C. V. RAMAN GLOBAL UNIVERSITY
ODISHA
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DEPARTMENT OF MATHEMATICS

MATHEMATICS - I

TEXT BOOK: ADVANCED ENGINEERING MATHEMATICS BY ERWIN KREYSZIG [8th EDITION]

LECTURES –21

Curves, Tangents and Arc Length using Vector Concepts
[Chapters –8.5]

Content:

- Introduction to Curves, Tangents and Arc Length using Vector Concepts
- Definition and Examples
- Properties
- Examples
- Test Knowledge
- Problem Solved
- Practice Problems

Curves. Arc Length. Curvature. Torsion

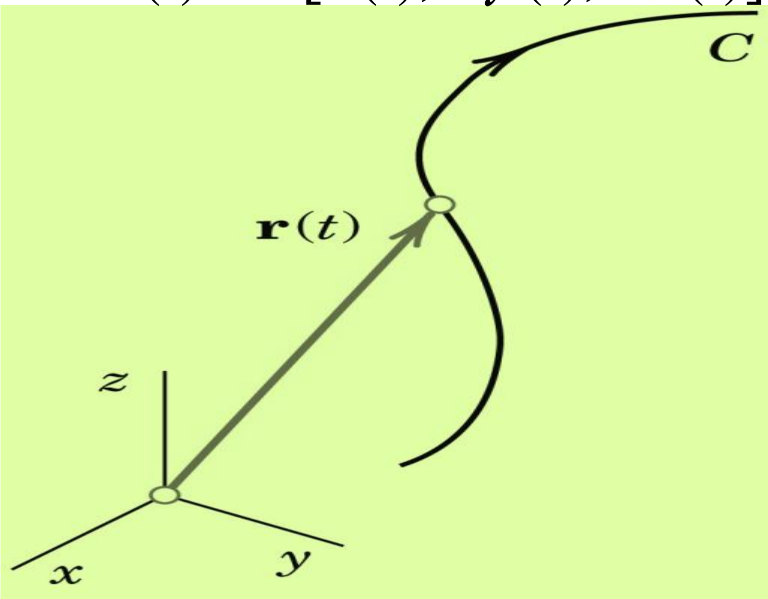
The application of vector calculus to geometry is a field known as **differential geometry**.

Bodies that move in space form paths that may be represented by curves C. This and other applications show the need for **parametric representations** of C with **parameter** t, which may denote time or something else (see Fig.).

Curves. Arc Length. Curvature. Torsion

A typical parametric representation is given by

(1) $$\vec{r}(t) = [x(t),\ \ y(t),\ \ z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$



Here t is the parameter and x, y, z are Cartesian coordinates, that is, the usual rectangular coordinates .

Fig. Parametric representation of a curve

Curves. Arc Length. Curvature. Torsion

To each value $t = t_0$, there corresponds a point of C with position vector $\mathbf{r}(t_0)$ whose coordinates are $x(t_0)$, $y(t_0)$, $z(t_0)$.

