DEPARTMENT OF MATHEMATICS

MATHEMATICS - I

FOR BTECH 1st SEMESTER COURSE [COMMON TO ALL BRANCHES OF ENGINEERING]

TEXT BOOK: ADVANCED ENGINEERING MATHEMATICS BY ERWIN KREYSZIG

LECTURES –29

Double Integrals
[Chapter – 9.3]

Content:

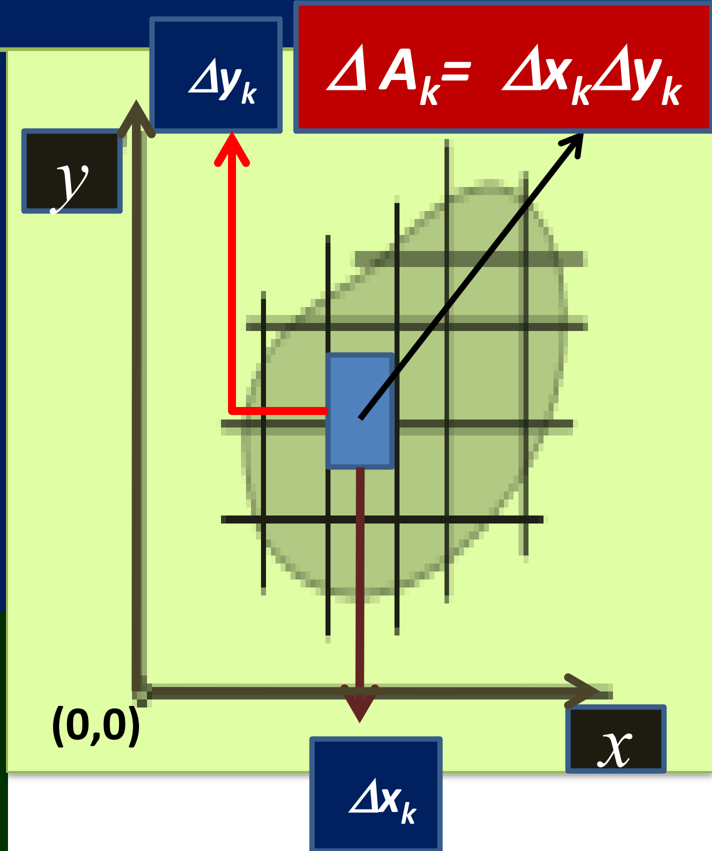
- Introduction to Double Integral
- Definition and Properties
- Computation of Double Integral
- Applications of Double Integrals
- Examples
- Change of Variables
- Examples
- Practice Problems

DOUBLE INTEGRAL

Let *f(x, y)* be any function. Let *R* be a region in a plane. Let us subdivide the region in to a large number of small rectangular subregions. Let the k^{th} region has the dimensions say Δx_k and Δy_k.

Let (x_k, y_k) be any arbitrary point on the k^{th} subregion and $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the k^{th} rectangular subregion. Next consider the sequence of real numbers $(J_1, J_2, \ldots,)$, where

$$J_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$



DOUBLE INTEGRAL

Assuming that ***f(x, y)*** is continuous in R and R is bounded by finitely many smooth curves. It can be shown that the sequence $(J_1, J_2, \ldots,)$ converges to a limit that is independent of the choice of subdivisions and corresponding points (x_k, y_k). This limit is called the ***double integral*** of ***f(x, y)*** over the region *R* and is denoted by

$$\iint_R f(x,y)\,dxdy \quad \text{or} \quad \iint_R f(x,y)\,dA$$

Applications of Double Integrals

- area of the region: finding area using double integral, volume of an elliptic paraboloid.
- average value of a function: calculating avarage strom rainfall, of a wire;
- Surface integral;
- double and volume integral.

SOME PROPERTIES OF DOUBLE INTEGRAL

1.Properties Linearity

$$1.\iint_A [f(x,y)+g(x,y)]dA = \iint_A f(x,y)dA + \iint_A g(x,y)dA$$

$$2.\iint_A cf(x,y)dA = c\iint_A f(x,y)dA$$

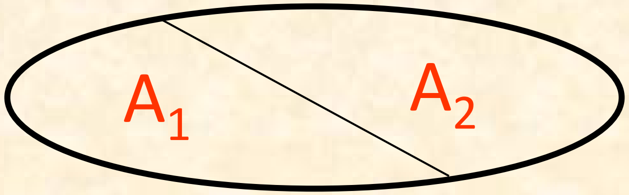
2.Comparison

If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

$$\iint_A f(x,y)dA \geq \iint_A g(x,y)dA$$

SOME PROPERTIES OF DOUBLE INTEGRAL

3.Additively



If A_1 and A_2 are non-overlapping regions then

$$\iint_{A_1 \cup A_2} f(x,y)\,dA = \iint_{A_1} f(x,y)\,dA + \iint_{A_2} f(x,y)\,dA$$

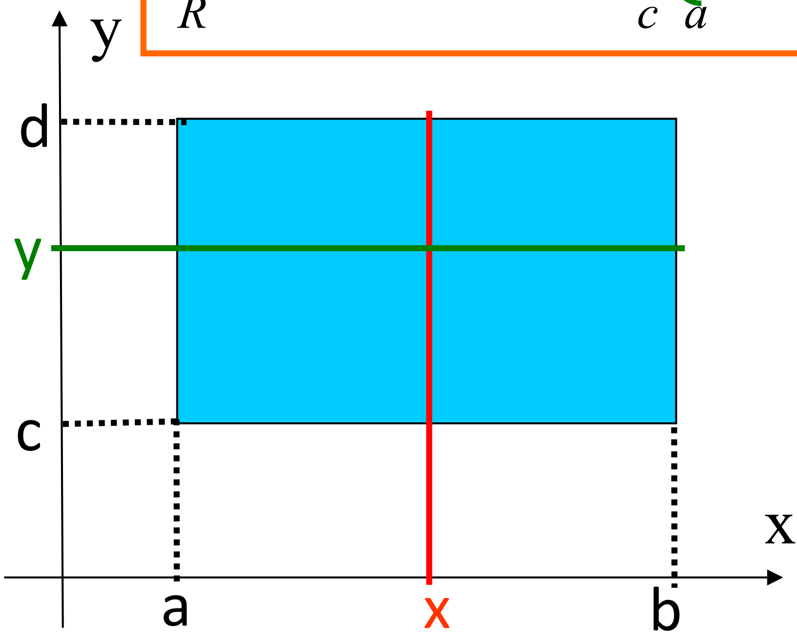
Computation of Double Integral

- If $f\,(x,y)$ is continuous on rectangle $R=[a,b]\times[c,d]$ then double integral is equal to iterated integral

$$\iint_R f(x,y)\,dA=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy=\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

