DEPARTMENT OF MATHEMATICs

[YEAR OF ESTABLISHMENT – 1997]

MATHEMATICS - I

TEXT BOOK: DIFFERENTIAL CALCULUS BY SHANTI NARAYAN & P.K.MITTAL

LECTURE

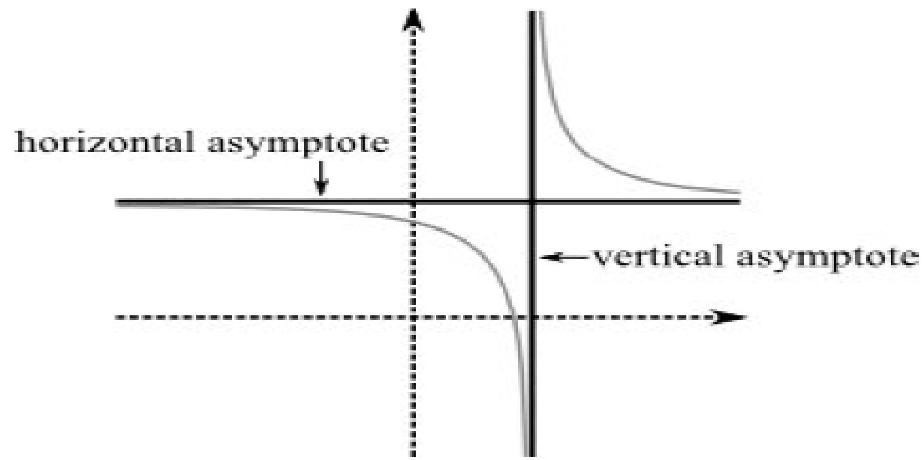
Asymptotes

4.1 Introduction:

An asymptote is a line that approaches closer to a given curve as one or both of x or y coordinates tend to infinity but never intersects or crosses the curve. There are two types of asymptotes viz. Rectangular asymptotes and Oblique asymptotes

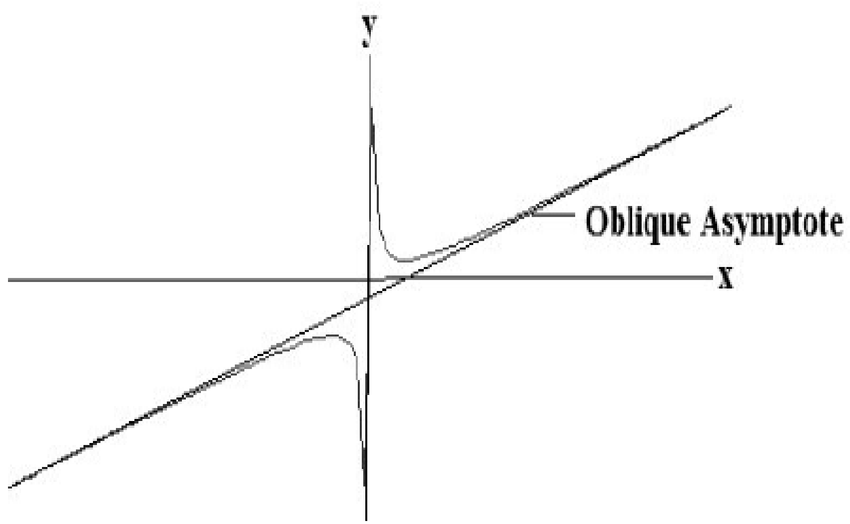
Rectangular Asymptote:

If an asymptote is parallel to x-axis or to y-axis, then it is called rectangular asymptote. An asymptote parallel to x-axis is called horizontal asymptote and the asymptote parallel to y-axis is called vertical asymptote.



Oblique Asymptote:

If an asymptote is neither parallel to x-axis nor to y-axis then it is called an oblique asymptote. An oblique asymptote occurs when the degree of polynomial in the numerator is greater than that of polynomial in the denominator. To find the oblique asymptote, numerator must be divided by the denominator by using either long division or synthetic division.



Determination of the Rectangular Asymptotes

1. Asymptotes parallel to Y-axes are obtained by equating to zero the real linear factors in the co-efficient of the highest power of y, in the equation of the curve.

