Small-Scale Fading II
(and basics about random processes)

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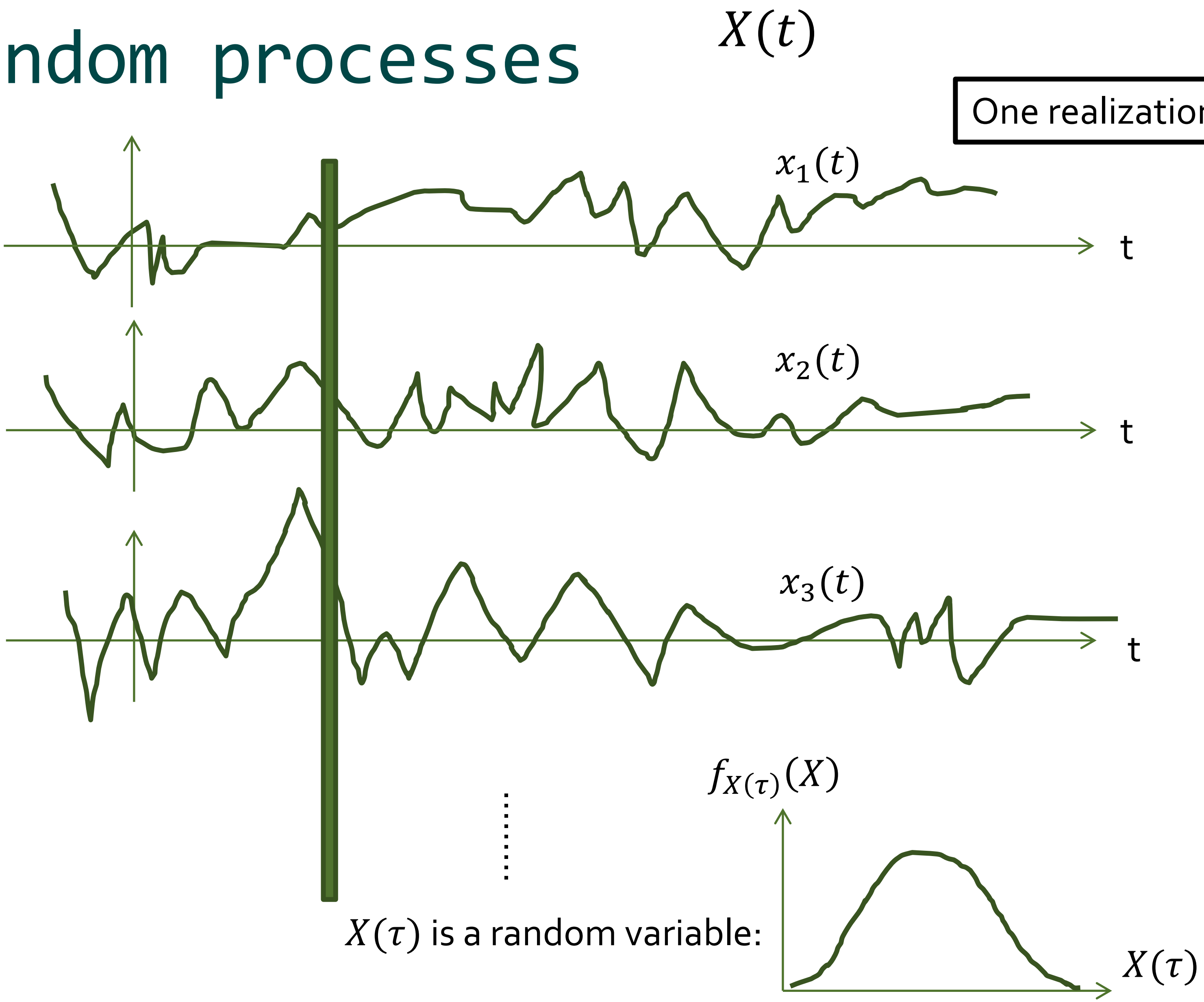
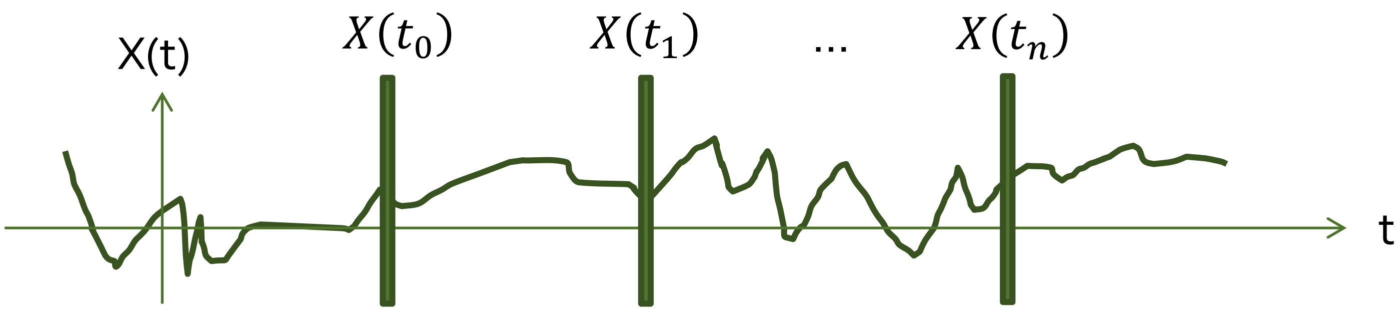
Random processes