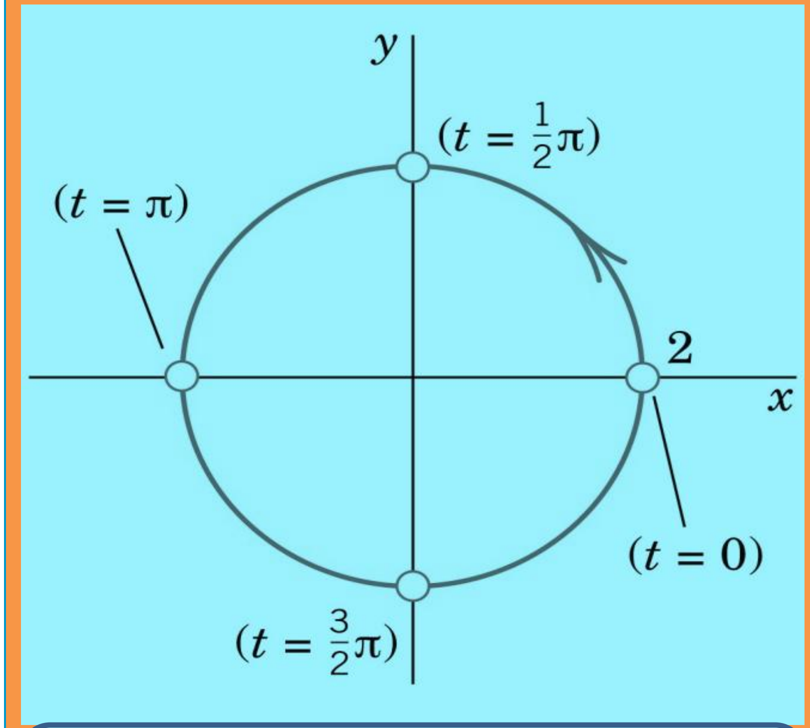


Fig. Circle in Example 1

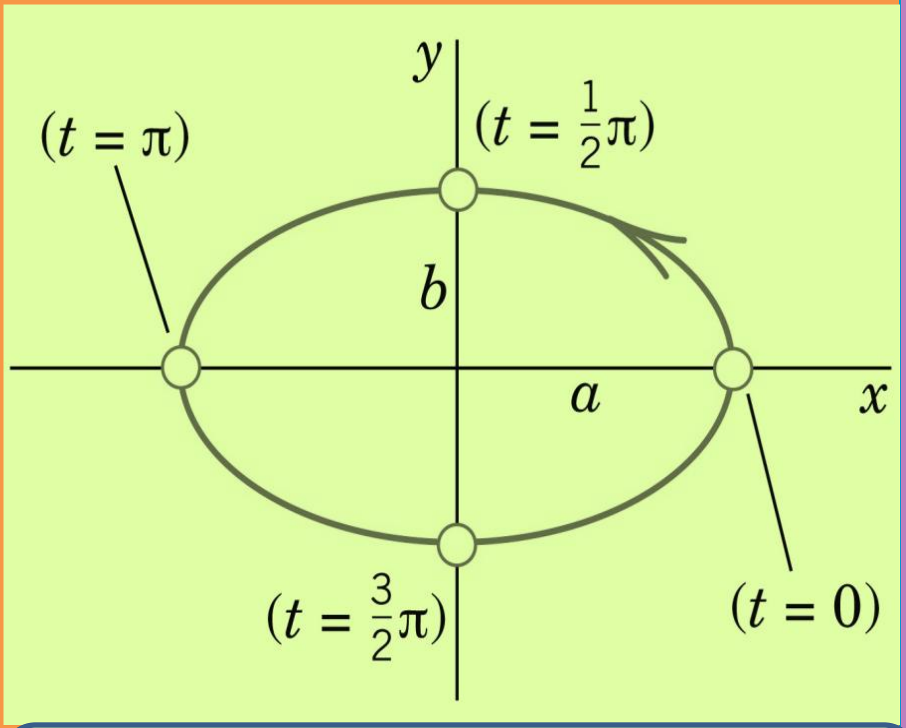


Fig. Ellipse in Example 2

The use of parametric representations has key advantages over other representations that involve projections into the xy-plane and xz-plane or involve a pair of equations with y or with z as independent variable. The projections have the representation

(2) $$y = f(x), \quad z = g(x).$$

The advantages of using (1) instead of (2) are that, in (1), the coordinates x, y, z all play an equal role, that is, all three coordinates are independent variables. Moreover, the parametric representation (1) induces an orientation on C. This means that as we increase t, we travel along the curve C in a certain direction. *The sense of increasing t is called the positive sense on C. The sense of decreasing t is then called the negative sense on C*, given by (1).

PARAMETRIC REPRESENTATION OF STRAIGHT LINE

We know that a straight line passing through a point, say (x_1, y_1, z_1) and having direction rations, say l, m, and n, can be represented as:

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

$$\text{Let } \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=t$$

$$\Rightarrow x=x_1+lt,\ \ y=y_1+mt,\ \text{and}\ z=z_1+nt$$

Parametric representation of the given straight line is

$$\vec{r}\left(t\right)=x\hat{i}+y\hat{j}+z\hat{k}=\left(x_1+lt\right)\hat{i}+\left(y_1+mt\right)\hat{j}+\left(z_1+nt\right)\hat{k}$$

PARAMETRIC REPRESENTATION OF STRAIGHT LINE

$$= x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \left(l\hat{i} + m\hat{j} + n\hat{k}\right)t$$

$$i.e.\ \vec{r}\,(t) = \vec{a} + \vec{b}t,$$

$$\text{where } \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\text{and } \vec{b} = l\hat{i} + m\hat{j} + n\hat{k}$$

$\therefore$ Parametric representation of the given straight line which passes through a point with position vector $\vec{a}$ and is in the direction of a vector $\vec{b}$ is $\vec{r}\left(t\right) = \vec{a} + \vec{b}t - - - (3)$

