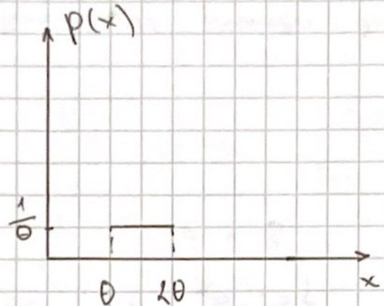
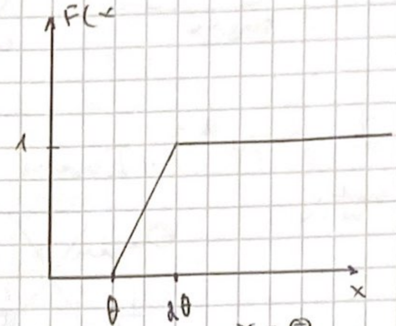
N 4

Случайная величина распределена равномерно на отрезке $[\theta, 2\theta]$





$F(x) = \frac{x-\theta}{\theta}$

а) ОММ : (методом моментов)

$$\alpha_1 = \int_{-\infty}^{+\infty} x\, p(x,\theta)\, dx = \int_{\theta}^{2\theta} x \frac{1}{\theta}\, dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = \frac{3}{2}\theta$$

$$\hat{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{\alpha}_1 = \alpha_1$$

$$\bar{x} = \frac{3}{2}\theta$$

$$\hat{\theta}_1 = \frac{2}{3}\bar{x}$$

ОМП :

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \begin{cases} 1/\theta^n, & x_i \in [\theta, 2\theta]\ \forall i \\ 0, & \exists\, x_i \mid x_i \notin [\theta, 2\theta] \end{cases}$$

$$L \to max \Rightarrow \begin{cases} \theta < x_{min} \\ x_{max} < 2\theta \end{cases} \Rightarrow \theta \in \left[\frac{x_{max}}{2}; x_{min}\right]$$

$$L \to max \Rightarrow \theta \to min$$

$$\hat{\theta}_2 = \frac{x_{max}}{2}$$

б)

1) $\tilde{\Theta}_1 = \frac{2}{3}\bar{x}$

$M[\hat{\Theta}_1] = \frac{2}{3} M[\bar{x}] = \frac{2}{3}\frac{1}{n}\sum M[x_i] =$

$= \frac{2}{3}\cdot\frac{1}{n}\cdot n M[\xi] = \frac{2}{3}\cdot\frac{3}{2}\cdot\Theta = \Theta \Rightarrow$ несмещ.

$D[\hat{\Theta}_1] = \frac{4}{9n^2} D[\sum x_i] = \frac{4}{9n^2}\cdot n\cdot D\xi =$

$= \frac{4}{9n} D\xi$

$M[\xi^2] = \int_{\Theta}^{2\Theta} x^2 \cdot \frac{1}{\Theta} dx = \frac{x^3}{3\Theta}\Big|_{\Theta}^{2\Theta} = \frac{7}{3}\Theta^2$

$D\xi = \frac{7}{3}\Theta^2 - \frac{9}{4}\Theta^2 = \frac{\Theta^2}{12}$

$D[\tilde{\Theta}_1] = \frac{4}{9\cdot 12 n}\Theta^2 = \frac{1}{27n}\Theta^2 \xrightarrow[n\to\infty]{} 0$ – состоят. по дост. условию

2) $\hat{\Theta}_2 = \frac{x_{max}}{2}$

$M[\hat{\Theta}_2] = \frac{1}{2} M[x_{max}] = \frac{1}{2}\int_{\Theta}^{2\Theta} y\, n\, D(y)^{n-1} g(y)\, dy =$

$= \frac{1}{2}\int_{\Theta}^{2\Theta} y\, n\left(\frac{y-\Theta}{\Theta}\right)^{n-1}\frac{1}{\Theta}\, dy = \frac{1}{2}\frac{n}{\Theta^n} = \left\{\frac{y}{\Theta} - 1 = t\right\} =$

$= \frac{n\Theta}{2}\int_0^1 (1+t)t^{n-1}\, dt = \frac{\Theta(2n+1)}{2(n+1)} \Rightarrow$ смещенная.

