C. V. RAMAN GLOBAL UNIVERSITY
ODISHA
ESTD - 2020

DEPARTMENT OF MATHEMATICS

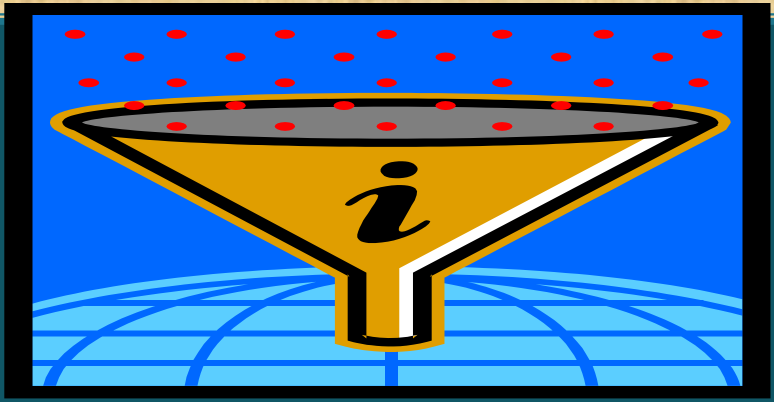
MATHEMATICS – I

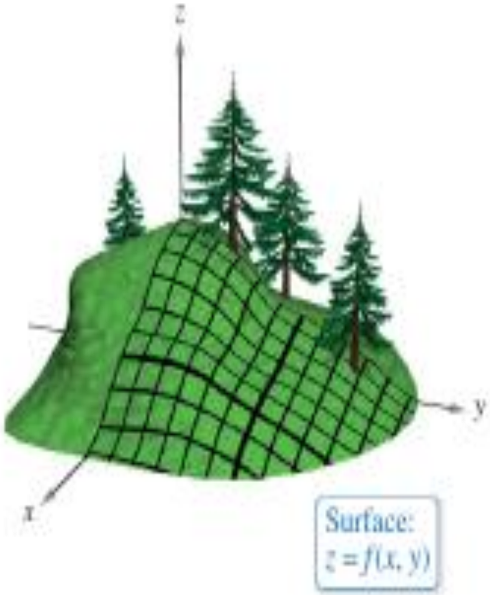
FOR BTECH FIRST SEMESTER COURSE [COMMON TO ALL BRANCHES OF ENGINEERING]

MATHEMATICS – I

LECTURE - 23 & 24: *DIRECTIONAL DERIVATIVES AND GRADIENT OF A SCALAR FIELD*

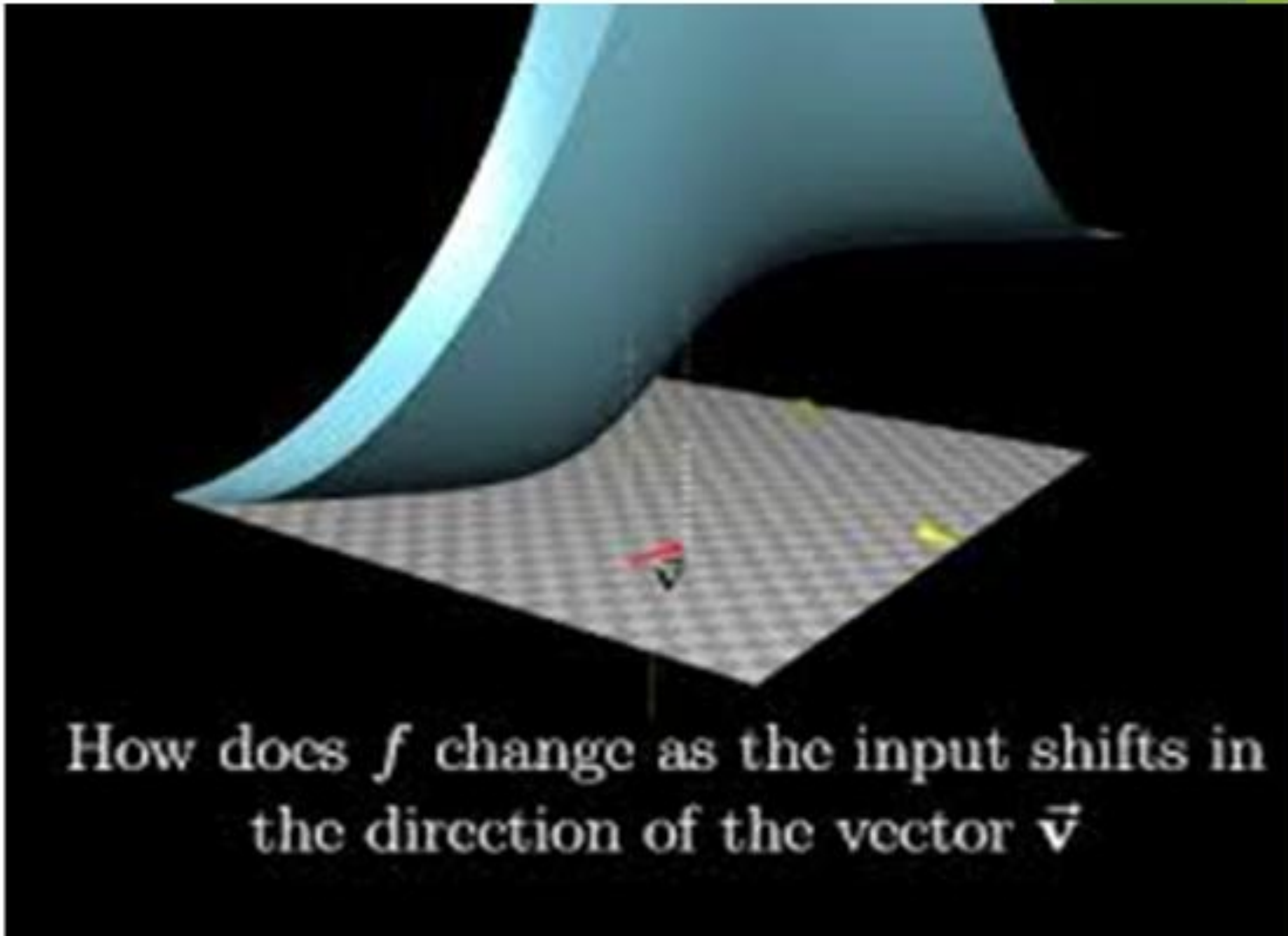
[Chapter – 8.9]