PARAMETRIC REPRESENTATION OF STRAIGHT LINE

We know that a straight line passing through two points, say (x_1, y_1, z_1) and (x_2, y_2, z_3) can be represented as:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\text{Let } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=t$$

$$\Longrightarrow x = x_1+(x_2-x_1)t,$$

$$y = y_1+(y_2-y_1)t,$$

$$\text{and } z = z_1+(z_2-z_1)t$$

PARAMETRIC REPRESENTATION OF STRAIGHT LINE

Parametric representation of the given straight line is

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = \left[x_1 + (x_2 - x_1)t\right]\hat{i}$$

$$+\left[y_1 + (y_2 - y_1)t\right]\hat{j} + \left[z_1 + (z_2 - z_1)t\ \right]\hat{k}$$

$$= x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \left((x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\right)t$$

$$i.e.\ \vec{r}(t) = \vec{a} + \left(\vec{b} - \vec{a}\right)t,$$

where $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

PARAMETRIC REPRESENTATION OF STRAIGHT LINE

$\therefore$ Parametric representation of the given straight line which passes through a point with position vector $\vec{a}$ and is in the direction of a vector $\vec{b}$ is

$$\vec{r}(t) = \vec{a} + \left(\vec{b} - \vec{a}\right)t \quad --- 3(A)$$

Curves. Arc Length. Curvature. Torsion

A *plane curve* is a curve that lies in a plane in space. A curve that is not plane is called a *twisted curve*.

A *simple curve* is a curve without *multiple points*, that is, without points at which the curve intersects or touches itself. Circle and helix are simple curves. Figure given below shows curves that are not simple.

An *arc* of a curve is the portion between any two points of the curve. For simplicity, we say "curve" for curves as well as for arcs.

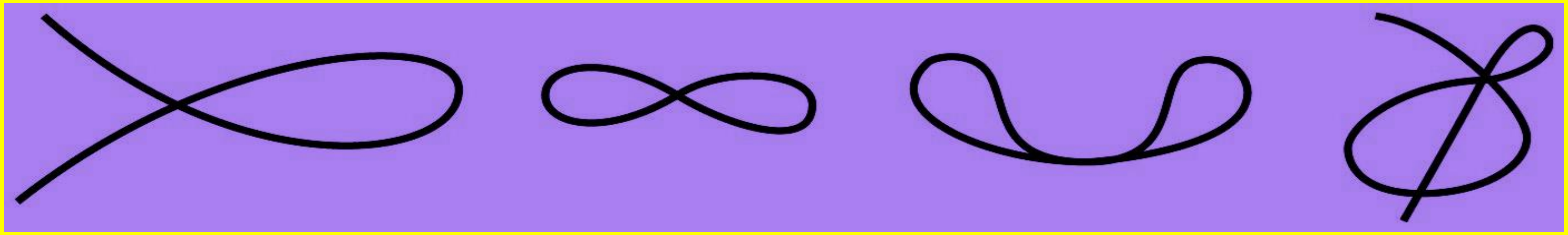


Fig. . Curves with multiple points

Tangent to a Curve

Tangents are straight lines touching a curve. The **tangent** to a simple curve C at a point P is the limiting position of a straight line L through P and a point Q of C as Q approaches P along C. If C is given by $\vec{\mathbf{r}}(t)$, and P and Q correspond to t and $t + \Delta t$, then a vector in the direction of L is

$$\boxed{\frac{1}{\Delta t}[\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)].}$$

In the limit this vector becomes the derivative

$$\boxed{\vec{\mathbf{r}}'(t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}[\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)],} \tag{4}$$

provided $\vec{\mathbf{r}}(t)$ is differentiable, as we shall assume from now on. If $\vec{\mathbf{r}}'(t) \neq \vec{0}$ we call $\vec{\mathbf{r}}'(t)$ a **tangent vector** of C at P because it has the direction of the tangent. The corresponding unit vector is the **unit tangent vector**

$$\vec{\mathbf{u}} = \frac{1}{|\vec{\mathbf{r}}'|}\vec{\mathbf{r}}' \tag{5}$$

Note that both $\vec{\mathbf{r}}'$ and $\vec{\mathbf{u}}$ point in the direction of increasing t. Hence their sense depends on the orientation of C. It is reversed if we reverse the orientation.

Tangent to a Curve

Therefore, the *tangent* to C at P is given by

(6) $$\boxed{\vec{q}(w) = \vec{r} + w\vec{r}'}$$

This is the sum of the position vector $\mathbf{r}$ of P and a multiple of the tangent vector $\mathbf{r}'$ of C at P. Both vectors depend on P. The variable w is the parameter in (6).

