C. V. Raman Global University
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MATHEMATICS-I

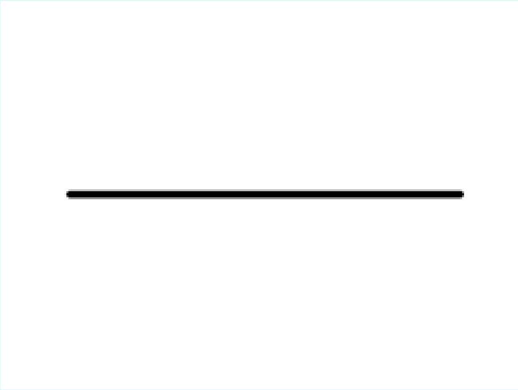
MATHEMATICS - I

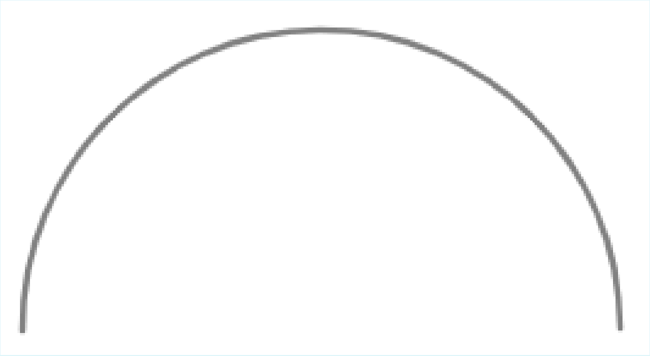
TEXT BOOK: DIFFERENTIAL CALCULUS BY SHANTI NARAYAN & P.K.MITTAL

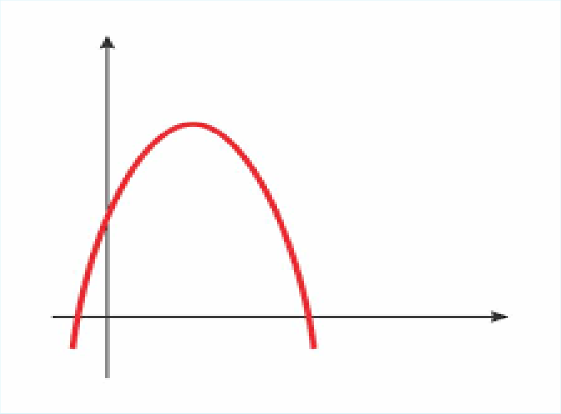
LECTURE

Curvature

Question: which figure is the most curvy.



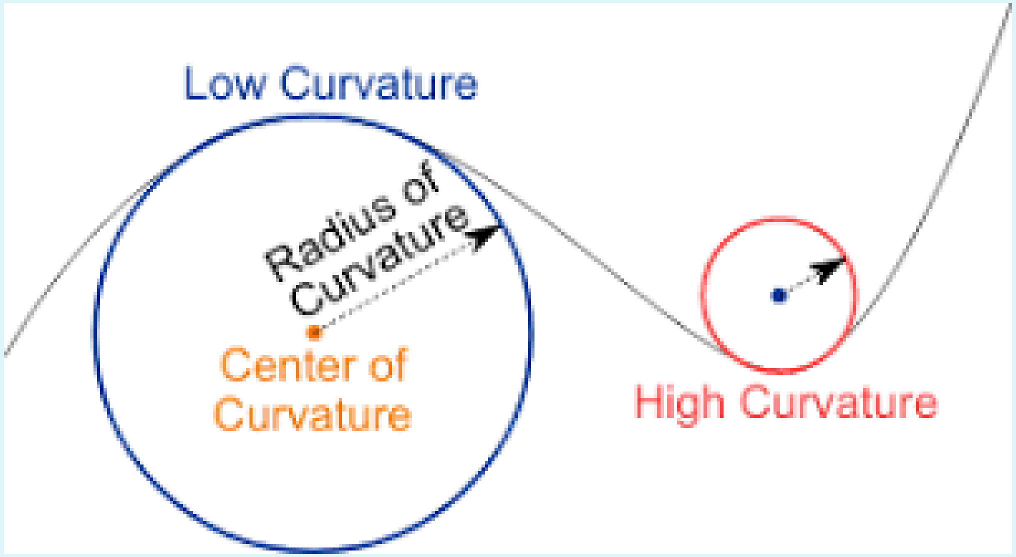




I) Line

ii) half circle

iii) parabola



Curvature of a curve at a point=1/radius of curvature the circle

Curvature and Radius of Curvature

Curvature is a numerical measure of bending of the curve. At a particular point on the curve , a tangent can be drawn. Let this line makes an angle Ψ with positive x- axis. Then curvature is defined as the magnitude of rate of change of Ψ with respect to the arc length s.

