Fluid Mechanics:

Mass Conservation and Bernoulli's Equation

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I – Conservation of Mass

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

$$\dot{m} = \rho A V$$

For steady system;

$$\dot{m}_{in} = \dot{m}_{out}$$

For incompressible flow;

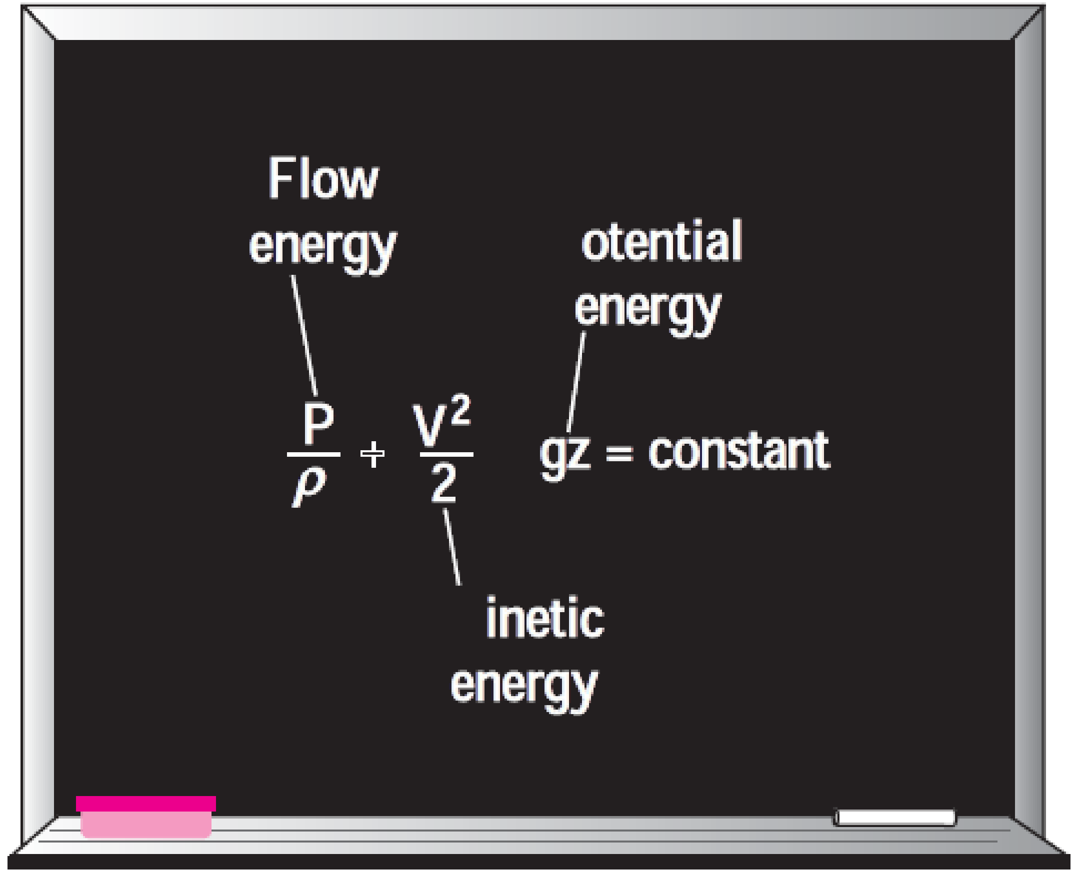
$$\rho = constant$$

$$A_1 V_1 = A_2 V_2$$

II – Bernoulli Equation

The Bernoulli equation is an approximate relation between pressure velocity and elevation and is valid in regions of steady incompressible flow where net frictional forces are negligible

$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{{V_2}^2}{2} + gz_2 = Constant$$