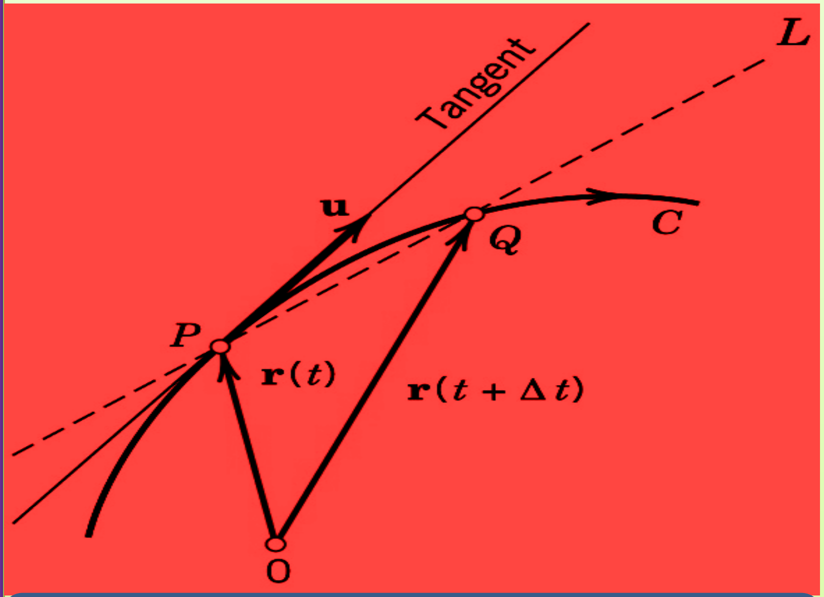


Fig. Tangent to a curve

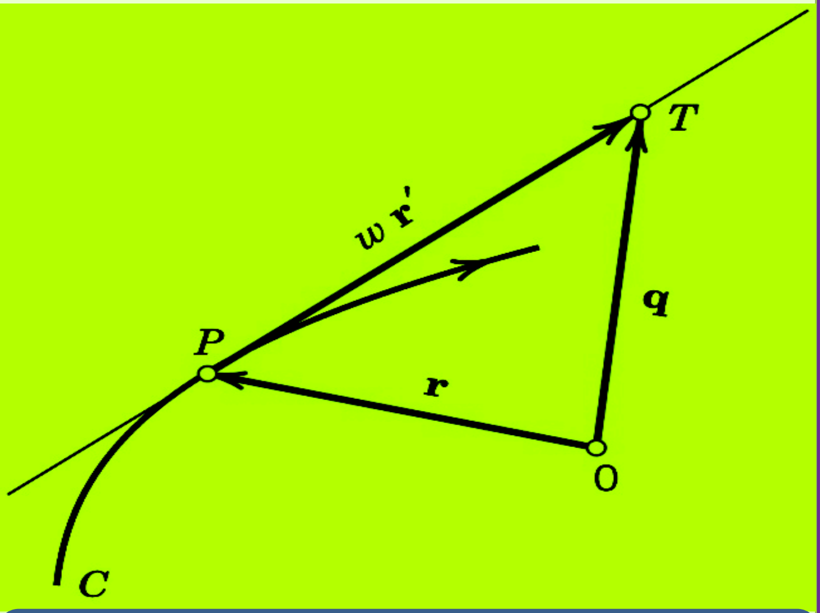


Fig. Formula (6) for the tangent to a curve

Length of a Curve

If $\vec{\mathbf{r}}(t)$ has a continuous derivative $\vec{\mathbf{r}}'$, it can be shown that the sequence $l_1, l_2, \ldots$ has a limit, which is independent of the particular choice of the representation of C and of the choice of subdivisions. This limit is given by the integral

(7) $$\boxed{l = \int_a^b \sqrt{\vec{\mathbf{r}}' \bullet \vec{\mathbf{r}}'}\,dt \qquad \left(\vec{\mathbf{r}}' = \frac{d\vec{\mathbf{r}}}{dt}\right).}$$

l is called the **length** of C, and C is called **rectifiable**.

