Chapter 3: Fuzzy Rules and Fuzzy Reasoning

Outline

- Extension principle
- Fuzzy relations
- Fuzzy if-then rules
- Compositional rule of inference
- Fuzzy reasoning

Extension Principle

- The extension principle is a basic concept of fuzzy set theory that provide a general procedure for extending crisp domains of mathematical expressions to fuzzy domains.

- Generalizes point-to-point mapping of a function $f(.)$ to a mapping between fuzzy sets. More specifically if f is a function from X to Y and

A is a fuzzy set on X defined as,

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

Then the extension principle states that the image of *A* under the mapping $f(.)$ *can be express as* a fuzzy set *B*,

$$B = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n$$

If $f(.)$ is a many-to-one mapping, then there exist x1, x2 $\in$ X, x1 $\neq$ x2
Such that $f(x_1) = f(x_2) = y^*, y^* \in Y$

where $y_i = f(x_i),\ for\ \ i = 1\ to\ n.$
If *f(.)* is a many-to-one mapping, then

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$$

Example: application of extension principle to fuzzy sets with discrete universe
Let $A = 0.1/{-2} + 0.4/{-1} + 0.8/0 + 0.9/1 + 0.3/2$
and $f(x) = x^2 - 3$
Upon applying the extension principle, we have

$$\begin{aligned} B &= 0.1/1 + 0.4/{-2} + 0.8/{-3} + 0.9/{-2} + 0.3/1 \\ &= 0.8/{-3} + (0.4 \vee 0.9)/{-2} + (0.1 \vee 0.3)/1 \\ &= 0.8/{-3} + 0.9/{-2} + 0.3/1, \end{aligned}$$

$\vee \rightarrow$ max

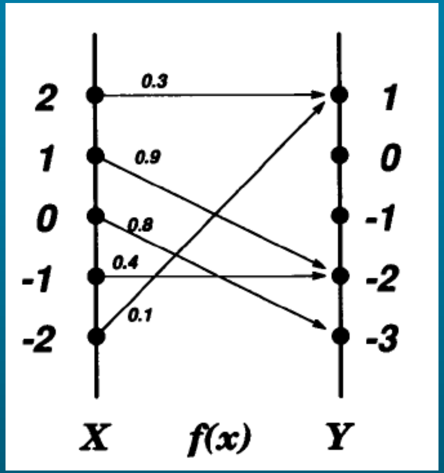
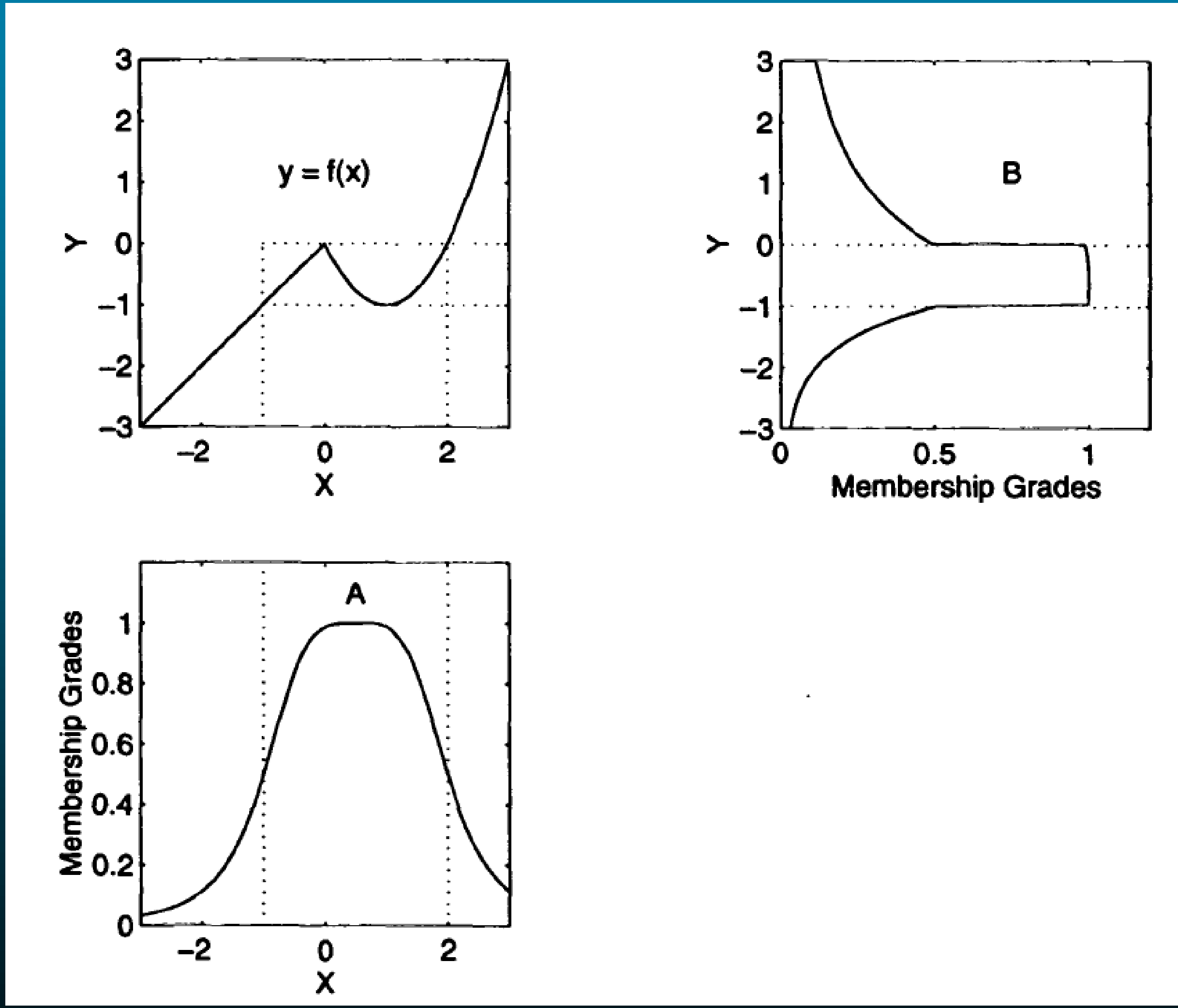