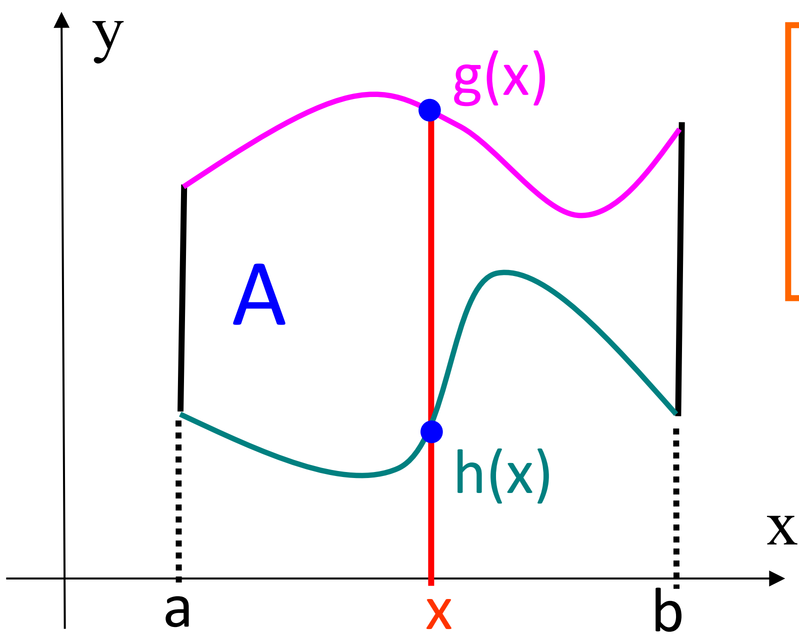
fixed

fixed



COMPUTATION OF DOUBLE INTEGRAL

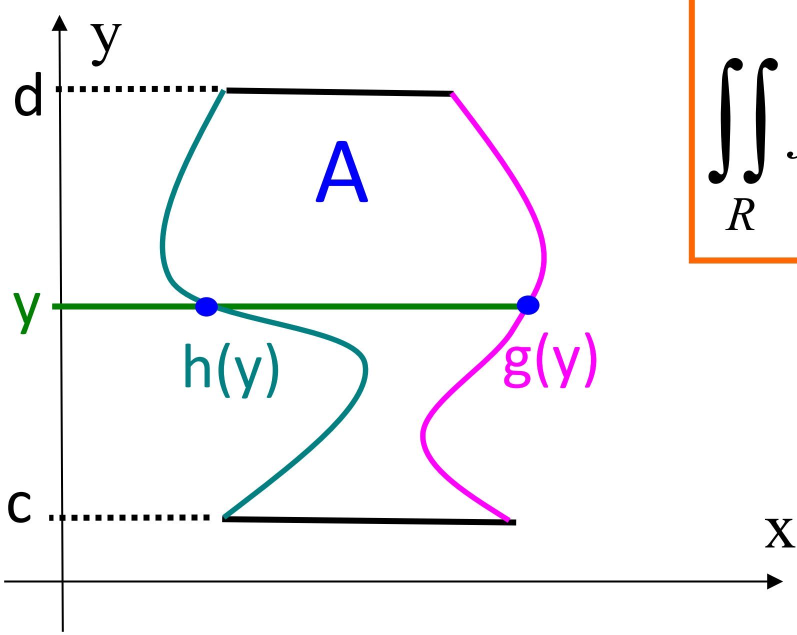
- If $f\,(x,y)$ is continuous on $A=\{(x,y)\mid x$ in $[a,b]$ and $h\,(x)\le y\le g\,(x)\}$ then double integral is equal to iterated integral



$$\iint\limits_A f(x,y)\,dA=\int_a^b\int_{h(x)}^{g(x)} f(x,y)\,dy\,dx$$

COMPUTATION OF DOUBLE INTEGRAL

- If $f(x,y)$ is continuous on $A=\{(x,y) \mid y$ in $[c,d]$ and $h(y) \le x \le g(y)\}$ then double integral is equal to iterated integral



$$\iint_R f(x,y)\,dA = \int_c^d \int_{h(y)}^{g(y)} f(x,y)\,dx\,dy$$

COMPUTATION OF DOUBLE INTEGRAL

If $f\ (x,\ y) = \varphi\ (x)\ \psi(y)$ is continuous on rectangle $R=[a,b]\times[c,d]$ then double integral is equal to iterated integral

$$\boxed{\iint\limits_R f(x,y)\,dA = \int\limits_c^d \int\limits_a^b \varphi(x)\psi(y)\,dx\,dy = \left[\int\limits_a^b \varphi(x)\,dx\right] \cdot \left[\int\limits_c^d \psi(y)\,dy\right]}$$

SOME APPLICATIONS OF DOUBLE INTEGRAL

1. Area A of a plane region R is given by

$$A = \iint_R dA$$

2. Volume V beneath the surface $z = f(x,y)$ (>0) and above the region R in the xy-plane is given by

$$V = \iint_R f(x,y)\, dxdy$$

SOME APPLICATIONS OF DOUBLE INTEGRAL

3. **Total Mass M of a mass distribution of the density *f(x,y)* (= mass per unit area) enclosing a region R in the xy-plane is given**

$$M = \iint_R f(x, y)\, dxdy$$

4. **Let $(\bar{x}, \bar{y})$ be the coordinates of the center of gravity of a plane region R enclosed in xy-plane with mass M of a mass distribution of the density *f(x,y)* (= mass per unit area). Then we have.**

$$\bar{x} = \frac{1}{M}\iint_R xf(x, y)\, dxdy \quad \text{and} \quad \bar{y} = \frac{1}{M}\iint_R yf(x, y)\, dxdy$$

