{x\to 0-} f(x) = \lim_{x\to 0}(-x) = -0 \\ &\Rightarrow \lim_{x\to 0} f(x) = f(0) = 0 \\ &\Rightarrow f(x) \text{ is continuous at } x = 0 \qquad (2) \end{aligned}$$

and hence in $[-1, 1]$, by Eqs. (1) and (2).

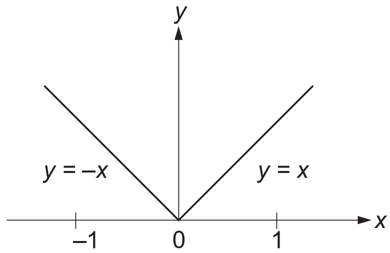

Figure 3.5 Modulus Function $y = (x)$

Derivability at $x = 0$

$$Rf'(0)=f'(0+) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{x-0}{x} = 1$$

$$Lf'(0)=f'(0-) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{-x-0}{x} = -1$$

$Rf'(0) \neq Lf'(0)$ or $f'(0+) \neq f'(0-)$

$\Rightarrow$ f is not derivable at $x = 0$ and hence not derivable at $(-1, 1)$. Also, $f(-1) = f(1) = 1$.

Since the second condition of derivability is not satisfied, Rolle's theorem is not applicable.

Example 3.6

Verify Rolle's theorem for $f(x) = e^{-x}\sin x$ in $[0, \pi]$. [JNTU 2000S]

Solution Since $\sin x$ and e^{-x} are continuous and derivable for all x, the function $f(x) = e^{-x}\sin x$ is continuous at $[0, \pi]$ and derivable at $(0, \pi)$,

$$f(0) = e^0 \sin 0 = 0 \quad \text{and}$$
$$f(\pi) = e^{-\pi}\sin\pi = e^{-\pi}\cdot 0 = 0$$
$$\Rightarrow f(0) = f(\pi) = 0$$

All the conditions of Rolle's theorem are satisfied. There exists $x \in (0, \pi)$ such that

$f'(x) = e^{-x}(\cos x - \sin x) = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow x = \dfrac{\pi}{4}$.

For $c = \dfrac{\pi}{4} \in (0, \pi)$, we have $f'(c) = 0$.

Example 3.7

Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$. [JNTU 2001]

Solution Functions $x(x+3)$ and $e^{-x/2}$ are continuous and derivable for all x. So, $f(x)$ is continuous and derivable. Also, $f(-3) = f(0) = 0$.

All the conditions of Rolle's theorem are satisfied. There exists $x \in (-3, 0)$ such that $f'(x) = 0$

$$\Rightarrow \quad f'(x) = \frac{e^{x/2}}{2}(x^2 - x - 6) = 0$$
$$\Rightarrow \quad x = 3 \text{ or } -2$$
$$f'(3) = 0 \quad \text{and} \quad f'(-2) = 0$$

c of Rolle's theorem is $c = -2 \in (-3, 0)$
$\because 3 \notin (-3, 0)$.

Example 3.8

If $f(x), g(x)$ and $h(x)$ are continuous in $[a, b]$ and are derivable in (a, b), then show that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

Solution Let $\phi\colon [a, b] \to R$ be defined by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \qquad (1)$$

$\Rightarrow$ $\phi(x) = pf(x) + qg(x) + rh(x)$, where

$$p = \begin{vmatrix} g(a) & h(a) \\ g(b) & h(b) \end{vmatrix}, \quad q = -\begin{vmatrix} f(a) & h(a) \\ f(b) & h(b) \end{vmatrix},$$

$$r = \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix}$$

f, g, h are continuous in $[a, b]$ and p, q and r are real constants.

$\Rightarrow$ $\phi(x)$ is continuous in $[a, b]$ and derivable in (a, b). Also, $\phi(a) = 0 = \phi(b)$ since two rows become identical in the determinant in Eq. (1).

$\therefore$ ϕ satisfies all the conditions of Rolle's theorem so that there exists $c \in (a, b)$ such that

$$\phi'(c) = pf'(c) + qg'(c) + rh'(c) = 0$$

$$\Rightarrow \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

Example 3.9

If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$, prove that the equation $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ has at least one root in (0, 1).