II – Bernoulli Equation

- **Bernoulli equation restrictions:**

1- Steady flow

2- Frictionless flow

3- Incompressible flow

4- Doesn't account for energy transfer via heat and work

$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{{V_2}^2}{2} + gz_2$$

III – Static, Dynamic, and Stagnation Pressures

$$P + \rho\frac{V^2}{2} + \rho gz = Constant$$

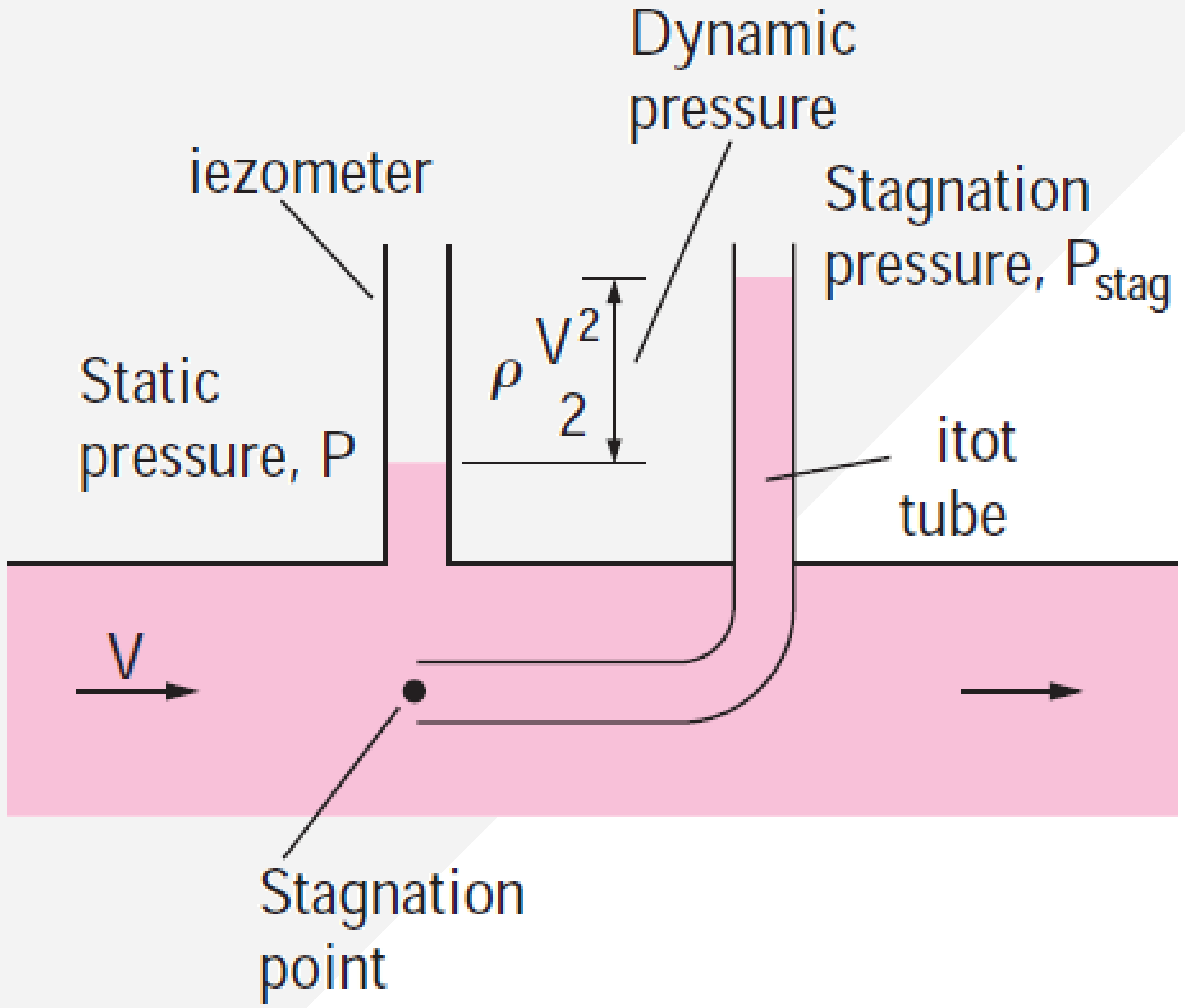
P ➔ *Static pressure*

$\rho\frac{V^2}{2}$ ➔ *Dynamic pressure, it represents the pressure rise when the fluid in motion is brought to a stop isentropically*

ρgz ➔ *Hydrostatic pressure, it accounts for the elevation effects*

$$P_{stagnation} = P + \rho\frac{V^2}{2}$$

III – Static, Dynamic, and Stagnation Pressures



$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

Problems

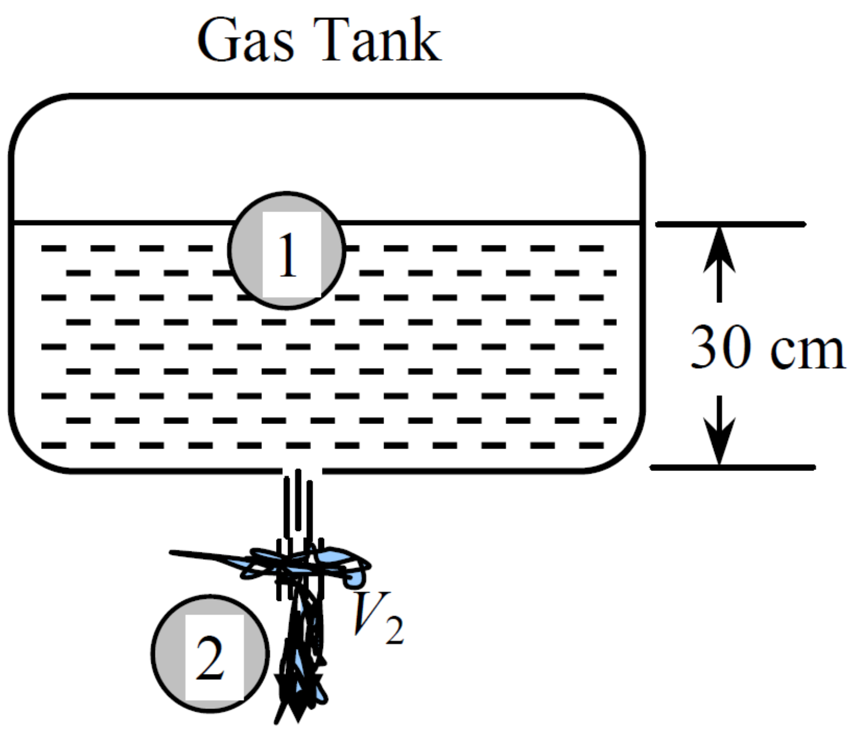
Problem 1

While traveling on a dirt road, the bottom of a car hits a sharp rock, and a small hole develops at the bottom of its gas tank. If the height of the gasoline in the tank is 30 cm, determine the initial velocity of the gasoline at the hole.

Discuss how the velocity will change with time and how the flow will be affected if the lid of the tank is closed tightly

Solution

Applying Bernoulli equation;



$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{{V_2}^2}{2} + gz_2$$

$$P_1 = P_2 = P_{atm}$$

$$V_1 = 0$$

$$z_1 = 0.3m, \qquad z_2 = 0$$

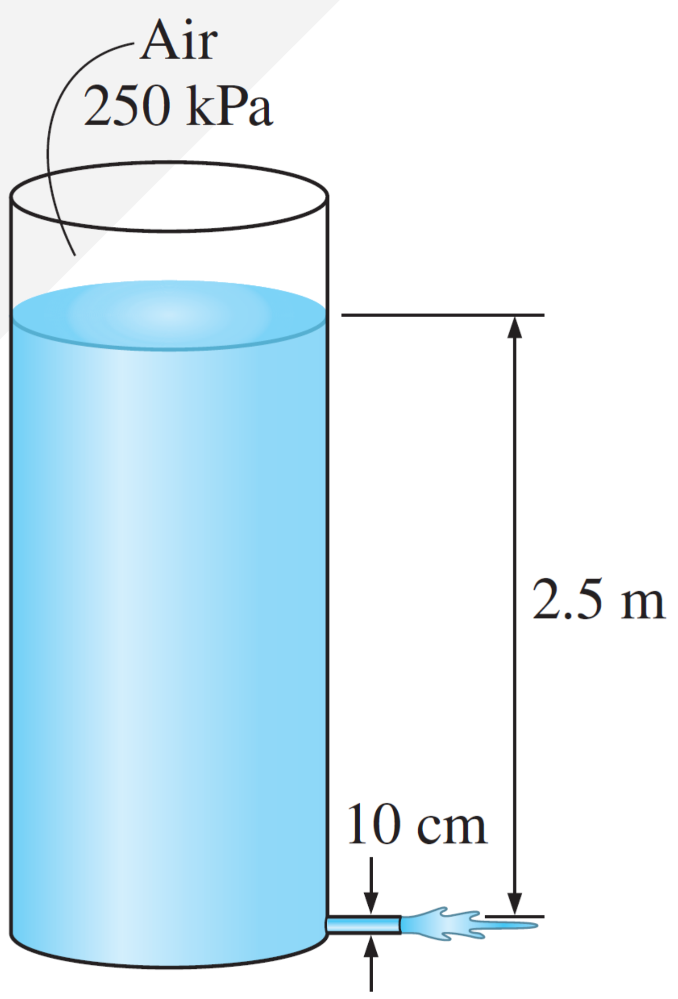
$$\therefore V_2 = \sqrt{2gz_1} = \sqrt{2 * 9.8 * 0.3}$$

$$\boxed{\therefore V_2 = 2.42\,\frac{m}{s}}$$

As the gasoline level decreases, the velocity at outlet will decrease as it's proportional to square root of the liquid level. Furthermore, if the lid is closed tightly, then as the liquid escapes, the pressure above the liquid decreases and the velocity at outlet will decrease further.

