Indian Association for the Cultivation of Science

(Deemed to be University under *de novo* Category)

Master's/Integrated Master's-PhD Program/Integrated Bachelor's-Master's Program/PhD Course

Final Examination-Autumn 2024

Subject: PHYSICS | *Subject Code(s): PHS 2101*

Full Marks: 50 | *Time Allotted: 3 h*

Answer any five questions.

1. Consider a system of $N = 3$ particles to be distributed among three discrete energy levels with enegies $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_0$ and $\epsilon_3 = 2\epsilon_0$. If the total energy of the macrostate is $E = 2\epsilon_0$, find the entropy of the system if the:

   (a) **5 marks** particles are indistinguishable;

   (b) **5 marks** paticles are distinguishable.

2. (a) **7 marks** Consider a system of fermions in 1 dimension with energy $\epsilon = cp$ where c is a constant and p is the momentum. The density of these fermions is n_0 and energy is E_0. If the density is now changed to $2n_0$, what will be the new value of energy in terms of E_0?

   (b) **3 marks** How will your answer change if the relation between energy and momentum is of the form $\epsilon = Kp^\alpha$, where K and α are constants? (No detailed calculation is necessary for this part. Only an analogy with the previous part would be enough.)

3. Consider radiation inside a container. All photons move with speed c, and their energy ϵ is related to their momentum p by the relation $\epsilon = cp$. Assume that they experience elastic collisions with the walls of the container.

   (a) **5 marks** Argue that the momentum transferred by photons coming from a solid angle $d\Omega$ to an area dA of the wall in time dt is given by

   $$\int dN_p \frac{d\Omega}{4\pi} \times \frac{c dt\, dA \cos\theta}{V} \times 2p\cos\theta\,,$$

   where dN_p is the number of photons whose momentum lies between p and $p + dp$.

   (b) **5 marks** By performing the integration over the solid angle, show that $P = \frac{1}{3}\rho$, where ρ is the energy density.

4. A function $F(\theta)$ satisfies the differential equation

$$\frac{d^2F}{d\theta^2} + \cot\theta \frac{dF}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta} \right] F(\theta) = 0\,,$$

and the function $G(\theta)$ is defined as

$$G(\theta) = \frac{dF(\theta)}{d\theta} + mF(\theta)\cot\theta\,.$$

(a) **4 marks** Show that

$$\frac{dG}{d\theta} = (m-1)\cot\theta\frac{dF}{d\theta} - l(l+1)F + \frac{m(m-1)}{\sin^2\theta}F.$$

(b) **6 marks** Now take another derivative of $G(\theta)$ to show that this function satisfies the differential equation

$$\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta} + \left[l(l+1) - \frac{(m-1)^2}{\sin^2\theta} \right] G(\theta) = 0\,.$$

5. (a) **4+1 marks** A function is defined as

$$S = \sum_{k=1}^{N} P_k \ln P_k\,,$$

where $\sum_k P_k = 1$. Using the method of Lagrange multipliers, show that the function is extremized if the P_k's are independent of k. Is this extremum a maximum or a minimum?

(b) **5 marks** Consider one of the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that

$i\theta\sigma_x$

$$e^{i\theta\sigma_x} = \cos\theta + i\sigma_x \sin\theta\,.$$

6. For a 1-dimensional simple harmonic oscillator, define the operator

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p - im\omega x)\,,$$

where the symbols on the right side have their usual significance.

(a) **2 marks** What is $a^\dagger$?

(b) **3 marks** Evaluate the commutator $[a, a^\dagger]$.

(c) **5 marks** Show that the Hamiltonian of the oscillator is given by

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega\,.$$