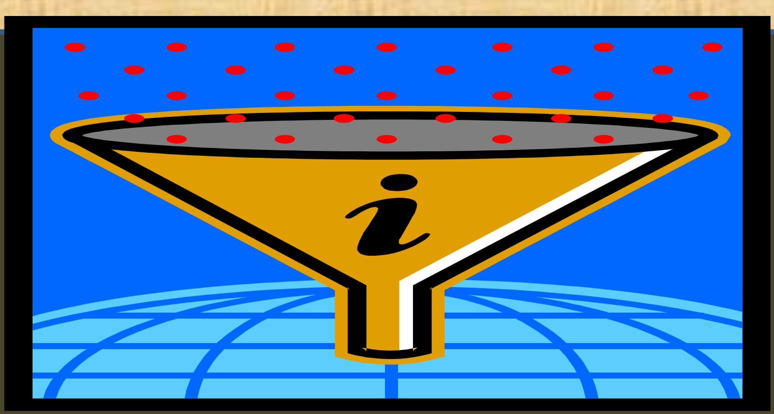
MATHEMATICS - I

FOR BTECH SECOND SEMESTER COURSE [COMMON TO ALL BRANCHES OF ENGINEERING]

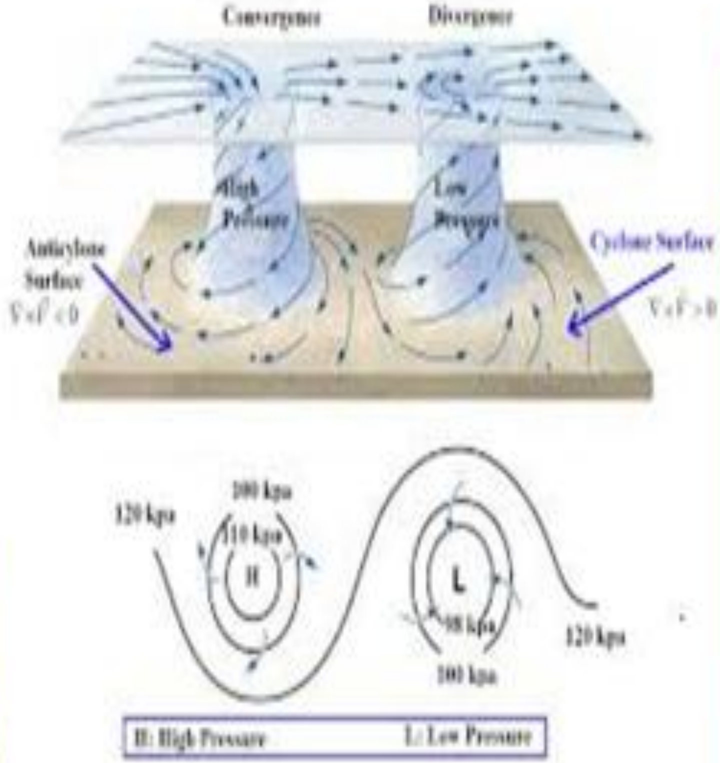
MATHEMATICS - I

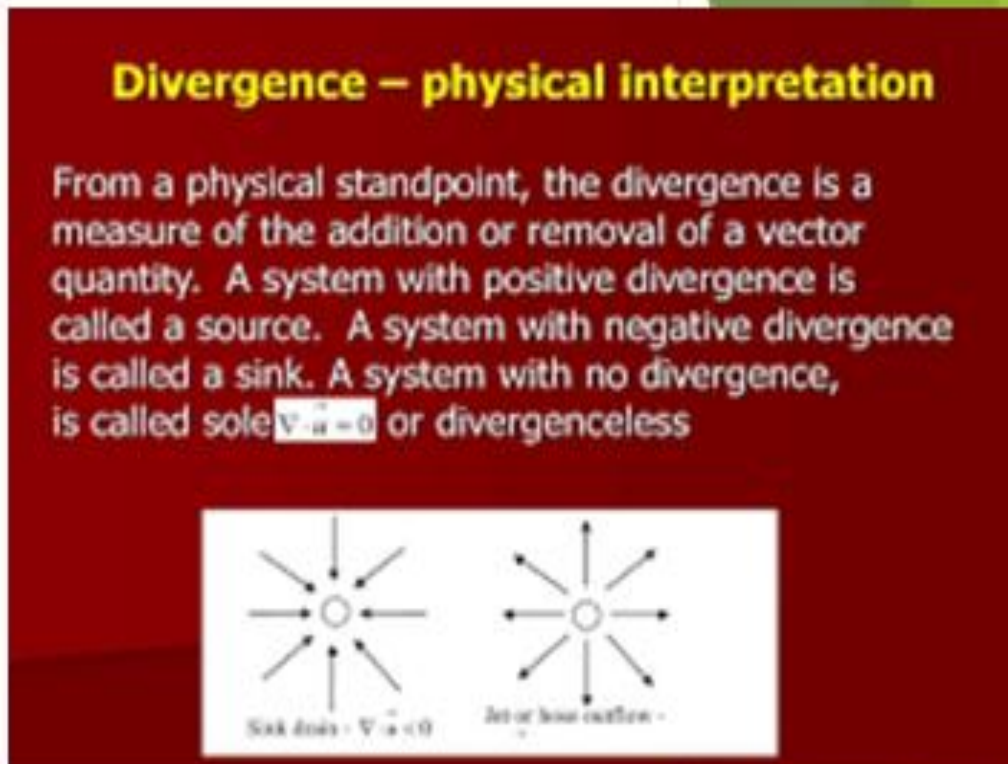
LECTURE - 25-*DIVERGENCE OF A VECTOR FIELD*