The curve will have no asymptote parallel to Y-axis, if the co-efficient of the highest power of y, is a constant or if its linear factors are all imaginary.

2. Asymptotes parallel to X-axes are obtained by equating to zero the real linear factors in the co-efficient of the highest power of x, in the equation of the curve.

The curve will have no asymptote parallel to X-axis, if the co-efficient of the highest power of x, is a constant or if its linear factors are all imaginary.

4.2 Method of finding rectangular asymptote:

- To find an asymptote parallel to x-axis equate to zero the coefficient of highest power of x in the equation of the curve.
- To find an asymptote parallel to y-axis equate to zero the coefficient of highest power of y in the equation of the curve.

Example1 Find the asymptotes parallel to coordinate axes of the curve

$$4x^2 + 9y^2 = x^2y^2$$

Solution: The equation of the given curve is $4x^2 + 9y^2 - x^2y^2 = 0$

Equating it to zero, coefficient of x^2 (which is highest power of x) we get,

$$4 - y^2 = 0 \Rightarrow y = \pm 2$$

$\therefore y = 2, y = -2$ are the two asymptotes parallel to x-axis

Equating to zero the coefficient of highest power of y, we get

$$9 - x^2 = 0 \Rightarrow x = \pm 3$$

$\therefore x = 3, x = -3$ are the asymptotes parallel to y – axis

Example 2. Find the Asymptotes parallel to co-ordinate axes of the following curve,

$$(x^2+y^2)x-ay^2=0$$

Solution

The coefficient of the highest power y^2 of y is x-a.

Hence, the asymptote parallel to y-axis is x-a = 0 .

The co-efficient of the highest power x^3 of x is 1 which is a constant. Hence there is no asymptote parallel to x-axis.

Exercise. Find the Asymptotes parallel to co-ordinate axes of the curves

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$

4.3 Method of finding oblique asymptote:

Let the asymptote be $y = mx + c$

Let the equation of the curve be

$$\phi_n(x, y) + \phi_{n-1}(x, y) + \dots\dots\dots + \phi_1(x, y) + k = 0 \quad -(1)$$

where $\phi_n(x, y)$ denotes the term of highest degree of the curve.

Step-1 Put x = 1, y = m in $\phi_n(x, y), \phi_{n-1}(x, y), \dots\dots, \phi_1(x, y)$

Step-2 Find all the real roots of $\phi_n(m) = 0$

Step-3 If m is a non-repeated root, then corresponding value of c is given by

$$c\ \phi_n'(m) + \phi_{n-1}(m) = 0, \quad (\phi_n'(m) \neq 0)$$

If $\phi_n'(m) = 0$ then there is no asymptote to the curve corresponding to this value of m.

Step-4 If m is a repeated root occurring twice, then the two values of c are given by

$$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0 \quad (\phi_n''(m) \neq 0)$$

Step-5 The asymptote of the curve is y = m x + c

Example 3. Find all the asymptotes of the curve

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$$

Solution: In the curve the highest degree term of x is x^3 and its coefficient is 1.Equating it to 0 we get 1= 0 which is absurd, thus the curve has no asymptote parallel to x – axis. Also the coefficient of highest degree term in y is 1, thus the curve has no asymptote parallel to y-axis.

Now finding oblique asymptote

Here $\phi_3(x, y) = x^3 - x^2y + y^3 - xy^2$

$\phi_2(x, y) = 2x^2 - 4y^2 + 2xy$, $\phi_1(x, y) = x + y$

Let the asymptote be given by $y = mx + c$

Step-1 Putting x = 1 & y = m in $\phi_3(x, y), \phi_2(x, y), \phi_1(x, y)$

$$\phi_3(m) = m^3 - m^2 - m + 1$$

$\phi_2(m) = 2 - 4m^2 + 2m$ and $\phi_1(m) = 1 + m$

Step-2 The values of m are obtained by solving $\phi_3(m) = 0$

$\therefore m^3 - m^2 - m + 1 = 0 \Rightarrow (m^2 - 1)(m - 1) = 0$

$\Rightarrow m = 1, 1, -1$

Here m = –1 is a non repeated root & m = 1 is a repeated root.

