MATHEMATICS-I

DEPARTMENT OF MATHEMATICS, CGU

MATHEMATICS - I

Unit-II

Functions of two and more variables and Special Functions: Functions of two or more several variables: limit, continuity and differentiability, homogenous functions and Euler's theorem, higher order partial derivatives and Taylor's series, maximum and minimum values, beta, gamma functions and error functions.

Functions of Several Variables:

- **Introduction**
- **Limits of a function of two variables**
- **Continuity of a function of two variables**

Introduction

This is part of multivariable calculus. In multivariable calculus, we study functions of two or more independent variables, e.g.,

- $z=f(x, y)$ or $w=f(x, y, z)$.

Many things depend on more than one independent variable. Here are just a few:

1. In thermodynamics pressure depends on volume and temperature.

2. In electricity and magnetism, the magnetic and electric fields are functions of the three space variables (x, y, z) and one-time variable t.

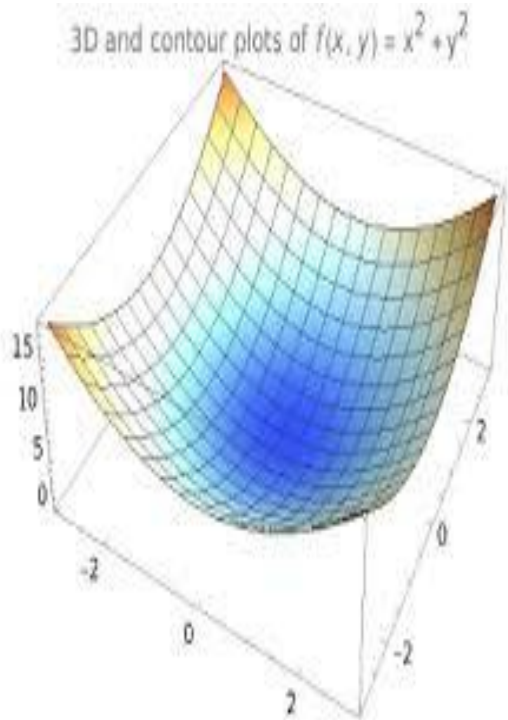
3. In economics, functions can depend on a large number of independent variables, e.g., a manufacturer's cost might depend on the prices of 27 different commodities.

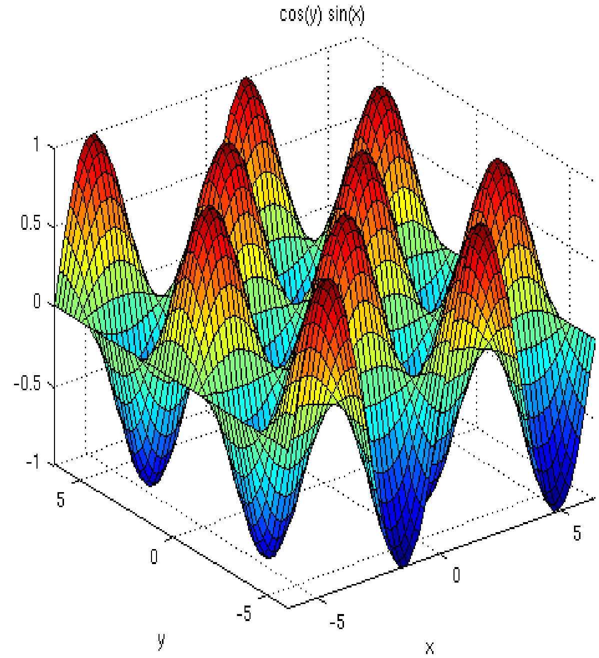
We have already studied the calculus of functions of a single real variable defined by $y = f(x)$. In this topic, we shall extend the concepts of functions of one variable to functions of two or more variables.