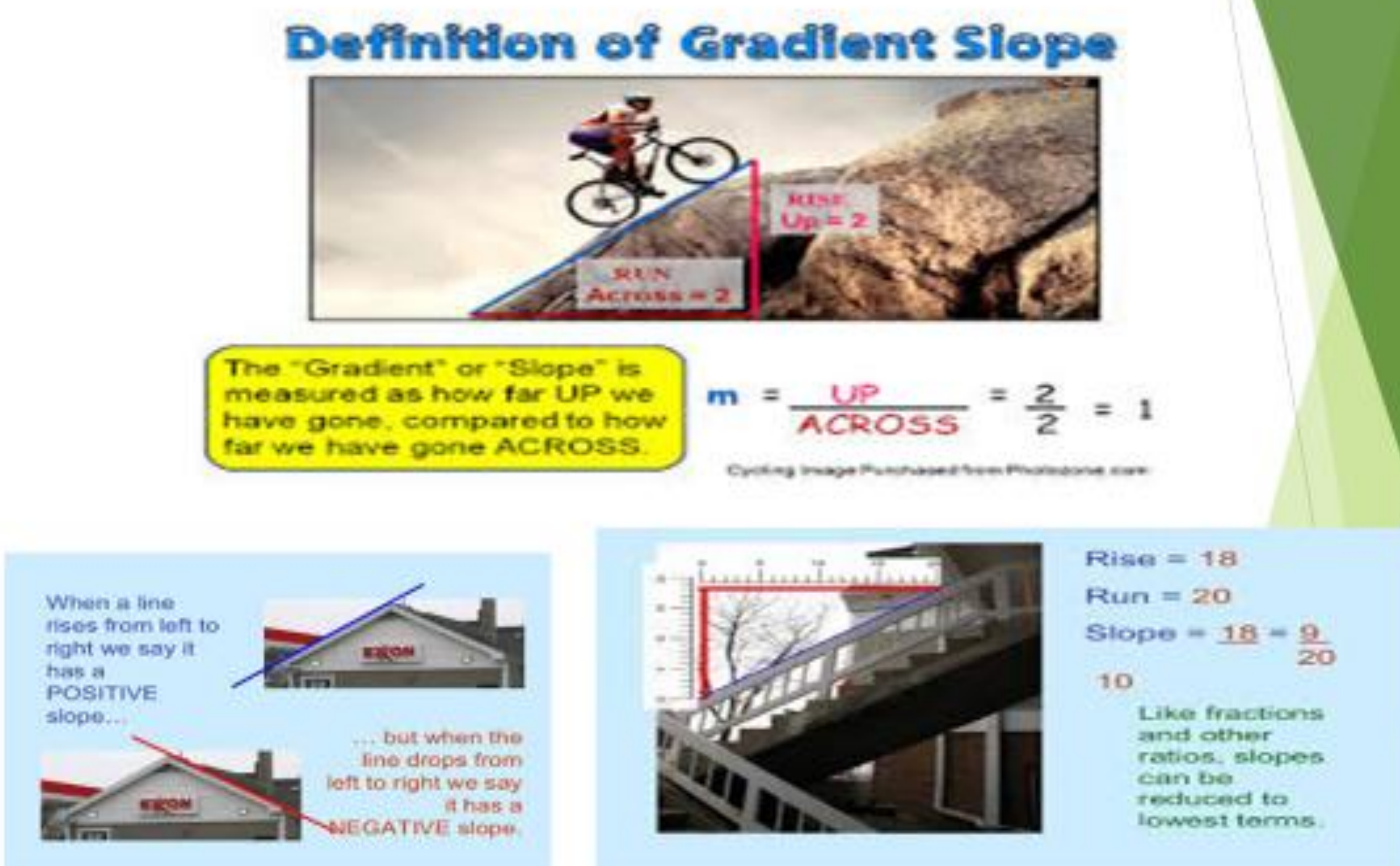


You are standing on the hillside represented by $z = f(x)$ in the above Figure and want to determine the hill's incline toward the z- axis.

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.**


How does f change as the input shifts in
the direction of the vector $\vec{\mathbf{v}}$

GRADIENT



Definition of Gradient Slope
RISE
Up = 2
RUN
Across = 2
The "Gradient" or "Slope" is measured as how far UP we have gone, compared to how far we have gone ACROSS.
m = UP / ACROSS = 2/2 = 1
When a line rises from left to right we say it has a POSITIVE slope...
... but when the line drops from left to right we say it has a NEGATIVE slope.
Rise = 18
Run = 20
Slope = 18 = 9
20
10
Like fractions and other ratios, slopes can be reduced to lowest terms.

EXPLANATION

- Gradients are an important part of life. The roof of a house is built with a gradient to enable rain water to run down the roof.

- An aeroplane ascends at a particular gradient after take off, flies at a different gradient and descends at another gradient to safely land.

- Tennis courts, roads, football and cricket grounds are made with a gradient to assist drainage.

GRADIENT OF A SCALAR FIELD

GRADIENT:-

Let $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ a differential operator.

The gradient of a scalar function f is defined as

$$\boxed{grad\ f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

GRADIENT OF A SCALAR FIELD

EXAMPLE-1:-Find the gradient of $f(x,y,z)=xyz$ at the point $P(1,2,3)$.

Solution:

Given function is $f(x,y,z) = xyz$

