Simulating Interferometric Visibilities - I

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This note describes how *uvw* coordinates for interferometer baselines are calculated from a distribution of antennas on the ground.

Context

Radio interferometers sample the celestial brightness distribution on angular scales defined by the separation of the individual receivers in the interferometric array. However, it is not the physical separation of antennas on the ground that defines the angular scales, but rather their separations projected onto the *uv*-plane, a co-ordinate system defined relative to the pointing direction of the array. Consequently, if we want to understand the sampling of an interferometer we need to be able to transform the locations of the antennas on the ground into their projected positions on the *uv*-plane as well as their offset from that plane, the *w*-direction.

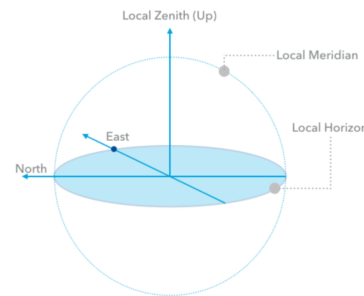

Figure 1: East-North-Up (ENU) co-ordinate system.

Array Description

The easiest and most commonly used way to describe the distribution of antennas in an array on the ground is to use the **East-North-Up (ENU)** co-ordinate system. In this system two perpendicular axes point East and North from a fixed reference point on the ground and the third points vertically up, towards the local zenith, see Figure 1.

In order to calculate our projected baseline co-ordinates in the standard UVW co-ordinate system we need to translate from this local system to a global system that we can reference to the celestial co-ordinate system.

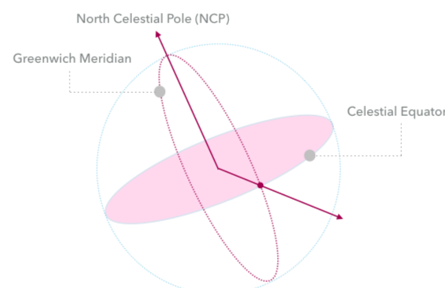

Figure 2: Earth-Centred-Earth-Fixed (ECEF) co-ordinate system.

One such system is known as the **Earth-Centred-Earth-Fixed (ECEF)** co-ordinate system and is defined with the vertical axis pointing not at zenith, but at the North Celestial Pole (NCP). The two other axes are defined in the plane perpendicular to the NCP, with the ECEF-x axis running through the Greenwich meridian[1], see Figure 2.

This means that the vertical axes of the coordinate systems are misaligned by an angle $(90 - \phi)$, where ϕ is the latitude of the array, and the ECEF-x axis is misaligned with the ENU-E axis by $(90 + \theta)$, where θ is the longitude of the array. These differences are shown in Figure 3.

[1] The Greenwich, or prime, meridian is the meridian (a line of longitude) on the Earth at which longitude is defined to be $0°$.

Both ENU and ECEF are Cartesian reference frames and can therefore be translated between by using rotations around three, or in this case two, axes. To go from ENU to ECEF the required translations are,

1. A clockwise rotation around ENU-E by an angle $(90 - \phi)$ in order to align the vertical axes, $R_1[-(\pi/2 - \phi)]$;

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi - \pi/2) & \sin(\phi - \pi/2) \\ 0 & -\sin(\phi - \pi/2) & \cos(\phi - \pi/2) \end{bmatrix} \tag{1}$$

2. A clockwise rotation around the new common vertical (z) axis by an angle $(90 + \theta)$ to align the ENU-E axis with the ECEF-x axis, $R_3[-(\pi/2 + \theta)]$.

$$R_3 = \begin{bmatrix} \cos(\pi/2 + \theta) & -\sin(\pi/2 + \theta) & 0 \\ \sin(\pi/2 + \theta) & \cos(\pi/2 + \theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

We can write the combination of these as a single rotation matrix:

$$R_3 R_1 = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi & \sin\phi \end{bmatrix} \tag{3}$$

and consequently can calculate,

$$ECEF_x = -ENU_x \sin\theta + ENU_y \cos\theta \tag{4}$$

$$ECEF_y = -ENU_x \cos\theta \sin\phi - ENU_y \sin\theta \sin\phi + ENU_z \cos\phi \tag{5}$$

$$ECEF_z = ENU_x \cos\theta \cos\phi + ENU_y \sin\theta \cos\phi + ENU_z \sin\phi \tag{6}$$

A simplification of the general ECEF co-ordinate system is the **XYZ system**. In this system the XYZ_z axis is aligned with the direction of the Earth's rotation axis, XYZ_y is aligned with the direction of East and XYZ_x completes a right-handed co-ordinate system. This system is broadly equivalent to the ECEF system, but instead of being aligned with the Greenwich meridian, it has its x-axis aligned with the *local meridian* instead, see Figure 4.