Fig. Extension principle on fuzzy sets with discrete universes

Example: Application of the extension principle to fuzzy sets with continuous universes

Let $\mu_A(x) = bell(x; 1.5, 2, 0.5)$ and

$$f(x) = \begin{cases} (x-1)^2, & if\ x \geq 0 \\ x, & if\ x \leq 0 \end{cases}$$


3
2
1
0
–1
–2
–3
Y
y = f(x)
–2
0
2
X
B
0
0.5
1
Membership Grades
A
1
0.8
0.6
0.4
0.2
Membership Grades

Now we consider a more general situation. Suppose that f is a mapping from an n-dimensional product space $X_1 \times \cdots X_n$ to a single universe Y such that $f(x_1, \ldots, x_n) = y$, and there is a fuzzy set A_i in each X_i, $i = 1, \ldots, n$. Since each element in an input vector $(x_1, \ldots, x_n)$ occurs *simultaneously*, this implies an AND operation. Therefore, the membership grade of fuzzy set B induced by the mapping f should be the minimum of the membership grades of the constituent fuzzy set A_i, $i = 1, \ldots, n$. With this understanding, we give a complete formal definition of the extension principle.

- ***Definition: Extension Principle***:

Suppose that function f is a mapping from an n-dimensional Cartesian product space $X_1 \times X_2 \times \ldots X_n$, to a one-dimensional universe Y such that $y = f(x_1, \ldots, x_n)$, and suppose $A_1, \ldots, A_n$ are n fuzzy sets in $X_1, X_2, \ldots, X_n$ respectively. Then the extension principle asserts that the fuzzy set B induced by the mapping f is defined by

$$\mu_B(y) = \begin{cases} \max\limits_{(x_1,\ldots,x_n),\ (x_1,\ldots,x_n)=f^{-1}(y)} [\min_i \ \mu_{A_i}(x_i)], & \text{if } f^{-1}(y) \neq \emptyset. \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

The foregoing extension principle assumes that $y = f(x_1, \ldots, x_n)$ is a crisp function. In cases where f is a fuzzy function [or, more precisely, when $y = f(x_1, \ldots, x_n)$ is a fuzzy set characterized by an $(n+1)$-dimensional MF], then we can employ the compositional rule of inference introduced in Section 3.4.1 (page 63) of the next chapter to find the induced fuzzy set B.

#Fuzzy Relations

Binary fuzzy relations [4, 6] are fuzzy sets in $X \times Y$ which map each element in $X \times Y$ to a membership grade between 0 and 1. In particular, unary fuzzy relations are fuzzy sets with one-dimensional MFs; binary fuzzy relations are fuzzy sets with two-dimensional MFs, and so on. Applications of fuzzy relations include areas such as fuzzy control and decision making. Here we restrict our attention to binary fuzzy relations; a generalization to n-ary relations is straightforward.

Definition: Binary Fuzzy Relation

Let X and Y be two universes of discourse. Then

$$\mathcal{R} = \{(\ (x, y),\ \mu_{\mathcal{R}}(x, y)\ ) \mid (x, y) \in X \times Y\} \tag{3.2}$$

is a **binary fuzzy relation** in $X \times Y$. [Note that $\mu_{\mathcal{R}}(x, y)$ is in fact a two-dimensional MF introduced in Section 2.4.2.]

Example: Binary fuzzy relation

Let $X = Y = R^+$ (the positive real line) and $\mathcal{R}$ = "y is much greater than x." The MF of the fuzzy relation $\mathcal{R}$ can be subjectively defined as

$$\mu_{\mathcal{R}}(x, y) = \begin{cases} \dfrac{y - x}{x + y + 2}, & \text{if } y > x. \\ 0, & \text{if } y \le x. \end{cases} \tag{3.3}$$

If $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then it is convenient to express the fuzzy relation $\mathcal{R}$ as a **relation matrix**:

$$\mathcal{R} = \begin{bmatrix} 0 & 0.111 & 0.200 & 0.273 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix}, \tag{3.4}$$

where the element at row i and column j is equal to the membership grade between the ith element of X and jth element of Y.

Fuzzy Relations

Other common Examples of binary relations are

- x is close to y (x and y are numbers)
- x depends on y (x and y are events)
- x and y look alike (x and y are persons or objects)
- If x is large, then y is small (x is an observed instrument reading and y is a corresponding control action)

- Fuzzy relations in different product spaces can be combined through a composition operation.
- Different composition operations have been suggested for binary relations ,
- The best known is the max-min composition proposed by Zadeh.

Max-min composition

The *max-min composition* of two fuzzy relations R_1 (defined on X and Y) and R_2 (defined on Y and Z):

$$R_1 \circ R_2 = \{[(x,z), \max_y \min(\mu_{R_1}(x,y), \mu_{R_2}(y,z))] \mid x \in X, y \in Y, z \in Z\}$$

$$\text{Or, } \mu_{R_1 \circ R_2}(x,z) = \max_y \min[\mu_{R_1}(x,y), \mu_{R_2}(y,z)]$$
$$= \vee_y [\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)]$$

- When $R1$ and $R2$ are expressed as relational matrices, the calculation of R1 ∘R2 is almost the same as matrix multiplication, except that × and + are replaced by ∧ and ∨ respectively. For this reason max-min composition is also called max-min product
- This operator is widely used but it is not easily subjected to mathematical analysis. For greater tractability max-product is proposed as an alternative.

Properties:

Several properties of max-min composition are given where R, S and T are binary relations on XxY, YxZ and ZxW, respectively.

