DEPARTMENT OF MATHEMATICs

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MATHEMATICS - I

TEXT BOOK: DIFFERENTIAL CALCULUS BY SHANTI NARAYAN & P.K.MITTAL

LECTURE - 3

Cauchy's mean value theorem (Chapter-8.5)

DEPARTMENT OF MATHEMATICS, CGU

Cauchy's Mean Value Theorem

If two functions $f(x)$ and $g(x)$ are

(i) continuous in a closed interval [a, b],
(ii) derivable in an interval (a, b), and
(iii) $g'(x) \neq 0$ for all $x \in (a, b)$,

Then there exists at least one c ∈ (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Ex-1: Verify Cauchy's Mean Value(CMV) theorem for the functions $f(x) = \ln x,\ g(x) = \frac{1}{x}$ in $[1, e]$.

Solution

$$f(x) = \ln x, \qquad g(x) = 1/x, \qquad [1, e]$$

$f'(x) = 1/x$, $g'(x) = -\dfrac{1}{x^2}$

By CMV theorem,

$$\frac{\ln e - \ln 1}{\frac{1}{e} - 1} = \frac{1}{c}.(-c^2) = -c$$

$$\therefore c = \frac{e}{e-1} \in (1, e)$$

Ex-2: If in the Cauchy's Mean Value Theorem, $f(x)=e^x$ and $F(x)=e^{-x}$ in the interval $[a,b]$, then show that the real number $c \in (a,b)$ is arithmetic mean between a and b.

Solution

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{e^b-e^a}{e^{-b}-e^{-a}}=-e^a e^b=-e^{a+b}$$

$$and \quad \frac{f'(x)}{F'(x)}=\frac{e^x}{-e^{-x}}$$

$$\Rightarrow \frac{f'(c)}{F'(c)}=-e^{2c}$$

$$\therefore -e^{a+b}=-e^{2c}$$

$$\Rightarrow c=\frac{a+b}{2}$$

Ex-3: If in the Cauchy's Mean Value Theorem, $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in the interval $[a, b]$, then show that $c \in (a, b)$ is geometric mean between a and b.

Solution

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = -\sqrt{ab}$$

$$and \quad \frac{f'(x)}{F'(x)} = \frac{\frac{1}{2}x^{-1/2}}{-\frac{1}{2}x^{-3/2}} = -x$$

$$\Rightarrow \frac{f'(c)}{F'(c)} = -c$$

$$\therefore -c = -\sqrt{ab}$$

$$\Rightarrow c = \sqrt{ab}$$

Ex-4: Use Cauchy's Mean Value Theorem to evaluate

$$\lim_{x\to 1}\left[\frac{\cos\frac{1}{2}\pi x}{\log(1/x)}\right]$$

Solution

From Cauchy's Mean Value theorem, we have

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(c)}{\varphi'(c)} \quad ; \quad \text{where } a<c<b \qquad \ldots(i)$$

$$\text{Let } f(x)=\cos\left(\frac{1}{2}\pi x\right); \qquad \varphi(x)=\log x$$

a=x, b = 1.

Then from (i)

$$\frac{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}\times x\right)}{\log 1-\log x}=\frac{-\frac{1}{2}\pi\sin\left(\frac{1}{2}\pi c\right)}{1/c}$$

where a < c< b,i.e., x<c<1.

Taking limits as x $\to$1 which implies that c $\to$1, we have

$$\lim_{x\to 1}\left\{\frac{-\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\lim_{x\to 1}\left\{\frac{-\frac{1}{2}\pi\sin\left(\frac{1}{2}\pi c\right)}{1/c}\right\}$$

$$\Rightarrow -\lim_{x\to 1}\left\{\frac{\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\frac{1}{2}\pi,\quad as\sin\left(\frac{1}{2}\pi c\right)\to 1\ \ as\, c\to 1$$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\frac{1}{2}\pi$$

Ex-5: If $0 \leq a < b < \pi/2$, then show that $0 < \cos a - \cos b < b - a$.

Solution

Apply CMV theorem to functions f(x) = cos x and g(x) = x in [a, b]. Then by CMV theorem

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\Rightarrow \cos a - \cos b = (b - a)\sin c$$

Since cos is decreasing in (0, π/2), so cos a > cos b for any a<b.

Thus cos a– cos b > 0. Also in (0, π/2) maximum value of sin x is 1. So since c ϵ(a, b), and sin c <1

So (b-a)sin c < (b-a). Thus 0 < cos a-cos b<(b-a).

Ex-6: Show that there exists a number c ϵ (a, b) such that

$$2c[f(a)-f(b)] = f'(c).[a^2-b^2]$$

When f is continuous in [a, b] and derivable in (a, b).

Solution

By applying CMV theorem to the two functions f(x) and $g(x)=x^2$

in [a, b], $\dfrac{f(b)-f(a)}{b^2-a^2} = \dfrac{f'(c)}{2c}$

for some c in (a, b). (proved)

Thank You