Step-3: For m = – 1 (non-repeated root),the corresponding value of c is given by

$$c\phi_3'(m) + \phi_2(m) = 0$$

Now $\phi_3(m) = m^3 - m^2 - m + 1$

$$\Rightarrow \phi_3' = 3m^2 - 2m - 1$$

$$\therefore c(3m^2 - 2m - 1) + (2 + 2m - 4m^2) = 0$$

$$\Rightarrow c\left(3m^2 - 2m - 1\right) = 4m^2 - 2m - 2 \quad \Rightarrow c = \frac{4m^2 - 2m - 2}{(3m^2 - 2m - 1)} = \frac{4}{4} = 1$$

Thus for $m = -1$, c= 1

Now the asymptote is $y = mx + c$ i.e. $y = -x + 1$ or $x + y = 1$

Step-4 For m = 1 (repeated root), the value of c is given by

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0\ldots\ldots..(2)$$

Now $\phi_3'(m) = 3m^2 - 2m - 1$

$$\Rightarrow \phi_3''(m) = 6m - 2$$

$$\phi_2(m) = 2 + 2m - 4m^2$$

$$\Rightarrow \phi_2''(m) = 2 - 8m$$

$$\phi_1(m) = 1 + m$$

Putting these values in (2) we get

$$\frac{c^2}{2}(6m-2) + c(-8m + 2) + 1 + m = 0$$

For $m = 1$

$$\frac{c^2}{2}(4) + c(-6) + 2 = 0 \Rightarrow c^2 - 3c + 1 = 0$$

$$\Rightarrow c = \frac{3 \pm \sqrt{5}}{2} = \frac{3 + \sqrt{5}}{2}, \quad \frac{3 - \sqrt{5}}{2}$$

For m = 1, c= $\frac{3+\sqrt{5}}{2}$, $\therefore$ the asymptote is y=mx + c i.e. $y = x + \frac{3+\sqrt{5}}{2}$

$$\Rightarrow 2y = 2x + (3 + \sqrt{5})$$

For m = 1, c= $\frac{3-\sqrt{5}}{2}$ $\therefore$ the asymptote is $2y = 2x + (3 - \sqrt{5})$

$\therefore$ Three asymptotes of the given curve is

$$x + y = 1,\ 2(y-x) = 3 + \sqrt{5},\ 2(y-x) = 3 - \sqrt{5}$$

Example 4: Find the asymptotes of the following cubic curve,

$$xy^2 - x^2y - 3x^2 - 2xy + y^2 + x - 2y + 1 = 0$$

Solution

Putting x=1, y=m in the third degree and second degree terms separately, we get

$$\phi_3(m) = m^2 - m, \qquad \phi_2(m) = -3 - 2m + m^2$$

The slopes of the asymptotes are given by,

$$\phi_3(m) = m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0,1$$

Again c is given by

$$c\phi_3'(m)+\phi_2(m)=0$$

i.e. $$c(2m-1)+(-3-2m+m^2)=0$$

Putting m=0,1, we get c = -3,4 respectively.

Therefore the asymptotes are

$$y=-3, y=x+4.$$

Asymptotes Parallel to co-ordinate axes

The coefficient of the highest power y^2 of y is x+1.

Hence, the asymptote parallel to y-axis is x+1 = 0 .

The co-efficient of the highest power x^2 of x is y+3

Hence, the asymptote parallel to x-axis is y+3 = 0

Exercise

1. Find the asymptotes to the following curves

(i) $y^3 + x^2y + 2xy^2 - y + 1 = 0$

(ii) $x^2y + xy^2 + xy + y^2 + 3x = 0$

2. Show that the following curve has no asymptotes :-

$$x^4 + y^4 = a^2(x^2 - y^2)$$

Intersection of a curve and its asymptotes

Note 1: A polynomial (of x, y) of degree n has at most n asymptotes.

Note 2: Every asymptote of polynomial of degree n, cut the curve at most $n-2$ points.