Problem 2

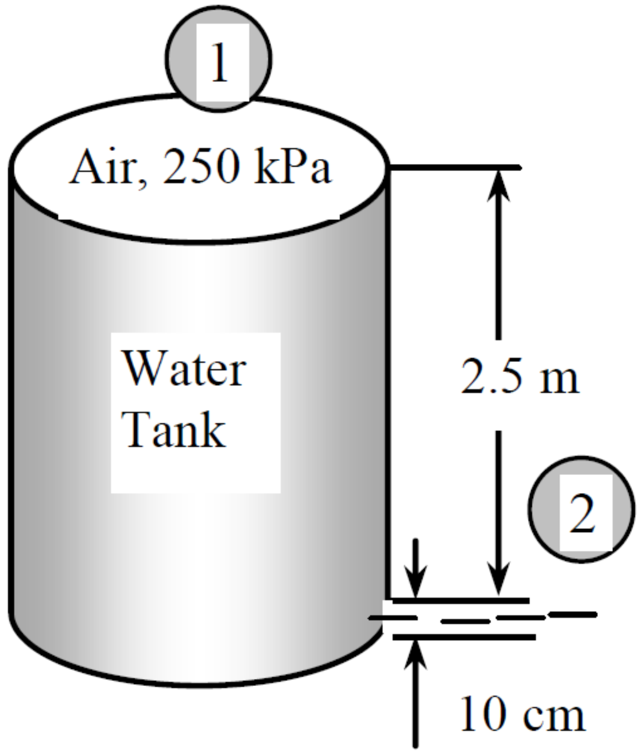
A pressurized tank of water has a 10-cm-diameter orifice at the bottom, where water discharges to the atmosphere. The water level is 2.5 m above the outlet. The tank air pressure above the water level is 250 kPa (absolute) while the atmospheric pressure is 100 kPa. Neglecting frictional effects, determine the initial discharge rate of water from the tank.



Solution

$$volume\ flowrate = V_2 A_{orifice}$$

$$A_{orifice} = \pi * \frac{0.1^2}{4} = 0.00785\, m^2$$



Applying Bernoulli equation to get V_2;

$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{{V_2}^2}{2} + gz_2$$

$$P_1 = 250\, kPa, \qquad P_2 = 100\, kPa$$

$$V_1 = 0$$

$$z_1 = 2.5m, \qquad z_2 = 0$$

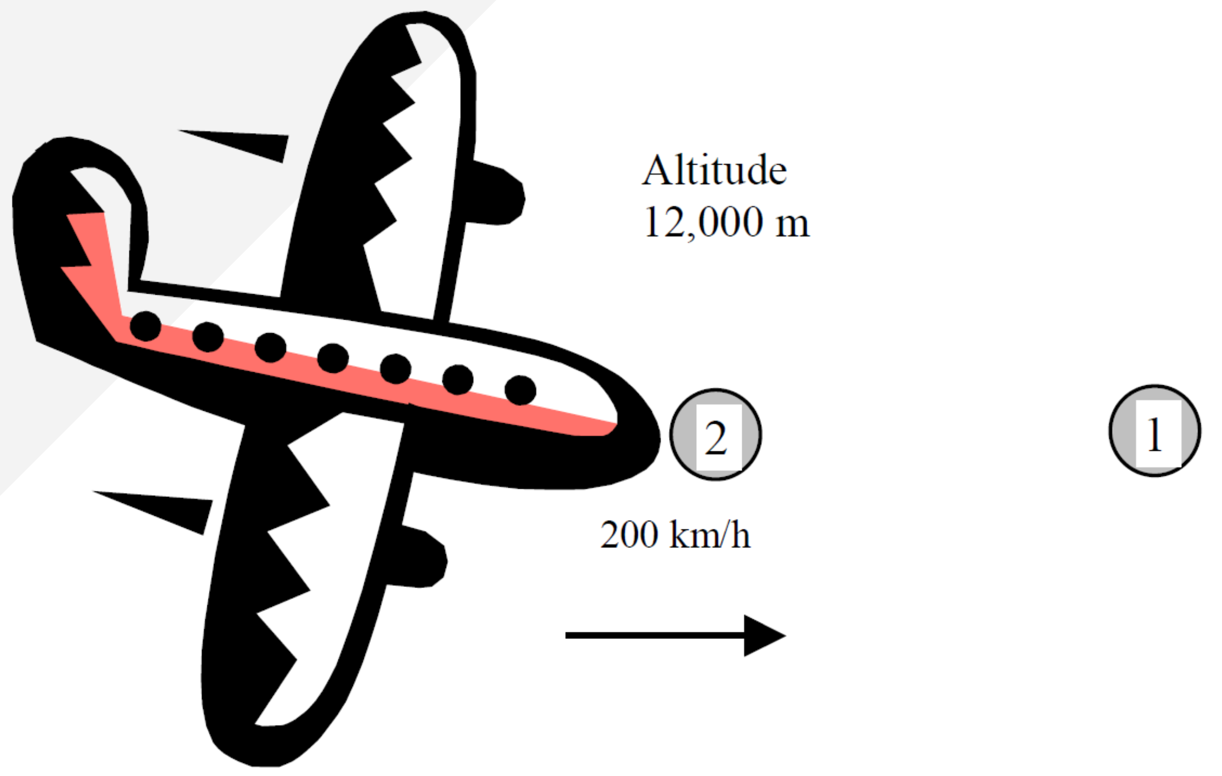
$$\frac{250 * 10^3}{1000} + 9.8 * 2.5 = \frac{100 * 10^3}{1000} + \frac{{V_2}^2}{2}$$

$$\therefore V_2 = \sqrt{2 * \left(\frac{250 * 10^3 - 100 * 10^3}{1000} + 9.8 * 2.5\right)} = 18.7\, m/s$$

$$\boxed{\therefore volume\, flowrate = 18.94 * 0.00785 = 0.147\, m^3/s}$$

Problem 3

An airplane is flying at an altitude of 12,000 m. Determine the gage pressure at the stagnation point on the nose of the plane if the speed of the plane is 200 km/h. How would you solve this problem if the speed were 1050 km/h? Explain.



