Modified Boris Push in Shearing Sheet — M. Kunz

$S<0$

$$\frac{v^{n+1}-v^n}{\Delta t} = \frac{q}{m}\left[E + \left(\frac{v^{n+1}+v^n}{2}\right)\times B\right] - S\left(\frac{v_x^{n+1}+v_x^n}{2}\right)\hat{y}$$

Define $v^{\mp} = v^{\substack{n \\ n+1}} \pm \frac{qE}{2m}\Delta t$. Then

$$v^+ = v^- + \frac{q\Delta t}{2m}(v^+ + v^-)\times B - \frac{S\Delta t}{2}(v_x^+ + v_x^-)\hat{y}$$

Define $b \equiv \frac{qB}{m}$, $\xi \equiv \frac{\Delta t}{2}$:

$$v^+ = v^- + \xi\,(v^+ + v^-)\times b - S\xi\,(v_x^+ + v_x^-)\hat{y}$$

Semi-implicit solve:

$$v^+ = v^- + 2\xi\,\Lambda^{-1}a(v^-) \qquad \text{w/} \quad \Lambda \equiv I - \xi\frac{\partial a}{\partial v}$$

$$a \equiv v\times b - S v_x \hat{y}$$

So, we need $\frac{\partial a}{\partial v}$, Λ, and its inverse:

$$a_i = \varepsilon_{ijk} v_j b_k - S v_x \delta_{iy}$$

$$\frac{\partial a_i}{\partial v_l} = \underbrace{\varepsilon_{ijk}\delta_{jl}}_{\varepsilon_{ilk}} b_k - S\,\delta_{lx}\delta_{iy} \rightarrow \frac{\partial a}{\partial v} = \begin{bmatrix} 0 & b_z & -b_y \\ -(S+b_z) & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & -\xi b_z & \xi b_y \\ \xi(S+b_z) & 1 & -\xi b_x \\ -b_y\xi & \xi b_x & 1 \end{bmatrix}$$

$$\det(\Lambda) = 1 + \xi^2 b_x^2 + \xi b_z \left(\xi(S+b_z) - \cancel{\xi^2 b_x b_y} \right)$$
$$+ \xi b_y \left(\xi^2 b_x (S+\cancel{b_z}) + b_y \xi \right)$$
$$= 1 + \xi^2 b^2 + \xi^2 S \left(b_z + b_x b_y \xi \right)$$
$$\text{w/ } b^2 \equiv b_x^2 + b_y^2 + b_z^2$$

$$\Lambda^{-1} = \frac{1}{\det \Lambda} \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & K \end{bmatrix}$$

w/
$$A = 1 + \xi^2 b_x^2$$
$$B = \xi^2 b_x b_y - \xi(S+b_z)$$
$$C = \xi^2 b_x (S+b_z) + \xi b_y$$
$$D = \xi^2 b_x b_y + \xi b_z$$
$$E = 1 + \xi^2 b_y^2$$
$$F = \xi^2 b_y b_z - \xi b_x$$
$$G = \xi^2 b_x b_z - \xi b_y$$
$$H = \xi^2 b_y (S+b_z) + \xi b_x$$
$$K = 1 + \xi^2 b_z (S+b_z)$$

$$a = \begin{bmatrix} (v\times b)_x \\ (v\times b)_y - S v_x \\ (v\times b)_z \end{bmatrix}$$

$$\Lambda^{-1} a = \left[1 + \xi^2 b^2 + \xi^2 S \left(b_z + b_x b_y \xi \right) \right]^{-1} \begin{bmatrix} A(v\times b)_x + D(v\times b)_y + G(v\times b)_z - DSv_x \\ B(v\times b)_x + E(v\times b)_y + H(v\times b)_z - ESv_x \\ C(v\times b)_x + F(v\times b)_y + K(v\times b)_z - FSv_x \\ \\ \end{bmatrix}$$

$$(\Lambda^{-1} a)_x = \left[1+\xi^2 b^2 + \xi^2 S(b_z + b_x b_y \xi)\right]^{-1}$$

$$\cdot \Big[(1+\xi^2 b_x^2)(v\times b)_x + (\xi^2 b_x b_y + \xi b_z)(v\times b)_y + (\xi^2 b_x b_z - \xi b_y)(v\times b)_z$$
$$- (\xi^2 b_x b_y + \xi b_z) S v_x \Big]$$

$$= (v\times b)_x + \xi^2 b_x \underbrace{\left[\cancel{b\cdot(v\times b)}\right]}_{0} + \xi\left[(v\times b)\times b\right]_x - \xi S v_x (b_z + \xi b_x b_y)$$

