W-Snapshots

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This note describes how the w-snapshots method works.

Context

East-West Arrays

If you have a bunch of antennas sitting on a plane then their projected separations will have a zero offset in the z-direction when viewed from zenith. That's the same as saying they all sit on a $w = 0$ plane in uvw space in a co-ordinate frame with the z/w-axis oriented towards the zenith.

The rotation necessary for rotating between the original uvw co-ordinate frame oriented towards the source and the $w = 0$ plane oriented towards the zenith varies as a function of time, because the separation between the two changes as the Earth rotates. That means we would need to perform a different rotation for every individual time integration step of an observation, which is laborious. However, for arrays that are purely oriented in an East-West configuration there is an alternative...

For arrays that lie in an exact East-West direction, antenna separations will remain constant in amplitude as a function of time when viewed from the North Pole. Furthermore they will also lie in a $w = 0$ plane. This is the same as saying that they all sit on a $w = 0$ plane in uvw space in a co-ordinate frame with the z/w-axis oriented towards the NCP. This is even better than the zenith projection because for widely separated arrays the curvature of the Earth will start to introduce non-zero w-values in a zenith projection.

You can demonstrate this relationship to yourself. Let's start with an array geometry (just three antennas) defined in the ENU co-ordinate frame:

$$\mathrm{ae1} = \begin{bmatrix} -54 \\ 0 \\ 0 \end{bmatrix}; \quad \mathrm{ae2} = \begin{bmatrix} 164 \\ 0 \\ 0 \end{bmatrix}; \quad \mathrm{ae3} = \begin{bmatrix} 102 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

and let's put it somewhere on the Earth:

$$\ell = +34\,\text{degrees}, \quad b = -107\,\text{degrees}. \tag{2}$$

If we use these inputs and the rotations defined in Memo XXX, we can convert these ENU positions to XYZ positions:

$$\mathrm{ae1} = \begin{bmatrix} 0 \\ -54 \\ 0 \end{bmatrix}; \;\; \mathrm{ae2} = \begin{bmatrix} 0 \\ 164 \\ 0 \end{bmatrix}; \;\; \mathrm{ae3} = \begin{bmatrix} 0 \\ 102 \\ 0 \end{bmatrix}. \tag{3}$$

and if we now define a source direction:

$$\mathbf{s_0} = (H, \delta) = (-3.49\,\mathrm{hr}, +21^\circ), \tag{4}$$

where H is the local Hour Angle, we can calculate the uvw co-ordinates:

$$\mathrm{ae1} = \begin{bmatrix} -46.82 \\ -9.6 \\ 25.2 \end{bmatrix}; \;\; \mathrm{ae2} = \begin{bmatrix} 142.8 \\ 29.5 \\ -77.0 \end{bmatrix}; \;\; \mathrm{ae3} = \begin{bmatrix} 89.0 \\ 18.4 \\ -47.9 \end{bmatrix}. \tag{5}$$

The first thing to notice is that w is not zero in these uvw co-ordinates. However, as described above, if we rotate to a co-ordinate frame oriented with w towards the NCP it should become zero. Let's test that.

The easiest way to calculate this rotation is to consider it in the opposite direction, i.e. rotating w from the NCP to the source direction. This is a rotation of $\pi/2 - \delta$ around the u axis:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \pi/2\delta & \sin \pi/2\delta \\ 0 & \sin \pi/2\delta & \cos \pi/2\delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\delta & \cos\delta \\ 0 & -\cos\delta & \sin\delta \end{bmatrix}, \tag{6}$$

and let's apply it such that

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = R \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}, \tag{7}$$

where uvw is the source oriented co-ordinate frame and $(uvw)'$ is the NCP oriented co-ordinate frame, which means that $w' = 0$. Taking this into account we can write:

$$u = u' \tag{8}$$

$$v = v' \sin\delta \tag{9}$$

$$w = -v' \cos\delta = -v \cot\delta. \tag{10}$$

You can verify this last relationship using the numbers in Eq. 5.

If we calculate uvw and $(uvw)'$ over a range of Hour Angle we can see how these values move in the uv-plane, see Figure 1.

