Indian Association for the Cultivation of Science

(Deemed to be University under the *de novo* category)

Integrated Bachelors-Masters program

End-Semester (Sem-II) Examination-Spring 2021

Subject: Linear Algebra & Multivariable Calculus ***Subject Code: MCS 1201A***

Full marks: 50 ***Time allotted: 3 hrs***

Question number 1 is ***compulsory****. Answer any* ***five*** *from the rest.*

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1. Let $T, S : V \to V$ be linear transformations from a finite dimensional real vector space V to V. Then **prove** or **disprove** the following:

   (i) $T \circ S$ is a linear transformation from V to V.

   (ii) $T \circ S - S \circ T$ is a linear transformation from V to V.

   (iii) Range of $T \circ S$ is always equal to range of $S \circ T$.

   (iv) Range of $T \circ S$=Range of T if S is injective.

   (v) If Range of $T \circ S$=Range of T then S is injective.

   (1+1+2+3+3)

2. Consider the following subsets of $\mathbb{R}^3$.

   $$V_1 = \{(x, y, z) \in \mathbb{R}^3 : x + y = 0\},\ V_2 = \{(x, y, z) \in \mathbb{R}^3 : x - y = 0\}.$$

   Prove that V_1, V_2 are vector subspaces of $\mathbb{R}^3$ and $V_1 \cup V_2$ is not a vector subspace of $\mathbb{R}^3$. Describe the vector subspaces $V_1 \oplus V_2$ and $V_1 \cap V_2$.

   (2+2+2)

3. Let $T : V \to V$ be a linear transformation on a finite dimensional real vector space V. Then prove that Null $(T) \cap$ Range $(T) = \{0\}$ **if and only if** Range $(T) =$ Range (T^2).

   Hint: To prove the only if part, first show Null (T) = Null (T^2) and then argue by contradiction.

   (2+4)

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined as: $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. Then answer the following **Questions 4 and 5**

4.(i). What are the matrices of the linear transformation T, $T \circ T$, $T \circ T \circ T$ and $T \circ T \circ T \circ T$ with respect to the standard basis $\mathcal{B} = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$.

4.(ii). What are the rank and nullity of the linear operators T, T^2, T^3 and T^4.

(4+2)

5. Find **all** the eigenvalues and eigenvectors corresponding to the linear operators T, T^2, T^3 and T^4.

(6)

6. Let $T : V \to V$ be an injective linear operator on a finite dimensional real vector space of dimension n. Prove that T^{-1} is a well-defined linear map on V. Also show that the matrix of the linear transformation

$$[T \circ T^{-1}]_{\mathcal{B}} = [T^{-1} \circ T]_{\mathcal{B}} = \text{Identity}_{n \times n}$$

with respect to any basis $\mathcal{B}$ of V. (6)

7. Let $f(x) = |x-1| + |x-2|$ in the interval $[0,3]$. Let

$$g(x) = \begin{cases} f'(x) & x \in [0,3] \setminus \{1,2\} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that that the Riemann integration, i.e., $\displaystyle\int_0^3 g'(x) = 0$.

(6)