Applying Bernoulli equation between point 1 and stagnation point (2);

$$\frac{P_1}{\rho}+\frac{{V_1}^2}{2}+gz_1=\frac{P_2}{\rho}+\frac{{V_2}^2}{2}+gz_2$$

$$P_1=P_{atm}, \qquad P_2=P_{stag}$$

$$V_1=200\frac{km}{hr}=55.55\frac{m}{s}, \qquad V_2=0$$

$$z_1=z_2$$

$$\rho=\rho_{air}\Big|_{Pressure\,at\,12000\,m}=0.312\ {}^{kg}\!/_{m^3}$$

$$\frac{P_{atm}}{0.312}+\frac{55.55^2}{2}=\frac{P_{stag}}{0.312}$$

$$\boxed{\therefore P_{stag,\,gage}=P_{stag}-P_{atm}=481\,Pascal}$$

At higher velocity of 1050 km/hr, the flow can no longer be considered incompressible and Bernoulli equation can no longer be applied. This problem can be solved using the modified Bernoulli equation that accounts for the effects of compressibility.

Problem 4

The water pressure in the mains of a city at a particular location is 270 kPa gage. Determine if this main can serve water to neighborhoods that are 25 m above this location.

Solution

$$P_{gage} = \rho g h$$

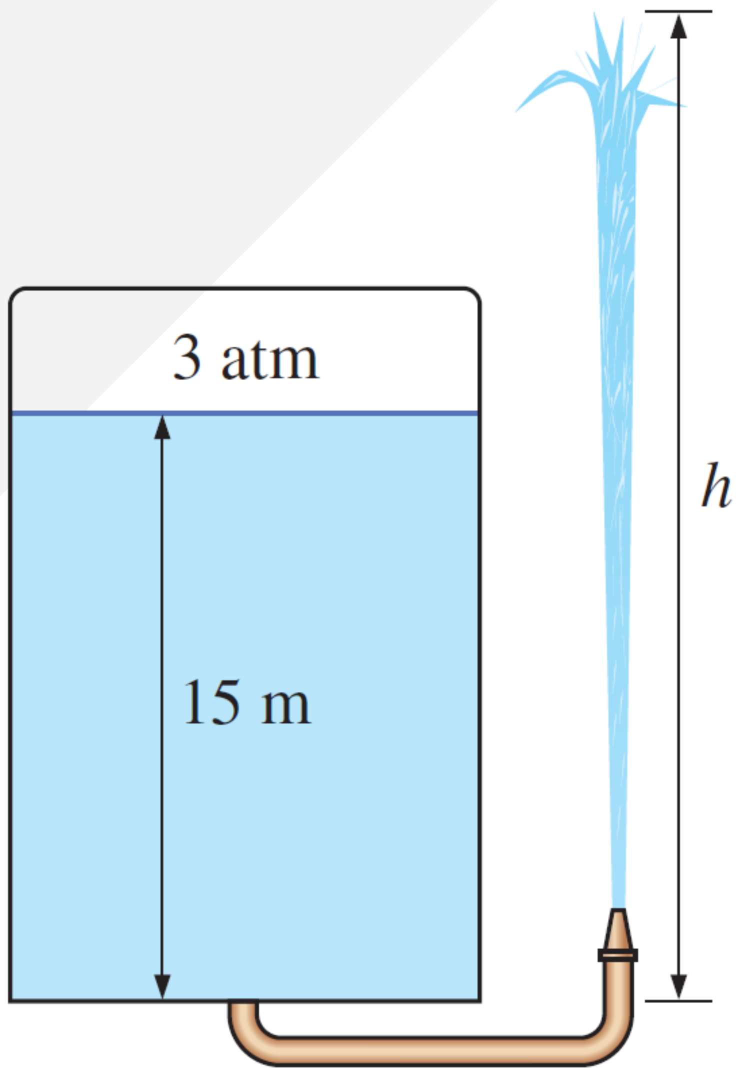
$$\therefore 270 * 10^3 = 1000 * 9.8 * h$$

$$\boxed{\therefore h = 27.55m}$$

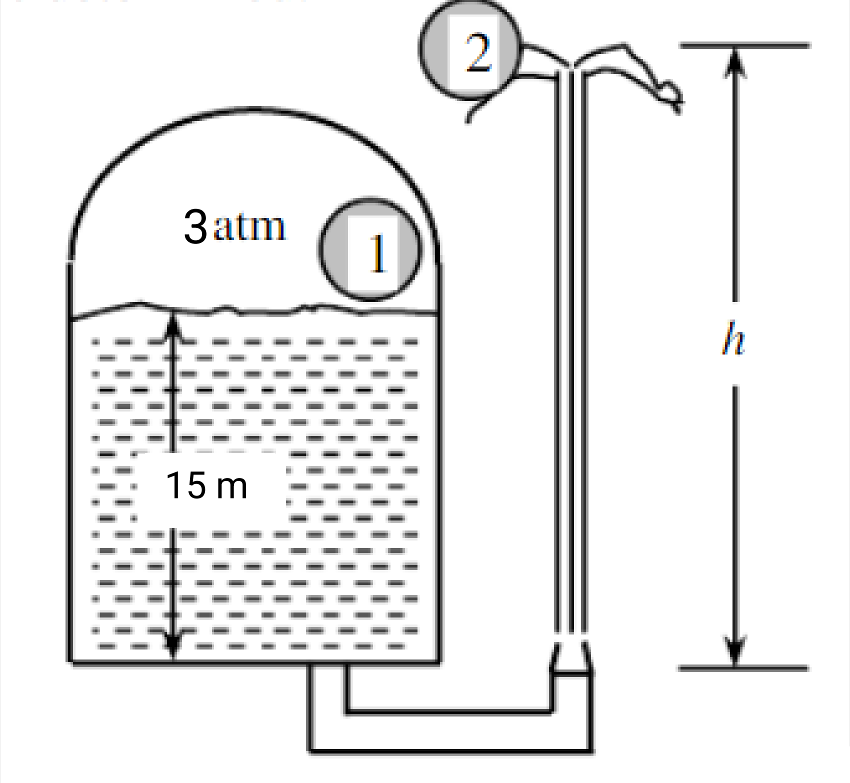
Then the applied pressure can raise water up to 27.55 meters. Meaning it can serve water to neighborhoods that are at 25 m height.

