$\frac{1}{2+z} = \frac{1}{2(1+\frac{z}{2})}$ $\frac{n(n-1)(n-2)}{6}$ $\frac{(-1)(-2)(-3)}{6}$ $\frac{\cancel{z}+3-\cancel{z}-1}{(z+1)(z+3)}$ $(a+b)^n = {}^nC_0\, a b^0 + {}^nC_1\, b^1 + \dots {}^nC_n\, b^n$

$-{}^1C_1$ $-{}^1C_2$ $\frac{n(n-1)}{2}$ $\frac{1(-1-1)}{2}$ $\left(1+\frac{1}{z}\right)^{-1} = 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3}$

Indian Association for the Cultivation of Science

(Deemed to be University under *de novo* Category)

Master's/Integrated Master's – PhD Program/Integrated Bachelor's-Master's Program/PhD Course

End-Semester Examination-Spring 2022

$\frac{1}{z+3} = \frac{1}{3+z} = \frac{1}{2+(1+z)}$

Subject: Mathematical Methods-II ***Subject Code(s): PHS-4202***

Full Marks: 50 ***Time Allotted: 3 h***

Answer all questions

Q.1 **(a)** Let $f(z)$ be complex analytic in a neighbourhood of a point $z_0 \in \mathbb{C}$ and z_0 is a **zero of order** m, for $m = 1, 2, \dots$ Show that $f(z)$ can be written as $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$. [2]

(b) Suppose a function $f(z)$ has a pole of order m at $z = z_0$. Show that the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of $f(z)$ is given by

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]_{z=z_0}$$

with

$$a_{-1} = [(z - z_0) f(z)]_{z=z_0},$$

when the pole is a simple pole ($m = 1$). [3]

Q.2 **(a)** Expand

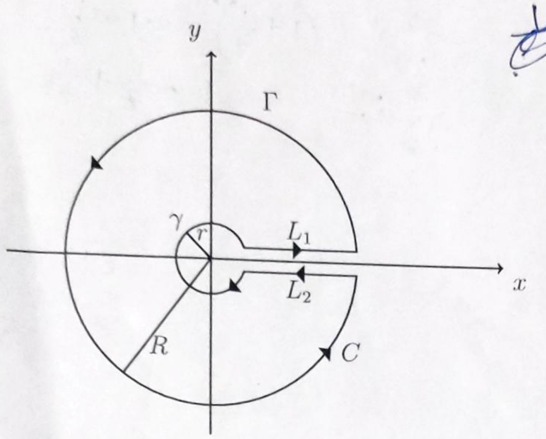
$$f(z) = \frac{1}{(z+1)(z+3)}$$

in a Laurent series valid for $1 < |z| < 3$. [3]

(b) Using the contour shown in the figure below, show that

$$\int_0^\infty \frac{x^{-a}}{x+1}dx = \frac{\pi}{\sin a\pi},$$

where $0 < a < 1$. [7]



$\rho^{a-1} = \frac{1}{\rho^{1-a}}$

Q.3 Evaluate

$$\int_0^\infty \frac{\ln x}{x^2+4}dx$$

by integrating

$$f(z) = \frac{(\log z)^3}{z^2+4}$$

around the contour as shown in **Q.2**.

[10]

Q.4 Let X_a $(a = 1, 2, 3)$ are the hermitian generators of a Lie group, which form an algebra under commutation

$$[X_a, X_b] = i f_{abc} X_c$$

f_{abc} are the structure constants of the group.

(a) Show that f_{abc} are real. **1**

(b) Show that the structure constants satisfy the following identity

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0.$$

2

(c) Define a set of matrices T_a

$$[T_a]_{bc} \equiv -i f_{abc}$$

and show that the identity in part (b) can be rewritten as

$$[T_a, T_b] = i f_{abc} T_c.$$

2

Q. 5 Define invariant subalgebra. Suppose X is any generator in the invariant subalgebra and Y is any generator in the whole algebra. Let $h = e^{iX}$ and $g = e^{iY}$. Show that

$$g^{-1}hg = e^{iX'}$$

where

$$X' = e^{-iY}Xe^{iY} = X - i[Y, X] - \frac{1}{2}[Y, [Y, X]] + ...$$

[2 + 8]

Q.6 (a) A homomorphism from the vector space $\mathbb{R}^3$ to the set of traceless Hermitian 2×2 matrices is defined by $\vec{x} \to \vec{x}\cdot\vec{\sigma}$. First show that $\det(\vec{x}\cdot\vec{\sigma}) = -|\vec{x}|^2$, where the Pauli matrices $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$. Second, prove the identity

$$x_i = \frac{1}{2}\mathrm{Tr}(\vec{x}\cdot\vec{\sigma}\ \sigma_i).$$

[1 + 2]

(b) Let $U \in SU(2)$,i.e., $U = \exp(i\vec{\sigma}.\hat{n}\theta/2)$. Show that $U\vec{x}\cdot\vec{\sigma}U^{-1} = \vec{y}\cdot\vec{\sigma}$ for some vector $\vec{y} \in \mathbb{R}^3$. [2]

Using the results of part (a), show that $\vec{y} = R(U)\vec{x}$ (i.e. find an expression for R_{ij} in terms of U). Also show that the linear transformation $\vec{y} = R(U)\vec{x}$ preserves the length of the vector. Show that $R(U)$ is a homomorphism, i.e., $R(U_1U_2) = R(U_1)R(U_2)$. What is the kernel of this homomorphism ? [3]

(c) Using the $SU(2)$ weight diagrams (i.e. pictorial way) find out the decomposition of the following tensor product of $SU(2)$ irreducible representations. You do not need to construct the precise linear combinations of states.

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{3}$$

[2]