Definition-1: If a point (x, y) lies in a certain part of the xy-plane or $(x, y) \in \mathbb{R}^2$, there corresponds a real value z according to some rule $f(x, y)$, then $f(x, y)$ is called a *real valued function of two real variables* x and y and is written as

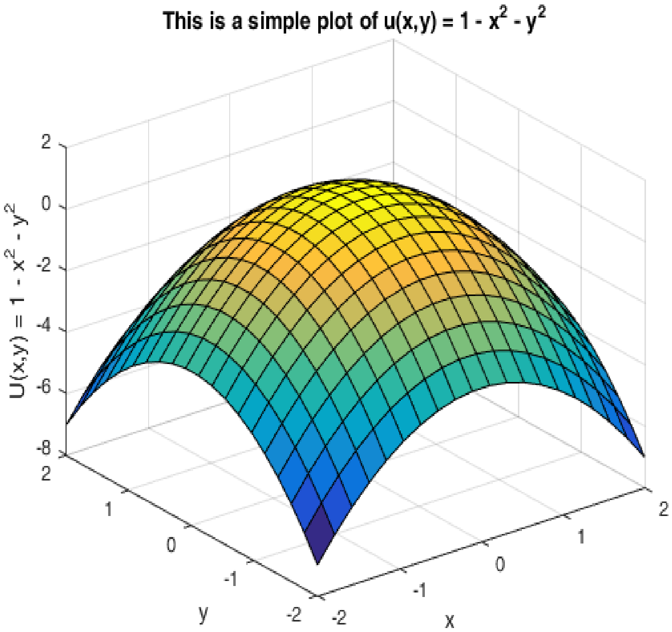
$$z = f(x, y); (x, y) \in \mathbb{R}^2, z \in \mathbb{R}.$$

Examples

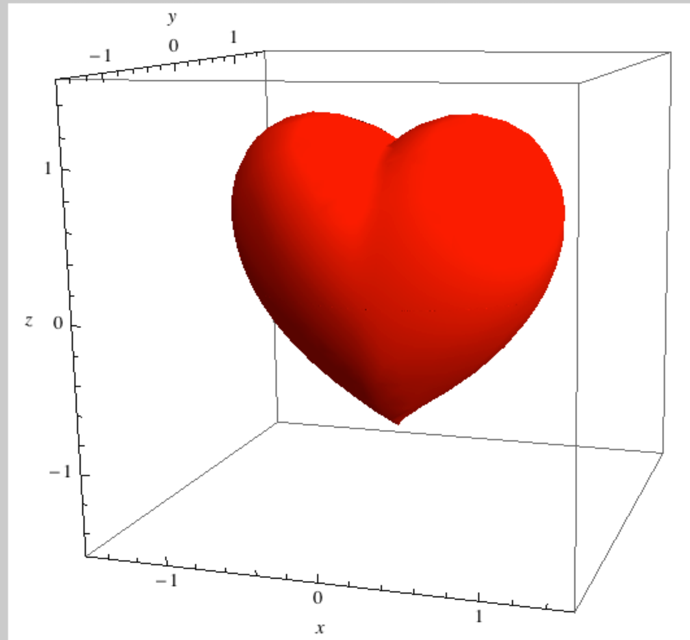

3D and contour plots of $f(x, y) = x^2 + y^2$
15
10
5
0
-2
0
2
2
0
-2


cos(y) sin(x)
1
0.5
0
-0.5
-1
5
0
-5
y
-5
0
5
x

Examples



$$(x^2 + \frac{9}{4}y^2 + z^2 - 1)^3 - x^2 z^3 - \frac{9}{200}y^2 z^3 = 0$$



In general, we define a real valued function of n variables as like the following:

$$z = f(x_1, x_2, \dots, x_n); (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, z \in \mathbb{R}$$

where $x_1, x_2, \dots, x_n$ are the n independent variables and z is the dependent variable.

Definition-2: (Neighborhood of a point): Let $P(x_0, y_0)$ be a point in $\mathbb{R}^2$. Then the neighborhood of the point $P(x_0, y_0)$ is the set of all points which lie inside a circle of radius δ with center at the point (x_0, y_0). It is denoted by $N_\delta(P)$ or $N(P, \delta)$. Therefore,

Open disc nbd

$$N_\delta(P) = \left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}.$$

$$N'_\delta(P) = \left\{(x, y) : 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$$

(deleted nbd)

Open square nbd

$$N_\delta(P) = \{(x, y) : |x - x_0| < \delta \ and \ |y - y_0| < \delta\}.$$

$$N'_\delta(P) = \{(x, y) : 0 < |x - x_0| < \delta \ and \ 0 < |y - y_0| < \delta\}$$

