Indian Association for Cultivation of Science
B. S. - First Year
Mid-Semester Examination 2024
Semester II
Subject: Linear Algebra and Multivariate Calculus
Subject Code: MCS 1201 A

Full Marks:—25 Time Allotted:—2 hours

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1. The paper carries 60 Marks. You can answer as many questions as you wish. If you score X, your final score will be $\frac{\min\{X,50\}}{2}$.

2. You are free to use any theorem that is taught to you by me. However you must state them at least once in your answer-script because they carry credits.

3. Partwise scores for each question is shown at the end of the question.

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(1) Let $\{z_n = (x_n, y_n)\}$ be a bounded sequence in $\mathbb{R}^2$, i.e. $\exists M > 0$ such that $|z_n| \leq M$ for all n. Show that there exists a convergent subsequence $\{z_{n_k} = (x_{n_k}, y_{n_k})\}$ of the sequence $\{z_n\}$. [7]

(2) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $a \in \mathbb{R}^n$. Show that the function f is continuous at a if and only if whenever a sequence $\{x_k\}$ in $\mathbb{R}^n$ converges to a, $f(x_k)$ converges to $f(a)$ as $k \to \infty$. [10]

(3) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\exists M, \alpha > 0$ such that for every $\overline{x}, \overline{y} \in \mathbb{R}^n$.

$$|f(\overline{x}) - f(\overline{y})| \leq M|\overline{x} - \overline{y}|^\alpha.$$

Show that f is uniformly continuous on $\mathbb{R}^n$. [5]

(4) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\overline{a} \in \mathbb{R}^n$. For every $\delta > 0$, define

$$O_f(\delta) = \sup\{|f(\overline{x}) - f(\overline{a})| : |\overline{x} - \overline{a}| < \delta\},$$

and

$$O_f(\overline{a}) = \inf\{O_f(\delta) : \delta > 0\}.$$

Show that f is continuous at $\overline{a}$ if and only if $O_f(\overline{a}) = 0$. [10]

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that $\frac{\partial f}{\partial x}(0, x)$ exists whenever $x \neq 0$ and $\lim_{x \to 0} \frac{\partial f}{\partial x}(0, x)$ exists and equals a. Show that $\frac{\partial f}{\partial x}(0, 0)$ exists and equals a.

[10]

(6) Let $f : [0, \infty) \to [0, \infty)$ be a continuous function.
  (a) State the Cauchy criterion for the existence of the improper Riemann integral $\int_0^\infty f(x)dx$.
  (b) Show that $\int_0^\infty |\sin x| dx$ does not exist.

[2 + 6]

(7) Let $C \subseteq \mathbb{R}^n$ be convex and $\bar{x}, \bar{y} \in C$. Let $f : C \to \mathbb{R}$ be continuous. ~~For~~ each $a \in \mathbb{R}$, lying between $f(\bar{x})$ and $f(\bar{y})$ show that $\exists\ \bar{z} \in C$ s.t $f(\bar{z}) = a$ [10]