To convert from ENU to XYZ we use the same rotation matrix as above (Equation 3), but with $\theta = 0$:

$$R = \begin{bmatrix} 0 & -\sin\phi & \cos\phi \\ 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \end{bmatrix}. \tag{7}$$

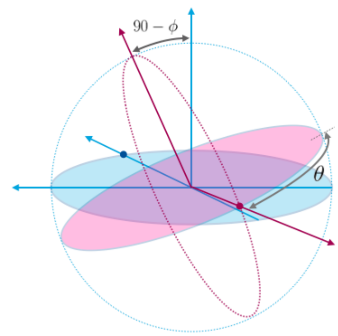

Figure 3: ENU and ECEF co-ordinate frames relative to each other.

Co-ordinate Rotations

One Cartesian co-ordinate frame can be rotated to another using three rotation matrices to rotate around the x, y and z axes, by angles ϕ, θ and ζ, respectively, where an anti-clockwise rotation corresponds to a positively valued angle. These three rotation matrices are:

$$R_1[\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$R_2[\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_3[\zeta] = \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

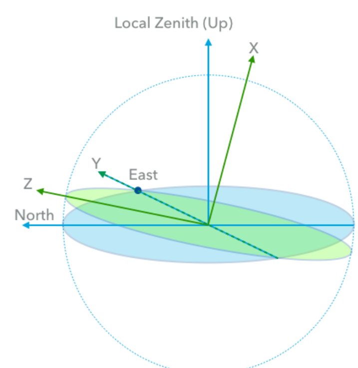

Figure 4: XYZ co-ordinate system, aligned with the Earth's rotation axis, relative to the ENU co-ordinate system.

So how do we turn this into UVW?

With the rotations above we can take antenna positions on the ground measured in metres or kilometres in the North-East-Up directions and translate them into ECEF or XYZ. We now need to translate those into UVW.

To do this we need to rotate frame once again. The UVW frame has its z-axis aligned towards the direction of observation, with the UVW x-axis running through the meridian defined by the *Hour Angle* of the observation direction. In the ECEF frame this Hour Angle (HA) is defined relative to the Greenwich meridian, regardless of where the telescope may be located; in the XYZ frame this HA is defined relative to the local meridian at the longitude of the telescope. In form, this rotation is basically the reverse of the rotation from ENU to ECEF, but the relevant angles are now $(90 - \delta)$ and HA, here denoted H:

$$R = \begin{bmatrix} -\sin H & \cos H & 0 \\ -\sin\delta\cos H & \sin\delta\sin H & \cos\delta \\ \cos\delta\cos H & -\cos\delta\sin H & \sin\delta \end{bmatrix}. \tag{8}$$

HOUR ANGLE

Local Hour Angle (HA) is defined as the difference between the Right Ascension of a source and the Local Sidereal Time:

$$LHA = RA - LST.$$

The General Hour Angle used by the ECEF system is the difference between the Right Ascension of a source and the Greenwich Sidereal Time:

$$GHA = RA - GST.$$

Worked Example

Following the CASA co-ordinate convention[2], we can demonstrate these calculations using a subset of antenna positions from the VLA telescope as an example. The positions of the VLA antennas are listed in Table 1 in ENU coordinates relative to a reference location for the array of $(\theta, \phi) = (-107.6184, +34.0790)$ in longitude and latitude.

[2] REF

Table 1: Antenna positions in ENU co-ordinates.

Name	East [m]	North [m]	Up [m]
ea06	-54.0649	263.8778	-4.2273
ea07	164.9788	-92.8032	-2.5268
ea11	102.8054	-63.7682	-2.6414

TO CONVERT THESE POSITIONS TO ECEF we must apply the rotations specified in Equation 3. If we do this we find that the positions of these antennas in ECEF are those listed in Table 2.

Table 2: Antenna positions in ECEF co-ordinates.