By definition, $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \frac{\partial}{\partial x}(xyz)\hat{i} + \frac{\partial}{\partial y}(xyz)\hat{j} + \frac{\partial}{\partial z}(xyz)\hat{k}$$

$$= yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

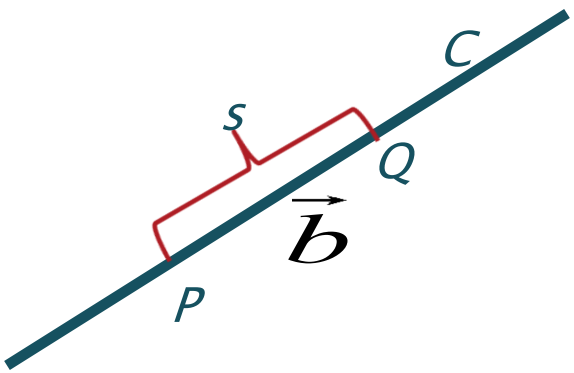
DIRECTIONAL DERIVATIVES

Let *f(x, y, z)* be any scalar function. Then its *directional derivatives* at the point *P* in the direction of a vector $\vec{b}$ is the rate of change of *f* at *P* in the direction of $\vec{b}$, which is denoted by

$$D_{\vec{b}} f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$

[s is the distance between P and Q]

Where Q is a variable point on any ray C in the direction of $\vec{b}$.



If $\vec{p}_0$ is the position vector of P and $\vec{b}$ is the unit vector, then the ray C is given by

$$\vec{r}(s) = x(s)i + y(s)j + z(s)k = \vec{p}_0 + s\vec{b} \quad --(1)$$

DIRECTIONAL DERIVATIVES

In the Cartesian system f is a function of the Cartesian co-ordinates – $x, y, z,$ say $f(x, y, z),$ and x, y, z depend on s

By Chain Rule,

$$D_{\vec{b}} f = \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\bullet\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$D_{\vec{b}} f = (grad\ f)\bullet\frac{d\vec{r}}{ds} \Rightarrow D_{\vec{b}} f = \frac{d\vec{r}}{ds}\bullet(grad\ f) - -(2)$$

DIRECTIONAL DERIVATIVES

$$(1) \Rightarrow \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} = \frac{d}{ds}\left(\vec{p}_0 + s\vec{b}\right) = \vec{b} --(3)$$

So, from (2) and (3), we get $D_{\vec{b}} f = \vec{b} \bullet \left(grad\ f \right) --(4)$