Arc Length s of a Curve

The length (7) of a curve C is a constant, a positive number. But if we replace the fixed b in (7) with a variable t, the integral becomes a function of t, denoted by $s(t)$ and called the *arc length function* or simply the **arc length** of C. Thus

(8) $$\boxed{s(t) = \int_a^t \sqrt{\vec{\mathbf{r}}' \bullet \vec{\mathbf{r}}'}\, d\tilde{t} \qquad \left(\vec{\mathbf{r}}' = \frac{d\vec{\mathbf{r}}}{d\tilde{t}}\right).}$$

Here the variable of integration is denoted by $\tilde{t}$ because t is now used in the upper limit.

Linear Element '*ds*'

We have $d\vec{r} = [dx, dy, dz] = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

And $ds^2 = d\vec{r} \bullet d\vec{r} = dx^2 + dy^2 + dz^2.$

ds is called the ***linear element*** of C.

ARC LENGTH AS PARAMETER.

Arc Length as Parameter.

The use of *s* in (1) instead of an arbitrary *t* simplifies various formulas. For the unit tangent vector (8) we simply obtain

$$(9) \qquad \vec{u}(s) = \vec{r}'(s).$$

Indeed,

$$(10) \qquad |\vec{r}'(s)| = \sqrt{\frac{d\vec{r}}{ds} \bullet \frac{d\vec{r}}{ds}} = \sqrt{\left(\frac{ds}{ds}\right)^2} = 1$$

in (9) shows that $\vec{r}'(s)$ is a unit vector.

ARC LENGTH AS PARAMETER.

PROBLEM:

If s is arc length of a curve $C : \vec{r}(t)$

then prove that $\dfrac{d\vec{r}}{ds}$ is a unit tangent of C.

SOLUTION:

$$\text{We have } s(t) = \int_a^t \sqrt{\vec{\mathbf{r}}' \bullet \vec{\mathbf{r}}'}\, d\tilde{t} \qquad \left(\vec{\mathbf{r}}' = \frac{d\vec{\mathbf{r}}}{d\tilde{t}} \right).$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \Box \frac{d\vec{r}}{dt}} = \left| \frac{d\vec{r}}{dt} \right| \text{---(A)}$$

(By Fundamental Theorem of Calculus)

ARC LENGTH AS PARAMETER.

By Chain Rule we have

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

Therefore, $\dfrac{d\vec{r}}{dt} \bullet \dfrac{d\vec{r}}{dt} = \left(\dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}\right) \bullet \left(\dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}\right)$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right|^2 = \left(\frac{d\vec{r}}{ds} \bullet \frac{d\vec{r}}{ds}\right)\left(\frac{ds}{dt}\right)^2$$

$$\Rightarrow \left|\frac{d\vec{r}}{dt}\right|^2 = \left|\frac{d\vec{r}}{ds}\right|^2 \left(\frac{ds}{dt}\right)^2$$

ARC LENGTH AS PARAMETER.

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \left|\frac{d\vec{r}}{ds}\right|^2 \left(\frac{ds}{dt}\right)^2$$ Using (A)

$$\Rightarrow \left|\frac{d\vec{r}}{ds}\right|^2 = 1 \text{ or } \left|\frac{d\vec{r}}{ds}\right| = 1$$

Moreover, since $\dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}$,

$\dfrac{d\vec{r}}{ds}$ is parallel to $\dfrac{d\vec{r}}{dt}$

Therefore, $\dfrac{d\vec{r}}{ds}$ is a unit vector which is a tangent of C,

i.e. $\dfrac{d\vec{r}}{ds}$ is a unit tangent of C.

PROBLEMS FOR PRACTICE

PROBLEM – 1:

Find a parameteric representation of the straight line through a point A in the direction of a vector $\vec{b}$, where $A:(3,1,5)$ and $\vec{b} = [4, \quad 7, \quad -1]$.

SOLUTION:

The parametric representation of the line through a point A with position vector $\vec{a}$ in the direction of a vector $\vec{b}$ is $\vec{r}(t) = \vec{a} + t\vec{b}$.

PROBLEMS FOR PRACTICE

Given that $\vec{a} = 3\hat{i} + \hat{j} + 5\hat{k}$ and $\hat{b} = \hat{4i} + 7\hat{j} - \hat{k}$

So a parametric epresentation of the given straigth line is $\vec{r}(t) = \vec{a} + t\vec{b}$

i.e. $\vec{r}(t) = (3\hat{i} + \hat{j} + 5\hat{k}) + t(\hat{4i} + 7\hat{j} - \hat{k})$

$= (3+4t)\hat{i} + (1+7t)\hat{j} + (5-t)\hat{k} = [3+4t, 1+7t, 5-t]$

PROBLEMS FOR PRACTICE

PROBLEM – 2:

Find a parametric representation of the straight line through the points $A(a,b,c)$ and $B(a+4,2-b,c-1)$.