SOME APPLICATIONS OF DOUBLE INTEGRAL

5. Moments of inertia I_x and I_y about x and y axes, respectively, of a mass with a mass distribution of the density $f(x,y)$ **(= mass per unit area)** in a region R in xy-plane are given as follows.

$$I_x = \iint_R y^2 f(x,y)\, dxdy \quad \text{and} \quad I_y = \iint_R x^2 f(x,y)\, dxdy$$

Polar Moment of inertia I_0 about *the origin* of a mass with a mass distribution of the density $f(x,y)$ **(= mass per unit area)** in a region R in xy-plane is given as follows.

$$I_0 = I_x + I_y = \iint_R \left(x^2 + y^2\right) f(x,y)\, dxdy$$

SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

1.Evaluate the following double integral: $\int_0^2 \int_0^4 (x^2 + y^2)dxdy$

Solution:

$$\int_0^2 \int_0^4 (x^2 + y^2)dxdy = \int_0^2 \left[\int_0^4 (x^2 + y^2)dx \right] dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + xy^2 \right]_0^4 dy = \int_0^2 \left(\frac{64}{3} + 4y^2 \right) dy = \left[\frac{64y}{3} + \frac{4y^3}{3} \right]_0^2$$

$$= \frac{128}{3} + \frac{32}{3} = \frac{160}{3}$$

ANS

SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

2.Evaluate the following double integral:

$$\int_0^2 \int_0^4 (x^2 + y^2)\,dx\,dy \text{ by order reversed.}$$

Solution:

$$\int_0^4 \int_0^2 (x^2 + y^2)\,dx\,dy = \int_0^4 \left[\int_0^2 (x^2 + y^2)\,dy \right] dx$$

$$= \int_0^4 \left[\frac{y^3}{3} + yx^2 \right]_0^2 dx = \int_0^4 \left(\frac{8}{3} + 2x^2 \right) dx = \left[\frac{8x}{3} + \frac{2x^3}{3} \right]_0^4$$

$$= \frac{32}{3} + \frac{128}{3} = \frac{160}{3}$$

ANS

SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

3.Evaluate the following double integral: $\int_0^3 \int_{-y}^{y} \left(x^2 + y^2\right) dxdy$

Solution:

$$\int_0^3 \int_{-y}^{y} \left(x^2 + y^2\right) dxdy = \int_0^3 \left[\int_{-y}^{y} \left(x^2 + y^2\right) dx \right] dy$$

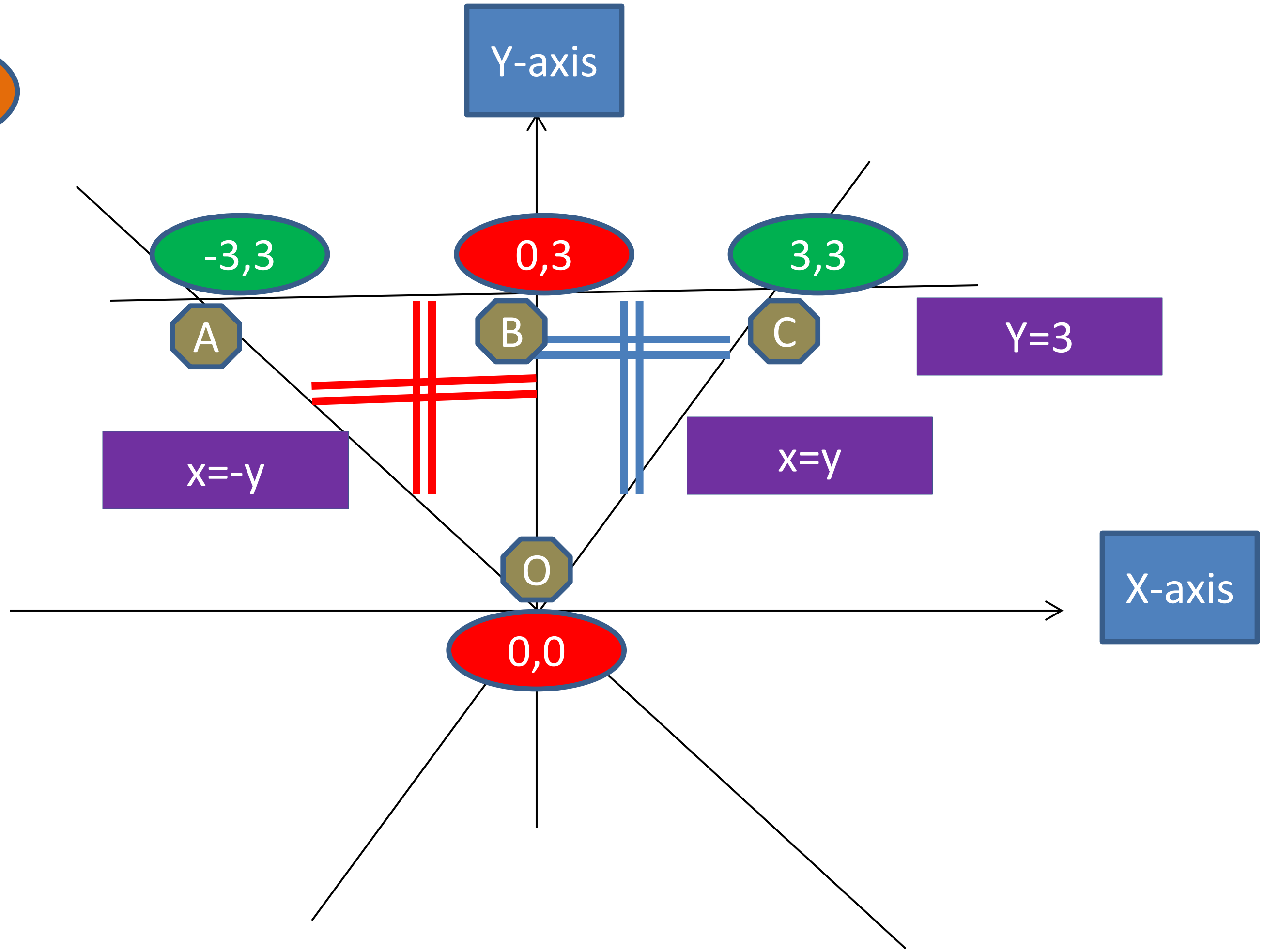
$$= \int_0^3 \left| \frac{x^3}{3} + xy^2 \right|_{-y}^{y} dy = \int_0^3 \left\{ \left(\frac{y^3}{3} + y^3 \right) - \left(\frac{-y^3}{3} - y^3 \right) \right\} dy$$