(deleted nbd)

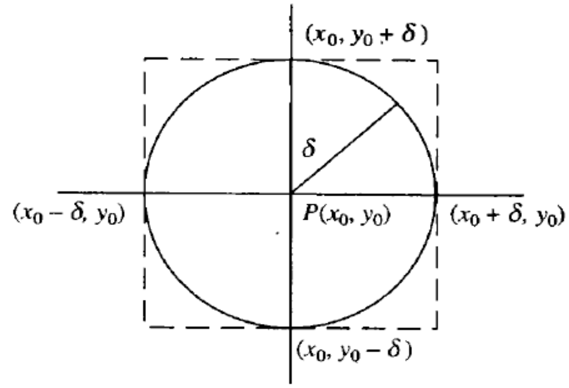


Fig. 2.2. Neighborhood of a point $P(x_0, y_0)$.

Definition-3: (Bounded function): A function f is said to be bounded in some domain D if there exists a real finite positive number M such that $|f(x, y)| \leq M$ for all $(x, y) \in D$.

Definition 4. (Limits): Let $z = f(x, y)$ be a function defined in a domain D. Let $P(x_0, y_0)$ be a point in D. A real number L is said to be limit of $f(x, y)$ at the point P, if for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon,$$

for every point (x, y) in the deleted δ-neighborhood of $P(x_0, y_0)$, i.e., whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

We then say that limit of the function $f(x, y)$ exists at P and the limit is L.

We write this mathematically as

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Definition 5. (Continuity): Let $z = f(x, y)$ be a function defined in a domain D. Let $P(x_0, y_0)$ be a point in D. The function $f(x, y)$ is continuous at $P(x_0, y_0)$ if for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that

$$|f(x, y) - f(x_0, y_0)| < \varepsilon,$$

For every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$

i.e., whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

We then say that the function $f(x,y)$ is continuous at $P(x_0, y_0)$.

And we write this mathematically as

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Example 1. Using $\delta - \varepsilon$ approach, show that

$$\lim_{(x,y)\to(2,1)} (3x + 4y) = 10$$

Solution: Let a positive real number $\varepsilon > 0$.

Here $f(x, y) = 3x + 4y$ is defined at $(2,1)$.

Consider $|f(x, y) - 10| = |3x + 4y - 10|$

$$= |3(x-2) + 4(y-1)| < 3|x-2| + 4|y-1|.$$

If we take $|x - 2| < \delta$ and $|y - 1| < \delta$, then we get

$$|f(x, y) - 10| < 7\delta < \varepsilon$$

which holds when $\delta < \varepsilon/7$.

Hence,

$$\lim_{(x,y)\to(2,1)} (3x + 4y) = 10.$$

Example 2.1 Using the δ-ε approach, show that

(i) $\lim_{(x, y) \to (2, 1)} (3x + 4y) = 10$, (ii) $\lim_{(x, y) \to (1, 1)} (x^2 + 2y) = 3$.

Solution

(i) Here $f(x, y) = 3x + 4y$ is defined at (2, 1). We have

$|f(x, y) - 10| = |3x + 4y - 10| = |3(x - 2) + 4(y - 1)| \le 3|x - 2| + 4|y - 1|$.

If we take $|x - 2| < \delta$ and $|y - 1| < \delta$, we get $|f(x, y) - 10| < 7\delta < \varepsilon$, which is satisfied when $\delta < \varepsilon/7$.

Hence, $\lim_{(x, y) \to (2, 1)} f(x, y) = 10$.

(ii) Here $f(x, y) = x^2 + 2y$ is defined at $(1, 1)$. We have

$$|f(x, y) - 3| = |x^2 + 2y - 3| = |(x - 1 + 1)^2 + 2(y - 1 + 1) - 3|$$

$$= |(x - 1)^2 + 2(x - 1) + 2(y - 1)| \le |x - 1|^2 + 2|x - 1| + 2|y - 1|$$

If we take $|x - 1| < \delta$ and $|y - 1| < \delta$, we get $|f(x, y) - 3| < \delta^2 + 4\delta < \varepsilon$ which is satisfied when

$$(\delta + 2)^2 < \varepsilon + 4 \text{ or } \delta < \sqrt{\varepsilon + 4} - 2.$$

Hence, $\lim\limits_{(x, y) \to (1, 1)} f(x, y) = 3.$

We can also write $\quad |f(x, y) - 3| < \delta^2 + 4\delta < 5\delta < \varepsilon$

which is satisfied when $\delta < \varepsilon/5$.