SOLUTION:

A straight line passing through two points , A , B with position vectors $\vec{a}$ *and* $\vec{b}$ *respectively is* $\vec{r}(t)=\vec{a}+t\left(\vec{b}-\vec{a}\right)$.

Given that $\vec{a}=a\hat{i}+b\hat{j}+c\hat{k}$ and $\hat{b}=(a+4)\hat{i}+(2-b)\hat{j}+(c-1)\hat{k}$

So a parametric epresentation of the given straigth line is $\vec{r}(t)=\vec{a}+t\left(\vec{b}-\vec{a}\right)$.

PROBLEMS FOR PRACTICE

$$\text{i.e. } \vec{r}(t) = (a\hat{i} + b\hat{j} + c\hat{k})$$

$$+t\left[\left(\left(a+4\right)\hat{i} + \left(2-b\right)\hat{j} + \left(c-1\right)\hat{k}\right) - (a\hat{i} + b\hat{j} + c\hat{k})\right]$$

$$\text{i.e. } \vec{r}(t) = (a\hat{i} + b\hat{j} + c\hat{k}) + t\left[4\hat{i} + 2\left(1-b\right)\hat{j} - \hat{k}\right]$$

$$\text{i.e. } \vec{r}(t) = \left(a+4t\right)\hat{i} + \left(b+2t-2bt\right)\hat{j} + \left(c-t\right)\hat{k}$$

PROBLEMS FOR PRACTICE

PROBLEM – 3:

What curve is represented by the parametric representation

$\left[t,\ t^3+2,0\right]$?

SOLUTION:

Let $x = t$.

So $y=t^3 \Rightarrow y=x^3$

So the required curve is $y=x^3$, $z = 0$

PROBLEMS FOR PRACTICE

PROBLEM – 4:

What curve is represented by the parametric representation $[\cosh t, \ \sinh t, 0]$?

SOLUTION:

Let $x = cosht$ and $y = sinht$

$\Rightarrow x^2 - y^2 = cosh^2 t - sinh^2 t = 1$

So the required curve is $x^2 - y^2 = 1$, $z = 0$, which is a hyperbola in the xy-plane.

PROBLEMS FOR PRACTICE

PROBLEMS– 5:

What curve is represented by the parametric representation $[3+6\cos t,\ -2+\sin t, 4]$?

SOLUTION:

Let $x = 3+6cost$ and $y = -2+sint$

We know that $cos^2t + sin^2t=1$

$$\Rightarrow \left(\frac{x-3}{6}\right)^2 + (y+2)^2 = 1$$

So the required curve is $\dfrac{(x-3)^2}{36} + \dfrac{(y+2)^2}{1} = 1, \quad z = 4$

PROBLEMS FOR PRACTICE

PROBLEM – 6:

Represent the curve $4x^2 - 3y^2 = 12, z = 1$ parametrically.

SOLUTION: Given curve is $4x^2 - 3y^2 = 12, z = 1.$

$$4x^2 - 3y^2 = 12 \quad \Rightarrow \frac{4x^2}{12} - \frac{3y^2}{12} = 1 \quad \Rightarrow \frac{x^2}{3} - \frac{y^2}{4} = 1$$

$$\Rightarrow \frac{x^2}{\left(\sqrt{3}\right)^2} - \frac{y^2}{2^2} = 1$$

PROBLEMS FOR PRACTICE

For a parameter t, the choice $x = \sqrt{3}\cosh t$ and $y = 2\sinh t$

satisfy the equation $\dfrac{x^2}{\left(\sqrt{3}\right)^2} - \dfrac{y^2}{2^2} = 1$

So a parametric representation of the given curve is

$$\vec{r}\left(t\right) = x\left(t\right)\hat{i} + y\left(t\right)\hat{j} + z\left(t\right)\hat{k},$$

i.e. $\vec{r}\left(t\right) = \sqrt{3}\cosh t\hat{i} + 2\sinh t\hat{j} + \hat{k}$

PROBLEMS FOR PRACTICE

PROBLEM – 7:

Represent the curve $x^2 + y^2 = 9, z = 5\tan^{-1}\left(y/x\right)$ parametrically.

SOLUTION:

Given curve is $x^2 + y^2 = 9, z = 5\tan^{-1}\left(y/x\right)$.

For a parameter t, the choice $x = 3\cos t$ and $y = 3\sin t$ satisfy the equation $x^2 + y^2 = 9$.