- Associativity: $R \circ (S \circ T) = (R \circ S) \circ T$
- Distributivity over union: $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
- Weak distributivity over intersection: $R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$
- Monotonicity: $S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$

Max-min Composition: Example

Let

R_1="x is relevant to y"

R_2="y is relevant to z"

where

$X=\{1\ 2\ 3\}$

$$R_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix}$$

$Y=\{\alpha\ \beta\ \chi\ \delta\}$

$$R_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

$Z=\{a\ b\}$

Max-min Composition: Example

Calculate:

2 is relevant to a

$$
R_1 = \begin{array}{c}
\begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ \boxed{0.4 \quad 0.2 \quad 0.8 \quad 0.9} \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix}
\begin{array}{l} X=\{ \\ 1 \\ 2 \\ 3\} \end{array} \\
Y=\{\alpha \quad \beta \quad \chi \quad \delta\}
\end{array}
\qquad
R_2 = \begin{array}{c}
\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \\
Z=\{a \quad b\}
\end{array}
$$

Ex) Derive the degree of relevance between 2 in X and a in Z based on R_1 and R_2

$$R_1 = \text{“}x \text{ is relevant to } y,\text{”} \quad X \times Y$$
$$R_2 = \text{“}y \text{ is relevant to } z,\text{”} \quad Y \times Z$$

where $X = \{1,2,3\}, Y = \{\alpha, \beta, \gamma, \delta\}, Z = \{a, b\}$

$$R_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

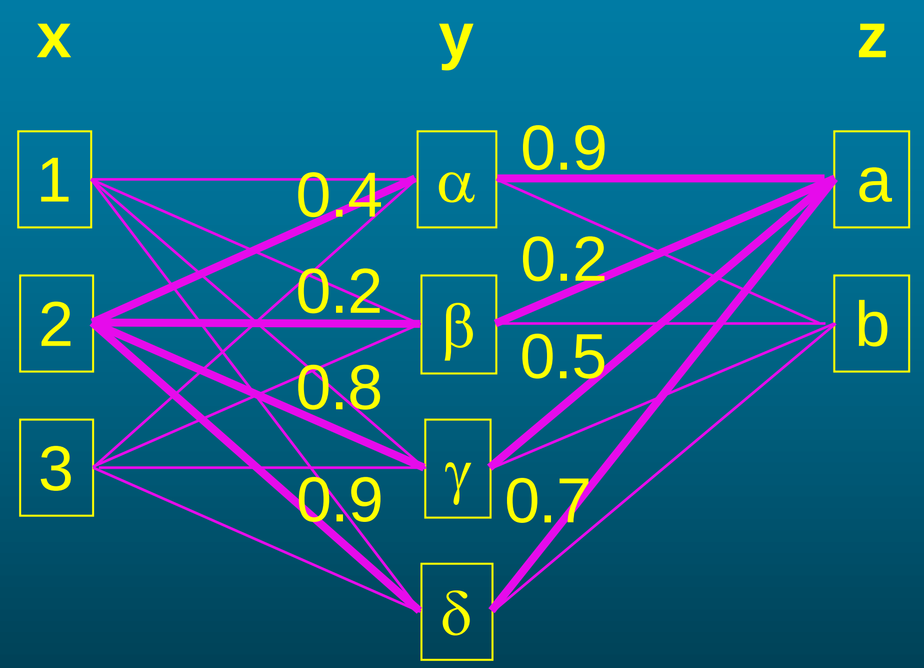
- *Max-min composition*:

$$\begin{aligned} \mu_{R_1 \circ R_2}(2, a) &= \max(0.4 \wedge 0.9, 0.2 \wedge 0.2, 0.8 \wedge 0.5, 0.9 \wedge 0.7) \\ &= \max(0.4, 0.2, 0.5, 0.7) \\ &= 0.7 \end{aligned}$$

- *Max-product composition*:

$$\begin{aligned} \mu_{R_1 \circ R_2}(2, a) &= \max(0.4 \times 0.9, 0.2 \times 0.2, 0.8 \times 0.5, 0.9 \times 0.7) \\ &= \max(0.36, 0.04, 0.40, 0.63) \\ &= 0.63 \end{aligned}$$