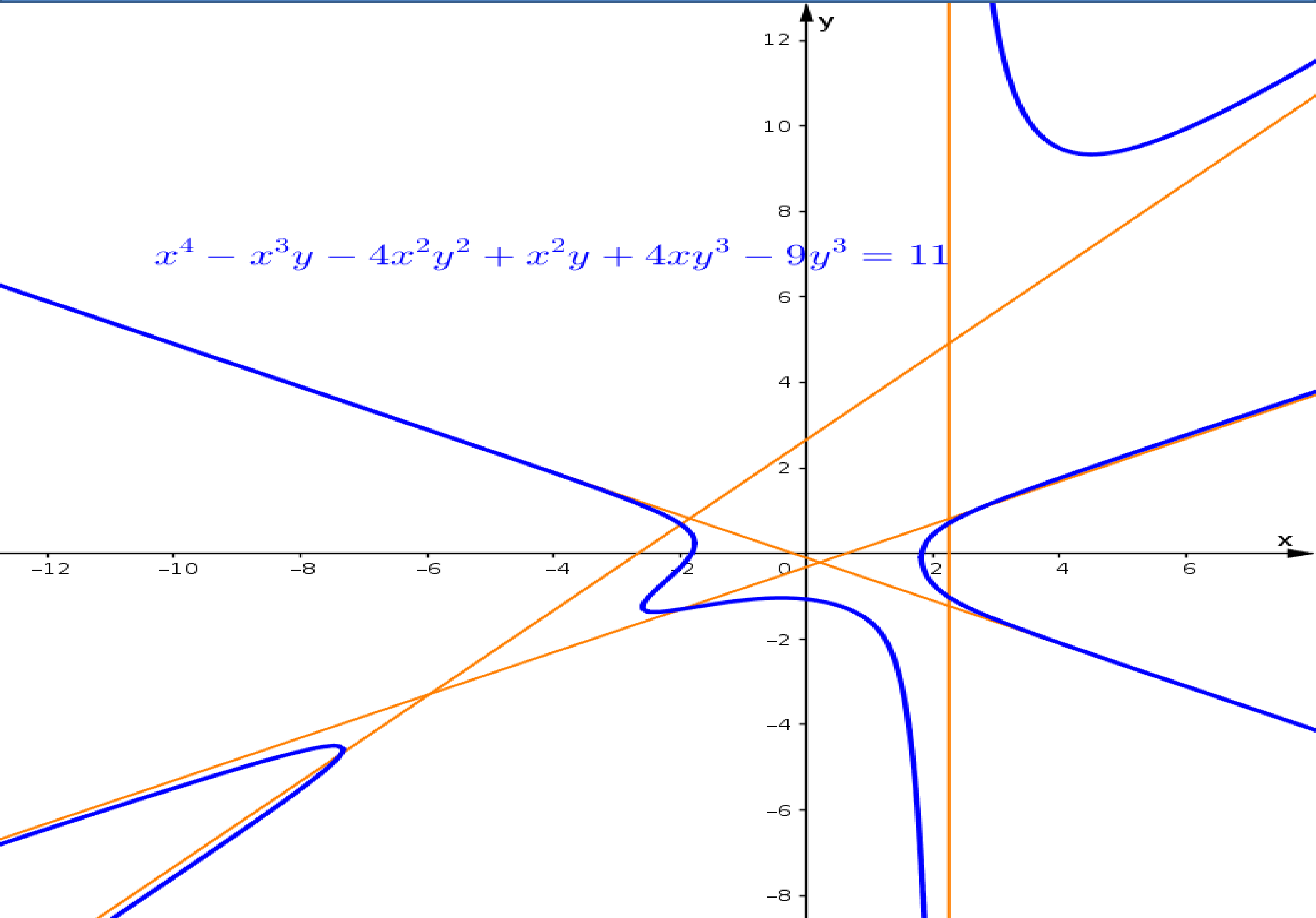
Note 3: Total number of points of intersection between asymptotes and the given curve is $n(n-2)$.

Note 4: The points of intersection lie on the curve given by

$$f(x,y) - g(x,y) = 0,$$

where $f(x,y)$ is the given curve and $g(x,y)$ the product of all asymptotes.