[Chapter – 8.10]

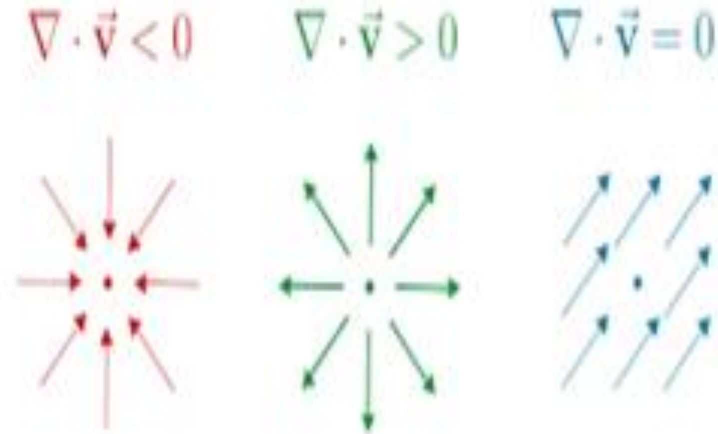


DIVERGENCE


Convergence
Divergence
High Pressure
Low Pressure
Anticylone Surface
Cyclone Surface
120 kpa
100 kpa
110 kpa
98 kpa
120 kpa
100 kpa
H: High Pressure
L: Low Pressure


Divergence – physical interpretation
From a physical standpoint, the divergence is a measure of the addition or removal of a vector quantity. A system with positive divergence is called a source. A system with negative divergence is called a sink. A system with no divergence, is called sole $\nabla \cdot \vec{a} = 0$ or divergenceless

APPLICATION OF DIVERGENCE

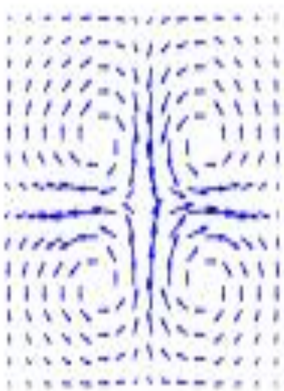


Goal: Reduce Scan Time

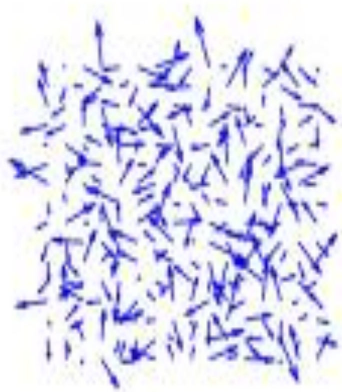
Utilize physical property of blood flow in compressed sensing reconstruction

- Blood flow is divergence-free

(What flows in flows out)



Divergence-free



Not divergence-free

DEFINITION OF DIVERGENCE

The divergence of a vector function $\vec{v}(x, y, z) = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$ is a scalar function which is defined as

$$div\,\vec{v} = \vec{\nabla}.\vec{v}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right).\left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

EXAMPLE OF DIVERGENCE

EXAMPLE:-Find the divergence of $2x^2 z\,\hat{i} - xy^2\,\hat{j} + 3yz^2\,\hat{k}$

