Parallactic Angle

Anna M. M. Scaife

September 16, 2016

This note describes how to calculate the Parallactic Angle (χ) of a source as well as showing where the commonly used expression for χ comes from.

Context

In radio astronomy, as well as other branches of astronomy, knowing the parallactic angle of your target source at any given point in time is important for a range of reasons - particularly when performing polarization observations. Calculating the parallactic angle, χ, isn't difficult and is normally done using the following formula:

$$\tan\chi = \frac{-\sin H}{\cos\delta\tan\phi - \sin\delta\cos H} \quad , \tag{1}$$

where H is the hour angle of the source, δ is the declination of the source and ϕ is the latitude of the observatory. Since H is a function of time[1], parallactic angle is also a function of time.

[1] **Hour Angle:** $H = RA - LST$

Quick worked example

For example, if we take the radio source 3C286 and observe it at an hour angle of -3.5 hrs with the KAIRA telescope ($\phi = +69.04°$), we can calculate

$$\begin{aligned} H &= -3.5 \times 15 \times \pi/180 = -0.92 \text{ rad} \\ \delta &= 30.5 \times \pi/180 = 0.53 \text{ rad} \\ \phi &= 69.04 \times \pi/180 = 1.20 \text{ rad} \end{aligned}$$

and that the source has a parallactic angle of

$$\begin{aligned} \tan\chi &= \frac{-\sin(-0.92)}{\cos(0.53)\tan(1.20) - \sin(0.53)\cos(-0.92)} \\ &= 0.42 \\ &\longrightarrow \ \chi = 0.39 \text{ rad} = 22.58 \text{ deg} \end{aligned}$$

If you're coding this in software, take care when you take the arctangent to return the angle. You need to retain the correct quadrant. Typically you'll do this with a function named `atan2`, rather than `atan`.

Nitty gritty

So what is this angle? It's easy to think of the parallactic angle as the projected angle of your receiver on the sky, but it's more useful to describe it as the angle between the local horizon and the celestial equator. This corresponds to the angle between the great circle joining your source position to the zenith in the local coordinate system and the great circle joining your source position to the NCP in the equatorial coordinate system. Sound confusing? Hopefully the picture in Fig. 1 will clear things up.

The green triangle in Fig. 1 joins up the source position (S), the local zenith (Z) and the NCP. From the way the two co-ordinate systems are defined this means that the angle $S - NCP - Z$ is simply the Hour Angle, H, of the source and the angle $NCP - S - Z$ is the parallactic angle, χ. You can see its relation to the angle between the celestial equator and the local horizon by considering similar triangles. The third angle in the triangle, $NCP - Z - S$, is $(360 - A)$, where A is the azimuth of the source in the local system. We're not going to use this angle, so it's not marked on the diagram.

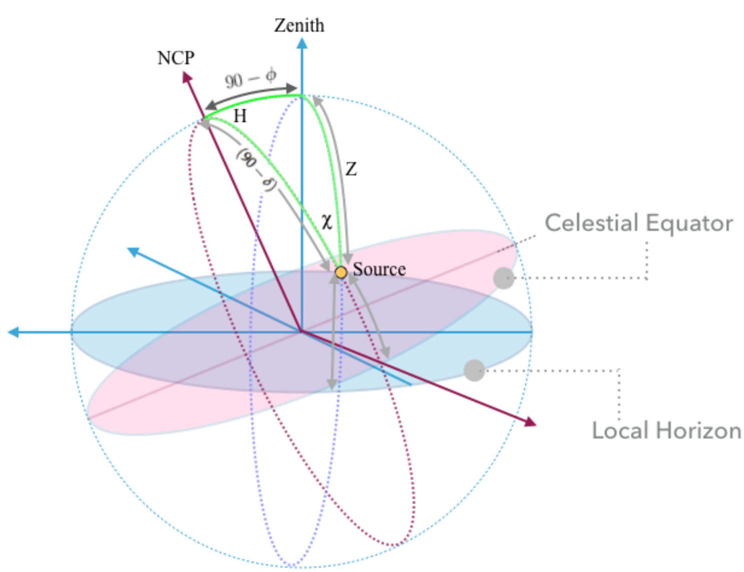

Figure 1: Intersection of local co-ordinate frame with the equatorial coordinate frame.

Spherical Triangles

IF WE WANT TO RELATE the different parameters (side length, angles) of this triangle to each other we need to use trigonometry. However, we also need to remember that we have a *curved* triangle so we need to use spherical trigonometry. The relationships for triangles in a spherical geometry can be subtlely different from those in a planar geometry.