$$\therefore \text{Curvature at P} = \left|\frac{d\Psi}{ds}\right|$$

Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

Radius of Curvature $= \boldsymbol{\rho}$, **and Curvature** $= \dfrac{\mathbf{1}}{\boldsymbol{\rho}}$

Curvature

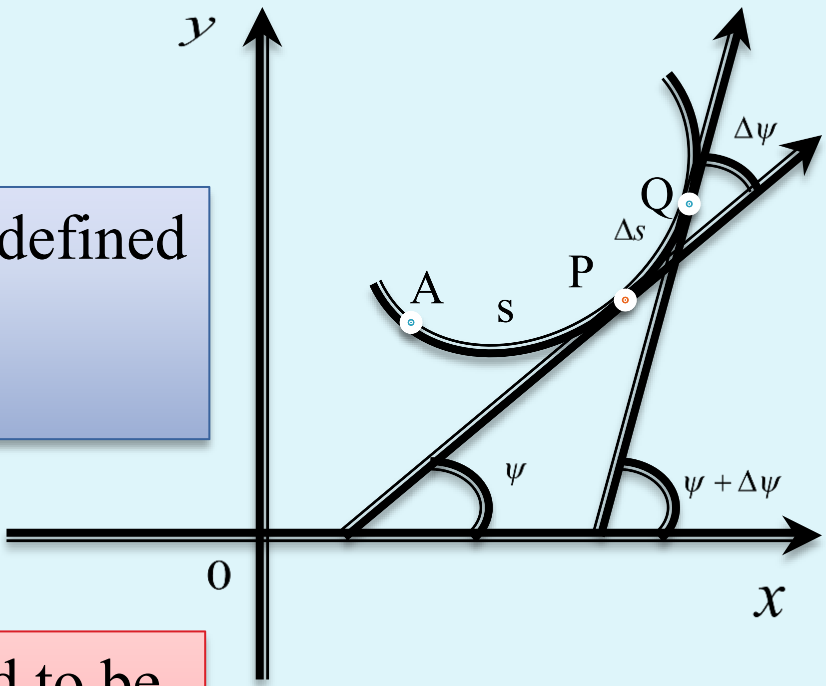
➢ Total bending or total curvature or the arc PQ is defined to be the angle $\Delta\psi$.

➢ Average curvature of the arc PQ is defined to be the ratio

$$\Delta\psi / \Delta s.$$

➢ Curvature of the curve at P is defined to be

$$\lim_{Q\to P}\frac{\Delta\Psi}{\Delta s}=\frac{d\Psi}{ds}.$$

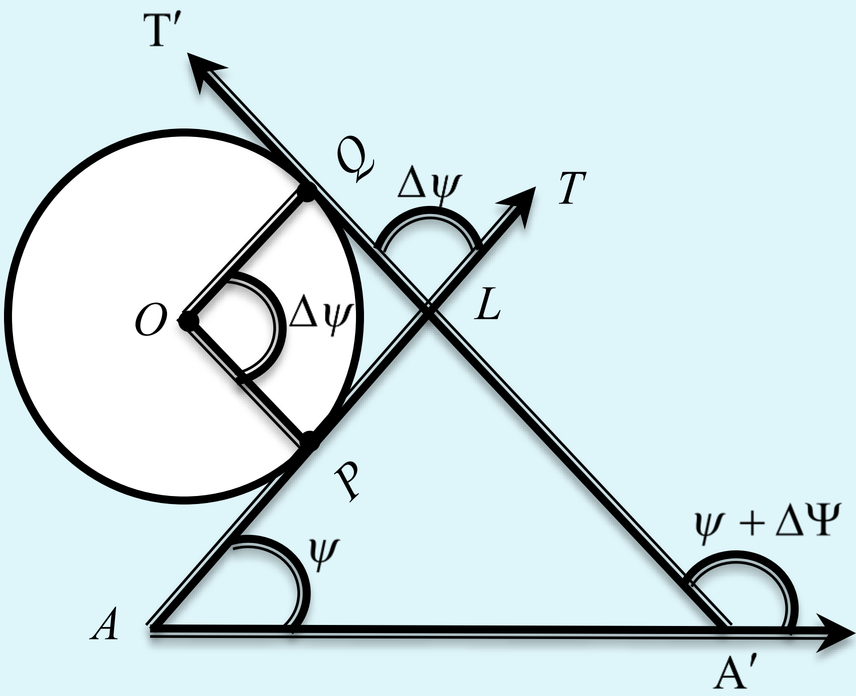


Curvature of a Circle

To prove that the curvature of a circle is constant.

Consider a circle with radius, r and centre O.

Let P, Q be two points on the circle and let arc $PQ = \Delta s$.



Let L be the point where the tangents PT , QT' at P and Q meet.