Intersection of a curve and its asymptotes



Particular cases:

(i) For a cubic, n=3, and therefore the asymptotes cut the curve in 3(3-2) = 3 points, which lie on a curve of degree 3 – 2 = 1,
i.e. on a straight line.

(ii) For a quadratic, n = 4, and, therefore the asymptotes cut the curve in 4 (4-2) = 8 points which lie on a curve of degree
4 – 2 = 2 i.e., on a conic.

Example 5: Show that the asymptotes of the cubic $x^3 - xy^2 - 2xy + 2x - y = 0$ cut the curve again in points which lie on the line $3x - y = 0$.

Solution

Putting x=1, y=m in the third degree and second degree terms separately, we get

$$\phi_3(m) = 1 - m^2, \qquad \phi_2(m) = -2m$$

The slopes of the asymptotes are given by

$$\phi_3(m) = 1 - m^2 = 0 \Rightarrow m = 1, -1$$

Again c is given by

$$c\phi_3'(m) + \phi_2(m) = 0$$

i.e. $$c(-2m) + (-2m) = 0$$

Putting m=1,-1 we get c = -1 in both the cases.

Therefore the asymptotes are

$$y = x - 1, y = -x - 1.$$

i.e. $x - y - 1 = 0 \quad and \quad x + y + 1 = 0.$

<u>Asymptotes Parallel to co-ordinate axes</u>

The coefficient of the highest power y^2 of y is -x.

Hence, the asymptote parallel to y-axis is x = 0 .

The co-efficient of the highest power x^3 of x is 1 which is a constant. Hence there is no asymptote parallel to x-axis.

Hence, the asymptotes of the given curve are,

x = 0, x – y -1= 0 and x + y +1 = 0.

The joint equation of the asymptotes is,

$$x(x - y - 1)(x + y + 1) = 0$$

$$\Rightarrow x^3 - xy^2 - 2xy - x = 0$$

The equation of the curve can be written as,

$$x^3 - xy^2 - 2xy + 2x - y = 0$$

$$\Rightarrow x^3 - xy^2 - 2xy - x + (3x - y) = 0$$

Here, $F_3 = x^3 - xy^2 - 2xy - x, \quad \mathrm{F}_1 = 3x - y.$

Hence, the points of intersection lie on the line,

$$\mathrm{F}_1 \equiv 3x - y = 0$$

Example-6. Find the equation of the quadratic curve which has $x = 0, y = 0, y = x$ and $y = -x$ for asymptotes and which passes through (a, b) and which cuts its asymptotes again in 8 points that lie on a circle whose centre is origin and radius a.

Solution

The combined equation of the asymptotes is

$$xy(x-y)(x+y)=0$$

$$\Rightarrow xy(y^2-x^2)=0. \qquad \ldots(i)$$

Also the equation of the circle with origin as centre and radius a is,

$$x^2+y^2=a^2.$$

Let the equation of the curve whose asymptotes are given by (i) be

$$xy(y^2 - x^2) + \lambda(x^2 + y^2 - a^2) = 0 \qquad ...(ii)$$

If it passes through (a, b), then

$$ab(b^2 - a^2) + \lambda(a^2 + b^2 - a^2) = 0 \Rightarrow \lambda = a(a^2 - b^2)/b.$$

So, the required equation of the curve is (from (ii))

$$bxy(y^2 - x^2) + a(a^2 - b^2)(x^2 + y^2 - a^2) = 0 \qquad (Ans)$$

Exercise

1. Find all the asymptotes of the curve

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5 = 0$$

Show that the asymptotes meet the curve again in three points which lie on a straight line, and find the equation of this line.

2. Find the asymptotes of the curve

$$4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

and show that they pass through the points of intersection of the curve with the ellipse

$$x^2 + 4y^2 = 4$$

Determination Of Asymptotes

1. **Equation of the Asymptotes which are not parallel to Y-axis can be obtained by following method:**

(i) Find $\lim_{x \to \infty} (y/x)$; let $\lim_{x \to \infty} (y/x) = m$.