Исправим: $\tilde{\Theta}_2' = \frac{2(n+1)}{2n+1}\cdot\hat{\Theta}_2$

$M[\tilde{\Theta}_2^2] = \frac{1}{4} M[x^2_{max}] = \frac{1}{4}\int_{\Theta}^{2\Theta} y^2 n\, F(y)^{n-1} g(y)\, dy =$

$= \frac{1}{4}\int_{\Theta}^{2\Theta} y^2 n\left(\frac{y-\Theta}{\Theta}\right)^{n-1}\frac{1}{\Theta}\, dy =$

$= \frac{n}{4\Theta^n}\int_{\Theta}^{2\Theta} y^2 (y-\Theta)^{n-1}\, dy = \frac{n}{4\Theta^n}\left(y^2\frac{(y-\Theta)^n}{n}\Big|_{\Theta}^{2\Theta} -\right.$

$\left. - \int_{\Theta}^{2\Theta}\frac{2y}{n}(y-\Theta)^n\, dy\right) = \frac{1}{4\Theta^n}\left(4\Theta^2\cdot\Theta^n - 2\left(\frac{y(y-\Theta)^{n+1}}{n+1}\Big|_{\Theta}^{2\Theta} -\right.\right.$

$\left.\left. - \int_{\Theta}^{2\Theta}\frac{(y-\Theta)^{n+1}}{n+1}\, dy\right)\right) = \Theta^2 - \frac{2\cdot 2\Theta\,\Theta^{n+1}}{4\Theta^n(n+1)} + \frac{2(y-\Theta)^{n+2}}{4\Theta^n(n+1)(n+2)}\Big|_{\Theta}^{2\Theta} =$

$= \frac{n\Theta^2}{n+1} + \frac{\Theta^2}{2(n+1)(n+2)}$

$$D[\hat{\Theta}_2] = \frac{n\,\theta^2}{n+1} + \frac{\theta^2}{2(n+1)(n+2)} - \frac{(2n+1)^2\,\theta^2}{4(n+1)^2} =$$

$$= \frac{n\,\theta^2}{4(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$ - состоятельн.

$$D[\hat{\Theta}_2'] = \frac{4(n+1)^2 n}{4(n+1)^2(n+2)(2n+1)^2} = \frac{n\,\theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$

c)

$$D[\hat{\Theta}_1] = \frac{1}{2+n}\,\theta^2 \sim \frac{1}{n}$$

$$D[\hat{\Theta}_2'] = \frac{n\,\theta^2}{(n+2)(2n+1)^2} \sim \frac{1}{n^2}$$

$\Rightarrow \hat{\Theta}_2$ - наиболее эффективная

g) Построить точный довер. интервал для параметра

$$\xi \in R[\theta, 2\theta] \qquad \begin{cases} h = \theta \\ \hat{h} = \dfrac{x_{max}}{2} \quad (\text{ОМП}) \end{cases}$$

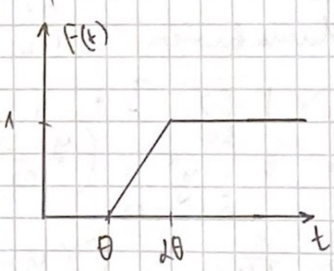
$$\exists\ f = \frac{\hat{h}}{h} = \frac{x_{max}}{2\theta}\ ; \quad f(h,\hat{h}) \sim g(t)$$ - не зависит от h

$$F = P(f<t) = P(x_{max} < 2\theta t) =$$

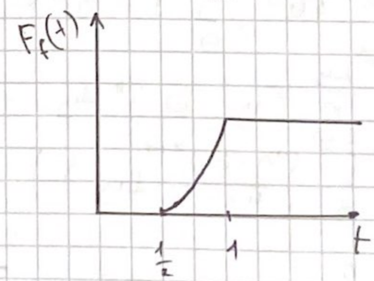
$$= P(x_i < 2\theta t,\ i = 1..n) = (P(x_i < 2\theta t))^n =$$