Solution Let $f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \cdots + \frac{a_{n-1}}{2}x^2 + a_nx$ f is continuous on $[0, 1]$ and derivable on (0, 1).

$$\text{Also,} \quad f(0) = 0 = f(1), \quad \text{(by hypothesis)}$$

f satisfies all the conditions of Rolle's theorem so that there exists $c \in (0, 1)$ such that

$$f'(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$
$$\Rightarrow \quad f'(c) = a_0c^n + a_1c^{n-1} + \cdots + a_n = 0$$
$$\Rightarrow \quad f'(x) = 0 \text{ has a root in } (0, 1).$$

EXERCISE 3.1

1. Verify Rolle's theorem for each of the functions in problems a–h.

(a) $f(x) = x^2 - 2x$ in $[0, 2]$. [JNTU 1996*S*]
Ans: $c = 1$

(b) $f(x) = \sin x$ in $[-\pi, \pi]$.
Ans: $c = \pm\pi/2$

(c) $f(x) = \sqrt{4 - x^2}$ in $[-2, 2]$.
Ans: $c = 0$

(d) $f(x) = (x - a)^m(x - b)^n$ in $[a, b]$, $m > 0, n > 0$.
Ans: $c = \frac{mb + na}{m + n}$

(e) $f(x) = x^2 - 5x + 6$ in $[2, 3]$.
Ans: $c = \frac{5}{2}$

(f) $f(x) = e^x \sin x$ in $[0, \pi]$.
Ans: $c = \frac{3\pi}{4}$

(g) $f(x) = x^3(1 - x)^2$ in $[0, 1]$.
Ans: $c = \frac{3}{5}$

(h) $f(x) = \log \frac{x^2 + ab}{(a + b)x}$ in $[a, b]$.
Ans: $c = \sqrt{ab}$

2. Examine if Rolle's theorem is applicable for the function $f(x) = \tan x$ in $[0, \pi]$.

Ans: Rolle's theorem is not applicable since f is discontinuous at $x = \pi/2$

3.1.3 Lagrange's Mean Value Theorem

Theorem 3.2 If $f : [a, b] \to R$ is (i) continuous in the closed interval $[a, b]$ and (ii) derivable in the open interval (a, b) then there exists a real number c such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof: Let $F(x) = f(x) + Ax$ where A is a constant such that $F(a) = F(b) \Rightarrow f(a) + Aa = f(b) + Ab$

$$\Rightarrow \quad -A = \frac{f(b) - f(a)}{b - a} \tag{3.5}$$

Clearly F is continuous in $[a, b]$ and derivable in (a, b) since $f(x)$ and Ax satisfy these conditions. Also, $F(a) = F(b)$

Since F satisfies all the conditions of Rolle's theorem there exists c in (a, b) such that $F'(c) = 0$.

$$\Rightarrow \quad F'(c) = f'(c) + A = 0 \tag{3.6}$$

$$\Rightarrow \quad -A = f'(c) = \frac{f(b) - f(a)}{b - a} \tag{3.7}$$

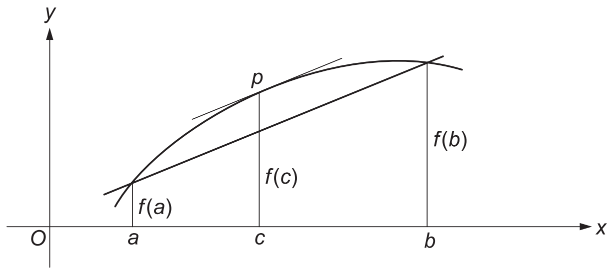

Figure 3.6 The Mean Value Theorem of Differential Calculus (Lagrange's Mean Value Theorem)

3.1.4 Geometrical Interpretation of Lagrange's Mean Value Theorem

Let $y = f(x)$ be a curve whose graph is continuous in $[a, b]$ and at each point in (a, b) a tangent can be drawn. The difference quotient $\dfrac{f(b)-f(a)}{b-a}$ gives the slope of the secant (chord) to the curve through the points $(a, f(a))$ and $(b, f(b))$. Then there must be at least one point c in the interval (a, b) for which the tangent to the curve is parallel to the secant. Both then have the same slope

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Another form

If the values of a and b are denoted by $a = x$ and $b = x + h$, then c can be expressed in the form $c = x + \theta h$ $(0 < \theta < 1)$.

