Indian Association for the Cultivation of Science
(Deemed to be University under the *de novo* category)
BS-MS Program
Final Examination-2022 (Autumn Semester-I)

Subject: Calculus of One Variable   *Subject Code(s): MCS 1101A*

Full marks: 50   *Time allotted: 3 hrs*

Answer **all** questions. Each question carries 5 marks.

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1. Prove that if f and g are differentiable at a then the functions $\max(f,g)$ and $\min(f,g)$ are differentiable at a, provided that $f(a) \neq g(a)$.

2. Suppose that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Prove Leibniz's formula:

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a).$$

3. Use Rolle's theorem to show that regardless of the value of b, there is at most one point $x \in [-1, 1]$ for which $x^3 - 3x + b = 0$.

4. Given n real numbers $a_1, a_2, \ldots, a_n$, let

$$f(x) = \sum_{i=1}^{n} (x - a_i)^2.$$

   Show that the least value of $f(x)$ is attained when x is the arithmetic mean of $a_1, a_2, \ldots, a_n$.

5. For a positive integer n, find the value of $\int_0^n [x]\, dx$.

6. Find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = 1 - x^2$.

7. Suppose that f is a one-one and continuous function and that f^{-1} has a derivative which is nowhere 0. Prove that f is differentiable.

8. Prove that if $f''(a)$ exists, then

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

9. Suppose $f(x) = \frac{\sin x}{x}$, $x \neq 0$, and $f(0) = 1$. Find $f^{(k)}(0)$.

10. Suppose z and w are two complex numbers. Prove that

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2),$$

and interpret this result geometrically.