(ii) Find $\lim_{x \to \infty} (y-mx)$; let $\lim_{x \to \infty} (y-mx) = c$.

Then, $y=mx+c$ is an asymptote.

2. **Equation of the Asymptotes which are not parallel to X-axis can be obtained by following method:**

(i) Find $\lim_{y \to \infty} \left(\frac{x}{y}\right)$; let $\lim_{y \to \infty} \left(\frac{x}{y}\right) = m$

(ii) Find $\lim_{y \to \infty} (x - my)$; let $(x - my) = d$

Then $x = my + d$ is an asymptote.

Example:

Find the horizontal and vertical asymptotes of the function.

$$f(x) = \frac{2x+1}{3x-5}$$

Solution:

Method 1:

Divide both numerator and denominator by x.

$$\lim_{x\to\infty} \frac{2x+1}{3x-5} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3-\frac{5}{x}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(2+\frac{1}{x}\right)}{\lim\limits_{x\to\infty}\left(3-\frac{5}{x}\right)}$$

$$= \frac{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 3 - 5\lim\limits_{x\to\infty} \frac{1}{x}}$$

$$= \frac{2+0}{3-5(0)} = \frac{2}{3}$$

The line $y = \frac{2}{3}$ is the horizontal asymptote.

Example:

Find the vertical asymptotes of $f(x) = \dfrac{4x}{x-3}$

Solution:

Method 1: Use the definition of Vertical Asymptote.

If x is close to 3 but larger than 3, then the denominator x – 3 is a small positive number and 2x is close to 8. So, $f(x) = \dfrac{4x}{x-3}$ is a large positive number.

Intuitively, we see that

$$\lim_{x \to 3^+} \frac{4x}{x-3} = \infty$$

Similarly, if x is close to 3 but smaller than 3, then x – 3 is a small negative number and 2x is close to 8. So, $\dfrac{4x}{x-3}$ is a large negative number.

$$\lim_{x \to 3^+} \frac{4x}{x-3} = \infty$$

The line x = 3 is the vertical asymptote.

Example: Calculate asymptotes and sketch the graph of the function $f(x) = \dfrac{x^2 - x - 2}{x - 3}$.

Solution: By equating the numerator with zero and solving for x we find the x-intercepts,

$$x^2 - x - 2 = (x+1)(x-2) = 0,$$

$$x_1 = -1 \text{ and } x_2 = 2.$$

We calculate $f(0)$ to find the y-intercept,

$$f(0) = 2/3.$$

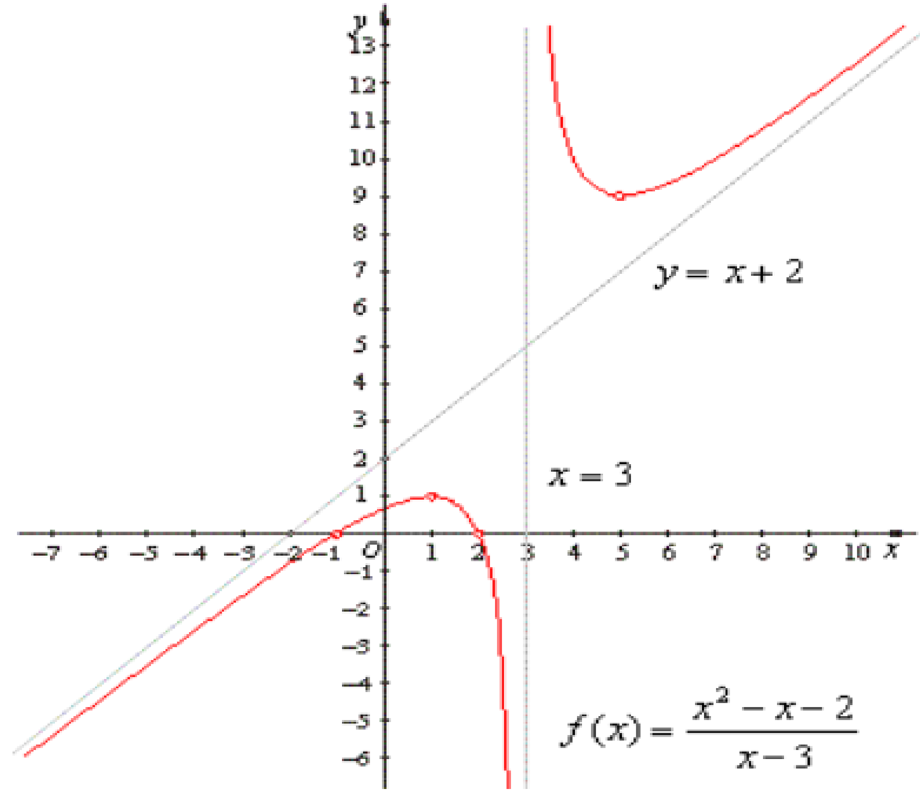
By equating the denominator with zero and solving for x we find the vertical asymptote,