The mean value theorem has the form

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h) \quad (0 < \theta < 1).$$

Corollary 1 If f is continuous in $[a, b]$ and derivable in (a, b), and $f'(x) = 0$ for all $x \in (a, b)$ then f is constant.

Proof: Let p be any point in (a, b).

f is continuous in $[a, p]$ and derivable in (a, p).

By Lagrange mean value theorem, there exists a point c in (a, p) such that $f(p) - f(a) = (p - a)f'(c)$.

But $f'(x) = 0$ for all $x \in (a, b) \Rightarrow f'(c) = 0$
$\Rightarrow f(p) = f(a)$.

Since p is any point in (a, b), f is constant in (a, b).

Corollary 2 A function f is said to be monotonically increasing in an interval if

$$x < y \;\Rightarrow\; f(x) < f(y) \quad \text{for all } x, y \in I$$

and monotonically decreasing in I if

$$x < y \;\Rightarrow\; f(x) > f(y) \quad \text{for all } x, y \in I$$

If f is such that $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is monotonically increasing in $[a, b]$.

Proof: Let p be any point in (a, b).

By mean value theorem,

$$\frac{f(p)-f(a)}{p-a} = f'(c) > 0 \quad (a < c < p)$$

$$p > a \;\Rightarrow\; f(p) > f(a)$$

$$\text{for all } p \in (a, b)$$

$\therefore$ f is monotonically increasing in (a, b).

If $f'(x) < 0$, then

$$\frac{f(p)-f(a)}{p-a} = f'(c) < 0$$

Therefore, we have

$$p > a \Rightarrow f(p) < f(a) \quad \text{for all } p \in (a, b)$$

$\therefore$ f is monotonically decreasing in (a, b).

Example 3.10

Verify Lagrange's mean value theorem for $f(x) = (x - 1)(x - 2)$ in $[1,3]$.

Solution $f(x) = (x - 1)(x - 2) = x^2 - 3x + 2$

f is a polynomial function. So, it is continuous and derivable for all x.

$a = 1, \quad b = 3;$

$f(1) = 0 \quad$ and $\; f(3) = 2; \quad f'(x) = 2x - 3$

The conditions of Lagrange's mean value theorem are satisfied.

$\therefore$ There exists c in $(1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow f'(c) = 2c - 3 = \frac{2 - 0}{3 - 1} = 1$$

$$\Rightarrow c = 2 \in (1, 3)$$

Hence Lagrange's mean value theorem is verified.

Example 3.11
Find c of Lagrange's mean value theorem for
$f(x) = x(x-1)(x-2)$ in $[0, 1/2]$.

Solution $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$.
f is a polynomial function. So, it is continuous and derivable for all x.
$a = 0, \quad b = \frac{1}{2};$
$f(a) = f(0) = 0, \quad f(b) = f\left(\frac{1}{2}\right) = \frac{3}{8}$

The conditions of Lagrange's mean value theorem are satisfied.

$$\text{Also,} \qquad f'(x) = 3x^2 - 6x + 2$$
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$\Rightarrow \quad 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{1/2} = \frac{3}{4}$$
$$12c^2 - 24c + 5 = 0$$
$$\Rightarrow c = \frac{6 \pm \sqrt{21}}{6}$$

Since $\dfrac{6 + \sqrt{21}}{6} \not\in (0, 1/2)$

the required $c = \dfrac{6 - \sqrt{21}}{6}$.

Example 3.12
Verify Lagrange's mean value theorem for
$f(x) = \log x$ in $[1, e]$.

Solution $f(x) = \log x, \quad a = 1, \quad b = e;$
$f(a) = f(1) = 0, \quad f(b) = f(e) = 1$
f is continuous and derivable for all $x > 0$.