Problem5

The water level in a tank is 15 m above the ground. A hose is connected to the bottom of the tank, and the nozzle at the end of the hose is pointed straight up. The tank cover is airtight, and the air pressure above the water surface is 3 atm gage. The system is at sea level. Determine the maximum height to which the water stream could rise.



Solution



Applying Bernoulli equation between point 1 and point (2);

$$\frac{P_1}{\rho}+\frac{{V_1}^2}{2}+gz_1=\frac{P_2}{\rho}+\frac{{V_2}^2}{2}+gz_2$$

$$P_1=300\,kPa\ gage,\qquad P_2=0\,gage$$

$$V_1=0=V_2$$

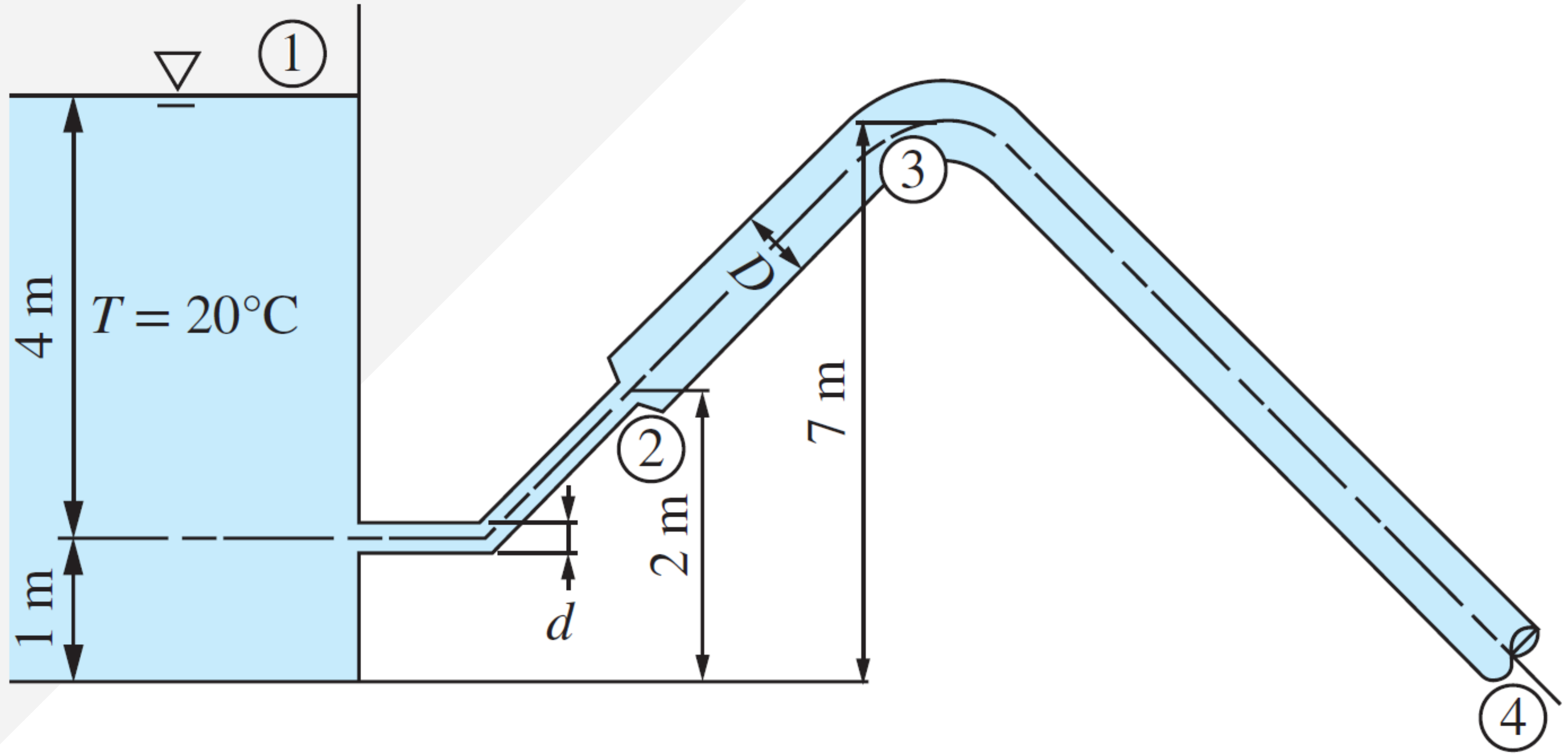
$$z_1=15m$$

$$\therefore\frac{300*10^3}{1000}+0+9.8*15=9.8*z_2$$

$$\boxed{\therefore z_2=45.6m}$$

Problem 6

Water at 20º is siphoned from a reservoir as shown in Fig. For d = 10 cm and D = 16 cm, determine (a) the minimum flow rate that can be achieved without cavitation occurring in the piping system and (b) the maximum elevation of the highest point of the piping system to avoid cavitation.



Solution

(a) To avoid cavitation, pressure at all points must be above the vapor pressure.

$$P_{vapor}@20^{o} = 2.338\ kPa$$

***Applying Bernoulli equation** between point (1) and point (2);*

***Note:** pressure at point (2) can't be below vapor pressure.*

$$let\ P_2 = P_v = 2338\,Pa$$

$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{{V_2}^2}{2} + gz_2$$

$$z_1 = 5m, \qquad z_2 = 2m$$

$$\therefore \frac{101325}{1000} + \frac{0^2}{2} + 9.8 * 5 = \frac{2338}{1000} + \frac{{V_2}^2}{2} + 9.8 * 2$$

$$\therefore V_2 = 16 m/s$$

$$\boxed{\therefore volume\, flowrate = A_2V_2 = \frac{\pi d^2}{4} * V_2 = \frac{\pi\, 0.1^2}{4} * 16 = 0.125\ m^3/s}$$

(a) To avoid cavitation at point(3), pressure must also be above the vapor pressure.