$$= (F(2\theta t))^n$$

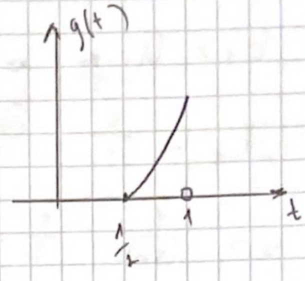
$$F(t) = \begin{cases} 0\ ; & t \le \theta \\ t/\theta - 1\ ; & \theta < t \le 2\theta \\ 1\ ; & t > 2\theta \end{cases}$$



$$F_f(t) = \begin{cases} 0\ ; & t \le \frac{1}{2} \\ (2t-1)^n\ ; & \frac{1}{2} < t \le 1 \\ 1\ ; & t > 1 \end{cases}$$



$$g(t)=\begin{cases}0; & t \le \frac{1}{2}\\ n(2t-1)^{n-1}; & \frac{1}{2} < t \le 1\\ 0; & t > 1\end{cases}$$



$$P\left(t_1 < \frac{X_{max}}{2\theta} < t_2\right) = \beta = 0{,}95$$

$n = 100$

$$\left\{ 100\int_a^b (2t-1)^{99}\,dt = \left\{\begin{matrix} 2t-1 = u \\ du = 2dt \end{matrix}\right\} = 100\int_a^b u^{99}\,du = \right.$$
$$\left. = 100\left(\frac{u^{100}}{100}\right)\Big|_a^b = b^{100} - a^{100} \right\}$$

$$\int_{1/2}^{t_1} g(t)\,dt = \int_{1/2}^{t_1} 2\cdot 100\cdot(2t-1)^{99}\,dt = 0{,}025 = t_1^{100} - \frac{1}{2^{100}}$$

$$\Rightarrow t_1 = \sqrt[100]{0{,}025 + (0{,}5)^{100}}$$

$$\int_{t_2}^{1} g(t)\,dt = \int_{t_2}^{1} 2\cdot 100(2t-1)^{99}\,dt = 0{,}025 = 1 - t_2^{100} \Rightarrow$$

$$\Rightarrow t_2 = \sqrt[100]{0{,}975}$$

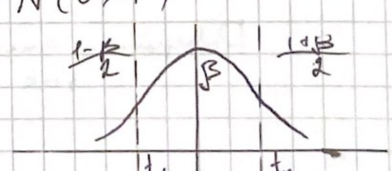
$$t_1 < \frac{X_{max}}{2\theta} < t_2 \Rightarrow \frac{X_{max}}{2t_2} < \theta < \frac{X_{max}}{2t_1}$$

e) Асимптотический дов. интервал для параметра θ

$\exists\ \beta = 0{,}95$

$\tilde{\theta}_1 = \frac{2}{3}\bar{x}$ – ОММ; $\bar{x} = \frac{3}{2}\tilde{\theta}_1$; $M[\xi] = \frac{3}{2}\theta$

ЦПТ: $\frac{\bar{x} - M\xi}{\sqrt{D\xi}}\sqrt{n} \rightsquigarrow N(0;1)$



$S \xrightarrow{F} \sqrt{D\xi}$

$\frac{3}{2}\,\frac{\tilde{\theta}_1 - \theta}{S}\sqrt{n} \rightsquigarrow N(0,1)$

$t_1 = u_{\frac{1-\beta}{2}} = -1{,}96$; $t_2 = u_{\frac{1+\beta}{2}} = 1{,}96$

$$t_1 < \frac{3}{2} \cdot \frac{\tilde{\Theta}_1 - \Theta}{s} \sqrt{n} < t_2$$

$$\Rightarrow \frac{1}{\sqrt{n}} \cdot \frac{2}{3} \cdot s t_1 + \tilde{\Theta}_1 < \Theta < \frac{1}{\sqrt{n}} \cdot \frac{2}{3} s t_2 + \tilde{\Theta}_1$$