Joint CDF for a random process

- **If we sample X(t) at times $t_0, \ldots t_n$, we can have a joint cdf of samples at those times:**

$$P_{X(t_0),\ldots,X(t_n)}(x_0, \ldots, x_n) = p(X(t_0) \leq x_0, X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n)$$



Stationary Random Process (Strict-sense)

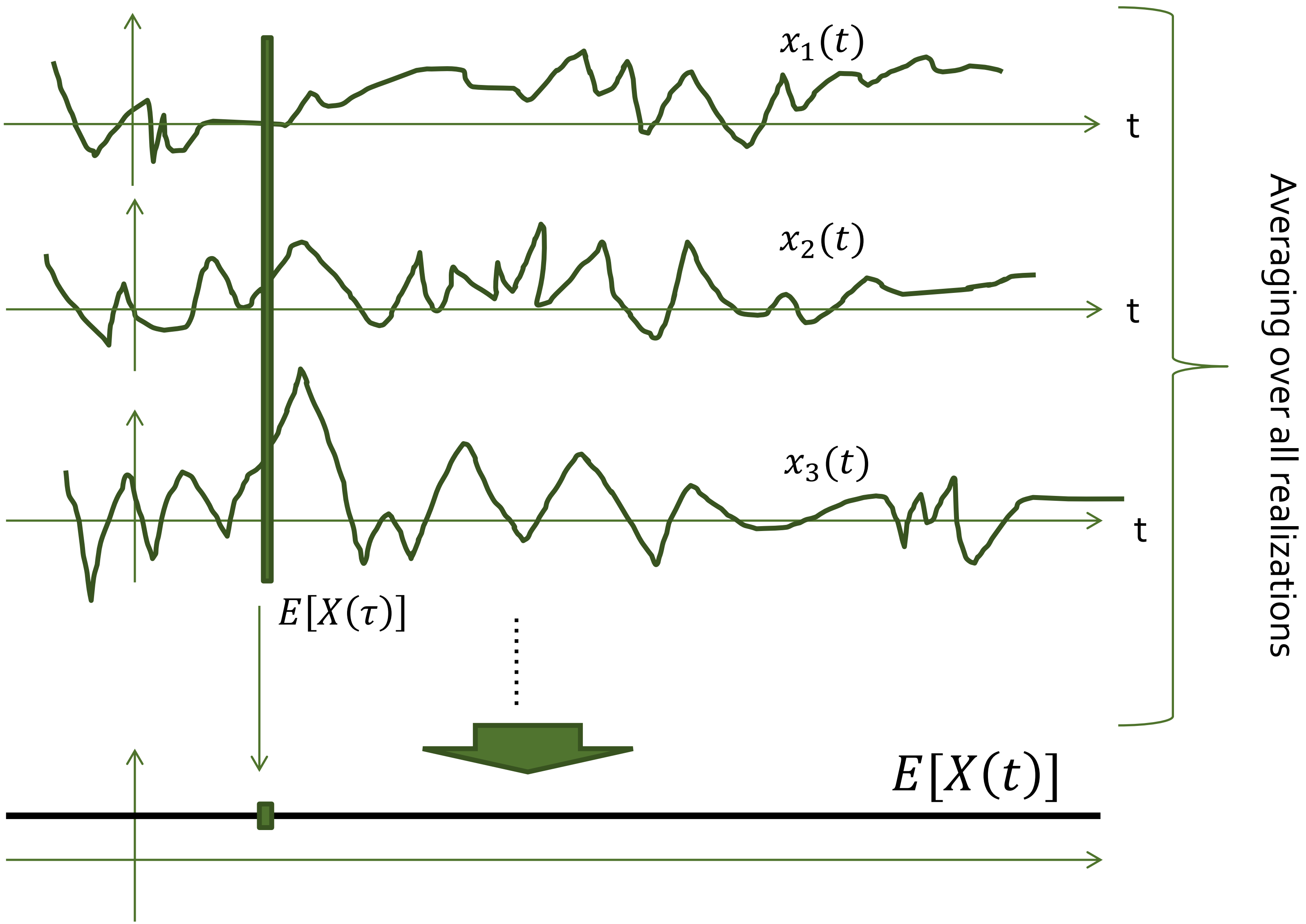
- **A random process X(t) is stationary if for all T, all n, and all sets of sample times $\{\boldsymbol{t_0}, \boldsymbol{t_1}, \dots, \boldsymbol{t_n}\}$ we have:**

$$p(X(t_0) \leq x_0, X(t_1) \leq x_1, \dots, X(t_n) \leq x_n) = p(X(t_0 + T) \leq x_0, X(t_1 + T) \leq x_1, \dots, X(t_n + T) \leq x_n)$$

