HW #2

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Problem 1

(a)

n is the number of measurements taken, $Y_{n\times 1}$ is the vector of observed measurements, and $X_{n\times p}$ is the matrix of 1's and 0's corresponding to whether or not each ball was weighed for the measurement.

$$n = 5, Y_{n\times 1} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \end{pmatrix}, X_{n\times p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$(X^T X) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$(X^T X)^{-1} = \frac{1}{3 \cdot 3 - 1 \cdot 1} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

$$X^T Y = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

(c)

$$\hat{\beta} = (X^T X)^{-1} X^T Y = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{17}{8} \\ \frac{13}{8} \end{pmatrix}$$

(d)

Yes it is the same.

(e)

$$H = X(X^T X)^{-1} X^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

(f)

$$\hat{\sigma^2} = \frac{RSS}{n-p}$$

$$\hat{Y} = HY = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{17}{8} \\ \frac{13}{8} \\ \frac{15}{4} \\ \frac{17}{8} \\ \frac{13}{8} \end{pmatrix}$$

$$RSS = \sum_i (\epsilon_i)^2 = (\epsilon_1)^2 + (\epsilon_2)^2 + (\epsilon_3)^2 + (\epsilon_4)^2 + (\epsilon_5)^2$$

$$= (\frac{17}{8} - 2)^2 + (\frac{13}{8} - 1)^2 + (\frac{15}{4} - 4)^2 + (\frac{17}{8} - 2)^2 + (\frac{13}{8} - 2)^2 = \frac{5}{8}$$

$$\hat{\sigma^2} = \frac{\frac{5}{8}}{5-2} = \frac{5}{24}$$

$$\hat{\sigma} = \sqrt{\frac{5}{24}} \approx 0.456$$

```
##
## Call:
## lm(formula = y_1 ~ a_1 + b_1 - 1)
##
## Residuals:
##      1      2      3      4      5
## -0.125 -0.625  0.250 -0.125  0.375
##
## Coefficients:
##     Estimate Std. Error t value Pr(>|t|)
## a_1   2.1250     0.2795   7.603  0.00472 **
## b_1   1.6250     0.2795   5.814  0.01013 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4564 on 3 degrees of freedom
## Multiple R-squared:  0.9784, Adjusted R-squared:  0.9641
## F-statistic:  68.1 on 2 and 3 DF,  p-value: 0.003164
```

Problem 2

(a)

$$n = 6, Y_{n\times 1} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \\ 5 \end{pmatrix}, X_{n\times p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(b)

$$(X^TX) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$(X^TX)^{-1} = \frac{1}{4 \cdot 4 - 2 \cdot 2} \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$X^TY = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

(c)

$$\hat{\beta} = (X^TX)^{-1}X^TY = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 13 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{11}{6} \end{pmatrix}$$

(d)

Yes it is the same.

(e)

$$H = X(X^TX)^{-1}X^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

(f)

$$\hat{\sigma^2} = \frac{RSS}{n-p}$$

$$\hat{Y} = HY = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{11}{6} \\ \frac{25}{6} \\ \frac{7}{3} \\ \frac{11}{6} \\ \frac{25}{6} \end{pmatrix}$$

$$RSS = \sum_i (\epsilon_i)^2 = (\epsilon_1)^2 + (\epsilon_2)^2 + (\epsilon_3)^2 + (\epsilon_4)^2 + (\epsilon_5)^2 + (\epsilon_6)^2$$

$$= (\frac{7}{3} - 2)^2 + (\frac{11}{6} - 1)^2 + (\frac{25}{6} - 4)^2 + (\frac{7}{3} - 2)^2 + (\frac{11}{6} - 2)^2 + (\frac{25}{6} - 5)^2 = \frac{5}{3}$$

$$\hat{\sigma^2} = \frac{\frac{5}{3}}{6-2} = \frac{5}{12}$$

$$\hat{\sigma} = \sqrt{\frac{5}{12}} \approx 0.645$$

```
##
## Call:
## lm(formula = y_2 ~ a_2 + b_2 - 1)
##
## Residuals:
##       1       2       3       4       5       6
## -0.3333 -0.8333 -0.1667 -0.3333  0.1667  0.8333
##
## Coefficients:
##     Estimate Std. Error t value Pr(>|t|)
## a_2   2.3333     0.3727   6.261  0.00332 **
## b_2   1.8333     0.3727   4.919  0.00793 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6455 on 4 degrees of freedom
## Multiple R-squared:  0.9691, Adjusted R-squared:  0.9537
## F-statistic:  62.8 on 2 and 4 DF,  p-value: 0.0009526
```

Problem 3

(a)

$$n = 7, Y_{n\times 1} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}, X_{n\times p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(b)

$$(X^T X) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$(X^T X)^{-1} = \frac{1}{5\cdot 5 - 3\cdot 3} \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{16} & -\frac{3}{16} \\ -\frac{3}{16} & \frac{5}{16} \end{pmatrix}$$

$$X^T Y = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ 17 \end{pmatrix}$$

(c)

$$\hat{\beta} = (X^TX)^{-1}X^TY = \begin{pmatrix} \frac{5}{16} & -\frac{3}{16} \\ -\frac{3}{16} & \frac{5}{16} \end{pmatrix} \begin{pmatrix} 18 \\ 17 \end{pmatrix} = \begin{pmatrix} \frac{39}{16} \\ \frac{31}{16} \end{pmatrix}$$

(d)

Yes it is the same.