SOLUTION:

$$\text{Let}\quad \vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k} = 2x^2 z\,\hat{i} - xy^2\,\hat{j} + 3yz^2\,\hat{k}$$

$$div\,\vec{v} = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}(2x^2 z) + \frac{\partial}{\partial y}(-xy^2) + \frac{\partial}{\partial z}(3yz^2)$$

$$= 4xz - 2xy + 6yz$$

PHYSICAL SIGNIFICANCE OF DIVERGENCE

The divergence of a vector function at a point P constitute the volume density of the flux of the vector function at that point that is divergence measures the change of flow. Now if the density is constant the fluid is incompressible and for that case $div\,\vec{v} = 0$.

USEFUL RESULTS INVOLVING DIVERGENCE

$$1.\ div(k\vec{v}) = k\,div\,\vec{v}$$

PROOF: Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\therefore k\vec{v} = k\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right) = \left(kv_1\right)\hat{i} + \left(kv_2\right)\hat{j} + \left(kv_3\right)\hat{k}$$

$$\therefore div(k\vec{v}) = \vec{\nabla}.(k\vec{v}) = \left(\frac{\partial}{\partial x}(kv_1) + \frac{\partial}{\partial y}(kv_2) + \frac{\partial}{\partial z}(kv_3)\right)$$

$$= k\frac{\partial v_1}{\partial x} + k\frac{\partial v_2}{\partial y} + k.\frac{\partial v_3}{\partial z} \qquad = k\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$$

$$= k\left(\vec{\nabla} \bullet \vec{v}\right) = k\,div\,\vec{v}$$

USEFUL RESULTS INVOLVING DIVERGENCE

$$2.\ div(\vec{\nabla} f) = \nabla^2 f$$

PROOF:

$$div(\vec{\nabla} f) = \vec{\nabla}.(\vec{\nabla} f)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right).\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \qquad = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f = \nabla^2 f$$

USEFUL RESULTS INVOLVING DIVERGENCE

$$3.\ div(f\vec{v}) = f\, div\vec{v} + \vec{v}.\vec{\nabla} f$$

PROOF: Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\therefore f\vec{v} = f\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right) = \left(fv_1\right)\hat{i} + \left(fv_2\right)\hat{j} + \left(fv_3\right)\hat{k}$$

$$\therefore div(f\vec{v}) = \vec{\nabla}.(f\vec{v}) = \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$

$$= \left(f\frac{\partial v_1}{\partial x} + v_1\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial v_2}{\partial y} + v_2\frac{\partial f}{\partial y}\right) + \left(f\frac{\partial v_3}{\partial z} + v_3\frac{\partial f}{\partial z}\right)$$

USEFUL RESULTS INVOLVING DIVERGENCE

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \left(v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y} + v_3\frac{\partial f}{\partial z}\right)$$

$$= f\, div\,\vec{v} + \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)\bullet\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= f\, div\,\vec{v} + \vec{v}\bullet\vec{\nabla} f$$

USEFUL RESULTS INVOLVING DIVERGENCE

$$4.\ div(f\vec{\nabla}g) = f\nabla^2 g + \left(\vec{\nabla}f.\vec{\nabla}g\right)$$

PROOF:

$$\vec{\nabla}g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$$

$$\therefore\ div(f\vec{\nabla}g) = \vec{\nabla}\bullet\left((f\vec{\nabla}g)\right) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$= \left(f\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial^2 g}{\partial y^2} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y}\right) + \left(f\frac{\partial^2 g}{\partial z^2} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial z}\right)$$

$$= f\nabla^2 g + \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial x}\hat{j} + \frac{\partial f}{\partial x}\hat{k}\right)\bullet\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right) = f\nabla^2 g + \left(\vec{\nabla}f.\vec{\nabla}g\right)$$

PROBLEMS FOR PRACTICE

1. Find the divergence of $3x^2\,\hat{i} + 5xy^2\,\hat{j} + xyz^3\,\hat{k}$ at the point $P(1,2,3)$.

SOLUTION:

$$\text{Let } \vec{v} = 3x^2\,\hat{i} + 5xy^2\,\hat{j} + xyz^3\,\hat{k}$$

$$\therefore div\,\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \frac{\partial}{\partial x}\left(3x^2\right) + \frac{\partial}{\partial y}\left(5xy^2\right) + \frac{\partial}{\partial z}\left(xyz^3\right)$$

$$= 6x + 10xy + 3xyz^2$$

$$\therefore \left[div\,\vec{v}\right]_{(1,2,3)} = 6(1) + 10(1)(2) + 3(1)(2)\left(3^2\right) = 80$$