If time shifts does not matter, then it is stationary

Mean (First Moment)

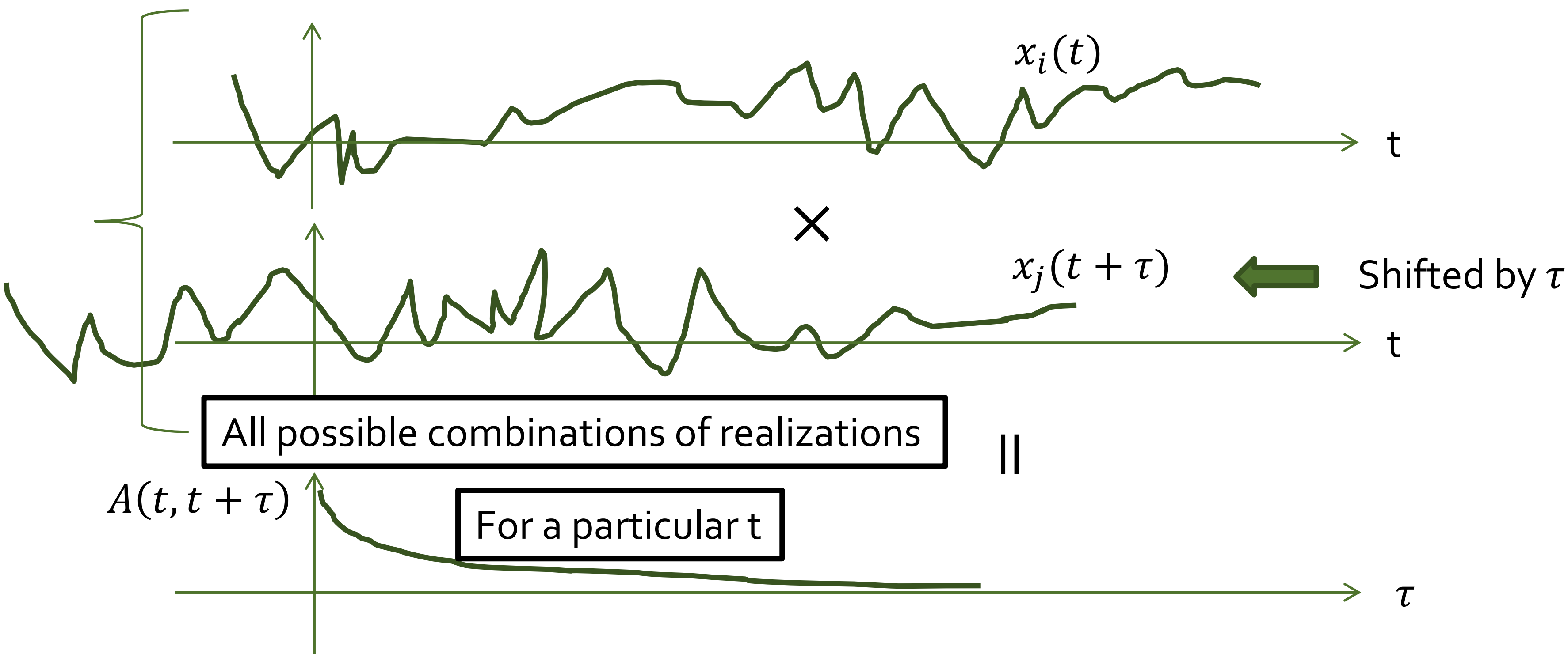
$E[X(t)]$



Autocorrelation (Second Moment)

- **"How similar a random process and a shifted version of itself is"**
- **Autocorrelation of a random process is defined as:**