(e)

$$H = X(X^TX)^{-1}X^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{16} & -\frac{3}{16} \\ -\frac{3}{16} & \frac{5}{16} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{1}{8} \\ -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{1}{8} \\ -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(f)

$$\hat{\sigma^2} = \frac{RSS}{n-p}$$

$$\hat{Y} = HY = \begin{pmatrix} \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{1}{8} \\ -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{5}{16} & -\frac{3}{16} & \frac{1}{8} & \frac{1}{8} \\ -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & -\frac{3}{16} & \frac{5}{16} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 2 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{39}{16} \\ \frac{31}{16} \\ \frac{35}{8} \\ \frac{39}{16} \\ \frac{31}{16} \\ \frac{35}{8} \\ \frac{35}{8} \end{pmatrix}$$

$$RSS = \sum_i (\epsilon_i)^2 = (\epsilon_1)^2 + (\epsilon_2)^2 + (\epsilon_3)^2 + (\epsilon_4)^2 + (\epsilon_5)^2 + (\epsilon_6)^2 + (\epsilon_7)^2$$

$$= (\frac{39}{16} - 2)^2 + (\frac{31}{16} - 1)^2 + (\frac{35}{8} - 4)^2 + (\frac{39}{16} - 2)^2 + (\frac{31}{16} - 2)^2 + (\frac{35}{8} - 5)^2 + (\frac{35}{8} - 5)^2 = \frac{35}{16}$$

$$\hat{\sigma^2} = \frac{\frac{35}{16}}{7-2} = \frac{7}{16}$$

$$\hat{\sigma} = \sqrt{\frac{7}{16}} \approx 0.661$$

```
##
## Call:
## lm(formula = y_3 ~ a_3 + b_3 - 1)
##
## Residuals:
##       1       2       3       4       5       6       7
## -0.4375 -0.9375 -0.3750 -0.4375  0.0625  0.6250  0.6250
##
## Coefficients:
##     Estimate Std. Error t value Pr(>|t|)
## a_3   2.4375     0.3698   6.592  0.00121 **
## b_3   1.9375     0.3698   5.240  0.00335 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Residual standard error: 0.6614 on 5 degrees of freedom
## Multiple R-squared:  0.9723, Adjusted R-squared:  0.9612
## F-statistic: 87.79 on 2 and 5 DF,  p-value: 0.0001276
```

Problem 4

(a)

$$n = 3, Y_{n\times 1} = \begin{pmatrix} 2 \\ 1.5 \\ 4.5 \end{pmatrix}, X_{n\times p} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(b)

$$(X^T X) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(X^T X)^{-1} = \frac{1}{2 \cdot 2 - 1 \cdot 1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$X^T Y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} 6.5 \\ 6 \end{pmatrix}$$

(c)

$$\hat{\beta} = (X^T X)^{-1} X^T Y = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 6.5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{11}{6} \end{pmatrix}$$

(d)

Yes it is the same.

(e)

$$H = X(X^T X)^{-1} X^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

(f)

$$\hat{\sigma^2} = \frac{RSS}{n-p}$$

$$\hat{Y} = HY = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 1.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{11}{6} \\ \frac{25}{6} \end{pmatrix}$$

$$RSS = \sum_i (\epsilon_i)^2 = (\epsilon_1)^2 + (\epsilon_2)^2 + (\epsilon_3)^2$$

$$= (\frac{7}{3} - 2)^2 + (\frac{11}{6} - 1.5)^2 + (\frac{25}{6} - 4.5)^2 = \frac{1}{3}$$

$$\hat{\sigma^2} = \frac{\frac{1}{3}}{3-2} = \frac{1}{3}$$

$$\hat{\sigma} = \sqrt{\frac{1}{3}} \approx 0.577$$

```
##
## Call:
## lm(formula = y_4 ~ a_4 + b_4 - 1)
##
## Residuals:
##       1       2       3
## -0.3333 -0.3333  0.3333
##
## Coefficients:
##     Estimate Std. Error t value Pr(>|t|)
## a_4   2.3333     0.4714   4.950    0.127
## b_4   1.8333     0.4714   3.889    0.160
##
## Residual standard error: 0.5774 on 1 degrees of freedom
## Multiple R-squared:  0.9874, Adjusted R-squared:  0.9623
## F-statistic: 39.25 on 2 and 1 DF,  p-value: 0.1122
```

(g)

The hat matrix is always symmetrical.