$$= \int_0^3 \frac{8y^3}{3} dy = \frac{8}{3} \left| \frac{y^4}{4} \right|_0^3 = \frac{8 \times 81}{3 \times 4} = 54$$

SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

4.Find the value of $\int_{0}^{3}\int_{-y}^{y}\left(x^2+y^2\right)dxdy$ by order reversed.

Solution:



SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

Given that the region of integration is bounded by $x = -y$, $x = y$, $y = 0$ and $y = 3$, *i.e.* the region of integration is portion OAC in the Figure.

For changing the order of integration, we divide the region of integration into vertical strips . The region of integration is divided into two parts OAB and OBC

For the region OAB, y varies from $(-x)$ to 3 and x varies from (-3) to 0. for the region OBC, y varies from x to 3 and x varies from 0 to 3

SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

$$\therefore \int_{0}^{3}\int_{-y}^{y}\left(x^2+y^2\right)dxdy = \iint_{OAB-region}\left(x^2+y^2\right)dydx + \iint_{OBC-region}\left(x^2+y^2\right)dydx$$

$$= \int_{x=-3}^{x=0}\left[\int_{y=-x}^{3}\left(x^2+y^2\right)dy\right]dx + \int_{x=0}^{3}\left[\int_{y=x}^{y=3}\left(x^2+y^2\right)dy\right]dx$$

$$= \int_{x=-3}^{x=0}\left|x^2y+\frac{y^3}{3}\right|_{-x}^{3}dx + \int_{x=0}^{3}\left|x^2y+\frac{y^3}{3}\right|_{x}^{3}dx$$

$$= \int_{x=-3}^{x=0}\left\{\left(3x^2+9\right)-\left(-x^3-\frac{x^3}{3}\right)\right\}dx + \int_{x=0}^{3}\left\{\left(3x^2+9\right)-\left(x^3+\frac{x^3}{3}\right)\right\}dx$$

SOME PROBLEMS INVOLVING DOUBLE INTEGRAL

$$=\int_{x=-3}^{x=0}\left(\frac{4x^3}{3}+3x^2+9\right)dx+\int_{x=0}^{3}\left(3x^2+9-\frac{4x^3}{3}\right)dx$$

$$=\left|\frac{x^4}{3}+x^3+9x\right|_{-3}^{0}+\left|x^3+9x-\frac{x^4}{3}\right|_{0}^{3}$$

$$=0-(27-27-27)+(27+27-27)=27+27=54$$

ANS

CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

The formula for a change of variables in double integrals from x,y to u,v is

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} f\big(x(u,v),y(u,v)\big)\left|J\right| dudv$$

That is, the integrand is expressed in terms of u and v, and $dxdy$ is replaced by $dudv$ times the absolute value of the ***Jacobian*** ***"J"*** which is defined as follows.

CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Here we assume that the functions $x = x(u,v)$ and y = y(u,v)

effecting the change are continuous and have continuous partial derivatives in some region R^* in the uv-plane such that for every (u, v) in R^* the corresponding point (x, y) lies in R and, conversely, to every (x, y) in R there corresponds one and only one (u, v) in R^*; furthermore, the Jacobian J is either positive throughout R^* or negative throughout R^*.

CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

Of particular practical interest are **polar coordinates** r and θ, which can be introduced by setting $x = r\cos\theta, \quad y = r\sin\theta$. Then

$$\boxed{J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r}$$

and

$$\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where R^* is the region in the $r\theta$-plane corresponding to R in the xy-plane.

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS:

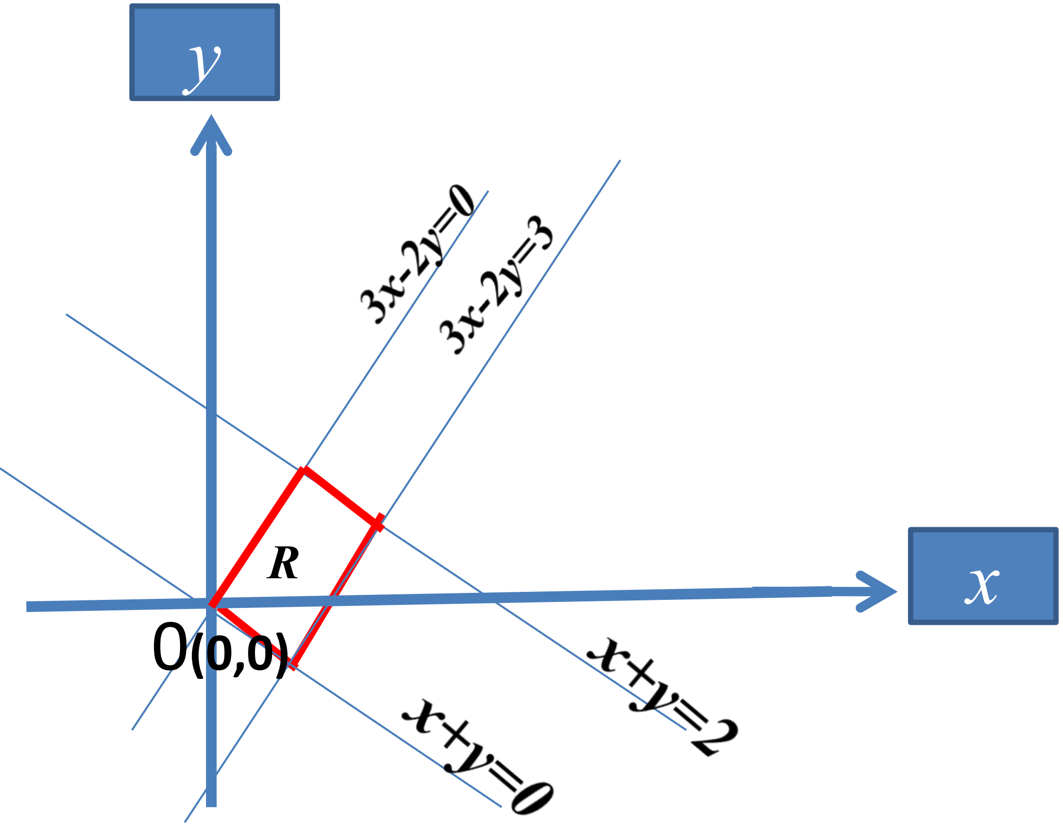
5. Evaluate $\iint\limits_R (x+y)^2 \, dxdy$, where R is region bounded by the parallelogram $x+y=0, x+y=2,$ $3x-2y=0, 3x-2y=3.$