$$A_X(t, t+\tau) \triangleq E[X(t)X(t+\tau)]$$



For stationary random processes…

- **Mean**

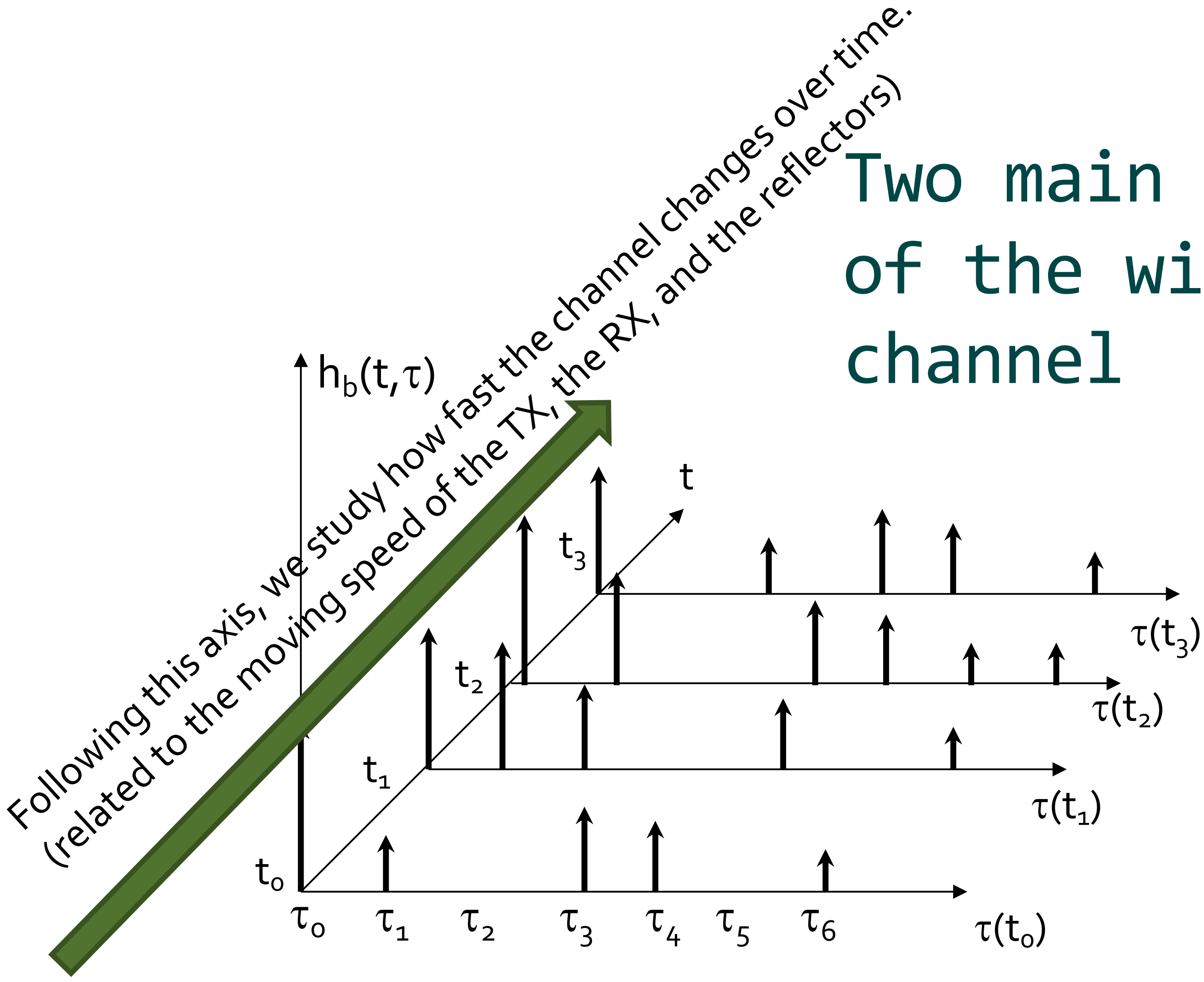
$$E[X(t)] = E[X(t-t)] = E[X(0)] = \mu_X$$