$$x = 3.$$

Let calculate following limits

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and} \quad c = \lim_{x\to\infty}[f(x) - mx]$$

to find the slant asymptote $y = mx + c$.



$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2 - x - 2}{x(x-3)} = \lim_{x\to\infty} \frac{x^2 - x - 2 \,|\div x^2}{x^2 - 3x \,|\div x^2} = \lim_{x\to\infty} \frac{1 - \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{3}{x}} = 1, \quad m = 1$$

$$c = \lim_{x\to\infty}[f(x) - mx] = \lim_{x\to\infty}\left[\frac{x^2 - x - 2}{x - 3} - 1 \cdot x\right] = \lim_{x\to\infty} \frac{2x - 2 \,|\div x}{x - 3 \,|\div x} = \lim_{x\to\infty} \frac{2 - \frac{2}{x}}{1 - \frac{3}{x}} = 2, \quad c = 2.$$

Therefore, the line $y = x + 2$ is the slant asymptote of the given function.

Asymptotes in polar co-ordinates

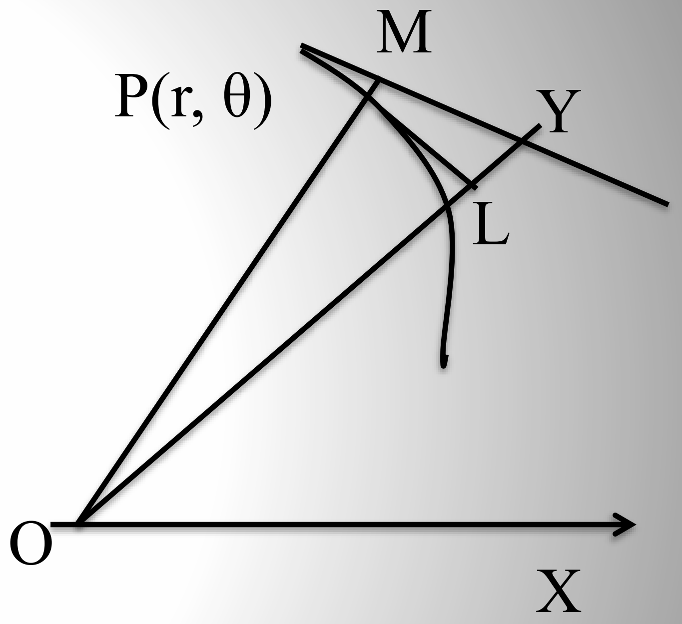
Consider a curve in polar form $r = f(\theta)$,

We have to obtain the constants p and,

α, so that the line

$$p = r \sin(\alpha - \theta),$$

is the asymptote of the given curve.



Working rule for obtaining asymptotes to polar curves:

Step-1: Change r to $\frac{1}{u}$ in the given equation of the curve.

Step-2: Find out limit of θ as $u \to 0$.

Let α be any of the several possible limits of θ.

Step-3: Determine $-\frac{d\theta}{du}$ and its limit as $u \to 0$ and $\theta \to \alpha$.

Let this limit be p.

Then $p = r\sin(\alpha - \theta)$ is the corresponding asymptote.

Example 1. Find the asymptotes of the hyperbolic spiral $r\,\theta = a$.