Solution:

Let $f(x,y)=(x+y)^2$

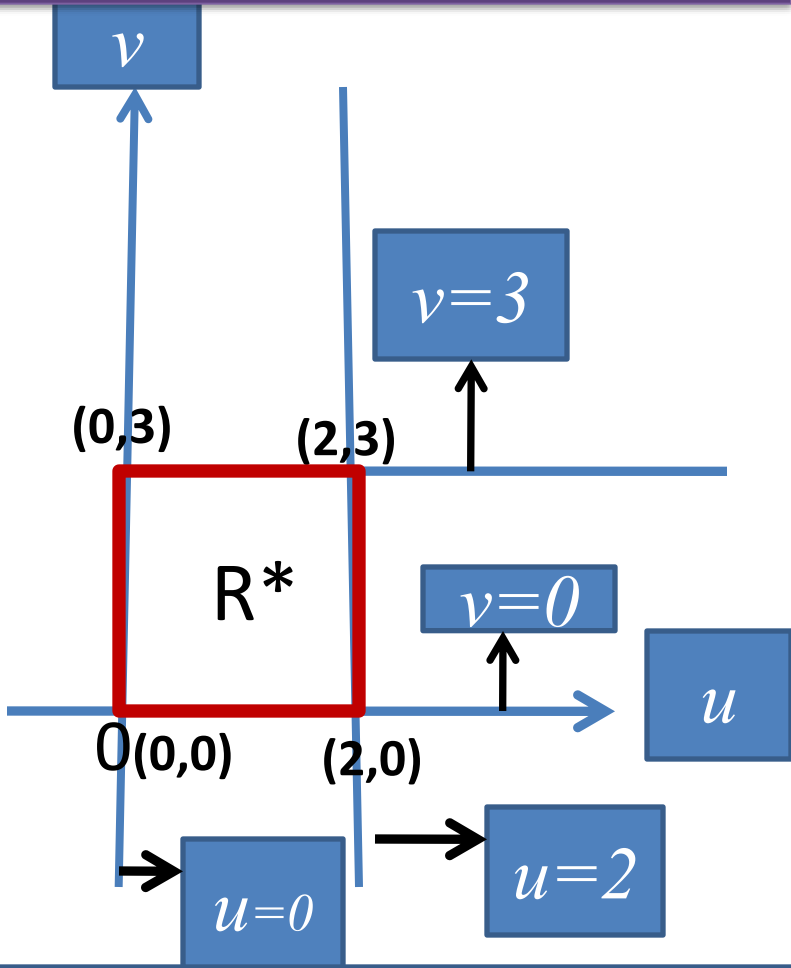
By changing the variables x,y to the new variables u, v, by the substitution (transformation) $x+y=u$, $3x-2y=v$, the given parallelogram R reduces to a rectangle R^* as shown in Figure given below.

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN



Given region

$R : 0 \leq x+y \leq 2,\ 0 \leq 3x\text{-}2y \leq 3$



Transformed region

$R^* : 0 \leq u \leq 2,\ 0 \leq v \leq 3$

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

Solving $x+y=u$ and $3x-2y=v$ for x and y we get.

$$x=\frac{1}{5}\left(2u+v\right) \qquad \text{and} \qquad y=\frac{1}{5}\left(3u-v\right).$$

$$\therefore \frac{\partial x}{\partial u}=\frac{\partial}{\partial u}\left(\frac{1}{5}\left(2u+v\right)\right)=\frac{2}{5},\ \frac{\partial x}{\partial v}=\frac{\partial}{\partial v}\left(\frac{1}{5}\left(2u+v\right)\right)=\frac{1}{5},$$

$$\frac{\partial y}{\partial u}=\frac{\partial}{\partial u}\left(\frac{1}{5}\left(3u-v\right)\right)=\frac{3}{5},\ \frac{\partial y}{\partial v}=\frac{\partial}{\partial v}\left(\frac{1}{5}\left(3u-v\right)\right)=-\frac{1}{5}.$$