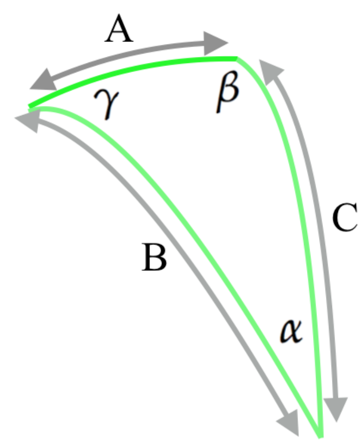

Figure 2: Generic spherical triangle.

The two key relationships we will need to use are the sine and cosine rules. In the spherical case the sine rule becomes,

$$\frac{\sin A}{\sin \alpha} = \frac{\sin B}{\sin \beta} = \frac{\sin C}{\sin \gamma} \ , \tag{2}$$

and the cosine rule becomes,

$$\cos A = \cos B \cos C + \sin B \sin C \cos \alpha \ , \tag{3}$$

where the generic quantities $(A, B, C, \alpha, \beta, \gamma)$ correspond to those indicated in Fig. 2.

Getting that formula

To get the expression for parallactic angle in Eq. 1 we can combine these two rules for our specific triangle, shown in Fig. 3. Firstly the sine rule gives us,

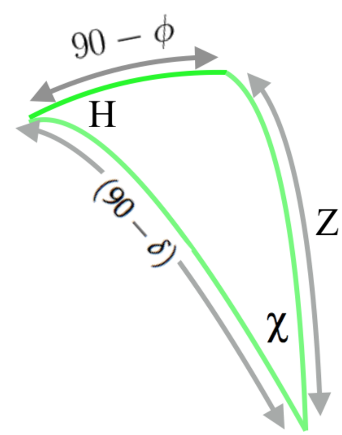

Figure 3: Specific spherical triangle.

$$\frac{\sin \chi}{\sin(90 - \phi)} = \frac{\sin H}{\sin Z},$$

which we can re-arrange to give

$$\sin \chi = \frac{\sin H \cos \phi}{\sin Z}, \tag{4}$$

where we have used the relationship $\sin(90 - \theta) = \cos \theta$.

Let's leave this expression for the moment and move on to the cosine rule. Here we can write,

$$\cos(90 - \phi) = \cos Z \cos(90 - \delta) + \sin z \sin(90 - \delta) \cos \chi \ .$$

Again we can replace $\cos(90 - \theta) = \sin \theta$ and $\sin(90 - \theta) = \cos \theta$ and if we do that we find,

$$\sin \phi = \cos Z \sin \delta + \sin Z \cos \delta \cos \chi$$

and hence

$$\cos \chi = \frac{\sin \phi - \cos Z \sin \delta}{\sin Z \cos \delta} \ . \tag{5}$$

From the expressions in Eq. 4 and Eq. 5, we can then write an expression for the tangent,

$$\tan\chi = \frac{\sin\chi}{\cos\chi} = \frac{\sin H\cos\phi\cos\delta}{\sin\phi - \cos Z\sin\delta} \quad . \tag{6}$$

For the moment this does not appear to be the same as our original expression in Eq. 1. Specifically it contains a reference to the zenith angle, Z, which doesn't appear anywhere in Eq. 1. We need to get rid of this, so we apply the cosine rule a second time, this time to get an expression for $\cos Z$:

$$\cos Z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H \tag{7}$$

which we can plug in to Eq. 6.

On first inspection this looks a bit of a mess,

$$\tan\chi = \frac{\sin H\cos\phi\cos\delta}{\sin\phi - \sin\phi\sin^2\delta - \cos\phi\cos\delta\cos H\sin\delta} \quad , \tag{8}$$

and if we then divide top and bottom by $\cos\phi\cos\delta$ it doesn't look much better:

$$\tan\chi = \frac{\sin H}{\frac{\tan\phi}{\cos\delta} - \frac{\tan\phi\sin^2\delta}{\cos\delta} - \cos H\sin\delta} \quad . \tag{9}$$

However, we can re-arrange this slightly to give,

$$\tan\chi = \frac{\sin H}{\tan\phi\left[\frac{1}{\cos\delta} - \frac{\sin^2\delta}{\cos\delta}\right] - \cos H\sin\delta} \quad . \tag{10}$$

The second term in the denominator is now starting to look familiar, we just need to sort out the first term. To do this we can expand unity such that $1 = \cos^2\delta + \sin^2\delta$ and the expression becomes

$$\begin{aligned} \tan\chi &= \frac{\sin H}{\tan\phi\left[\frac{\sin^2\delta+\cos^2\delta-\sin^2\delta}{\cos\delta}\right] - \cos H\sin\delta} && (11) \\ &= \frac{-\sin H}{\cos\delta\tan\phi - \sin\delta\cos H} \quad , && (12) \end{aligned}$$

which is exactly what we want.