We have $\angle POQ = \angle TLT' = \Delta\Psi$.

We have $\dfrac{arc\ PQ}{OP} = \angle POQ,$

$$\Rightarrow \frac{\Delta s}{r} = \Delta\Psi \Rightarrow \frac{\Delta\Psi}{\Delta s} = \frac{1}{r}.$$

Problems

Find the radius of curvature at any point of the following:-

$$(i)\ s = c\tan\Psi$$

Solution

$$\rho = \frac{ds}{d\Psi} = c\sec^2\Psi.$$

$$(ii) s = a\log(\tan\Psi + \sec\Psi) + a\tan\Psi\sec\Psi$$

Solution

$$\rho = \frac{ds}{d\Psi} = \frac{a\sec\Psi(\sec\Psi + \tan\Psi)}{\tan\Psi + \sec\psi} + a\sec^3\Psi + a\tan^2\Psi\sec\Psi$$

$$= a\sec\Psi(1 + \tan^2\Psi) + a\sec^3\Psi$$

$$= a\sec\Psi \cdot \sec^2\Psi + a\sec^3\Psi$$

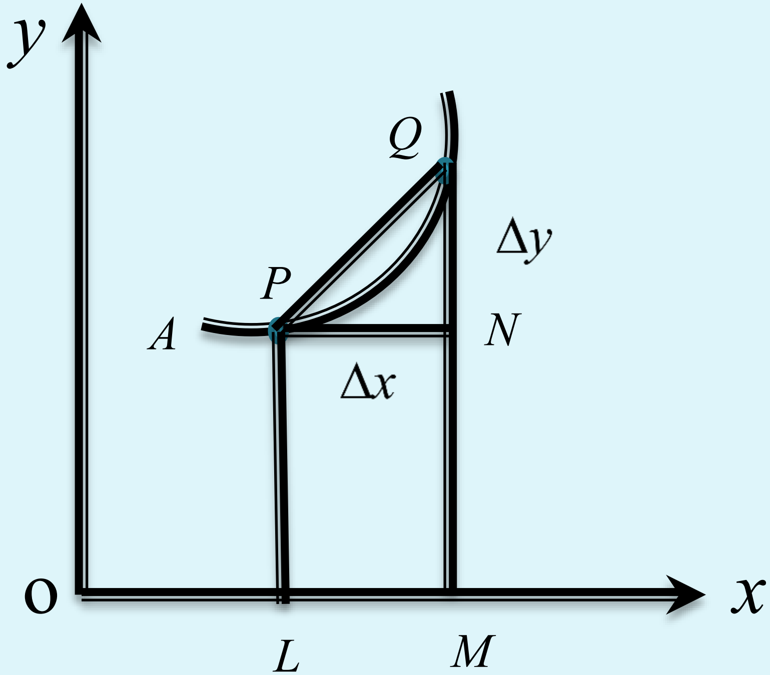
$$= 2a\sec^3\Psi$$

Length of arc as a function. Derivative of arc

Let us consider a point A on the curve $y = f(x)$.

Let $P(x, y)$ be a point on the curve such that $AP = s$.

Let $Q(x+\Delta x, y+\Delta y)$ be another point on the curve.

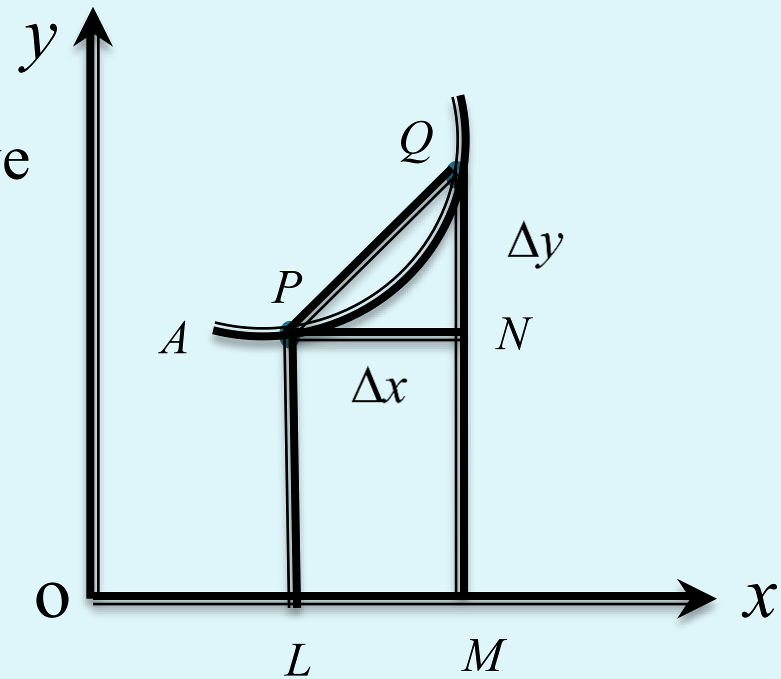


We shall now prove that

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

Let $arc\ AQ = s + \Delta s$ so that arc $PQ = \Delta s$.