Name	x [m]	y [m]	z [m]
ea06	-5.7154	160.6257	216.1922
ea07	142.134	-97.5021	-78.2815
ea11	87.8302	-63.0871	-54.2970

Just as the ENU co-ordinates are relative to a local zero point at the specified longitude and latitude of the array, these ECEF co-ordinates are also relative to a reference point, which is the position of the same reference point in ECEF coordinates. We can calculate this using the following equations:

$$x_0 = (v+h)\cos\phi\cos\theta \tag{9}$$

$$y_0 = (v+h)\cos\phi\sin\theta \tag{10}$$

$$z_0 = \left[v(1+e^2)+h\right]\sin\phi \tag{11}$$

where,

$$v = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \quad (12)$$

$$e^2 = 1 - \frac{b^2}{a^2} \quad (13)$$

and $a = 6378137.000$ m is the equatorial radius of the Earth (semi-major axis), $b = 6356752.314$ m is the polar axis of the Earth (semi-minor axis) and h is the altitude above sea-level. In practice we would probably only need relative co-ordinates for our antennas (because it's the baseline we're interested in for calculating the *uvw*) but some calculations for other purposes may need to know the absolute ECEF coordinates of the antennas. For our test sample the offsets are listed in Table 3, assuming an altitude of $h = 0$ m.

Table 3: Array reference position in ECEF co-ordinates.

x_0	=	-1600657.49392 m
y_0	=	-5040295.10668 m
z_0	=	3553707.97701 m

IF INSTEAD WE WANTED TO CONVERT OUR ENU CO-ORDINATES TO XYZ then we would use the rotation specified in Equation 7. If we do this we find that the positions of these antennas in XYZ are those listed in Table 4.

Table 4: Antenna positions in XYZ co-ordinates.

Name	x [m]	y [m]	z [m]
ea06	-151.3614	-54.0649	216.1922
ea07	49.9080	164.9788	-78.2815
ea11	33.5438	102.8054	-54.2971

TO CALCULATE UVW CO-ORDINATES we need to specify our observing direction, $\mathbf{s_0}$. Astronomical sources have positions defined in a number of co-ordinate systems, but the one we are going to use is the Equatorial co-ordinate system. In this system the position of a source is specified as a Right Ascension (RA) and a declination (δ). However, in order to calculate *uvw* we need to convert the RA into an Hour Angle (see above). For this example we will assume that this has already been done and specify our observing direction in terms of (H, δ), where H is the *local* hour angle. For this example we will assume $\mathbf{s_0} = (H, \delta) = (-3.49\text{h}, +21^\circ)$.

Let's first consider calculating UVW from the XYZ array positions. We can do this using the rotation specified in Equation 8 and we find that the positions of these antennas in UVW are those listed in Table 5.

Table 5: Antenna positions in UVW co-ordinates.

Name	u [m]	v [m]	w [m]
ea06	86.8166	250.3068	-48.8028
ea07	61.2599	-130.8183	122.3543
ea11	36.2387	-87.2036	75.6610

If we want to calculate UVW directly from the ECEF co-ordinates we need to translate our *local* hour angle into the hour angle relative to the Greenwich meridian. Hour angles are negative at all points before transit and positive after transit. We can do this using the longitude of the array. In this case our array is at $\theta = 107.6184^\circ$ W, which means that it is separated from the Greenwich meridian by $\Delta H = 107.6184/15 = 7.1746$ h. Our pointing direction is at a local hour angle of $LHA = -3.49$ h relative to the local meridian at the telescope, which means that it points to the East of the array, i.e. back towards the Greenwich meridian. This makes the HA of our pointing direction $GHA = +3.6846\,\mathrm{h}$ relative to the Greenwich meridian. Using this value of GHA as our hour angle we can then calculate UVW co-ordinates from the ECEF co-ordinates directly by applying the rotation in Equation 8. This again results in the values listed in Table 5.

Finally, to calculate the *uvw* co-ordinates that define the sampling of the array, we must take the difference between the antenna positions in the UVW co-ordinate frame. If we do this for our three antennas relative to $ea06$, we find the values listed in Table 6.

Table 6: Baseline co-ordinates in UVW space.

Name	u [m]	v [m]	w [m]
ea06	0	0	0
ea07	61.2599	-130.8183	122.3543
ea11	36.2387	-87.2036	75.6610