$\boxed{\text{Constant. Does not change with t.}}$

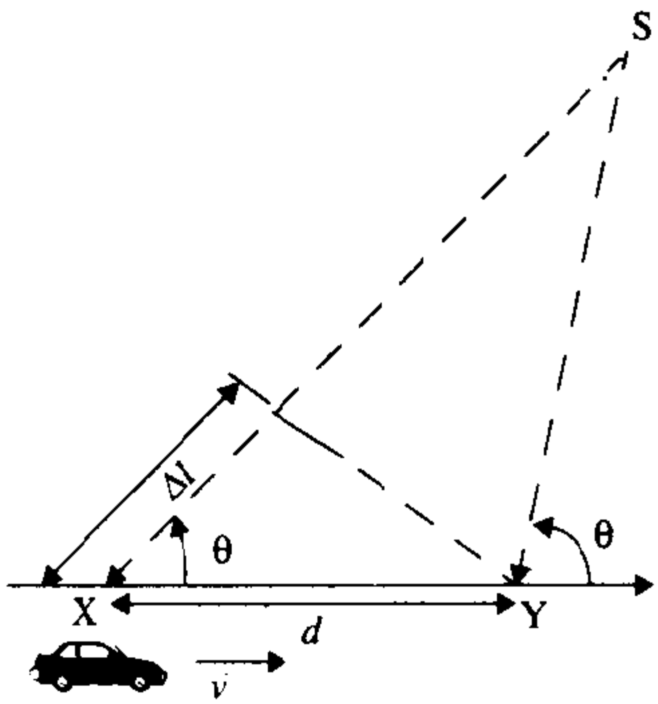
- **Autocorrelation**

$$A_X(t, t+\tau) = E[X(t-t)X(t+\tau-t)] = E[X(0)X(\tau)] \triangleq A_X(\tau)$$

Two main aspects of the wireless channel



Doppler Effect



- Difference in path lengths $\Delta \mathrm{l} = d\ cos\theta = v\Delta\mathrm{t}\ \cos\theta$
- Phase change $\Delta\phi = \frac{2\pi\Delta \mathrm{l}}{\lambda} = \frac{2\pi v\Delta\mathrm{t}}{\lambda}\cos\theta$
- Frequency change, or Doppler shift,

$$f_d = \frac{1}{2\pi}\ \frac{\Delta\phi}{\Delta\mathrm{t}} = \frac{v}{\lambda}cos\theta$$

Example

$$f_d = \frac{1}{2\pi}\frac{\Delta\phi}{\Delta t} = \frac{v}{\lambda}cos\theta$$

- **Consider a transmitter which radiates a sinusoidal carrier frequency of 1850 MHz. For a vehicle moving 60 mph, compute the received carrier frequency if the mobile is moving**

  1. directly toward the transmitter.
  2. directly away from the transmitter
  3. in a direction which is perpendicular to the direction of arrival of the transmitted signal.

- **Ans:**

  - Wavelength=$\lambda = \frac{c}{f_c} = \frac{3\times10^8}{1850\times10^6} = 0.162\ (m)$
  - Vehicle speed $v = 60mph = 26.82\ \left(\frac{m}{s}\right)$

  1. $f_d = \frac{26.82}{0.162}\cos(0) = 160\ (Hz)$
  2. $f_d = \frac{26.82}{0.162}\cos(\pi) = -160\ (Hz)$
  3. Since $\cos\left(\frac{\pi}{2}\right) = 0$, there is no Doppler shift!

Doppler Effect

- If the car (mobile) is moving toward the direction of the arriving wave, the Doppler shift is positive
- Different Doppler shifts if different θ (incoming angle)
- Multi-path: all different angles
- Many Doppler shifts → Doppler spread

Narrow-band Fading Model

- **Sending an unmodulated carrier wave with random phase offset $\boldsymbol{\phi_0}$:**

$$s(t) = Re\{\exp\big(j(2\pi f_c t + \phi_0)\big)\} = \cos(2\pi f_c t + \phi_0)$$

- **Received signal becomes**

$$r(t) = Re\left\{\underbrace{\left[\sum_{n=1}^{N(t)} \alpha_n(t)\exp\big(-j\phi_n(t)\big)\right]}_{\boxed{\text{Sum of many MPC}}}\underbrace{\exp(j2\pi f_c t)}_{\boxed{\text{Carrier with frequency } f_c}}\right\}$$

$$= r_I(t)\cos(2\pi f_c t) - r_Q(t)\sin(2\pi f_c t)$$

$$r(t) = Re\left\{\left[\sum_{n=1}^{N(t)} \alpha_n(t)\exp\left(-j\phi_n(t)\right)\right]\exp(j2\pi f_c t)\right\}$$

$$= r_I(t)\cos(2\pi f_c t) - r_Q(t)\sin(2\pi f_c t)$$

$$r_I(t) = \sum_{n=1}^{N(t)} \alpha_n(t)\cos\phi_n(t) \qquad r_Q(t) = \sum_{n=1}^{N(t)} \alpha_n(t)\sin\phi_n(t)$$

$$\phi_n(t) = 2\pi f_c \tau_n(t) - \phi_{D_n} - \phi_0$$

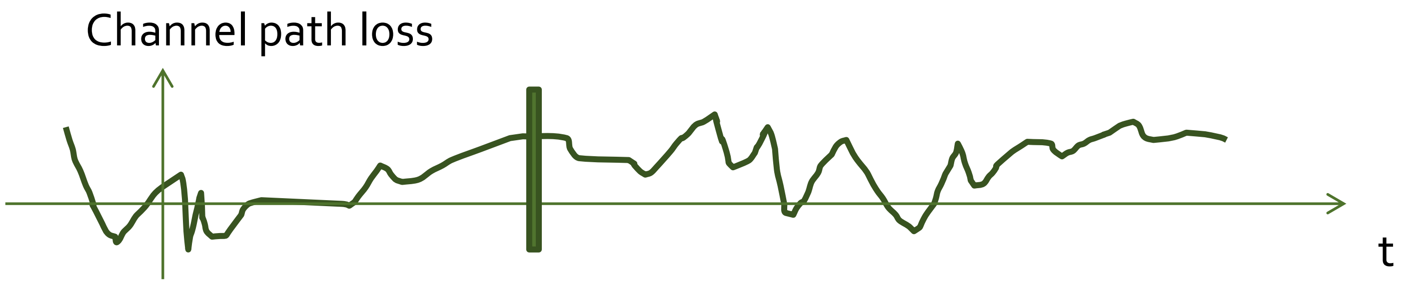
Phase shift due to delay

Doppler Shift

Carrier phase shift (same for all MPC)

Since N(t) is large & we assume $\alpha_n(t)$ and $\phi_n(t)$ are independent for different MPC, we can approximate $r_I(t)$ and $r_Q(t)$ as **jointly Gaussian random processes**.