Example 3.4 (cont'd.)



$$\mu_{R_1 \circ R_2}(2, a) = 0.7 \quad \text{(max-min composition)}$$

$$\mu_{R_1 \circ R_2}(2, a) = 0.63 \quad \text{(max-product composition)}$$

Max-Star Composition

Max-product composition:

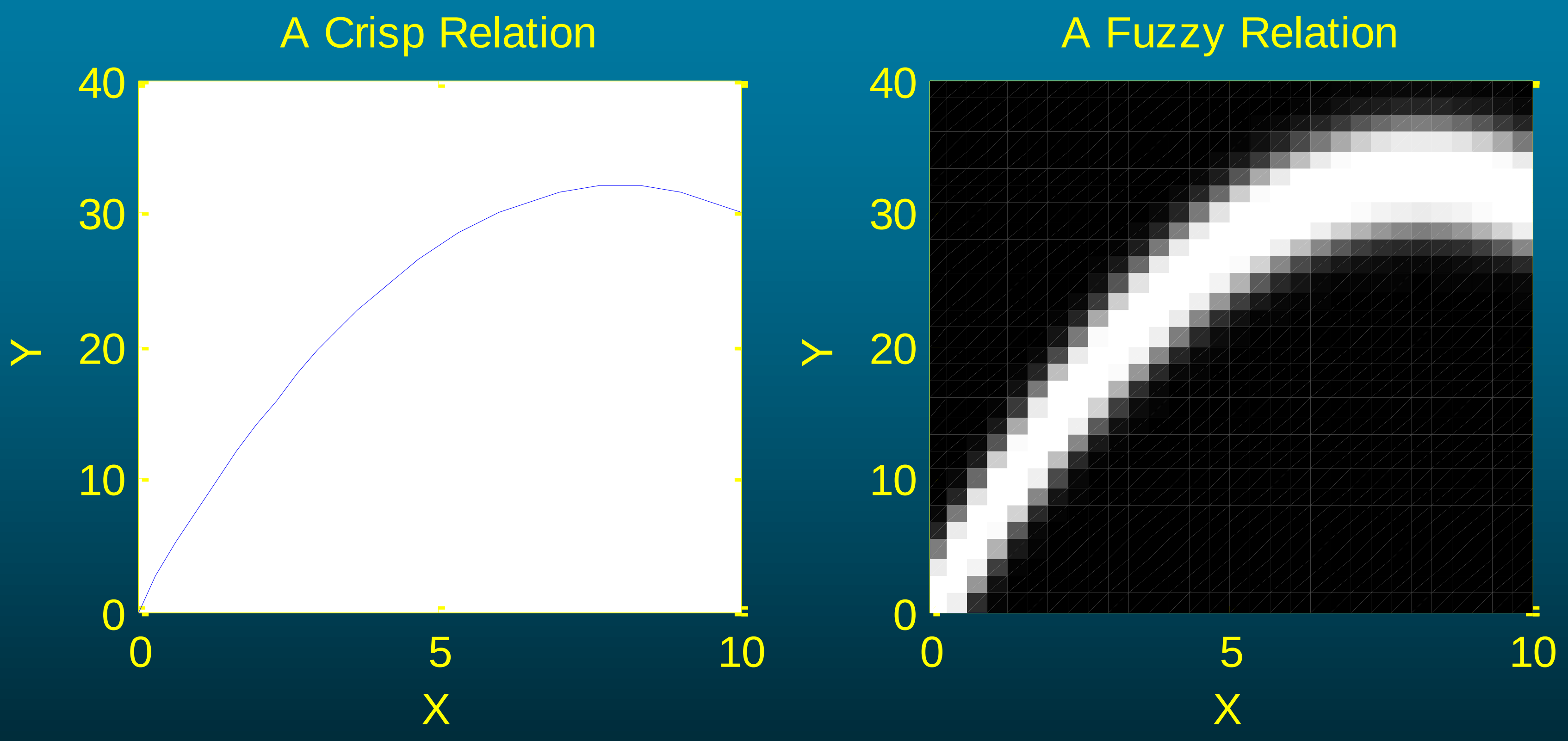
$$\mu_{R_1 \circ R_2}(x,z) = \vee_y [\mu_{R_1}(x,y)\mu_{R_2}(y,z)]$$

