Indian Association for the Cultivation of Science

(Deemed to be University under the *de novo* category)

Integrated Bachelor's-Master's Program

End-Semester Examination-2019 (Semester I/III)

Subject: Calculus of One Variable *Subject Code(s): Mathematics I*

Full marks: 50 *Time allotted: 3 hrs*

Instruction: Answer **Group A** and any **THREE** from each of Groups B and C.

GROUP A

1. Answer **ANY FIVE** from the following: $(5 \times 2) = [10]$
   (a) State clearly the Archimedean property.
   (b) Define clearly what is meant by a sequence $\{x_n\}$ converging to a limit x.
   (c) Define what is meant by a subsequence of a sequence $\{x_n\}$.
   (d) Define clearly what is meant by the convergence of an infinite series $\sum_n x_n$.
   (e) State clearly the comparison test for an infinite series.
   (f) For a function f on an open interval I, how is $\lim_{x \to a} f(x)$ defined at a point $a \in I$?
   (g) State the intermediate value theorem for a continuous function on an interval.

2. For each of the following statements, you are to only state whether it is **TRUE** or **FALSE**. Attempt **ANY FIVE ONLY** $(5 \times 2) = [10]$
   (a) If A is a bounded non-empty set of rational numbers, then $\sup A$ is rational.
   (b) Any sequence that converges to a limit must be bounded.
   (c) Any sequence that is not bounded above must diverge to $+\infty$.
   (d) If $x_n y_n \to 0$, then at least one of $\{x_n\}$ and $\{y_n\}$ must converge to 0.
   (e) Every bounded sequence must have a subsequence that converges.
   (f) An infinite series $\sum x_n$, with $|x_n| < 1/n$ for all n, must converge.
   (g) If $x_n \geq 0, y_n \geq 0$ and $\sum(x_n + y_n)$ converges, then both $\sum x_n$ and $\sum y_n$ converge.
   (h) If f is a function such that f^2 is continuous at a, then f is continuous at a.

GROUP B: Attempt ANY THREE ONLY (Each Carries 5 Marks)

3. Suppose B and C are two non-empty subsets of $\mathbb{R}$, both bounded above. Show that, if $\sup B < \sup C$, then there must be some $c \in C$, which is an upper bound for B.

4. Let $\{x_n\}$ and $\{y_n\}$ be two real sequences. Prove that, if $x_n \to 0$ and $\{y_n\}$ bounded, then $x_n y_n \to 0$.

5. Let Let $\{x_n\}$ and $\{y_n\}$ be two real sequences converging to x and y respectively. Prove that, if $x < y$, then the sequence $z_n = \min\{x_n, y_n\}$ must converge to x.

6. Let $\{x_n\}$ be a sequence of non-negative real numbers. Prove that, if the series $\sum x_n$ converges, then the series $\sum x_n^2$ also converges.

7. Let $\{s_n\}$ denote the sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} n/2^n$.
   (a) Show that, for all $n \geq 2$, $s_n = \frac{1}{2}\left[s_n - n/2^n + (1 + 2^{-1} + \cdots + 2^{-(n-1)})\right]$.
   (b) From (a), deduce that the series $\sum\limits_{n=1}^{\infty} n/2^n$ converges and find the sum.

8. Let f be a continuous function on an interval I. Prove that, if f is not a constant function, then it is not possible for $f(x)$ to be rational for every $x \in I$.

GROUP C: Attempt ANY THREE ONLY (Each Carries 5 Marks)

9. For each of the following sequences, decide, stating reasons, whether it converges or not and. in case it does, identify the limit.
   (a) $\left\{\dfrac{9^n + (-5)^n + 10n^{20}}{9^{n+1} + 7^{n+5}}\right\}$ (b) $\left\{\dfrac{\sqrt{n}+3}{\sqrt{3n+10}-2}\cos^2(n\pi/2)\right\}$ (c) $\{\sqrt{4n^2+3n} - 2n\}$

10. For an irrational number $x \in (0,1)$, let $x = .a_1a_2a_3\ldots$ denote its decimal expansion.
   (a) Consider the sequence $\{x_n\}$ where x_n is the number with the terminating decimal expansion $x_n = .a_1 \ldots a_n$. Show that $\{x_n\}$ is an increasing sequence of rationals.
   (b) Show that $0 < x - x_n \leq (10)^{-n}$ and hence conclude that $x_n \to x$.

11. For each of the following infinite series, decide, stating reasons, whether the series converges or not.
   (a) $\sum\limits_{n=1}^{\infty} \dfrac{2^n n!}{n^n}$ (b) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{3n+1}}{n\log(5n+1)}$ (c) $\sum\limits_{n=1}^{\infty} n^7 \cdot (5)^{-n^2/5}$

12. Assume that the exponents 2^r have been defined for every rational r and assume that they satisfy the usual properties of exponents. For every real x, denote $S(x)$ to be the set $S(x) = \{2^r : r \text{ rational}, r \leq x\}$.
   (a) Show that $S(x)$ is a non-empty set which is bounded above and that, for any rational r, $2^r = \sup S(r)$.
   (b) Defining $2^x = \sup S(x)$ for every real x, prove that $2^{x+y} = 2^x \cdot 2^y$.

13. Find if the following limits exist and if so, identify the limits. Your answers must be justified.
   (a) $\lim\limits_{x\to a} \dfrac{\cos 3x - \cos 3a}{x - a}$ (b) $\lim\limits_{x\to 0+} x \log x$ (c) $\lim\limits_{x\to\pi/2} \dfrac{\sin|x - \pi/2|}{x - \pi/2}$

14. Consider the function $f(x) = [1/x] =$ integer part of $1/x$ for $x \in (0,2)$. Identify, with justification, the points $a \in (0,2)$ where f is continuous and the points $a \in (0,2)$ where f is not continuous.