Also, $f'(x) = \dfrac{1}{x}$; all the conditions of Lagrange's mean value theorem are satisfied.

$$\therefore \quad \frac{1}{c} = \frac{1 - 0}{e - 1} \Rightarrow c = (e - 1) \in (1, e)$$

Hence Lagrange's mean value theorem is verified.

Example 3.13
Find c of Lagrange's mean value theorem if
$f(x) = e^x$ in [0, 1].

Solution $f(x) = e^x \quad a = 0, \quad b = 1;$
$f(a) = f(0) = 1, \quad f(b) = f(1) = e^1 \quad f'(x) = e^x$

Clearly f is continuous in $[0, 1]$ and derivable in $(0, 1)$. So, all the conditions of Lagrange's mean value theorem are satisfied.

$$\therefore \quad f'(c) = e^c = \frac{e - 1}{1 - 0} \Rightarrow c = \log(e - 1) \in (0, 1).$$

Example 3.14
Examine if Lagrange's mean value theorem can be applied for

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \text{in } [-1, 1].$$

Solution For $x \neq 0$, f is clearly continuous and derivable

Continuity at $x = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$$
$$\because \quad \left| \sin \frac{1}{x} \right| \leq 1$$

So, f is continuous at $x = 0$ also.

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{h \sin 1/h}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

does not exist.

So f is not derivable at $x = 0$.

Hence Lagrange's mean value theorem cannot be applied.

Example 3.15
If $f(x) = x^2$ find $\theta \in (0, 1)$ such that
$f(x + h) = f(x) + hf'(x + \theta h)$.

Solution $f(x) = x^2$ is continuous and derivable for all x. $f'(x) = 2x$

By Lagrange's mean value theorem,

$$f'(x + \theta h) = \frac{f(x + h) - f(x)}{h}$$
$$\Rightarrow 2(x + \theta h) = 2x + h \Rightarrow \theta = \frac{1}{2}$$

Example 3.16
Prove that

$$x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1 + x)} \qquad \text{for } x > 0. \qquad (1)$$

Solution Let $f(x) = \log(1+x) - \left(x - \frac{x^2}{2}\right)$. f is continuous and derivable for all $x > 0$.

$$f'(x) = \frac{1}{1+x} - (1-x) = \frac{x^2}{1+x} > 0 \text{ for } x > 0$$

$\Rightarrow$ f is monotonically increasing for $x > 0$.

$$\Rightarrow \quad x - \frac{x^2}{2} < \log(1+x) \qquad (2)$$

Let $g(x) = x - \dfrac{x^2}{2(1+x)} - \log(1+x)$

g is continuous and derivable for all $x > 0$.

$$g'(x) = 1 - \frac{x}{1+x} + \frac{x^2}{2(1+x)^2} - \frac{1}{1+x}$$
$$= \frac{x^2}{2(1+x)^2} > 0$$

$\Rightarrow$ $g(x)$ is monotonically increasing for $x > 0$.

$$\Rightarrow \quad \log(1+x) < x - \frac{x^2}{2(1+x)} \qquad (3)$$

From Eqs. (2) and (3), we obtain Eq. (1).

Note
Using Lagrange's mean value, the approximate solutions of equations of the form $f(x) = 0$ can be obtained by Newton's method.

Suppose $(a+h)$ is an exact root of $f(x) = 0$,

we have
$$\begin{aligned} 0 &= f(a+h) \\ &= f(a) + hf'(a+\theta h) \quad (0 < \theta < 1) \\ \Rightarrow h &\simeq -\frac{f(a)}{f'(a)} \quad \text{(Newton's formula).} \end{aligned}$$

Example 3.17
Calculate the approximate root near 2 for the equation $f(x) = x^4 - 12x + 7 = 0$.