Amplitude distribution - Rayleigh



- $z(t) = |r(t)| = \sqrt{r_I^2(t) + r_Q^2(t)}$
- $r_I(t)$ **and** $r_Q(t)$ **are both zero-mean Gaussian random process (so at a given time, two Gaussian random variables).**
- **z(t)'s distribution - the amplitude distribution of r(t):**

$$p_Z(z) = \frac{2z}{\overline{P}_r}\exp\left[-\frac{z^2}{\overline{P}_r}\right] = \frac{z}{\sigma^2}\exp\left[-\frac{z^2}{2\sigma^2}\right], z \geq 0$$

This is the famous Rayleigh distribution!

Power distribution: Rayleigh

- **We can obtain the power distribution by making the change of variables $z^2(t) = |r(t)|^2$ to obtain**

$$p_{Z^2}(x) = \frac{1}{\overline{P}_r} \exp\left(-\frac{x}{\overline{P}_r}\right) = \frac{1}{2\sigma^2} \exp\left(-\frac{x}{2\sigma^2}\right), x \geq 0$$

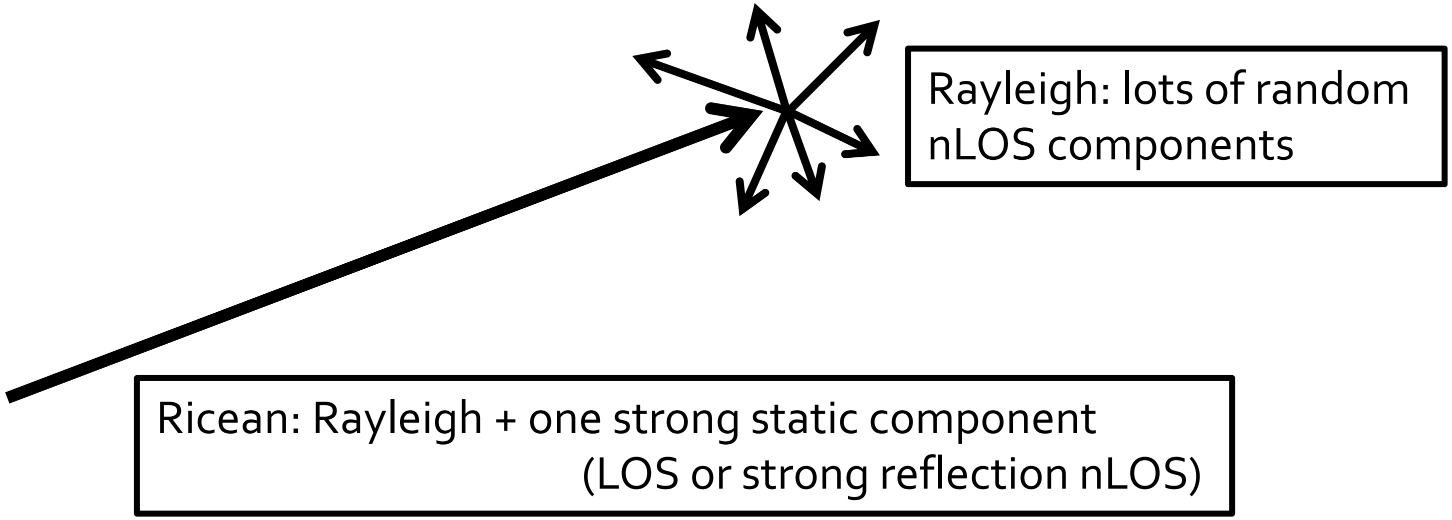
Example: Rayleigh fading

- **Consider a channel with Rayleigh fading (no LOS!) and average received power $\overline{P_r} = 20$ dBm. Find the probability that the received power is below 10 dBm.**
- **We want to find the probability that $Z^2 < 10\ dBm = 10\ mW$.**

$$p(Z^2 < 10) = \int_0^{10} \frac{1}{100} \exp\left(-\frac{x}{100}\right) dx = 0.095$$

With a LOS component – Ricean (or Rician)

