HW3

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2024-09-30

Problem 1

(a)

$$CI = \hat{A} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{d_1}$$
$$\alpha = 0.2, n = 5, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$d_1 = x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{8}$$
$$\hat{A} = \frac{17}{8}, t_{\frac{0.2}{2},5-2} = t_{0.1,3} = 1.638, \hat{\sigma} = \sqrt{\frac{5}{24}}$$
$$CI = \frac{17}{8} \pm 1.638\sqrt{\frac{5}{24}}\sqrt{\frac{3}{8}} = [1.667, 2.583]$$

(b)

$$CI = \hat{B} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{d_1}$$
$$\alpha = 0.2, n = 5, p = 2$$
$$x_{new} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$d_1 = x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{8}$$
$$\hat{B} = \frac{13}{8}, t_{\frac{0.2}{2},5-2} = t_{0.1,3} = 1.638, \hat{\sigma} = \sqrt{\frac{5}{24}}$$
$$CI = \frac{13}{8} \pm 1.638\sqrt{\frac{5}{24}}\sqrt{\frac{3}{8}} = [1.167, 2.083]$$

(c)

$$CI = \hat{A} + \hat{B} \pm t_{\frac{\alpha}{2}, n-p} \hat{\sigma} \sqrt{d_1}$$
$$\alpha = 0.2, n = 5, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$d_1 = x_{new}^T (X^T X)^{-1} x_{new} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}$$
$$\hat{A} = \frac{17}{8}, \hat{B} = \frac{13}{8}, t_{\frac{0.2}{2}, 5-2} = t_{0.1,3} = 1.638, \hat{\sigma} = \sqrt{\frac{5}{24}}$$
$$CI = \frac{17}{8} + \frac{13}{8} \pm 1.638 \sqrt{\frac{5}{24}} \sqrt{\frac{1}{2}} = [3.221, 4.279]$$

(d)

$$CI = \hat{A} - \hat{B} \pm t_{\frac{\alpha}{2}, n-p} \hat{\sigma} \sqrt{d_1}$$
$$\alpha = 0.2, n = 5, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$d_1 = x_{new}^T (X^T X)^{-1} x_{new} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$$
$$\hat{A} = \frac{17}{8}, \hat{B} = \frac{13}{8}, t_{\frac{0.2}{2}, 5-2} = t_{0.1,3} = 1.638, \hat{\sigma} = \sqrt{\frac{5}{24}}$$
$$CI = \frac{17}{8} - \frac{13}{8} \pm 1.638 \sqrt{\frac{5}{24}} \sqrt{1} = [-0.248, 1.248]$$

(e)

$$CI = \hat{A} \pm t_{\frac{\alpha}{2}, n-p} \hat{\sigma} \sqrt{1 + x_{new}^T (X^T X)^{-1} x_{new}}$$
$$\alpha = 0.2, n = 5, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$x_{new}^T (X^T X)^{-1} x_{new} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{8}$$
$$\hat{A} = \frac{17}{8}, t_{\frac{0.2}{2}, 5-2} = t_{0.1,3} = 1.638, \hat{\sigma} = \sqrt{\frac{5}{24}}$$
$$CI = \frac{17}{8} \pm 1.638 \sqrt{\frac{5}{24}} \sqrt{1 + \frac{3}{8}} = [1.248, 3.002]$$

(f)

```r
y_1 <- c(2, 1, 4, 2, 2)
a_1 <- c(1, 0, 1, 1, 0)
b_1 <- c(0, 1, 1, 0, 1)

mod_1 <- lm(y_1 ~ a_1 + b_1 -1)
predict(mod_1, data.frame(a_1 = 1, b_1 = 0), interval = 'confidence', level = 0.8)
```

```
##     fit      lwr      upr
## 1 2.125 1.667237 2.582763
```

```r
predict(mod_1, data.frame(a_1 = 0, b_1 = 1), interval = 'confidence', level = 0.8)
```

```
##     fit      lwr      upr
## 1 1.625 1.167237 2.082763
```

```r
predict(mod_1, data.frame(a_1 = 1, b_1 = 1), interval = 'confidence', level = 0.8)
```

```
##    fit     lwr     upr
## 1 3.75 3.22142 4.27858
```

```r
predict(mod_1, data.frame(a_1 = 1, b_1 = -1), interval = 'confidence', level = 0.8)
```

```
##   fit        lwr      upr
## 1 0.5 -0.2475246 1.247525
```

```r
predict(mod_1, data.frame(a_1 = 1, b_1 = 0), interval = 'prediction', level = 0.8)
```

```
##     fit     lwr     upr
## 1 2.125 1.24845 3.00155
```