$$let\ P_3 = P_v = 2338\, Pa$$

***Applying Bernoulli equation** between point (1) and point (3);*

$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_3}{\rho} + \frac{{V_3}^2}{2} + gz_3$$

$$z_1 = 5m,$$

$$A_3V_3 = A_2V_2$$

$$\pi * \frac{0.16^2}{4} V_3 = \pi * \frac{0.1^2}{4} * 16$$

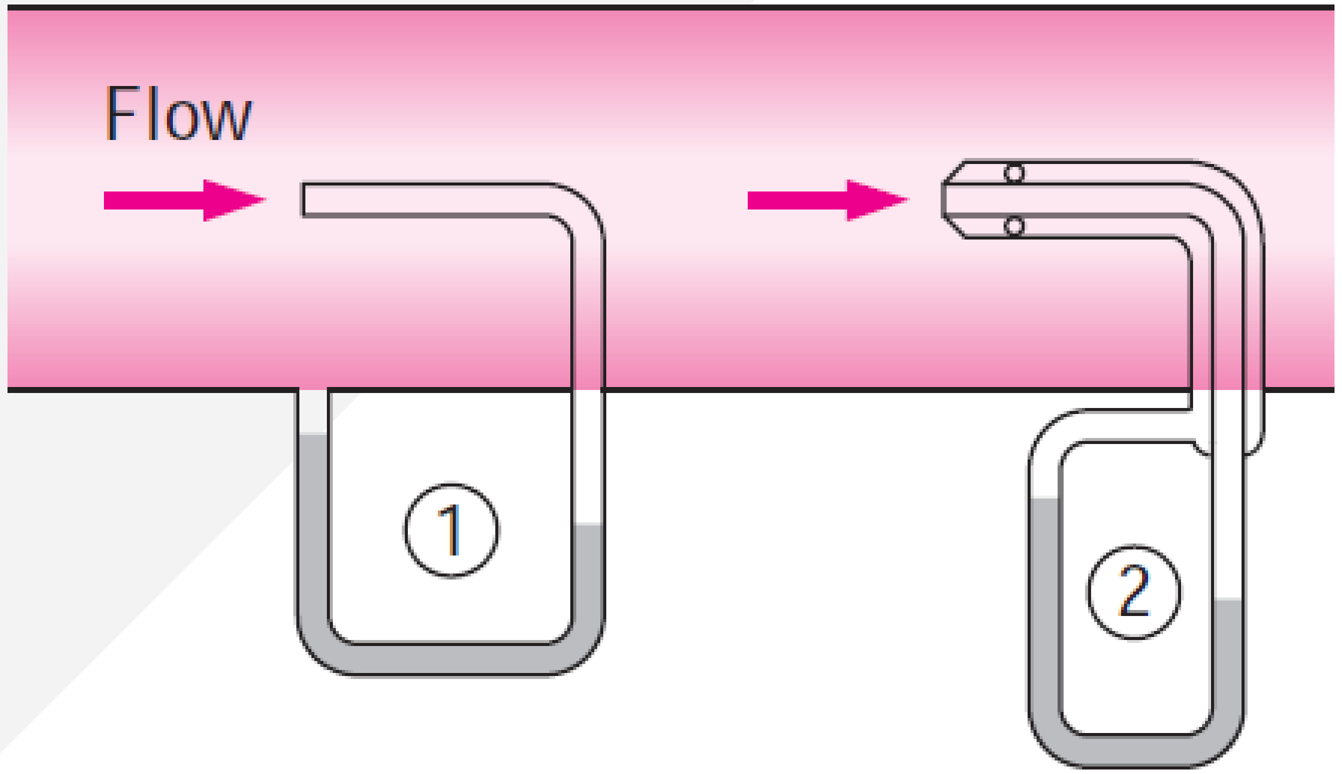
$$\therefore V_3 = 6.25 m/s$$

$$\therefore \frac{101325}{1000} + \frac{0^2}{2} + 9.8 * 5 = \frac{2388}{1000} + \frac{6.25^2}{2} + 9.8 * z_3$$

$$\boxed{\therefore z_3 = 13m}$$

Problem 7

The velocity of a fluid flowing in a pipe is to be measured by two different Pitot-type mercury manometers shown in Fig. Would you expect both manometers to predict the same velocity for flowing water? If not, which would be more accurate? Explain. What would your response be if air were flowing in the pipe instead of water?



Solution

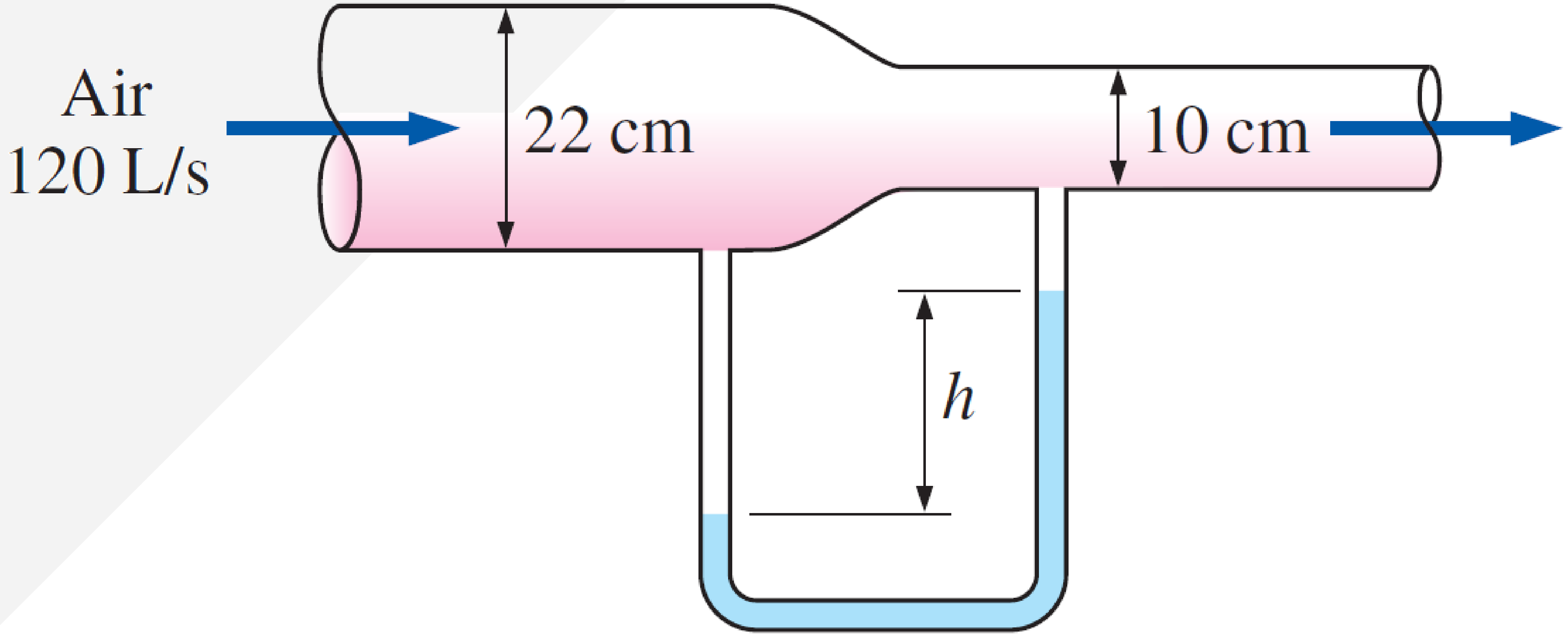
Arrangement 1 consists of a Pitot probe that measures the stagnation pressure at the pipe centerline, along with a static pressure tap that measures static pressure at the bottom of the pipe. Arrangement 2 is a Pitot-static probe that measures both stagnation pressure and static pressure at nearly the same location at the pipe centerline. Because of this, arrangement 2 is more accurate.