Solution

$$f(x) = x^4 - 12x + 7 = 0$$
$$f'(x) = 4x^3 - 12$$

Take $a = 2$ $\quad f(2) = 16 - 24 + 7 = -1$

$$f'(2) = 32 - 12 = 20$$

By Lagrange's mean value theorem,

$$f(a+h) \simeq f(a) + hf'(a+\theta h)$$
$$\Rightarrow \quad h = -\frac{f(a)}{f'(a)}$$
$$\Rightarrow \quad h = -\frac{-1}{20} = \frac{1}{20} = 0.05$$

An approximate root is $a + h = 2 + 0.05 = 2.05$. Applying Lagrange's mean value theorem again and taking $a = 2.05$

$$h = -\frac{f(a)}{f'(a)} = -\frac{f(2.05)}{f'(2.05)} = -\frac{0.061}{22.46} = -0.0027$$

A second approximation to the root is

$$2.05 - 0.0027 = 2.0473.$$

Example 3.18
Calculate approximately $\sqrt[5]{245}$ using Lagrange's mean value theorem.

Solution Let $f(x) = x^{1/5}$, $a = 243 = 3^5$, $b = 245$

$$f'(x) = \left(\frac{1}{5}\right) x^{-4/5}$$
$$f(a+h) = f(a) + hf'(c)$$

c is a point in $(a, b) = (243, 245)$.

Since we are interested in an approximate value of $f(a+h)$, we may take c as 243 so that

$$\begin{aligned} f'(c) &= f'(243) = f'(3^5) = \frac{1}{5}(3^5)^{-4/5} \\ &= \frac{1}{5} \cdot 3^{-4} = \frac{1}{5} \cdot \frac{1}{81} \\ f(245) &= f(243) + (245 - 243) f'(243) \\ \sqrt[5]{245} &= (243)^{1/5} + 2 \cdot \frac{1}{5} \cdot \frac{1}{81} \\ &= 3 + 2 \cdot \frac{1}{5} \cdot \frac{1}{81} \\ &= 3 + 0.0049 = 3.0049. \end{aligned}$$

EXERCISE 3.2

1. Verify Lagrange's mean value theorem for each of the functions in problems (a) to (e):

(a) $f(x) = x - x^3$ in $[-2, 1]$ [JNTU 2002]
Ans: $c = -1$

(b) $f(x) = 5x^2 + 7x + 6$ in $[3, 4]$
Ans: $c = \frac{7}{2}$

(c) $f(x) = \frac{1}{x}, \quad a = 1, \quad b = 4$
Ans: $c = 2$

(d) $f(x) = e^x$ in $[0, 1]$
Ans: $c = e - 1$

(e) $f(x) = \log x$ in $[e^2, e^3]$
Ans: $c = (e-1)e^2$

2. Show that

(a) $\cos x > 1 - \frac{x^2}{2}$

(b) $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{x}{3} - \frac{1}{8}$
[JNTU 1998S]

(c) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$
[JNTU 1996]

3. Calculate an approximate value of $\sqrt[6]{65}$ using Lagrange's mean value theorem.
[**Hint:** Take $f(x) = x^{1/6}$ in $(64, 65)$
$f(a+h) = f(a) + hf'(a)$]
Ans: : 2.0052

4. Find an approximate value for the root of $x^3 - 2x - 5$ in $(2, 3)$.

[**Hint:** $f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$ Take $a = 2$, at $h = 2.1$

$$= -\frac{f(a)}{f'(a)} = -\frac{f(2)}{f'(2)} = 0.1$$

Take $a = 2.1$ $\quad h = -\frac{f(2.1)}{f'(2.1)} = -0.0053$

root $= 2.1 - 00543 = 2.0946$]

Ans: root=2.0946

5. Using Lagrange's theorem show that

$$x > \log(1+x) > \frac{x}{1+x}.$$

6. Test if Lagrange's mean value theorem holds for $f(x) = x - x^3$ in $[-2, 1]$. If so, find an approximate value of c. [JNTU 2002]
Ans: $c = 1$ or -1

3.1.5 Cauchy's Mean Value Theorem

Theorem 3.3 Let $f : [a, b] \to R$, $g : [a, b] \to R$ be

(i) continuous in the closed interval $[a, b]$
(ii) derivable in the open interval (a, b)
and (iii) $g'(x) \neq 0$ for all x in (a, b); then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof: Let

$$\phi(x) = f(x) + Ag(x) \tag{3.7}$$

where A is a constant such that

$$\phi(a) = \phi(b) \Rightarrow f(a) + Ag(a) = f(b) + Ag(b)$$
$$\Rightarrow -A = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{3.8}$$

If $g(b) = g(a)$ then by Rolle's theorem there exists an x in (a, b) such that $g'(x) = 0$, which contradicts condition (3) of the hypothesis.