Solution

Here, $\theta = \frac{a}{r} = au$

So that $\theta \to 0$ as $u \to 0$. Hence, $\alpha = \lim \theta = 0.$

Since $u = \frac{\theta}{a}$, we have $du/d\theta = 1/a$ or $d\theta\,/\,du = a$

Therefore $p = -d\theta\,/du$ at $\theta = \alpha$ is $p = -a.$

Thus, the asymptotes is $p = r\sin(\alpha - \theta)$

This implies $-a = r\sin(0 - \theta)$ or $a = r\sin\theta$.

Example 2. Find the asymptotes of the curve $r\left(1-e^{\theta}\right)=a$

Solution

Here given that $r=\frac{a}{1-e^{\theta}}$

Then $u=\frac{1}{r}=\frac{1-e^{\theta}}{a}$

And $u=0$ implies $\theta=0$, so $\alpha=0$.

$$\frac{du}{d\theta}=-\frac{e^{\theta}}{a}$$

Then $p=-\frac{d\theta}{du}$ at $\alpha=0$, is given by $p=a$.

Therefore the required asymptote is $p=r\sin(\alpha-\theta)$

This implies $-a=r\sin(0-\theta)$ or $a=r\sin\theta$.

Example 3. Find the asymptotes of the curve $r = \dfrac{2a}{1-2\cos\theta}$

Here

$$u = \frac{1}{r} = \frac{1}{a}\left(\tfrac{1}{2} - \cos\theta\right).$$

When $u \to 0$, $(\tfrac{1}{2} - \cos\theta) \to 0$ so that $\cos\theta \to \tfrac{1}{2}$.

$\therefore \quad \theta_1 = \pm\pi/3.$

Now,

$$\frac{du}{d\theta} = \frac{1}{a}\sin\theta \text{ or } -\frac{d\theta}{du} = -\frac{a}{\sin\theta}.$$

$\therefore$
$$-\frac{d\theta}{du} \to -\frac{2a}{\sqrt{3}} \text{ as } \theta \to \frac{\pi}{3},$$

and

$$\frac{-d\theta}{du} \to \frac{2a}{\sqrt{3}} \text{ as } \theta \to -\frac{\pi}{3}.$$

$\therefore \quad -\dfrac{2a}{\sqrt{3}} = r\sin\left(\dfrac{\pi}{3} - \theta\right)$, i.e., $4a = r(\sqrt{3}\sin\theta - 3\cos\theta)$,

and $\quad \dfrac{2a}{\sqrt{3}} = r\sin\left(-\dfrac{\pi}{3} - \theta\right)$, i.e., $-4a = r(\sqrt{3}\sin\theta + 3\cos\theta)$,

are the two asymptotes.

Example 4. Find the asymptotes of the curve $r = a\tan\theta$

Given r = a tanθ, i.e. 1/r = cos θ/a sin θ ......(1)

Let $u = \frac{1}{r}$ then $u \to 0$ implies $\frac{\cos\theta}{a\sin\theta} \to 0$.

i.e. $\cos\theta \to 0$ impling $\theta \to (2n+1)\frac{\pi}{2}$

Now $$\frac{du}{d\theta} = \frac{1}{a}(-\text{cosec}^2\theta) = -\frac{1}{a\sin^2\theta}$$

$$p = \underset{\theta\to\theta_1}{\text{Lt}}\left(-\frac{d\theta}{du}\right) = -\underset{\theta\to\theta_1}{\text{Lt}}(-a\sin^2\theta) = a(\sin\theta_1)^2$$

$$= a[\sin(2n+1)\pi/2]^2 = a(-1)^{2n} = a \quad .....(3)$$

By definition,

$$p = r\sin(\theta_1 - \theta) \quad \text{....(4)}$$

From (3) and (4)

$$a = r\sin((2n\pi + \pi)/2 - \theta))$$

$$a = r\sin(n\pi + \pi/2 - \theta) = r(-1)^n \sin(\pi)/2 - \theta)$$

$$a = \pm r\cos\theta \text{ or } r\cos\theta = \pm a$$

Exercise

1. Find the asymptotes of the following curves,

(i) $r\log\theta = a$

(iii) $r\sin n\theta = a$