However, it turns out that static pressure in a pipe varies with elevation across the pipe cross section in much the same way as in hydrostatics. Therefore, arrangement 1 is also very accurate, and the elevation difference between the Pitot probe and the static pressure tap is nearly compensated by the change in hydrostatic pressure.

There is no change in our analysis when the water is replaced by air.

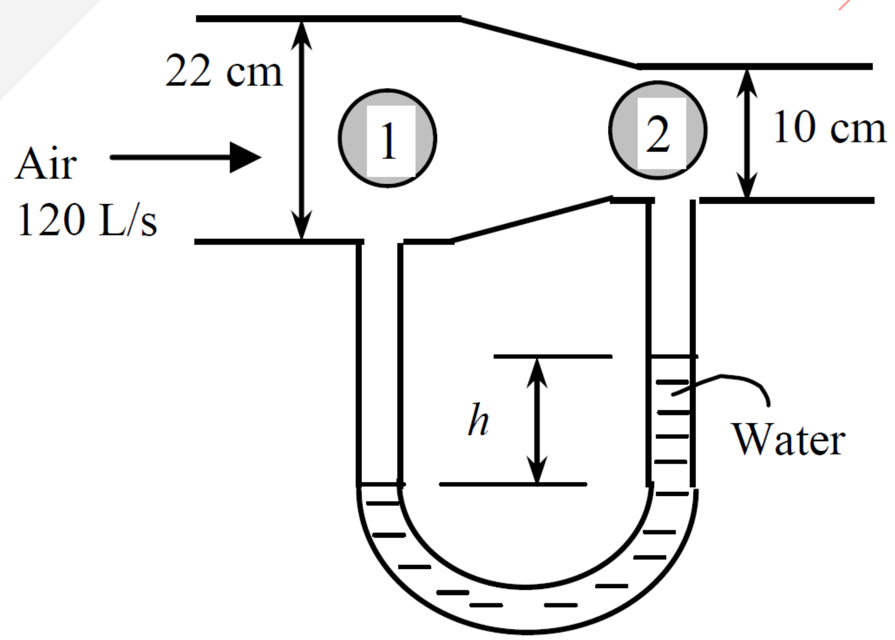
Problem 8

Air flows through a pipe at a rate of 120 L/s. The pipe consists of two sections of diameters 22 cm and 10 cm with a smooth reducing section that connects them. The pressure difference between the two pipe sections is measured by a water manometer. Neglecting frictional effects, determine the differential height of water between the two pipe sections. Take the air density to be 1.20 kg/m3.



Solution

For incompressible flow;



$$volume\,flowrate = A_1V_1 = A_2V_2 = 120\frac{L}{s} = 0.12\frac{m^3}{s}$$

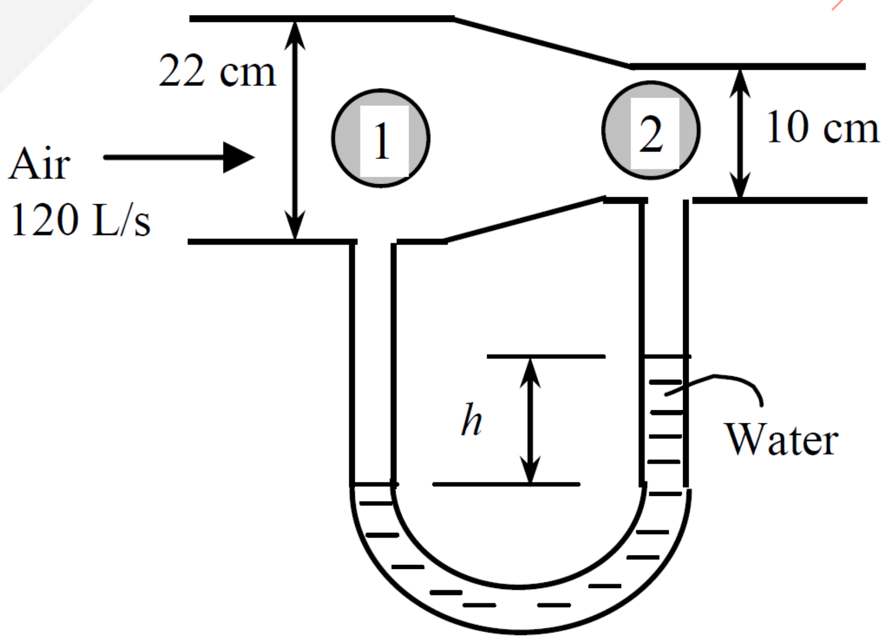
$$\pi * \frac{0.22^2}{4}V_1 = \pi * \frac{0.1^2}{4}V_2 = 0.12\frac{m^3}{s}$$

$$V_1 = 3.157\frac{m}{s}, \qquad V_2 = 15.28\frac{m}{s}$$

Applying Bernoulli equation between point 1 and point (2);

$$\frac{P_1}{\rho} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{{V_2}^2}{2} + gz_2$$

$$z_1 = z_2$$



$$\therefore P_1 - P_2 = \rho_{air}\left(\frac{V_2{}^2 - V_1{}^2}{2}\right)$$

$$\therefore P_1 - P_2 = 1.2\left(\frac{15.28^2 - 3.157^2}{2}\right) = 134.1\ Pa$$