№ 5

a) ОМП

$\xi \sim p(x, \Theta)$

$$L(\vec{x}_n, \Theta) = \prod_{i=1}^{n} p(x_i, \Theta) = \begin{cases} \prod_{i=1}^{n} \frac{(\Theta - 1)}{x_i^{\Theta}}; & x_i \geq 1 \\ 0; & x_i < 1 \end{cases}$$

$$\ln L = n \ln(\Theta - 1) - \Theta \cdot \sum_{i=1}^{n} \ln x_i \to \max$$

$$\frac{\partial \ln L}{\partial \Theta} = \frac{n}{\Theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow$$

$$\Rightarrow \hat{\Theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

Проверим: $\frac{\partial^2 \ln L}{\partial \Theta^2} = -\frac{n}{(\Theta-1)^2} < 0$ - след. точка max.

б) $\exists\ \beta = 0{,}95$

$$\int_1^{med} \frac{\Theta - 1}{x^{\Theta}} dx = \int_1^{med} (\Theta - 1) x^{-\Theta} dx = \frac{\Theta - 1}{-(\Theta - 1)} x^{-\Theta + 1} \Big|_1^{med} \textcircled{=}$$

$$\textcircled{=} \frac{1}{2}$$

$1 - med^{-\Theta + 1} = \frac{1}{2} \Rightarrow med = 2^{\frac{1}{\Theta - 1}}$ (при $\Theta = 10$ $med \approx 1{,}0801$)

Доверит. интервал будем считать bootstrap'ом.

c) $\sqrt{n I(\theta)}\,(\theta^2 - \theta) \sim N(0,1)$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = M\left[\left(\frac{\partial \ln \frac{\theta-1}{x^\theta}}{\partial\theta}\right)^2\right] =$$

$$= M\left[\frac{1}{(\theta-1)^2}\right] - \frac{2}{\theta-1} M[\ln x] + M[\ln^2 x]$$

$$M[\ln x] = \int_1^{+\infty} \ln x \, \frac{\theta-1}{x^\theta}\, dx = (\theta-1)\int_1^{+\infty} \ln x \cdot x^{-\theta}\, dx =$$

$$= (\theta-1)\left(\frac{\ln x \cdot x^{-\theta+1}}{-\theta+1}\Big|_1^{+\infty} + \int_1^{+\infty} \frac{x^{-\theta}}{\theta-1}\, dx\right) = 0 + \frac{x^{-\theta+1}}{\theta+1}\Big|_1^{+\infty} =$$

$$= \frac{1}{\theta-1}$$

$$M[\ln^2 x] = \int_1^{+\infty} \ln^2 x \, \frac{\theta-1}{x^\theta}\, dx = (\theta-1)\int_1^{+\infty} \ln^2 x \cdot x^{-\theta}\, dx =$$

$$= (\theta-1)\left(\frac{\ln^2 x \cdot x^{-\theta+1}}{-\theta+1}\Big|_1^{+\infty} - \int_1^{+\infty} \frac{2\ln x \cdot x^{-\theta}}{-\theta+1}\, dx\right) =$$

$$= 0 + 2(\theta-1)\int_1^{+\infty} \ln x \cdot x^{-\theta}\, dx = \frac{2}{(\theta-1)^2}$$

$$I(\theta) = \frac{1}{(\theta-1)^2} - \frac{2}{(\theta-1)^2} + \frac{2}{(\theta-1)^2} = \frac{1}{(\theta-1)^2}$$

$$\frac{\sqrt{n}\,(\hat\theta - \theta)}{\theta - 1} \sim N(0,1)$$

$$\hat\theta - \frac{\hat\theta - 1}{\sqrt{n}}\, t_1 < \theta < \hat\theta - \frac{\hat\theta - 1}{\sqrt{n}}\, t_2$$

$$\hat\theta - \frac{\hat\theta - 1}{\sqrt{n}}\, u_{\frac{1-\beta}{2}} < \theta < \hat\theta - \frac{\hat\theta - 1}{\sqrt{n}}\, u_{\frac{1+\beta}{2}}$$