PROBLEMS FOR PRACTICE

2.Find the value of

$$\vec{\nabla}.\left(r^3\vec{r}\right),\ \text{where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \text{and } r = \left|\vec{r}\right|$$

SOLUTION:

$$\vec{\nabla}.\left(r^3\vec{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(r^3\vec{r}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(r^3 x\hat{i} + r^3 y\hat{j} + r^3 z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(r^3 x) + \frac{\partial}{\partial y}(r^3 y) + \frac{\partial}{\partial z}(r^3 z)$$

$$\Rightarrow \vec{\nabla}.\left(r^3\vec{r}\right) == \left(r^3\frac{\partial x}{\partial x} + x\frac{\partial}{\partial x}\left(r^3\right)\right) + \left(r^3\frac{\partial y}{\partial y} + y\frac{\partial}{\partial y}\left(r^3\right)\right) + \left(r^3\frac{\partial z}{\partial z} + z\frac{\partial}{\partial z}\left(r^3\right)\right)$$

PROBLEMS FOR PRACTICE

$$\Rightarrow \vec{\nabla}.\left(r^3\vec{r}\right)=\left(r^3(1)+3xr^2\frac{\partial r}{\partial x}\right)+\left(r^3(1)+3yr^2\frac{\partial r}{\partial y}\right)+\left(r^3(1)+3zr^2\frac{\partial r}{\partial z}\right)$$

$$\Rightarrow \vec{\nabla}.\left(r^3\vec{r}\right)=3r^3+3r^2\left(x\frac{\partial r}{\partial x}+y\frac{\partial r}{\partial y}+z\frac{\partial r}{\partial z}\right)------(1)$$

Given that $\mathbf{r}=\left|\vec{r}\right|=\sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$=\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$$

PROBLEMS FOR PRACTICE

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2 + z^2}\right)$$

$$= \frac{2y}{2\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r}$$

and

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\sqrt{x^2 + y^2 + z^2}\right)$$

$$= \frac{2z}{2\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r}$$

$$\text{Equation } (1) \Rightarrow \vec{\nabla}.\left(r^3 \vec{r}\right) = 3r^3 + 3r^2\left(x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right)$$

$$= 3r^3 + 3r^2\frac{1}{r}\left(x^2 + y^2 + z^2\right) \quad = 3r^3 + 3r\left(r^2\right) = 6r^3$$

PROBLEMS FOR PRACTICE

3.Find $\nabla^2 f$ if $f = \dfrac{xy}{z}$

SOLUTION:

Given function is $f = \dfrac{xy}{z}$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{z}\right) = \frac{y}{z}, \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy}{z}\right) = \frac{x}{z} \text{ and } \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{xy}{z}\right) = -\frac{xy}{z^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{z}\right) = 0, \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{z}\right) = 0, \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial y}\left(-\frac{xy}{z^2}\right) = \frac{2xy}{z^3}$$

$$So \quad \nabla^2 f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0 + 0 + \frac{2xy}{z^3} = \frac{2xy}{z^3}$$

PROBLEMS FOR PRACTICE

4.Find the divergence of $e^{x}\hat{i}+e^{-x}y\hat{j}+2z\sinh x\hat{k}$

SOLUTION:

$$div\left(e^{x}\hat{i}+e^{-x}y\hat{j}+2z\sinh x\hat{k}\right)$$

$$=\frac{\partial}{\partial x}\left(e^{x}\right)+\frac{\partial}{\partial y}\left(e^{-x}y\right)+\frac{\partial}{\partial z}(2z\sinh x)$$

$$=e^{x}+e^{-x}+2\sinh x \quad =e^{x}+e^{-x}+e^{x}-e^{-x}$$

$$=2e^{x}$$

PROBLEMS FOR PRACTICE

5.Find the divergence of $xyz(x\hat{i}+y\hat{j}+z\hat{k})$.

SOLUTION:

$$div\left(xyz(x\hat{i}+y\hat{j}+z\hat{k}) \right)$$

$$= \frac{\partial}{\partial x}\left(x^2yz\right)+\frac{\partial}{\partial y}\left(xy^2z\right)+\frac{\partial}{\partial z}\left(xyz^2\right)$$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz$$