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

$$\therefore J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{vmatrix}$$

$$= -\frac{2}{25} - \frac{3}{25} = -\frac{5}{25} = -\frac{1}{5}$$

$$\therefore |J| = \left| -\frac{1}{5} \right| = \frac{1}{5}$$

Problems Involving Change of Variables in Double Integrals: Jacobian

Also we have $f\left(x(u,v), y(u,v)\right) = u^2$

Since we have $u = x + y = 0$ and $u = x + y = 2$, u varies from 0 to 2. Since we have $v = 3x - 2y = 0$ and $v = 3x - 2y = 3$, v varies from 0 to 3. Thus the given integral in terms of the new variables u, v is given by

$$\iint_R (x+y)^2 \, dxdy = \iint_R f(x,y) \, dxdy$$

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

$$= \iint_{R^*} f\left(x(u,v), y(u,v)\right)|J|\,du\,dv$$

$$= \int_0^3 \int_0^2 u^2\left(\frac{1}{5}\right)du\,dv = \frac{1}{5}\int_0^3 \int_0^2 u^2\,du\,dv \;\; = \frac{1}{5}\int_0^3 \left(\int_0^2 u^2\,du\right)dv$$

$$= \frac{1}{5}\int_0^3 \left[\frac{u^3}{3}\right]_0^2 dv \;=\frac{1}{5}\int_0^3 \left[\frac{2^3}{3} - 0\right]dv \;\; = \frac{8}{15}\int_0^3 dv$$

$$= \frac{8}{15}[v]_0^3 = \frac{24}{15} = \frac{8}{5}.$$

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS:

6. Evaluate $\iint_R e^{-\left(x^2+y^2\right)}dxdy$, where R is region $0 < x < \infty,\quad 0 < y < \infty$ using ploar substitution.

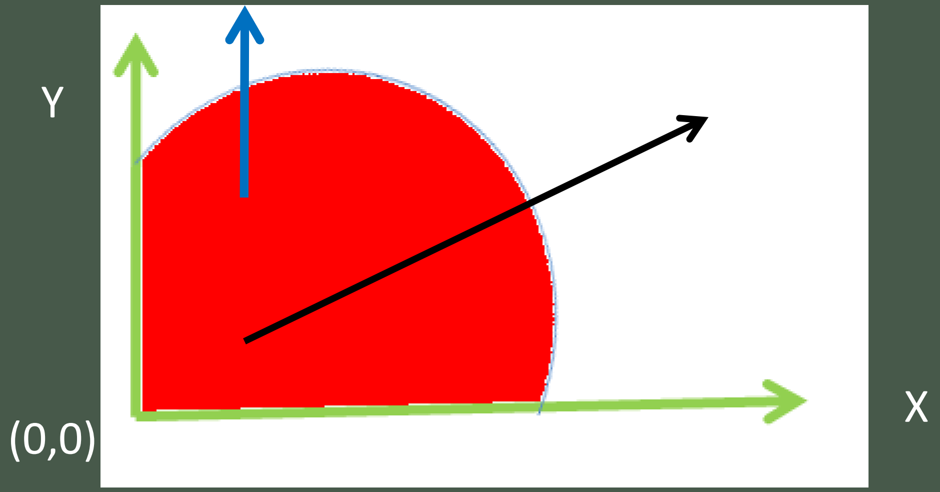
Solution:

$$\text{Let } f\left(x, y\right)=e^{-\left(x^2+y^2\right)}$$

Let us change the variables x,y to the new variables $u,\ v,$ by the polar substitution (transformation) $x = rcos\theta,\ y = rsin\theta.$

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

The region of double integration R is R: $0 < x < \infty$, $0 < y < \infty$, i.e. the first quadrant as shown in the figure given below. By changing the variables x,y to the new variables r, θ, by the polar substitution (transformation) $x = r\cos\theta$, $y = r\sin\theta$, the transformed region of integration R^* is R^* : $0 < r < \infty$, $0 < \theta < \pi/2$.



PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

$$\therefore \frac{\partial x}{\partial r}=\frac{\partial}{\partial r}\left(r\cos\theta\right)=\cos\theta,\ \frac{\partial x}{\partial \theta}=\frac{\partial}{\partial \theta}\left(r\cos\theta\right)=-r\sin\theta,$$

$$\frac{\partial y}{\partial r}=\frac{\partial}{\partial r}\left(r\sin\theta\right)=\sin\theta,\ \frac{\partial y}{\partial \theta}=\frac{\partial}{\partial \theta}\left(r\sin\theta\right)=r\cos\theta.$$

Therefore, the Jacobian of transformation is given by

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta}\\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r$$

Problems Involving Change of Variables in Double Integrals: Jacobian

$$\therefore |J| = |r| = r$$

Also we have

$$f\left(x(r,\theta), y(r,\theta)\right) = e^{-\left(r^2\cos^2\theta + r^2\sin^2\theta\right)} = e^{-r^2}.$$

Thus the given integral in terms of the polar variables r,θ is given by.

$$\iint_R e^{-\left(x^2+y^2\right)}dxdy = \iint_R f\left(x,y\right)dxdy$$

PROBLEMS INVOLVING CHANGE OF VARIABLES IN DOUBLE INTEGRALS: JACOBIAN

$$= \iint_{R^*} f\left(x(r,\theta), y(r,\theta)\right)|J|\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r\,dr\,d\theta \qquad = \int_0^{\frac{\pi}{2}} \left(\int_0^{\infty} e^{-r^2} r\,dr \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} d\theta$$

Problems involving change of variables in double integrals: Jacobian

$$= \int_0^{\frac{\pi}{2}} \left[0 + \frac{1}{2} \right]_0^{\infty} d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2} \left[\theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

APPLICATION OF DOUBLE INTEGRATION

1. VOLUME OF A REGION

Find the volume of the region of the beneath z=$4x^2 + 9y^2$ and above the rectangle with vertices (0,0), (3,0),(3,2) and (0,2)

SOLUTION: We know that the volume V beneath the surface $z=f(x,y)$ (>0) and above the region R in the XY plane is given by $V = \iint_R f(x, y)dxdy$

APPLICATION OF DOUBLE INTEGRATION

Given that f(x,y)=$4x^2 + 9y^2$ the region R in the XY plane is bounded by the lines x=0,x=3,y=0 and y=2

Hence volume

$$V = \iint_R f(x,y)\,dxdy$$

$$= \int_{y=0}^{2}\left(\int_{x=0}^{3}(4x^2+9y^2)dx\right)dy = \int_{y=0}^{2}\left|\frac{4x^3}{3}+9xy^2\right|_0^3 dy$$

$$= \int_{y=0}^{2}\left(36+27y^2\right)dy = \left|36y+\frac{27y^3}{3}\right|_0^2 = 72+72 = 144$$

