Q3

a. Prove that two-dimensional convolution is equivariant to the shift transformation, i.e., T: x → x + c ∀c. (with padding of c)

$$f(x): I'(i,j) = \sum_{m=-\frac{L+1}{2}}^{\frac{L+1}{2}} \sum_{n=-\frac{L+1}{2}}^{\frac{L+1}{2}} I(i-m, j-n)F(m,n)$$

$$T: I(i,j) \to I(i, j+c)$$

Need to show that:

$$I'(i, j+c) = \sum_{m=-\frac{L+1}{2}}^{\frac{L+1}{2}} \sum_{n=-\frac{L+1}{2}}^{\frac{L+1}{2}} I(i-m, j-n+c)F(m,n)$$

Shifting the image with c:

$$\sum_{m=-\frac{L+1}{2}}^{\frac{L+1}{2}} \sum_{n=-\frac{L+1}{2}}^{\frac{L+1}{2}} I(i-m, j-n+c)F(m,n) = \sum_{m=-\frac{L+1}{2}}^{\frac{L+1}{2}} \sum_{n=-\frac{L+1}{2}}^{\frac{L+1}{2}} I(i-m, t-n)F(m,n) = I'(i,t)$$
$$= I'(i, j+c)$$

b. Given an image I of size W,H, and a filter F of size dx, dy. Give a necessary and sufficient condition for the I to be equivariant to convolution with F.

With any of the following conditions:

- Stride = 1
- Padding >= Filter + shift (the size of the padding is larger or equal the length of the filter plus the number of pixels of the shifting)
- If there is dilation: filter*dilation + shift <= padding (the product of the length of the filter with the dilation size plus the shift size)

c. Is MAX-POOLING 3X3 shift invariant to one pixel shift? Assume you apply the MAX-POOLING with stride 3 so the windows are disjoint.

No! example: assuming after the shift there is a column of zeros.

Before shift:

4	1	1	1	1	2
0	0	0	0	0	0
1	1	1	1	1	1
2	1	3	4	1	1
1	1	1	1	1	1
0	0	0	0	0	0

$\longrightarrow$

4	5
3	4

After shift:

1	1	1	1	2	0
0	0	0	0	0	0
1	1	1	1	1	0
1	3	4	1	1	0
1	1	1	1	1	0
0	0	0	0	0	0

$\longrightarrow$

1	2
4	1

As you can see, they are not equal.

d. Given inputs x1, x2, … , xd to a fully connected layer give a sufficient condition for the layer to be permutation invariant.

A sufficient condition would be that all weights for each node are equal.

Bonus: Find a necessary and sufficient condition for permutation invariance.

If at least one of the weights is different and one of the inputs is different than the others, then there is at least one permutation where it won't be invariant. Hence, the condition above is necessary and sufficient.