- **If the channel has a fixed LOS component then $r_I(t)$ and $r_Q(t)$ are no longer zero-mean variables.**
- **The received signal becomes the superposition of a complex Gaussian component and a LOS component.**



Ricean distribution

- **Amplitude distribution:**

$$p_Z(z) = \frac{z}{\sigma^2} \exp\left[-\frac{z^2 + s^2}{2\sigma^2}\right] I_0\left(\frac{zs}{\sigma^2}\right), \qquad z \geq 0$$

- $2\sigma^2 = \sum_{n,n\neq 0} E[\alpha_n^2]$ **is the average power in the nLOS MPCs.**
- $s^2 = \alpha_0^2$ **is the power in the dominant strong component.**
- $I_0(x)$**: the modified Bessel function of zero-th order.**

Ricean distribution

- **The average power in the Ricean fading is**

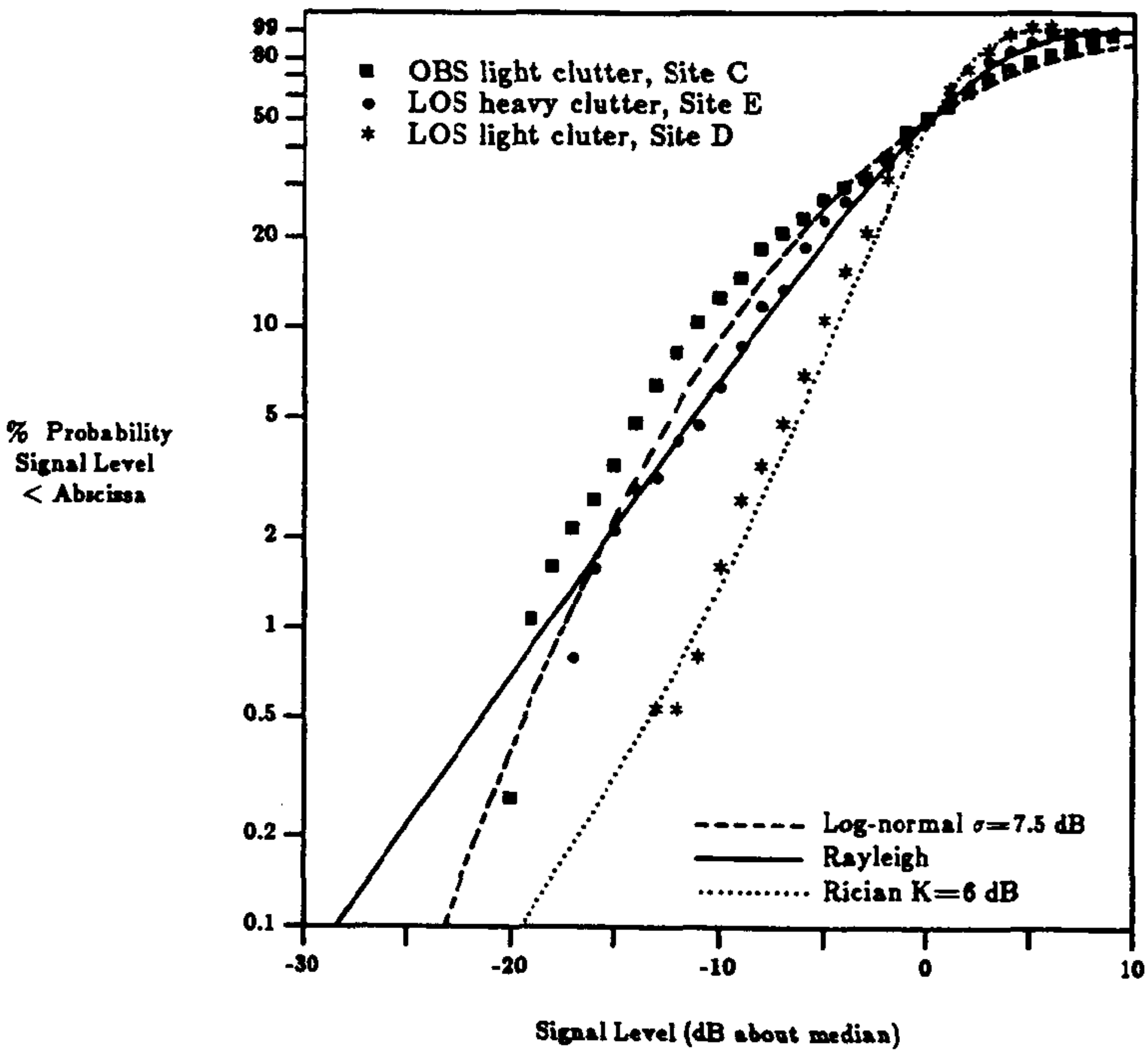
$$\overline{P_r} = \int_0^\infty z^2 p_Z(z)dz = s^2 + 2\sigma^2$$

- **The Ricean distribution is often described in terms of a fading parameter K, defined by**

$$K = \frac{s^2}{2\sigma^2}$$

- **K is the ratio of the power in the dominant component to the power in the other random MPCs.**
  - K=0, then Ricean degenerates to Rayleigh
  - K=∞, then Ricean becomes a non-fading LOS channel.