Problem 2

(a)

$$CI = \hat{A} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{d_1}$$

$$\alpha = 0.1, n = 6, p = 2$$

$$x_{new} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$d_1 = x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$

$$\hat{A} = \frac{7}{3}, t_{\frac{0.1}{2},6-2} = t_{0.05,4} = 2.132, \hat{\sigma} = \sqrt{\frac{5}{12}}$$

$$CI = \frac{7}{3} \pm 2.132\sqrt{\frac{5}{12}}\sqrt{\frac{1}{3}} = [1.539, 3.128]$$

(b)

$$CI = \hat{B} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{d_1}$$
$$\alpha = 0.1, n = 6, p = 2$$
$$x_{new} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$d_1 = x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3}$$
$$\hat{B} = \frac{11}{6}, t_{\frac{0.1}{2},6-2} = t_{0.05,4} = 2.132, \hat{\sigma} = \sqrt{\frac{5}{12}}$$
$$CI = \frac{11}{6} \pm 2.132\sqrt{\frac{5}{12}}\sqrt{\frac{1}{3}} = [1.039, 2.628]$$

(c)

$$CI = \hat{A} + \hat{B} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{d_1}$$
$$\alpha = 0.1, n = 6, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$d_1 = x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3}$$
$$\hat{A} = \frac{7}{3}, \hat{B} = \frac{11}{6}, t_{\frac{0.1}{2},6-2} = t_{0.05,4} = 2.132, \hat{\sigma} = \sqrt{\frac{5}{12}}$$
$$CI = \frac{7}{3} + \frac{11}{6} \pm 2.132\sqrt{\frac{5}{12}}\sqrt{\frac{1}{3}} = [3.372, 4.961]$$

(d)

$$CI = \hat{A} - \hat{B} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{d_1}$$
$$\alpha = 0.1, n = 6, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$d_1 = x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$$
$$\hat{A} = \frac{7}{3}, \hat{B} = \frac{11}{6}, t_{\frac{0.1}{2},6-2} = t_{0.05,4} = 2.132, \hat{\sigma} = \sqrt{\frac{5}{12}}$$
$$CI = \frac{7}{3} - \frac{11}{6} \pm 2.132\sqrt{\frac{5}{12}}\sqrt{1} = [-0.876, 1.876]$$

(e)

$$CI = \hat{A} \pm t_{\frac{\alpha}{2},n-p}\hat{\sigma}\sqrt{1 + x_{new}^T(X^TX)^{-1}x_{new}}$$
$$\alpha = 0.1, n = 6, p = 2$$
$$x_{new} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$x_{new}^T(X^TX)^{-1}x_{new} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}$$
$$\hat{A} = \frac{7}{3}, t_{\frac{0.1}{2},6-2} = t_{0.05,4} = 2.132, \hat{\sigma} = \sqrt{\frac{5}{12}}$$
$$CI = \frac{7}{3} \pm 2.132\sqrt{\frac{5}{12}}\sqrt{1 + \frac{1}{3}} = [0.744, 3.922]$$

(f)

```
y_2 <- c(2, 1, 4, 2, 2, 5)
a_2 <- c(1, 0, 1, 1, 0, 1)
b_2 <- c(0, 1, 1, 0, 1, 1)

mod_2 <- lm(y_2 ~ a_2 + b_2 -1)
predict(mod_2, data.frame(a_2 = 1, b_2 = 0), interval = 'confidence', level = 0.9)
```

```
##        fit      lwr      upr
## 1 2.333333 1.538841 3.127826
```

```
predict(mod_2, data.frame(a_2 = 0, b_2 = 1), interval = 'confidence', level = 0.9)
```

```
##        fit      lwr      upr
## 1 1.833333 1.038841 2.627826
```

```
predict(mod_2, data.frame(a_2 = 1, b_2 = 1), interval = 'confidence', level = 0.9)
```

```
##        fit      lwr      upr
## 1 4.166667 3.372174 4.961159
```

```
predict(mod_2, data.frame(a_2 = 1, b_2 = -1), interval = 'confidence', level = 0.9)
```

```
##   fit        lwr      upr
## 1 0.5 -0.8761012 1.876101
```

```
predict(mod_2, data.frame(a_2 = 1, b_2 = 0), interval = 'prediction', level = 0.9)
```

```
##        fit       lwr      upr
## 1 2.333333 0.7443486 3.922318
```