PROBLEMS FOR PRACTICE

Moreover, $x = 3\cos t$ and $y = 3\sin t$

$$\Rightarrow z = 5\tan^{-1}\left(y/x\right) = 5\tan^{-1}\left(\frac{3\sin t}{3\cos t}\right) = 5\tan^{-1}\left(\tan t\right) = 5t$$

So a parametric representation of the given curve is

$$\vec{r}\left(t\right) = x\left(t\right)\hat{i} + y\left(t\right)\hat{j} + z\left(t\right)\hat{k},$$

i.e. $\vec{r}\left(t\right) = 3\cos t\hat{i} + 3\sin t\hat{j} + 5t\hat{k}$

PROBLEMS FOR PRACTICE

PROBLEM – 8:

Given a curve $C:\vec{r}(t)=t\hat{i}+t^3\hat{j},$ find a tangent vector $\vec{r}'(t)$ and the corresponding unit tangent vector $\vec{u}(t)$, $\vec{r}'$ and $\vec{u}$ at the point $P:(1,1,0)$, and the tangent at P.

Solution:

Given that $\vec{r}(t)=t\hat{i}+t^3\hat{j}$

$$\therefore \vec{r}'(t)=\frac{d}{dt}\left(t\hat{i}+t^3\hat{j}\right)=\hat{i}+3t^2\hat{j}$$

$$\therefore \vec{u}(t)=\frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}=\frac{\hat{i}+3t^2\hat{j}}{\left|\hat{i}+3t^2\hat{j}\right|}=\frac{\hat{i}+3t^2\hat{j}}{\sqrt{1+9t^4}}$$

PROBLEMS FOR PRACTICE

$$\therefore \vec{r}'_{\text{at } P(1,1,0)} = \hat{i} + 3(1)^2\,\hat{j} = \hat{i} + 3\hat{j}$$

$$\text{and } \vec{u}_{\text{at } P(1,1,0)} = \frac{\hat{i} + 3(1)^2\,\hat{j}}{\sqrt{1+9(1)^4}} = \frac{\hat{i}+3\hat{j}}{\sqrt{10}} = \frac{1}{\sqrt{10}}\hat{i} + \frac{3}{\sqrt{10}}\hat{j}$$

$$\text{Also } \vec{r}_{\text{at } P(1,1,0)} = (1)\hat{i} + (1)^3\,\hat{j} = \hat{i} + \hat{j}$$

Parametric representation of the tangent is

$$\vec{q}(w) = \vec{r} + w\vec{r}' = \hat{i} + \hat{j} + w\left(\hat{i} + 3\hat{j}\right)$$

$$\text{i.e. } \vec{q}(w) = (1+w)\hat{i} + (1+3w)\hat{j}$$

PROBLEMS FOR PRACTICE

PROBLEM – 9:

Given a curve $C : \vec{r}(t) = 2\cos t\hat{i} + 2\sin t\hat{j} + t\hat{k}$, find a tangent vector $\vec{\mathrm{r}}'(\mathrm{t})$ and the corresponding unit tangent vector $\vec{\mathrm{u}}(\mathrm{t})$, $\vec{\mathrm{r}}'$ and $\vec{\mathrm{u}}$ atthe point $P:(2,0,0)$, and the tangent at P.

Solution:

Given that $\vec{r}(t) = 2\cos t\hat{i} + 2\sin t\hat{j} + t\hat{k}$

$$\therefore \vec{r}'(t) = \frac{d}{dt}\left(2\cos t\hat{i} + 2\sin t\hat{j} + t\hat{k}\right)$$

$$= -2\sin t\hat{i} + 2\cos t\hat{j} + \hat{k}$$

PROBLEMS FOR PRACTICE

$$\therefore \vec{u}(t) = \frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|} = \frac{-2\sin t\hat{i} + 2\cos t\hat{j} + \hat{k}}{\left|-2\sin t\hat{i} + 2\cos t\hat{j} + \hat{k}\right|}$$

$$= \frac{-2\sin t\hat{i} + 2\cos t\hat{j} + \hat{k}}{\sqrt{(-2)^2 + 2^2 + 1^2}} = \frac{-2\sin t\hat{i} + 2\cos t\hat{j} + \hat{k}}{3}$$

$$= -\frac{2}{3}\sin t\hat{i} + \frac{2}{3}\cos t\hat{j} + \frac{1}{3}\hat{k}$$

$$\text{At } P:(2,0,0),\ \left(2\text{cost},2\text{sint},\text{t}\right) = (2,0,0)$$