In general the directional derivative of the function f in the direction of a vector $\vec{a} \neq \vec{0}$ of any length is given by

$$\boxed{D_{\vec{a}} f = \frac{\vec{a}}{|\vec{a}|} \bullet \left(grad\ f \right)}$$

or

$$\boxed{D_{\vec{a}} f = \frac{1}{|\vec{a}|}\left(\vec{a} \bullet grad\ f\right)} --(5)$$

DIRECTIONAL DERIVATIVES

Example 2:-**Find directional derivative of** $f(x,y,z)=2x^2+3y^2+z^2$ ***at the point (2, 1, 3) in* the direction of** $\vec{a}=\hat{i}-2\hat{k}$

Solution:

Given function is $f=2x^2+3y^2+z^2$

Now $grad\ f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$

$$=\frac{\partial}{\partial x}\left(2x^2+3y^2+z^2\right)\hat{i}+\frac{\partial}{\partial y}\left(2x^2+3y^2+z^2\right)\hat{j}+\frac{\partial}{\partial z}\left(2x^2+3y^2+z^2\right)\hat{k}$$

$\Rightarrow grad\ f=4x\,\hat{i}+6y\,\hat{j}+2z\,\hat{k}$

Therefore, $\left[grad\ f\right]_{\text{at }(2,1,3)}=4(2)i+6(1)j+2(3)j=8\hat{i}+6\hat{j}+6\hat{k}$

Given vector is $\vec{a}=\hat{i}-2\hat{k}$

Therefore, $|\vec{a}|=\left|\hat{i}-2\hat{k}\right|=\sqrt{(1)^2+(-2)^2}=\sqrt{5}$

DIRECTIONAL DERIVATIVES

$$\text{So}, D_a f = .\frac{1}{\left|\vec{a}\right|}\left(\vec{a} \bullet grad\ f\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\hat{i} - 2\hat{k}\right) \bullet \left(8\hat{i} + 6\hat{j} + 6\hat{k}\right)\right)$$

$$= \frac{1}{\sqrt{5}}\left((1)(8) + (0)(6) + (-2)(6)\right)$$

$$= \frac{1}{\sqrt{5}}(-4) = -\frac{4}{\sqrt{5}}$$

SOME RESULTS CONCERNING GRADIENT

THEOREM – 1:

Let *f(P)* = *f(x, y, z)* be a scalar function having continuous first partial derivatives. Then *grad f* exists and its length and direction are independent of the particular choice of Cartesian coordinates in space. If a point *P* the *grad f* is not the zero vector, it has the direction of maximum increase of *f* at *P*.

Proof:

The direction derivative of f in the direction of a vector $\vec{a}$ is given by $D_{\vec{a}} f = \frac{1}{|\vec{a}|}\left(\vec{a} \bullet grad\ f\right)$

$$= \frac{1}{|\vec{a}|}\left(|\vec{a}||grad\ f|\cos\gamma\right) = |grad\ f|\cos\gamma \text{---} (1)$$

$(\gamma$ is the angle between $\vec{a}$ and $grad\ f)$

SOME RESULTS CONCERNING GRADIENT

As f is a scalar function its value at a point P depends on P but not on the particular choice of coordinates. The same holds for the arc length s of any ray C, hence for $D_{\vec{a}} f$. So from (1), we find that $D_{\vec{a}} f$ is maximum when $cos\gamma = 1$, i.e. $\gamma = 0$, and then $D_{\vec{a}} f = |gradf|$. This implies that length and direction of $grad\ f$ are independent of the coordinates. Since $\gamma = 0$ if and only if $\vec{a}$ has the direction of $grad\ f$, hence $grad\ f$ is the direction of maximum increase of f at P, provided

$grad\ f \neq \vec{0}$ at P.

SOME RESULTS CONCERNING GRADIENT

THEOREM 2:- Let f be a differentiable scalar function that represents a surface $S: f(x,y,z) = c = constant$. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

Proof: Let $f(x,y,z)=c$ ------(1) represents a surface, say S.

$$\text{Let } \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} --- (2)$$

be the position vector any curve , say C on S.

$$\text{Now, } \frac{df}{dt} = \frac{d}{dt}(c) = 0$$

$$\Rightarrow \frac{d}{dt}\left[f(x(t), y(t), z(t))\right] = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) = 0$$

SOME RESULTS CONCERNING GRADIENT

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) = 0$$

$$\Rightarrow \vec{\nabla} f \cdot \frac{d\vec{r}}{dt} = 0 \qquad \Rightarrow (grad\ f) \cdot \vec{r'} = 0$$

As $\vec{r'}$ represents the tangent vector to the surface and $(grad\ f) \cdot \vec{r'} = 0$

$\Rightarrow grad\ f$ is perpendicular to the tangent vector at the point P.