ANS

APPLICATION OF DOUBLE INTEGRATION

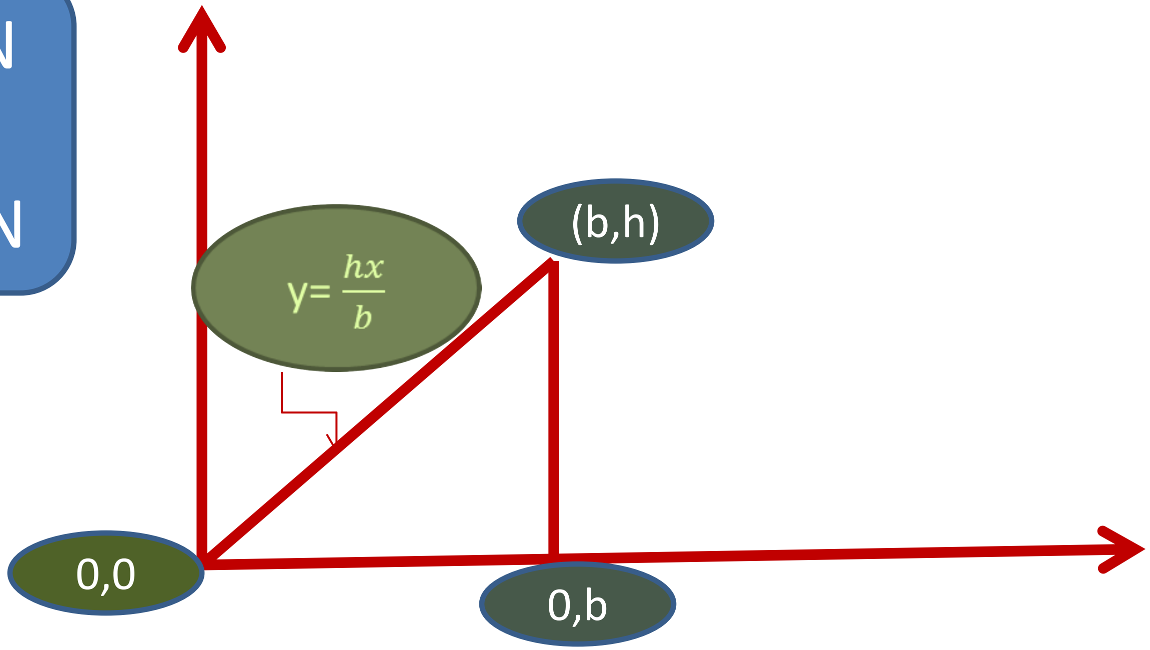
2. Center of Gravity

Find the coordinates $\bar{x}, \bar{y}$ of the center of gravity of a mass of density ***f(x ,y)=1*** in a region ***R***. where ***R*** is the triangle with the vertices ***(0,0), (b,0), (b, h)***

<u>SOLUTION:</u> We know that if f(x,y) be the density of a distribution of mass in the in the XY plane. Then the total mass M in a region R is given by

$$M = \iint_R f(x, y)dxdy.$$

APPLICATION OF DOUBLE INTEGRATION



Given that $f(x,y)=1$ in the given region R in xy plane x varies from 0 to b and y varies from 0 to $\frac{hx}{b}$

$$\text{So } M = \int_{x=0}^{b} \int_{y=0}^{y=\frac{hx}{b}} 1.dydx = \int_{x=0}^{b} dx \int_{y=0}^{y=\frac{hx}{b}} dy$$

APPLICATION OF DOUBLE INTEGRATION

$$M = \int_{x=0}^{b} \frac{hx}{b}\,dx = \left|\frac{hx^2}{2b}\right|_0^b = \frac{hb^2}{2b} = \frac{hb}{2}$$

Let the coordinates of the center of gravity of the mass in R

$$\overline{x} = \frac{1}{M}\iint_R xf(x,y)\,dxdy \text{ and } \overline{y} = \frac{1}{M}\iint_R yf(x,y)\,dxdy$$

$$\Rightarrow \overline{x} = \frac{2}{hb}\int_0^b \int_0^{\frac{hx}{b}} x\,dydx = \frac{2}{hb}\int_0^b \left(\int_0^{\frac{hx}{b}} x\,dy\right) dx = \frac{2}{hb}\int_0^b \left[xy\right]_0^{\frac{hx}{b}} dx$$

APPLICATION OF DOUBLE INTEGRATION

$$= \frac{2}{hb}\int_0^b \frac{hx^2}{b}dx = \frac{2}{hb}\left(\frac{h}{b}\right)\int_0^b x^2 dx = \frac{2}{b^2}\left[\frac{x^3}{3}\right]_0^b = \frac{2}{b^2}\left[\frac{b^3}{3}\right] = \frac{2b}{3}$$

$$\Rightarrow \bar{y} = \frac{2}{hb}\int_0^b \int_0^{\frac{hx}{b}} y\,dy\,dx = \frac{2}{hb}\int_0^b \left(\int_0^{\frac{hx}{b}} y\,dy\right)dx = \frac{2}{hb}\int_0^b \left[\frac{y^2}{2}\right]_0^{\frac{hx}{b}} dx$$

$$= \frac{2}{hb}\int_0^b \frac{h^2x^2}{2b^2}dx = \frac{2}{hb}\left(\frac{h^2}{2b^2}\right)\int_0^b x^2 dx = \frac{h}{b^3}\left[\frac{x^3}{3}\right]_0^b = \frac{h}{b^3}\left[\frac{b^3}{3}\right] = \frac{h}{3}$$