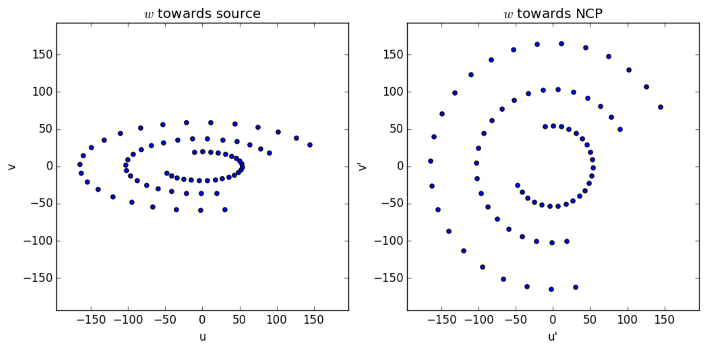

Figure 1:

So this is all great. The rotation is straightforward and we've improved our uv coverage at the same time. You may be asking, "What's the down-side?"

Well, the down-side is that whatever we do in the uvw-plane has an effect in the reciprocal (Fourier) lmn-plane. A rotation is a linear transformation, Fourier transforms are linear, and so we've also rotated our lmn co-ordinate frame. The result of this rotation relative to the original coordinate system is generally described as a *warp*.

Non-East-West Arrays

To generalise the rotation principle above to an array that isn't completely East-West we need to add in one extra rotation. For an East-West array, all antennas lie on the Y-axis in the corresponding XYZ frame, which means that when we transform to uvw coordinates everything lies in a plane that pivots into $w \neq 0$ around $v = 0$, i.e. around the u-axis. In this case we just need a single rotation back around that axis to bring the w-axis in line with the NCP.

For an array that has extent in a North-South direction as well as East-West, the plane in uvw space will not pivot purely around $v = 0$, but rather in a plane which is a function of both u and v. This means that we need *two* rotations to bring the w-axis in line with the NCP.

Rotations

The rotation we need in this case is an $R_3 R_1$ rotation around the zenith angle, Z, for the R_1 rotation and around the parallactic angle,

χ, for the R_3 rotation. The combined matrix looks like:

$$
\begin{aligned}
R_3 R_1 &= \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos Z & \sin Z \\ 0 & -\sin Z & \cos Z \end{bmatrix} \\
&= \begin{bmatrix} \cos\chi & \sin\chi\cos Z & \sin\chi\sin Z \\ -\sin\chi & \cos\chi\cos Z & \cos\chi\sin Z \\ 0 & -\sin Z & \cos Z \end{bmatrix}. \qquad (11)
\end{aligned}
$$

If we make the same assumption as before and apply this rotation matrix, we find the following relations,

$$u = u'\cos\chi + v'\sin\chi\cos Z \qquad (12)$$

$$v = -u'\sin\chi + v'\cos\chi\cos Z \qquad (13)$$

$$w = -v'\sin Z \qquad (14)$$

From Equations 12 & 13, we can write

$$u' = \frac{u - v'\sin\chi\cos Z}{\cos\chi} \text{ and} \qquad (15)$$

$$u' = \frac{v - v'\cos\chi\cos Z}{-\sin\chi}. \qquad (16)$$

Equating these two equations we can then write

$$\frac{u - v'\sin\chi\cos Z}{\cos\chi} = \frac{v - v'\cos\chi\cos Z}{-\sin\chi} \qquad (17)$$

which we can re-arrange as

$$-u\sin\chi + v'\sin^2\chi\cos Z = v\cos\chi - v'\cos^2\chi\cos Z \qquad (18)$$

and then

$$v\cos\chi + u\sin\chi = v'\cos Z. \qquad (19)$$

Hence we can write

$$v' = \frac{v\cos\chi + u\sin\chi}{\cos Z}. \qquad (20)$$

We can now insert this quantity in the equation for w:

$$
\begin{aligned}
w &= -\frac{v\cos\chi + u\sin\chi}{\cos Z}\sin Z \qquad (21) \\
&= u\sin\chi\tan Z - v\cos\chi\tan Z. \qquad (22)
\end{aligned}
$$

Warped Co-ordinates

Non-Zero "Up" Values