Indian Association for Cultivation of Science
B. S. - First Year
End-Semester Examination 2023, Semester I
Subject: Calculus of Single Variables
Subject Code: MAT 1101 A

Full Marks:—50 **Time Allotted:—3 hours**

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1. *The paper carries 60 Marks. You can answer as many questions as you wish. If you score X, your final score will be* $\frac{\min\{X,50\}}{\ }$.

2. *You are free to use any theorem that is taught to you by me. However you must state them at least once in your answer-scipt because they carry credits.*

3. $\mathbb{R}$ *will denote the set of all real numbers and* $\mathbb{Q}$ *the set of all rational numbers.*

4. *Part-wise scores for each question is shown at the end of the question.*

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(1) (a) Compute (giving a brief justification)

$$\lim_{n \to \infty}(n!)^{\frac{1}{n}}.$$

(b) Giving brief justifications, show that for every real number x, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is absolutely convergent.

[4 + 4]

(2) Using the formula

$$e^{ix} = \cos x + i \sin x, \quad x \in \mathbb{R}$$

and giving brief justifications, for real numbers x and y prove the following:

(a)

$$\sin^2 x + \cos^2 x = 1.$$

(b)

$$\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y.$$

(c)

$$\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y.$$

[2 + 4 + 4]

(3) For positive real numbers x and y, show that

$$\sup\{r \in \mathbb{Q} : r \leq x\} \cdot \sup\{s \in \mathbb{Q} : s \leq y\} = \sup\{t \in \mathbb{Q} : t \leq x \cdot y\}.$$

[14]

(4) Let

$$f(x) = e^{\frac{x+1}{x-1}}, \; x \in \mathbb{R} \setminus \{1\}.$$

Show that there is ~~a~~ no continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \neq 1$. (~~You are required to~~ briefly prove that g ~~is continuous at all real number~~ x.)

[10]

(5) Consider the function $f(x) = |x^3|$, $-\infty < x < \infty$.

(a) Is the function f differentiable at all real numbers x? If yes, find its derivative at all x. If not, determine (giving a brief justification) the set of all real numbers x at which f is not differentiable

(b) Does

$$\frac{d^n f}{dx^n}$$

exist at all real number x and for all positive integer n? Justify your answer.

[9 + 2]

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For a real number x, define

$$F(x) = \int_0^x f(y)dy.$$

Show that F is differentiable and $\frac{dF}{dx} \equiv f$ on $\mathbb{R}$.

[7]