$\Rightarrow grad\ f$ is normal to the surface S at P.

LAPLACIAN OF A FUNCTION

$$\text{Let } \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \text{ a differential operator.}$$

$$\text{Then } \nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ is called Laplacian}$$

i.e. for a scalar function $f(x, y, z)$, we have

$$\nabla^2 f = \left(\vec{\nabla} \cdot \vec{\nabla}\right) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

SOME PROBLEMS INVOLVING GRADIENT

Example 3: - Find a unit normal vector of the cone of revolution $z^2 = 4(x^2 + y^2)$ **at the point *(1, 0, 2)*.**

SOLUTION:

Given surface is $z^2 = 4(x^2 + y^2)$

Let $f = 4(x^2 + y^2) - z^2 = 0$

$$So,\ grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\left(4(x^2 + y^2) - z^2\right)\hat{i} + \frac{\partial}{\partial y}\left(4(x^2 + y^2) - z^2\right)\hat{j} + \frac{\partial}{\partial z}\left(4(x^2 + y^2) - z^2\right)\hat{k}$$

$$= 8x\,\hat{i} + 8y\,\hat{j} - 2z\,\hat{k}$$

SOME PROBLEMS INVOLVING GRADIENT

$$\Rightarrow \left[grad\ f\right]_{(1,0,2)} = 8(1)\hat{i} + 8(0)\hat{j} - 2(2)\hat{k} = 8\hat{i} - 4\hat{k}$$

So, a unit normal vector to the given surface is $\dfrac{grad\ f}{\left|grad\ f\right|}$

$$= \frac{8\hat{i} - 4\hat{k}}{\left|8\hat{i} - 4\hat{k}\right|} = \frac{8\hat{i} - 4\hat{k}}{\sqrt{64+16}} = \frac{4(2\hat{i} - \hat{k})}{4\sqrt{5}} = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{k}$$

SOME PROBLEMS INVOLVING GRADIENT

Example 4: -Find the potential field *f* of the vector field

$$\vec{V} = [yz \quad zx \quad xy]$$

SOLUTION:

Given that $\vec{V} = grad f = [yz \quad zx \quad xy]$

$$\Rightarrow \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = [yz \quad zx \quad xy]$$

$$\frac{\partial f}{\partial x} = yz - - - (1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = zx - - - - (2)$$

$$\frac{\partial f}{\partial z} = xy - - - - (3)$$

Integrating (1) w.r.t. x, we get $f(x, y, z) = xyz + k(y, z) - - - (4)$

Where k is an arbitrary function of y and z.

Differentiating (4) w.r.t. y, we get $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xyz + k(y,z)\right)$

$$\Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial k}{\partial y} ---(5)$$

From (2) and (5), we have $zx = xz + \frac{\partial k}{\partial y} \Rightarrow \frac{\partial k}{\partial y} = o$

Since k is a function of y and z and $\frac{\partial k}{\partial y} = o$

On integration with respect to y we get

$k(y,z)$ = A function of $z = c(z)$ $\left(\text{say}\right)$

So $(4) \Rightarrow f(x,y,z) = xyz + c(z) ---(6)$

SOME PROBLEMS INVOLVING GRADIENT

Differentiating (6) w.r.t. z, we get $\dfrac{\partial f}{\partial z}=\dfrac{\partial}{\partial z}\left(xyz+c(z)\right)$

$$\Rightarrow \frac{\partial f}{\partial z}=xy+\frac{dc}{dz}---(7)$$

From (3) and (7), we have $xy=xy+\dfrac{dc}{dz} \Rightarrow \dfrac{dc}{dz}=o$

$$\Rightarrow c=A\ \text{Constant}=\text{C}\ \left(\text{say}\right)$$

Therefore, $(6)\Rightarrow f(x,y,z)=xyz+C$

The required potential field is $f(x,y,z)=xyz+C$

SOME USEFUL RESULTS

$$1.\ \vec{\nabla}(fg) = f\overrightarrow{\nabla g} + g\overrightarrow{\nabla f}$$

Proof:

$$\vec{\nabla}(fg) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(fg)$$

$$= \frac{\partial}{\partial x}(fg)\,\hat{i} + \frac{\partial}{\partial y}(fg)\,\hat{j} + \frac{\partial}{\partial z}(fg)\,\hat{k}$$

$$= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\hat{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\hat{j} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= f\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right) + g\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = f\overrightarrow{\nabla g} + g\overrightarrow{\nabla f}$$

