HW #2

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Problem 1

(a)

n is the number of measurements taken, $Y_{n\times 1}$ is the vector of observed measurements, and $X_{n\times p}$ is the matrix of 1's and 0's corresponding to whether or not each ball was weighed for the measurement.

$$n = 5, Y_{n\times 1} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \end{pmatrix} \quad X_{n\times p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$(X^T X) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$(X^T X)^{-1} = \frac{1}{3 \cdot 3 - 1 \cdot 1} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

$$X^T Y = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

(c)

$$\hat{\beta} = (X^T X)^{-1} X^T Y = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{17}{8} \\ \frac{13}{8} \end{pmatrix}$$

(d)

Yes it is the same.

(e)

$$H = X(X^TX)^{-1}X^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

(f)

$$\hat{\sigma^2} = \frac{RSS}{n-p}$$

$$\hat{Y} = HY = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{17}{8} \\ \frac{13}{8} \\ \frac{15}{4} \\ \frac{17}{8} \\ \frac{13}{8} \end{pmatrix}$$

$$RSS = \sum_i (\epsilon_i)^2 = (\epsilon_1)^2 + (\epsilon_2)^2 + (\epsilon_3)^2 + (\epsilon_4)^2 + (\epsilon_5)^2$$

$$= (\frac{17}{8} - 2)^2 + (\frac{13}{8} - 1)^2 + (\frac{15}{4} - 4)^2 + (\frac{17}{8} - 2)^2 + (\frac{13}{8} - 2)^2 = \frac{5}{8}$$

$$\hat{\sigma^2} = \frac{\frac{5}{8}}{5-2} = \frac{5}{24}$$

$$\hat{\sigma} = \sqrt{\frac{5}{24}} \approx 0.456$$

```
##
## Call:
## lm(formula = y_1 ~ a_1 + b_1 - 1)
##
## Residuals:
##      1      2      3      4      5
## -0.125 -0.625  0.250 -0.125  0.375
##
## Coefficients:
##     Estimate Std. Error t value Pr(>|t|)
## a_1   2.1250     0.2795   7.603  0.00472 **
## b_1   1.6250     0.2795   5.814  0.01013 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4564 on 3 degrees of freedom
## Multiple R-squared:  0.9784, Adjusted R-squared:  0.9641
## F-statistic:  68.1 on 2 and 3 DF,  p-value: 0.003164
```

Problem 2