Now $\phi(x)$ satisfies all the conditions of Rolle's theorem and hence there exists a point c in (a, b) such that $\phi'(c) = 0$

$$\Rightarrow -A = \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Another form

Put $b = a + h$, $c = a + \theta h$ where $(0 < \theta < 1)$, we have

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)} \quad (0 < \theta < 1).$$

Note
Cauchy's mean value theorem is an extension (generalisation) of Lagrange's mean value theorem and

if we take $g(x) = x$, Cauchy's mean value theorem reduces to Lagrange's mean value theorem.

Example 3.19
Verify Cauchy's mean value theorem for the following pairs of functions:

(i) $f(x) = x^3, \quad g(x) = x^2$ in $[1, 2]$

(ii) $f(x) = \log x, \quad g(x) = \dfrac{1}{x}$ in $[1, e]$

(iii) $f(x) = \sin x, \quad g(x) = \cos x$ in $[a, b]$

Solution

(i) $f(x) = x^3, \quad g(x) = x^2 \quad$ in $[1, 2]$

$$f'(x) = 3x^2, \quad g'(x) = 2x$$

$$a = 1, \quad b = 2$$

$$f(a) = f(1) = 1, \quad g(a) = g(1) = 1$$

$$f(b) = f(2) = 8, \quad g(b) = g(2) = 4$$

$$\frac{f'(c)}{g'(c)} = \frac{3c^2}{2c} = \frac{3}{2}c$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{8-1}{4-1} = \frac{7}{3} = \frac{3c^2}{2c} = \frac{3}{2}c$$

$$\Rightarrow c = \frac{14}{9}.$$

(ii) $f(x) = \log x, \; g(x) = \dfrac{1}{x} \quad$ in $[1, e]$

$$f'(x) = \frac{1}{x}, \; g'(x) = -\frac{1}{x^2}$$

$$a = 1, \; b = e \qquad \frac{f'(c)}{g'(c)} = \frac{1/c}{-1/c^2} = -c$$

$$f(a) = f(1) = 0, \quad g(a) = g(1) = 1$$

$$f(b) = f(e) = 1, \quad g(b) = g(e) = \frac{1}{e}$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{1-0}{\frac{1}{e} - 1} = -c$$

$$\Rightarrow c = \frac{e}{e-1} \in (1, e).$$

(iii) $f(x) = \sin x, \; g(x) = \cos x$

$$f'(x) = \cos x, \quad g'(x) = -\sin x$$

$$f(a) = \sin a, \quad g(a) = \cos a;$$

$$\frac{f'(c)}{g'(c)} = \cot c$$

$$f(b) = \sin b, \quad g(b) = \cos b;$$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{\sin b - \sin a}{\cos b - \cos a} = -\cot c$$

$$\Rightarrow \frac{2\cos\frac{b+a}{2}\sin\frac{b-a}{2}}{-2\sin\frac{b+a}{2}\sin\frac{b-a}{2}} = -\cot c$$

$$\Rightarrow \cot\frac{a+b}{2} = \cot c \Rightarrow c = \frac{a+b}{2}.$$

Example 3.20
Let $f : [a, b] \to R$ be continuous in $[a, b]$ and derivable in (a, b). Then show that there exists a number c in (a, b) such that $2c[f(a) - f(b)] = f'(c)(a^2 - b^2)$.

Solution By taking $g(x) = x^2$ in Cauchy's mean value theorem, we have

$$\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$$

$$\Rightarrow \quad 2c(f(b) - f(a)) = f'(c)(b^2 - a^2).$$

Example 3.21
If f and g are differentiable on $[0, 1]$ such that $f(0) = 2$ and $g(0) = 0$; $f(1) = 6$ and $g(1) = 2$ then show that there exists $c \in (0, 1)$ such that $f'(c) = 2g'(c)$.