SOME USEFUL RESULTS

$$2.\vec{\nabla}(f^n) = nf^{n-1}\vec{\nabla}f$$

Proof:

$$\vec{\nabla}(f^n) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(f^n)$$

$$= \frac{\partial}{\partial x}(f^n)\hat{i} + \frac{\partial}{\partial y}(f^n)\hat{j} + \frac{\partial}{\partial z}(f^n)\hat{k}$$

$$= \left(nf^{n-1}\frac{\partial f}{\partial x}\right)\hat{i} + \left(nf^{n-1}\frac{\partial f}{\partial y}\right)\hat{j} + \left(nf^{n-1}\frac{\partial f}{\partial z}\right)\hat{k}$$

$$= nf^{n-1}\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = nf^{n-1}\vec{\nabla}f.$$

SOME USEFUL RESULTS

$$3.\nabla^2(fg) = g\nabla^2 f + f\nabla^2 g + 2\vec{\nabla} f.\vec{\nabla} g$$

Proof:

$$\nabla^2(fg) = \frac{\partial^2}{\partial x^2}(fg) + \frac{\partial^2}{\partial y^2}(fg) + \frac{\partial^2}{\partial z^2}(fg)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(fg)\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(fg)\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}(fg)\right)$$

$$= \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)$$

$$= f\frac{\partial^2 g}{\partial x^2} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + g\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}$$

$$+ f\frac{\partial^2 g}{\partial y^2} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + g\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\frac{\partial^2 g}{\partial z^2} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + g\frac{\partial^2 f}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}$$

SOME USEFUL RESULTS

$$= f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)$$

$$+g\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)$$

$$+2\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)$$

$$= g\nabla^2 f + f\nabla^2 g + 2\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right).\left(\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}\right)$$

$$= g\nabla^2 f + f\nabla^2 g + 2\vec{\nabla} f.\vec{\nabla} g$$

SOME USEFUL RESULTS

$$4.\ \vec{\nabla}\left(\frac{f}{g}\right)=\frac{1}{g^2}\left(g\vec{\nabla}f-f\vec{\nabla}g\right)$$

Proof:

$$\vec{\nabla}\left(\frac{f}{g}\right)=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left(\frac{f}{g}\right)$$

$$=\hat{i}\,\frac{\partial}{\partial x}\left(\frac{f}{g}\right)+\hat{j}\,\frac{\partial}{\partial y}\left(\frac{f}{g}\right)+\hat{k}\,\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

$$=\hat{i}\left(\frac{g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}}{g^2}\right)+\hat{j}\left(\frac{g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}}{g^2}\right)+\hat{k}\left(\frac{g\frac{\partial f}{\partial z}-f\frac{\partial g}{\partial z}}{g^2}\right)$$

$$=\frac{1}{g^2}\left[\left(g\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)-f\left(\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}\right)\right)\right]=\frac{1}{g^2}\left(g\vec{\nabla}f-f\vec{\nabla}g\right)$$

PROBLEMS FOR PRACTICE

1. Find grad f at the point $P(2,0)$ where $f = ln\,(x^2+y^2)$.

Solution:

Given function is $f = \ln(x^2 + y^2)$

So we have $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\ln(x^2 + y^2)\right) = \dfrac{2x}{x^2 + y^2}$,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x^2 + y^2)\right) = \frac{2y}{x^2 + y^2}$$

and $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left(\ln(x^2 + y^2)\right) = 0$

PROBLEMS FOR PRACTICE

Therefore, we have $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= \frac{2x}{x^2+y^2}\hat{i} + \frac{2y}{x^2+y^2}\hat{j} + 0\hat{k}$ So, $\left[grad\ f\right]_{(2,0)} = \frac{4}{4+0}\hat{i} + \frac{0}{4+0}\hat{j} = 4\hat{i}$

2. Find the unit normal vector to the surface $z = \sqrt{x^2+y^2}$ at the point *(6,8,10)*.

Solution:

Given surface is $z = \sqrt{x^2+y^2}$

Let $f(x,y,z) = \sqrt{x^2+y^2} - z$

PROBLEMS FOR PRACTICE

So $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\sqrt{x^2+y^2} - z\right) = \dfrac{x}{\sqrt{x^2+y^2}}$,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2} - z\right) = \frac{y}{\sqrt{x^2+y^2}}$$

and $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left(\sqrt{x^2+y^2} - z\right) = -1$

$$\overrightarrow{\nabla f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial x}\hat{k} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j} - \hat{k}$$

PROBLEMS FOR PRACTICE

$$\Rightarrow \left[\overrightarrow{\nabla f}\right]_{(6,8,10)} = \frac{6}{\sqrt{6^2+8^2}}\hat{i} + \frac{8}{\sqrt{6^2+8^2}}\hat{j} - \hat{k} \qquad = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} - \hat{k}$$