ANS

APPLICATION DOUBLE INTEGRATION

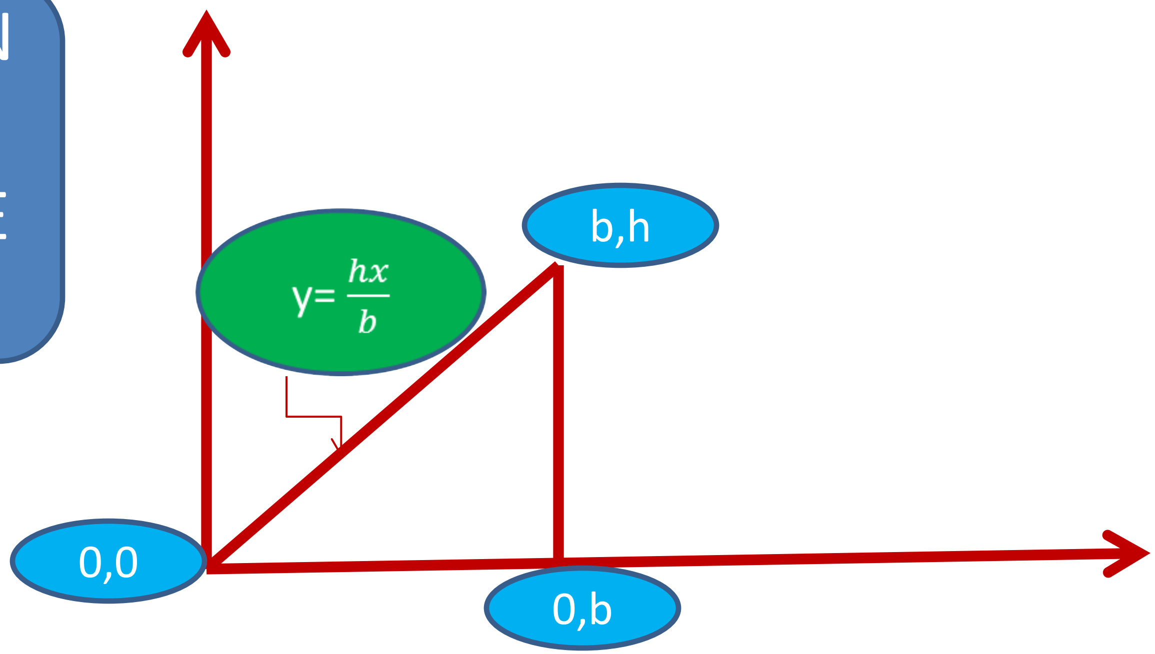
3. Moment of Inertia

Find the moment of inertia and polar moment of inertia I_x, I_y, I_0 of a mass of density f(x ,y)=1 in a region R. where R is the triangle with the vertices (0,0), (b,0), (b, h).

SOLUTION: We know that if $f(x, y)$ be the density of a distribution of mass in the in the XY plane. Then moment of inertia in a region R in x axis and y axis is given by

$$I_x = \iint_R y^2 f(x, y)\,dxdy\ ,\ I_y = \iint_R x^2 f(x, y)\,dxdy \text{ and } I_0 = I_x + I_y$$

APPLICATION OF DOUBLEINTEGRATION



Given that $f(x,y)=1$ in the given region R in XY- plane y varies from 0 to $\frac{\mathbf{hx}}{\mathbf{b}}$ and x varies from 0 to b

$$I_x = \int_{x=0}^{b} \int_{y=0}^{y=\frac{hx}{b}} y^2 .dydx = \int_0^b \left(\int_{y=0}^{y=\frac{hx}{b}} y^2 dy \right) dx = \int_0^b \left[\frac{y^3}{3} \right]_0^{\frac{hx}{b}} dx$$

APPLICATION OF DOUBLE INTEGRATION

$$= \int_0^b \left[\frac{h^3 x^3}{3b^3} \right] dx = \frac{h^3}{3b^3} \int_0^b x^3 dx = \frac{h^3}{3b^3} \left| \frac{x^4}{4} \right|_0^b = \frac{b^4 h^3}{12b^3} = \frac{bh^3}{12}$$

$$I_y = \iint_R x^2 dxdy = \int_0^b x^2 \left[\int_0^{\frac{hx}{b}} dy \right] dx = \int_0^b x^2 \left| y \right|_0^{\frac{hx}{b}} dx = \frac{h}{b} \int_0^b x^3 dx$$

$$= \frac{h}{b} \left| \frac{x^4}{4} \right|_0^b = \frac{b^4 h}{4b} = \frac{b^3 h}{4}$$

APPLICATION OF DOUBLE INTEGRATION

The polar moment of inertia I_0 about origin of the mass in R is

$$I_0 = I_x + I_y = \iint_R \left(x^2 + y^2\right) f(x,y)\,dxdy$$

$$= \iint_R \left((x^2 + y^2\right) dxdy$$

$$= \int_0^b \left[\int_0^{\frac{hx}{b}} \left(x^2 + y^2\right) dy \right] dx = \int_0^b \left| x^2 y + \frac{y^3}{3} \right|_0^{\frac{hx}{b}} dx = \int_0^b \left(\frac{hx^3}{b} + \frac{h^3 x^3}{3b^3} \right) dx$$

$$= \left| \frac{hx^4}{4b} + \frac{h^3 x^4}{12b^3} \right|_0^b = \frac{hb^4}{4b} + \frac{h^3 b^4}{12b^3} = \left(\frac{b^3 h}{4} + \frac{h^3 b}{12} \right)$$

ANS

TEST YOUR KNOWLEDGE

$Evaluate\ the\ following\ double\ \text{int}\,egral$

$$Q.1:\int_{0}^{\frac{\pi}{4}}\int_{0}^{y}\frac{\sin y}{y}dxdy$$

$$Q.2:\int_{1}^{5}\int_{0}^{x^2}\left(1+2x\right)e^{x+y}dydx$$

$Q.3:$ *Find the volume of the following regions in space.*

The first octant region bounded by the coordinate planes and surfaces $y=1-x^2, z=1-x^2.$

$Q.4:$ *Find the center of the Gravity of a mass of density* $f(x,y)=1$ *in a region R. where R is the region* $x^2+y^2 \le a^2$ *in the first quadrant.*