Solution f and g are derivable in $[0, 1]$.
$\Rightarrow$ f and g are continuous in $[0, 1]$.
By Cauchy's mean value theorem, there exists c such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

$$a = 0, \quad b = 1 \quad f(0) = 2, \quad f(1) = 6$$

$$g(0) = 0, \quad g(1) = 2$$

$$\therefore \quad 4g'(c) = 2f'(c) \Rightarrow f'(c) = 2g'(c).$$

Example 3.22
Find c of Cauchy's mean value theorem for

$$f(x) = \sqrt{x}, \quad g(x) = \frac{1}{\sqrt{x}} \qquad \text{in} \quad [a, b].$$

Solution x^n is continuous for all $x > 0$ and $n \in Q$. f and g are continuous on $[a, b] \subseteq R$.
Also, $f'(x) = \frac{1}{2\sqrt{x}}, \quad g'(x) = -\frac{1}{2x\sqrt{x}}$ exist for all $x > 0$.
$\Rightarrow$ f and g are derivable on $(a, b) \subseteq R^+$ and $g'(x) \neq 0$.

By Cauchy's mean value theorem, there exists $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = -\sqrt{ab} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c\sqrt{c}}}$$

$$\Rightarrow c = \sqrt{ab} \in (a, b).$$

3.1.6 Generalised Mean Value Theorem—Taylor's Theorem

Theorem 3.4 If $f : [a, a+h] \to R$ is a function whose $(n-1)$th derivative $f^{(n-1)}$ is

(i) continuous in the closed interval $[a, a+h]$

(ii) derivable in the open interval $(a, a+h)$

and (iii) p is a given positive integer, then there exists at least one number θ in (0,1) such that

$$f(a+h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a) + R_n \qquad (3.9)$$

$$(f^{(0)} \equiv f)$$

$$\text{where } R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(a+\theta h) \qquad (3.10)$$

$$(0 < \theta < 1)$$

Proof: $f^{(r)}$ $(r = 0, 1, 2, \cdots (n-1))$ are continuous in $[a, a+h]$ and derivable in $(a, a+h)$.

$$\text{Let } \phi(x) = \sum_{r=0}^{n-1} \frac{(a+h-x)^r}{r!} f^{(r)}(x) + A(a+h-x)^p \qquad (3.11)$$

where the constant A is to be determined such that

$$\phi(a+h) = \phi(a)$$

$$\Rightarrow f(a+h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a) + h^p A, \qquad (3.12)$$

[using Eq. (3.11)]

ϕ satisfies all the conditions of Rolle's theorem so that there exists $c \in (a, a+h)$ such that

$$\phi'(c) = \phi'(a+\theta h) = 0.$$

Differentiating (3.11) with respect to x,

$$\phi'(x) = \sum_{r=0}^{n-1} \frac{(a+h-x)^r}{r!} f^{(r+1)}(x) - \sum_{r=1}^{n-1} \frac{(a+h-x)^{r-1}}{(r-1)!} f^{(r)}(x) - Ap(a+h-x)^{p-1}$$

(Except the nth term all the terms of the first series cancel out with the corresponding terms of the second series.)

$$\Rightarrow \phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - Ap(a+h-x)^{p-1}$$

$$\Rightarrow 0 = \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - Ap(1-\theta)^{p-1}h^{p-1}$$

$$\Rightarrow A = \frac{\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)}{p(1-\theta)^{p-1}h^{p-1}} = \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(a+\theta h); \qquad (3.13)$$

$$h \neq 0,\ \theta \neq 1$$

Substituting the value of A from Eq. (3.13) into Eq. (3.12)

$$f(a+h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(a+\theta h) \qquad (3.14)$$

$$(0 < \theta < 1).$$

Note

Taylor's theorem is a generalisation of Cauchy's mean value theorem. Cauchy's mean value theorem is obtained from Taylor's theorem for $n = 1$. Here, R_n denotes the remainder after n terms of Taylor's series expansion. The various forms of remainders are

(a) Schlomilch and Roche form of R_n:

$$\frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(a+\theta h)$$