Therefore, a Unit normal vector to the given surface is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} - \hat{k}}{\left|\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} - \hat{k}\right|} = \frac{\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} - \hat{k}}{\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + (-1)^2}}$$

$$= \frac{3}{5\sqrt{2}}\hat{i} + \frac{4}{5\sqrt{2}}\hat{j} + \frac{5}{5\sqrt{2}}\hat{k}$$

PROBLEMS FOR PRACTICE

3.Find the directional derivative of $f(x,y)=e^x cosy$ at the point p(2,π,0) in the direction of the vector $\vec{a} = 2\hat{i} + 3\hat{j}$

Solution:

Given function is $f(x,y,z) = e^x \cos y$

$$\therefore\ grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial y}\hat{k} = \frac{\partial}{\partial x}\left(e^x \cos y\right)\hat{i} + \frac{\partial}{\partial y}\left(e^x \cos y\right)\hat{j} + \frac{\partial}{\partial y}\left(e^x \cos y\right)\hat{k}$$

$$= e^x \cos y\,\hat{i} - e^x \sin y\,\hat{j}$$

$$Now \quad [\nabla f]_{(2,\pi,0)} = \left(e^2 \cos \pi\right)\hat{i} + \left(e^2 \sin \pi\right)\hat{j} = -e^2\,\hat{i}$$

$$D_a f = \frac{\vec{a}}{\left|\vec{a}\right|} \bullet \overrightarrow{\nabla f} = \left(\frac{2\hat{i}+3\hat{j}}{\left|2\hat{i}+3\hat{j}\right|}\right) \bullet \left(-e^2\,\hat{i}\right) = \left(\frac{2\hat{i}+3\hat{j}}{\sqrt{13}}\right) \bullet \left(-e^2\,\hat{i}\right) = -\frac{2}{\sqrt{13}}e^2$$

PROBLEMS FOR PRACTICE

4. Find grad f at the point $P(2,1,3)$ where $f=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$

Solution: Given function is $f=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$

So we have $\dfrac{\partial f}{\partial x}=\dfrac{\partial}{\partial x}\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$,

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=-y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

and $\dfrac{\partial f}{\partial z}=\dfrac{\partial}{\partial z}\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=-z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$

PROBLEMS FOR PRACTICE

Therefore, we have $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \left[-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]\hat{i} + \left[-y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]\hat{j} + \left[-z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]\hat{k}$$

So, $$[grad\ f]_{(2,1,3)} = \left[-2\left(2^2+1^2+3^2\right)^{-\frac{3}{2}}\right]\hat{i}$$

$$+\left[-\left(2^2+1^2+3^2\right)^{-\frac{3}{2}}\right]\hat{j} + \left[-3\left(2^2+1^2+3^2\right)^{-\frac{3}{2}}\right]\hat{k}$$

$$= -\frac{1}{7\sqrt{14}}\hat{i} - \frac{1}{14\sqrt{14}}\hat{j} - \frac{3}{14\sqrt{14}}\hat{k} = -(14)^{-\frac{3}{2}}\left(2\hat{i}+\hat{j}+3\hat{k}\right)$$

PROBLEMS FOR PRACTICE

5. Find unit normal to the surface $x^2+y^2+2z^2=26$ at $P(2,2,3)$.

Solution: Given surface is $x^2+y^2+2z^2=26$

Let $f(x,y,z)=x^2+y^2+2z^2=26$

So $\dfrac{\partial f}{\partial x}=\dfrac{\partial}{\partial x}\left(x^2+y^2+2z^2\right)=2x,$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^2+y^2+2z^2\right)=2y$$

and $\dfrac{\partial f}{\partial z}=\dfrac{\partial}{\partial z}\left(x^2+y^2+2z^2\right)=4z$

PROBLEMS FOR PRACTICE

$$\overrightarrow{\nabla f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial x}\hat{k} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$$

$$\Rightarrow \left[\overrightarrow{\nabla f}\right]_{(2,2,3)} = 4\hat{i} + 4\hat{j} + 12\hat{k}$$

Therefore, a Unit normal vector to the given surface is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{4\hat{i} + 4\hat{j} + 12\hat{k}}{\left|4\hat{i} + 4\hat{j} + 12\hat{k}\right|} = \frac{4\hat{i} + 4\hat{j} + 12\hat{k}}{\sqrt{(4)^2 + (4)^2 + (12)^2}}$$

$$= \frac{4\hat{i} + 4\hat{j} + 12\hat{k}}{4\sqrt{11}} = \frac{1}{\sqrt{11}}\hat{i} + \frac{1}{\sqrt{11}}\hat{j} + \frac{3}{\sqrt{11}}\hat{k} = \frac{1}{\sqrt{11}}\left(\hat{i} + \hat{j} + 3\hat{k}\right)$$

PROBLEMS FOR PRACTICE

6. Find f if $\vec{V} = grad\ f = \left[\frac{y}{z}, \quad \frac{x}{z}, \quad -\frac{xy}{z^2}\right]$

SOLUTION:

Given that $\vec{V} = grad\ f = \left[\frac{y}{z}, \quad \frac{x}{z}, \quad -\frac{xy}{z^2}\right]$

$$\Rightarrow \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right] = \left[\frac{y}{z}, \quad \frac{x}{z}, \quad -\frac{xy}{z^2}\right]$$

$$\frac{\partial f}{\partial x} = \frac{y}{z} - - - \qquad (1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x}{z} - - - - (2)$$

$$\frac{\partial f}{\partial z} = -\frac{xy}{z^2} - - (3)$$

PROBLEMS FOR PRACTICE

Integrating (1) w.r.t. x, we get $f(x,y,z)=\dfrac{xy}{z}+k(y,z)---(4)$

Where k is an arbitrary function of y and z.

Differentiating (4) w.r.t. y, we get $\dfrac{\partial f}{\partial y}=\dfrac{\partial}{\partial y}\left(\dfrac{xy}{z}+k(y,z)\right)$

$$\Rightarrow \frac{\partial f}{\partial y}=\frac{x}{z}+\frac{\partial k}{\partial y}--(5)$$

From (2) and (5), we have $\dfrac{x}{z}=\dfrac{x}{z}+\dfrac{\partial k}{\partial y}$

Since k is a function of y and z and $\dfrac{\partial k}{\partial y}=o$

On integration with respect to y we get $k(y,z)=$ A function of $z=c(z)$ (say)

So $(4) \Rightarrow f(x,y,z)=\dfrac{xy}{z}+c(z)---(6)$

PROBLEMS FOR PRACTICE

Differentiating (6) w.r.t. z, we get $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{xy}{z} + c(z)\right)$

$$\Rightarrow \frac{\partial f}{\partial z} = -\frac{xy}{z^2} + \frac{dc}{dz} - - - (7)$$

From (3) and (7), we have $-\dfrac{xy}{z^2} = -\dfrac{xy}{z^2} + \dfrac{dc}{dz}$ $\Rightarrow \dfrac{dc}{dz} = o$

$$\Rightarrow c = A \text{ Constant} = \text{C } (\text{say})$$

Therefore, $(6) \Rightarrow f(x, y, z) = \dfrac{xy}{z} + C$

The required potential field is $f(x, y, z) = \dfrac{xy}{z} + C$

PROBLEMS FOR PRACTICE

7. Find the directional derivative of $f(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$

at P(3,0,4) in the direction of $\vec{a}=\hat{i}+\hat{j}+\hat{k}$.

Solution: Given function is $f(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$

So we have $\dfrac{\partial f}{\partial x}=\dfrac{\partial}{\partial x}\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$,

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=-y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

and $\dfrac{\partial f}{\partial z}=\dfrac{\partial}{\partial z}\left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)=-z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$

PROBLEMS FOR PRACTICE

Therefore, we have $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \left[-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]\hat{i} + \left[-y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]\hat{j} + \left[-z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]\hat{k}$$

So, $[grad\ f]_{(3,0,4)} = \left[-3\left(3^2+0^2+4^2\right)^{-\frac{3}{2}}\right]\hat{i}$

$$+\left[-0\left(3^2+0^2+4^2\right)^{-\frac{3}{2}}\right]\hat{j} + \left[-4\left(3^2+0^2+4^2\right)^{-\frac{3}{2}}\right]\hat{k}$$

$$= -\frac{3}{125}\hat{i} - \frac{4}{125}\hat{k}$$

PROBLEMS FOR PRACTICE

Given vector is $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

$$\therefore \left|\vec{a}\right| = \left|\hat{i} + \hat{j} + \hat{k}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$D_a f = \frac{1}{\left|\vec{a}\right|}\left(\vec{a} \bullet \overrightarrow{\nabla f}\right) = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right) \bullet \left(-\frac{3}{125}\hat{i} - \frac{4}{125}\hat{k}\right)$$

$$= \frac{1}{\sqrt{3}}\left(-\frac{3}{125} - \frac{4}{125}\right) = -\frac{7}{125\sqrt{3}}$$

ASSIGNMENTS

1. Find the directional derivative of $f(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$

   at P(3,0,4) in the direction of $\vec{a}=\hat{i}+\hat{j}+\hat{k}$.

2. $Find\ f\ if\ v = grad\ f = \begin{bmatrix} ye^x & e^x & 1 \end{bmatrix}$

3. Pr $ove\ that \quad \nabla^2(fg)=g\nabla^2 f+f\nabla^2 g+2\vec{\nabla} f.\vec{\nabla} g$

4. Find the unit normal vector to the surface

   $x^2+y^2+2z^2=26$ at $P(2,2,3)$.