Ricean, Rayleigh, and Log-Normal



Coherence Time

- **Coherence Time:**
  Coherence time is a statistical measure of the range of time over which the channel can be considered "static".

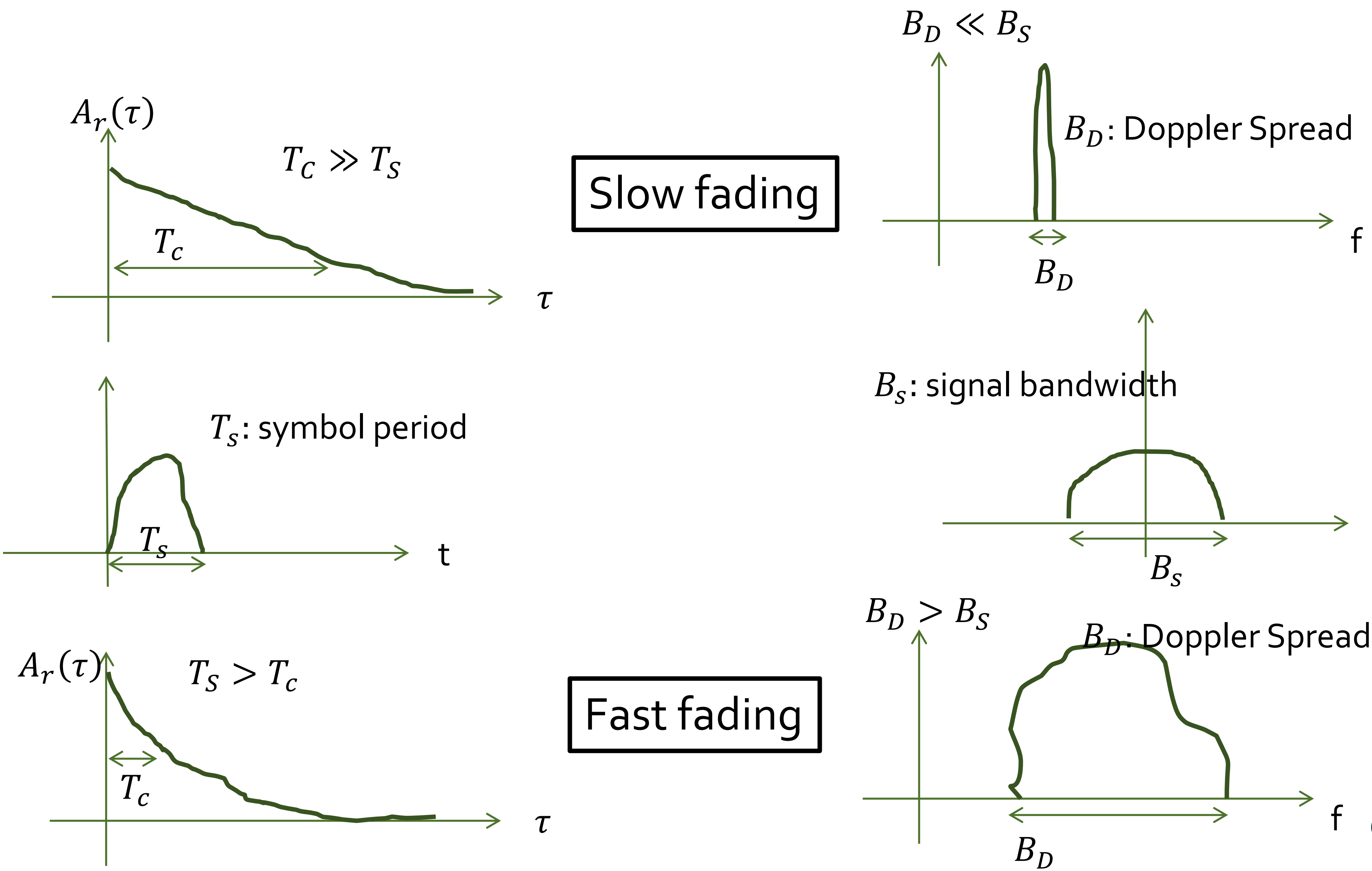
- **90% coherence time:**

$$T_{c,0.9} = argmin_{\tau}\left(\frac{A_r(\tau)}{A_r(0)} < 0.9\right)$$

The first time interval that normalized autocorrelation drops below the threshold.

- **We can define 50% coherence time in a similar way too.**

Fast and slow fading channel



Recap: Two Important / Practical Aspects

- **Fading Distribution**

- **Time Correlation**