$$\Longrightarrow 2\cos t = 2, 2\sin t = 0, t = 0$$

$$\Longrightarrow t = 0$$

PROBLEMS FOR PRACTICE

$$\therefore \vec{r}'_{\text{at } P(1,1,0)} = \vec{r}'(0) = (-2\sin 0)\hat{i} + (2\cos 0)\hat{j} + \hat{k}$$

$$= 2\hat{j} + \hat{k}$$

and $\vec{u}_{\text{at } P(2,0,0)} = \vec{u}(0)$

$$= \left(-\frac{2}{3}\sin 0\right)\hat{i} + \left(\frac{2}{3}\cos 0\right)\hat{j} + \frac{1}{3}\hat{k}$$

$$= \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

PROBLEMS FOR PRACTICE

$$\text{Also } \vec{r}_{\text{at } P(2,0,0)} = \vec{r}(0)$$

$$= (2\cos 0)\hat{i} + (2\sin 0)\hat{j} + (0)\hat{k} = 2\hat{i}$$

Parametric representation of the tangent is

$$\vec{q}(w) = \vec{r} + w\vec{r}' = 2\hat{i} + w\left(2\hat{j} + \hat{k}\right)$$

$$\text{i.e. } \vec{q}(w) = 2\hat{i} + 2w\hat{j} + w\hat{k}$$

PROBLEMS FOR PRACTICE

PROBLEM – 10:

Find the length of the curve $\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + ct\hat{k}$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$.

Solution:

Given curve is $\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + ct\hat{k}$

$$\therefore \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(a\cos t\hat{i} + a\sin t\hat{j} + ct\hat{k}\right) = -a\sin t\hat{i} + a\cos t\hat{j} + c\hat{k}$$

PROBLEMS FOR PRACTICE

$$\therefore \frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt}$$

$$= \left(-a\sin t\hat{i} + a\cos t\hat{j} + c\hat{k}\right) \bullet \left(-a\sin t\hat{i} + a\cos t\hat{j} + c\hat{k}\right)$$

$$= a^2 \sin^2 t + a^2 \cos^2 t + c^2 = a^2 + c^2$$

$$\Rightarrow \sqrt{\frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt}} = \sqrt{a^2 + c^2}$$

At $(a, 0, 0)$, $\left(a\cos t, a\sin t, ct\right) = (a, 0, 0)$

$\Longrightarrow a\cos t = a, a\sin t = 0, ct = 0$

$\Longrightarrow t = 0$

PROBLEMS FOR PRACTICE

At $(a, 0, 2\pi c)$, $\left(a\cos t, a\sin t, ct\right) = (a, 0, 2\pi c)$

$$\Rightarrow a\cos t = a, a\sin t = 0, ct = 2\pi c$$

$$\Rightarrow t = 2\pi$$

So the required length of the given curve from $\left(a, 0, 0\right)$ to $\left(a, 0, 2\pi c\right)$ is

$$= \int_0^{2\pi} \sqrt{\frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt}} dt = \int_0^{2\pi} \sqrt{a^2 + c^2} dt = \sqrt{a^2 + c^2} \int_0^{2\pi} dt$$

$$= \sqrt{a^2 + c^2} \left[t\right]_0^{2\pi} = 2\pi\sqrt{a^2 + c^2}$$

PROBLEMS FOR PRACTICE

PROBLEM – 11:

Find the length of the curve $\vec{r}(t) = t\hat{i} + \cosh t\hat{j}$ from $t = 0$ to $t = 1$

Solution:

Given curve is $\vec{r}(t) = t\hat{i} + \cosh t\hat{j}$

$$\therefore \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t\hat{i} + \cosh t\hat{j}\right) = \hat{i} + \sinh t\hat{j}$$

$$\therefore \frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt} = \left(\hat{i} + \sinh t\hat{j}\right) \bullet \left(\hat{i} + \sinh t\hat{j}\right) = 1 + \sinh^2 t = \cosh^2 t$$

PROBLEMS FOR PRACTICE

$$\Rightarrow \sqrt{\frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt}} = \cosh t$$

So the required length of the given curve from $t = 0$ to $t = 1$ is

$$= \int_0^1 \sqrt{\frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt}}\, dt = \int_0^1 \cosh t\, dt = \left|\sinh t\right|_0^1$$

$$= \sinh(1) - \sinh(0)$$

$$= \sinh 1$$