From the right angled ΔPQN, we have



$$PQ^2 = PN^2 + NQ^2,$$
$$=(\Delta x)^2 + (\Delta y)^2$$
$$\Rightarrow \left(\frac{PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$
$$\Rightarrow \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

Assuming, on an intuitive basis that

$$\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1,$$

We obtain in the limit

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

Thus we have

$$\boxed{\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.}$$

Taking positive sign before the radical

- **For parametric Cartesian equations** $x = f(t),\ y = F(t)$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

- **For polar equations** $r = f(\theta),$

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$$

Problems

Ex:1 Find ds/dx for the following curves

$(i)\ y = c\cosh\ x/c$

Solution

We have $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad (1)$$

$$\frac{dy}{dx} = \sinh\ x/c$$

Substituting the above value in Eq. (1), we have

$$\frac{ds}{dx} = \sqrt{1+\sinh^2\ x/c}$$

$$= \cosh\ x/c$$

$$(ii)\, y = a\log\left[a^2/\left(a^2 - x^2\right)\right]$$

Solution

$$\frac{dy}{dx} = \frac{2ax}{(a^2 - x^2)}$$

$$So, \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{ds}{dx} = \sqrt{1 + \frac{4a^2x^2}{(a^2 - x^2)^2}}$$

$$\Rightarrow \frac{ds}{dx} = \frac{(a^2 + x^2)}{(a^2 - x^2)}$$

Ex:2 Find ds/dt for the following curves

$(i)\, x = a\cos^3 t,\; y = b\sin^3 t$

Solution

We have $$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad (1)$$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \qquad \frac{dy}{dt} = 3b\sin^2 t\cos t$$

Substituting the above value in Eq. (1), we have

$$\frac{ds}{dt} = 3\sin t\cos t\sqrt{a^2\cos^2 t + b^2\sin^2 t}$$

$$(ii)\, x = ae^t \sin t,\ y = ae^t \cos t$$

Solution

$$\frac{dx}{dt} = ae^t(\sin t + \cos t),$$

$$\frac{dy}{dt} = ae^t(\cos t - \sin t)$$

So, $$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{2}ae^t$$

Ex:2 Find $ds/d\theta$ for the following curves

$(i) r = a(1+\cos\theta)$

Solution

We have $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad (1)$$

and $$\frac{dr}{d\theta} = -a\sin\theta,$$

Substituting the above value in Eq. (1), we have

$$\frac{ds}{d\theta} = \sqrt{r^2 + a^2\sin^2\theta}$$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{2}a\sqrt{1+\cos\theta} = \sqrt{2}a\sqrt{2\cos^2\theta/2}$$

$$= 2a\cos\theta/2$$

5.2

- **Radius of curvature of Cartesian curve: $y = f(x)$**

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+y_1^2\right)^{3/2}}{|y_2|} \text{ (When tangent is parallel to x – axis)}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} \text{ (When tangent is parallel to y – axis)}$$

- **Radius of curvature of parametric curve:**

$\mathbf{x = f(t), y = g(t)}$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y''-y'x''|}, \quad \text{where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

Example 2.

Find the curvature and radius of curvature of the parabola

$$y = x^2$$

at the origin.

Solution.

Write the derivatives of the quadratic function:

$$y' = \left(x^2\right)' = 2x; \;\; y'' = (2x)' = 2.$$

Then the curvature of the parabola is defined by the following formula:

$$K = \frac{y''}{\left[1 + (y')^2\right]^{\frac{3}{2}}} = \frac{2}{\left[1 + (2x)^2\right]^{\frac{3}{2}}} = \frac{2}{(1 + 4x^2)^{\frac{3}{2}}}.$$

At the origin (at $x = 0$), the curvature and radius of curvature, respectively, are

$$K\left(x = 0\right) = \frac{2}{(1 + 4 \cdot 0^2)^{\frac{3}{2}}} = 2, \;\; R = \frac{1}{K} = \frac{1}{2}.$$

Exercises

1. Find the radius of curvature at any point of the following:-

$(i)\ s = 4a\sin\Psi$ $\qquad$ $(ii)\ s = 4a\sin\frac{1}{3}\Psi$

$(iii)\ s = c\log\sec\Psi$

2. Find ds/dt for the following curves

$(i) x = a(t - \sin t),\ y = a(1 - \cos t)$

$(ii) x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$

3. Find $ds/d\theta$ for the following curves

$(i) r^2 = a^2\cos 2\theta$