For Water manometer,

$$\therefore P_1 - P_2 = \rho_W\, g\, h$$

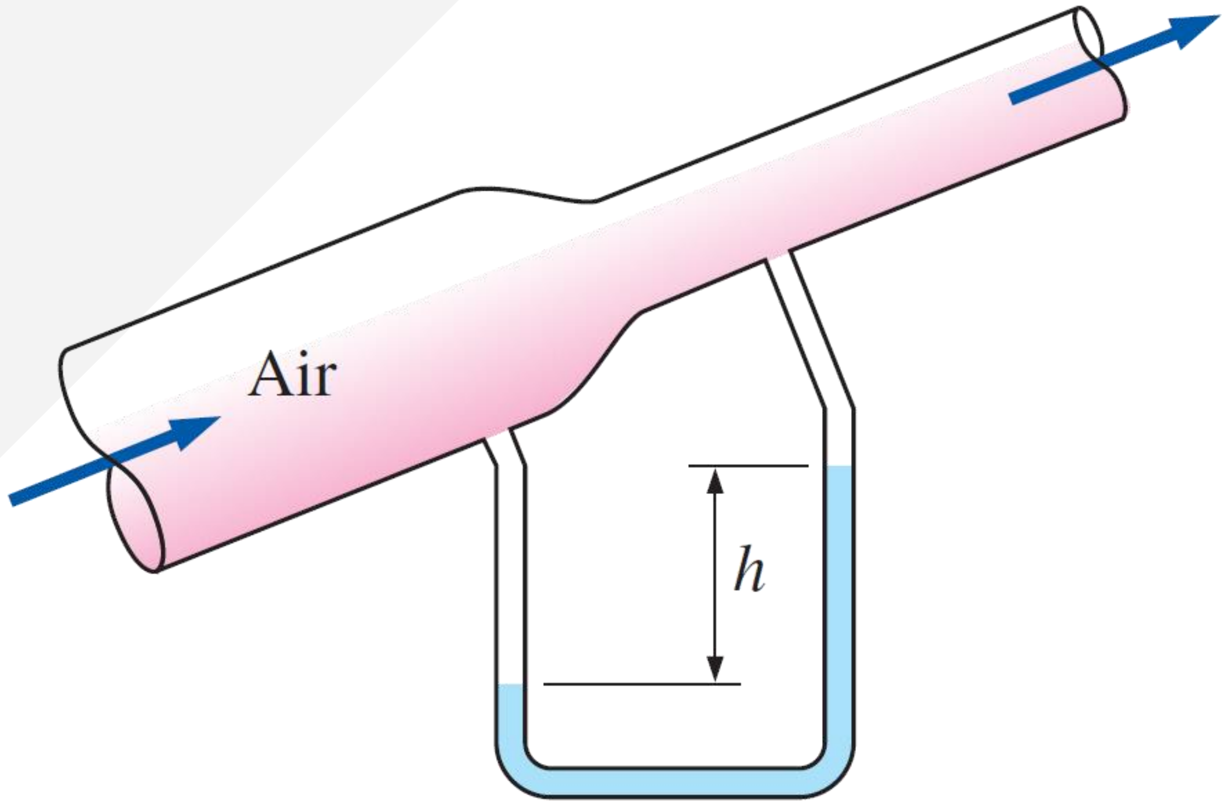
$$\therefore 134.1 = 1000 * 9.8 * h$$

$$\boxed{\therefore h = 0.0136 m}$$

Problem 9

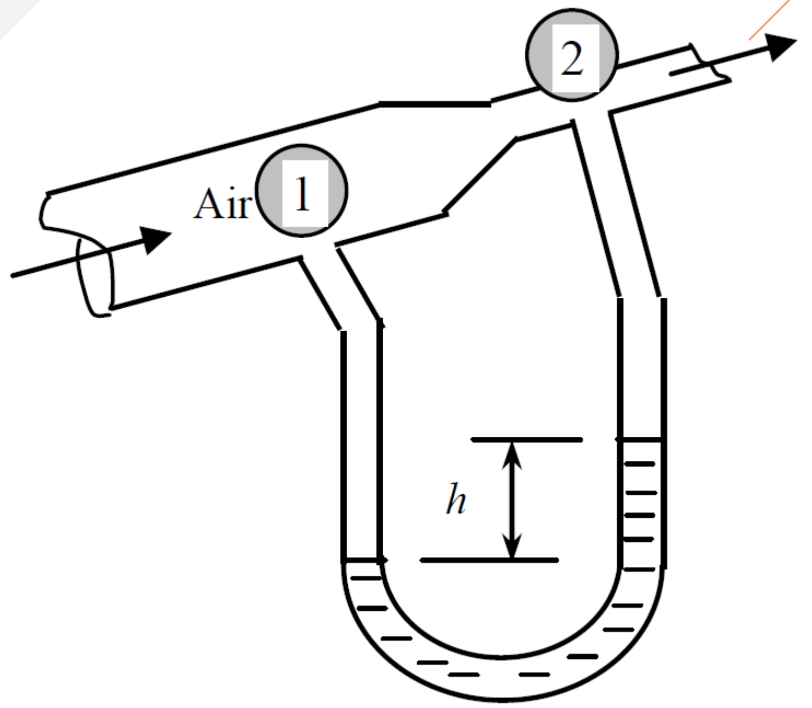
Air at 105 kPa and 37°C flows upward through a 6-cm diameter inclined duct at a rate of 65 L/s. The duct diameter is then reduced to 4 cm through a reducer. The pressure change across the reducer is measured by a water manometer.

The elevation difference between the two points on the pipe where the two arms of the manometer are attached is 0.20 m. Determine the differential height between the fluid levels of the two arms of the manometer



Solution

Assuming incompressible flow;



$$volume\ flowrate = A_1V_1 = A_2V_2 = 0.065m^3/s$$

$$\pi * \frac{0.06^2}{4}V_1 = \pi * \frac{0.04^2}{4}V_2 = 0.065m^3/s$$

$$V_1 = 23\frac{m}{s}, \qquad V_2 = 51.75\frac{m}{s}$$

Applying Bernoulli equation between point 1 and point (2);

$$\frac{P_1}{\rho_{air}} + \frac{{V_1}^2}{2} + gz_1 = \frac{P_2}{\rho_{air}} + \frac{{V_2}^2}{2} + gz_2$$

$$z_2 - z_1 = 0.2m$$

$$\rho_{air} = \frac{P}{RT} = \frac{105}{0.287 * (37 + 273)} = 1.18\ {}^{kg}/_{m^3}$$

$$\therefore P_1 - P_2 = \rho_{air}\left(\frac{V_2{}^2 - V_1{}^2}{2} + g(z_2 - z_1)\right)$$

$$\therefore P_1 - P_2 = 1.18\left(\frac{51.75^2 - 23^2}{2} + 9.8 * (0.2)\right) = 1270\, Pascal$$

For Water manometer,

$$\therefore P_1 - P_2 = \rho_{water}\, g h$$

$$\therefore 1270 = 1000 * 9.8 * h$$

$$\boxed{\therefore h = 0.13m = 13cm}$$

Thank You