.

Exercise: Using $\delta - \varepsilon$ approach prove that

(i) $\lim_{(x,y)\to(1,1)} (10x - 18y) = -8,$

(ii) $\lim_{(x,y)\to(0,-1)} (5x - 17y) = 17.$

Example-2: Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

Solution: The limit does not exist if it is not finite, or if it depends on a particular path.

Consider the path $y = mx$. As $(x, y) \to (0,0)$, we get $x \to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence the limit does not exist.

Exercise: Prove that limit does not exist for the following function

(i) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ (ii) $\lim_{(x,y)\to(0,0)} \frac{x^2-y}{x^2+y}$ (iii) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$

Example 2.3 Show that the following limits

(i) $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy}{x^2 + y^2}$, (ii) $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x + \sqrt{y}}{x^2 + y^2}$,

(iii) $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^3 y}{x^6 + y^2}$. (iv) $\lim\limits_{(x, y) \to (0, 1)} \tan^{-1}\left(\dfrac{y}{x}\right)$.

do not exist.

Continuity: A function $z = f(x,y)$is said to ***continuous*** at a point (x_0, y_0), if

- f is defined as at the point (x_0, y_0),
- $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exits, and
- $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

If any one of the above conditions is not satisfied, then the function is said to be discontinuous at that point (x_0, y_0). Therefore, a function $f(x,y)$ is continuous at (x_0, y_0) if

$$|f(x,y) - f(x_0, y_0)| < \varepsilon,$$

whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

.

Note-1: If $f(x_0, y_0)$ is defined and $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ but $f(x_0, y_0) \neq L$, then the point (x_0, y_0) is called a point of *removable continuity*.

Note-2: If the function $f(x, y)$ is continuous at every point in a domain D, then it is continuous in D.

Note-3: A continuous functions has the following four properties:

- A continuous function in a closed and bounded domain D attains at least once its maximum value M and minimum value m at some point inside or on the boundary of D.
- For any number μ that satisfies $m < \mu < M$, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \mu$.

- If it attains both positive and negative values in a closed and bounded domain D, then it will have the value zero at some point in D.
- If $z = f(x, y)$ is continuous at some point $P(x_0, y_0)$ and $w = g(z)$ is a composite function defined at $z_0 = f(x_0, y_0)$, then the composite function $g(f(z))$ is also continuous at P.

Example-3: Show that the following function is continuous at (0,0).

$$f(x,y)=\begin{cases}\dfrac{2x^4+3y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Solution: Let $x=r\cos\theta\,,y=r\sin\theta$. Then $r=\sqrt{x^2+y^2}\neq 0.$ we have

$$\begin{aligned}&|f(x,y)-f(0,0)|\\ &=\left|\frac{2x^4+3y^4}{x^2+y^2}\right|=\left|\frac{r^4(2\cos^4\theta+3\sin^4\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\right|\\ &< r^2[2|\cos^4\theta|+3|\sin^4\theta|\,]<5r^2<\varepsilon\end{aligned}$$

If we choose $\delta < \sqrt{\varepsilon/5}$, then we get $|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2 + y^2} < \delta$.

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

Example-4: Show that the following function is discontinuous at (0,0).

$$f(x,y)=\begin{cases}\dfrac{x^2-x\sqrt{y}}{x^2+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Solution: Let us choose the path $y=m^2x^2$. As $(x,y)\to(0,0)$, we get $x\to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)}\frac{x^2-x\sqrt{y}}{x^2+y}=\lim_{x\to 0}\frac{x^2(1-m)}{x^2(1+m^2)}=\frac{(1-m)}{(1+m^2)}$$

which depends on m and the limit does not exist and resulting the function is not continuous at $(0,0)$.

Practice Problems

1. Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

2. Show that the following function is discontinuous at (0,0).

$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$