$$(\Lambda^{-1} a)_y = \left[1+\xi^2 b^2 + \xi^2 S(b_z + b_x b_y \xi)\right]^{-1}$$

$$\cdot \Big[(\xi^2 b_x b_y - \xi S - \xi b_z)(v\times b)_x + (1+\xi^2 b_y^2)(v\times b)_y$$
$$+ (\xi^2 b_y S + \xi^2 b_y b_z + \xi b_x)(v\times b)_z - (1+\xi^2 b_y^2) S v_x \Big]$$

$$= (v\times b)_y + \xi^2 b_y \underbrace{\left[\cancel{b\cdot(v\times b)}\right]}_{0} + \xi\left[(v\times b)\times b\right]_y - S v_x$$
$$+ \xi S\Big(-(v\times b)_x + \underbrace{\xi b_y (v\times b)_z - \xi b_y^2 v_x}_{-v_y \xi b_y b_x + \xi \cancel{b_y v_x b_y} - \xi \cancel{b_y^2 v_x}}\Big)$$

$$= (v\times b)_y + \xi\left[(v\times b)\times b\right]_y \cancel{+ 2\xi S v_y (b_z + \xi b_x b_y) - S v_x}$$
$$- S(v_x + \xi(v\times b)_x) - S\xi v_y \xi b_y b_x \cancel{+ \xi S v_z b_y}$$

$$(\Lambda^{-1} a)_z = \left[1+\xi^2 b^2 + \xi^2 S(b_z + b_x b_y \xi)\right]^{-1}$$

$$\cdot \Big[(\xi^2 b_x S + \xi^2 b_x b_z + \xi b_y)(v\times b)_x + (\xi^2 b_y b_z - \xi b_x)(v\times b)_y$$
$$+ \underset{1+\xi^2 b_z S + \xi^2 b_z^2}{(\cancel{\xi^2 b_y S + \xi^2 b_y b_z + \xi b_x})}(v\times b)_z - (\xi^2 b_y b_z - \xi b_x) S v_x \Big]$$

$=$ (next page)

$y \downarrow$

$$= (v\times b)_z + \xi^2 b_z \cancelto{0}{\left[b\cdot(v\times b)\right]} + \xi\left[(v\times b)\times b\right]_z$$

$$+ \cancel{\xi^2 b_x S (v\times b)_x} + \cancel{\xi^2 b_z S (v\times b)_z} - \xi^2 b_y b_z S v_x + \xi b_x S v_x$$

$$+ \cancel{\xi^2 b_y S (v\times b)_y} - \xi^2 b_y S (v\times b)_y$$

$$= (v\times b)_z + \xi\left[(v\times b)\times b\right]_z + \xi S v_x\left[b_x - \cancel{\xi b_y b_z} + \cancel{\xi b_y b_z}\right]$$

$$- \xi^2 b_y S v_z b_x$$

$$= (v\times b)_z + \xi\left[(v\times b)\times b\right]_z + \xi S\left[b_x v_x - \xi b_x b_y v_z\right]$$

So,

$$\boxed{\begin{aligned} v^+ = v^- &+ 2\xi\left[\frac{(v^-\times b) + \xi(v^-\times b)\times b}{1+\xi^2 b^2 + \xi^2 S(b_z + b_x b_y \xi)}\right] \\ &+ 2\xi\Big[\hat{x}\left(-\xi S v_x^-(b_z + \xi b_x b_y)\right) \\ &\quad + \hat{y}\left(-S(v_x^- + \xi(v\times b)_x) - \xi S v_y^-(\cancel{b_z} + \xi b_x b_y)\right) \\ &\quad + \hat{z}\left(\xi S b_x (v_x^- - \xi b_y v_z^-)\right)\Big] \Big/ \left[1 + \xi^2 b^2 + \xi^2 S(b_z + b_x b_y \xi)\right] \end{aligned}}$$

Note: w/o shear, $v^+ = v^- + 2\xi\left[\dfrac{(v^-\times b) + \xi(v^-\times b)\times b}{1+\xi^2 b^2}\right]$,

which is the Boris push.

w/o magnetic field, $v^+ = v^- - 2\xi S v_x^- \hat{y}$,

which is just a shear.

Therefore, ① Do ½ E push.

② Do modified Boris rotation (in box)

③ Do ½ E push.

Then, we need to update position due to shear: $x \to x - q\Omega x(2\xi)\hat{y}$

For position, do

Ⓐ $x' = x^n + \xi v^n$ before ①

Ⓑ $x^{n+1} = x' + \xi v^{n+1}$ after ③

Ⓒ $x = x^{n+1} - 2q\Omega\xi\hat{y}\left(\dfrac{x^n + x^{n+1}}{2}\right)$