In general, we have max * compositions:

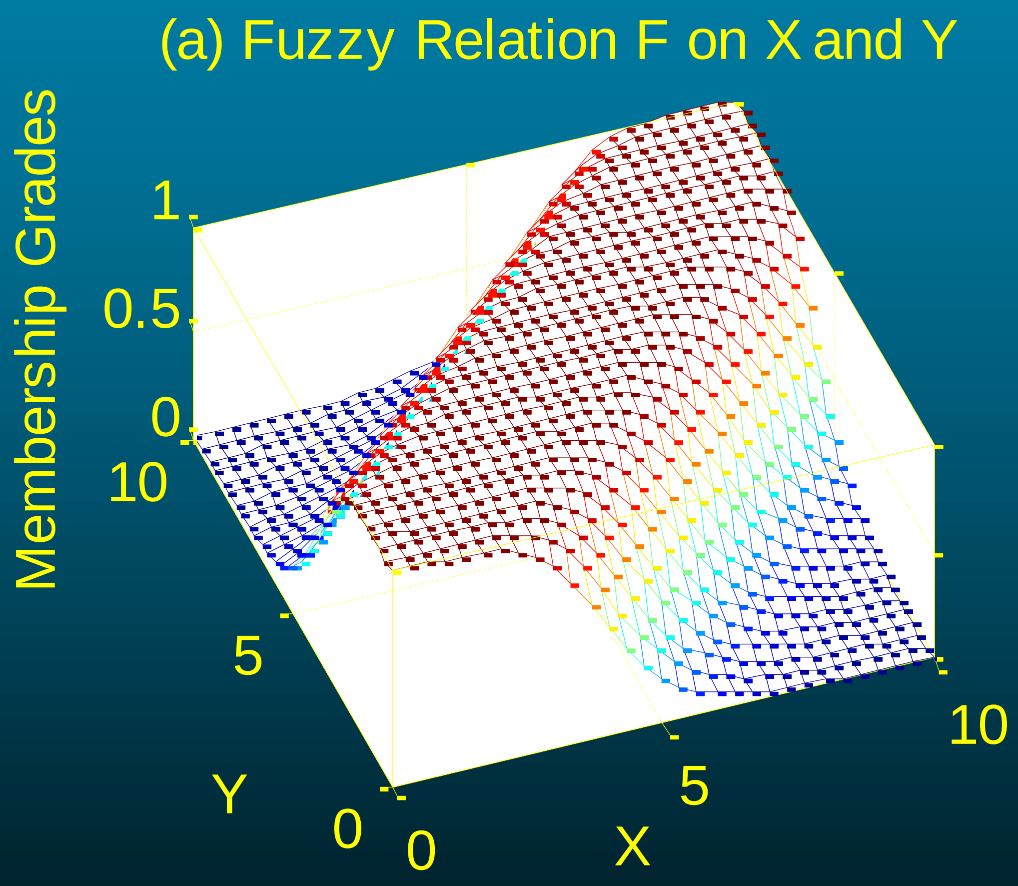
$$\mu_{R_1 \circ R_2}(x,z) = \vee_y [\mu_{R_1}(x,y) * \mu_{R_2}(y,z)]$$

where * is a T-norm operator.

Example: x is close to y



Example: X is close to Y



Example 3.4 – Max * Compositions

R_1: x is relevant to y

	y=α	y=β	y=γ	y=δ
x=1	0.1	0.3	0.5	0.7
x=2	0.4	0.2	0.8	0.9
x=3	0.6	0.8	0.3	0.2

R_2: y is relevant to z

	z=a	z=b
y=α	0.9	0.1
y=β	0.2	0.3
y=γ	0.5	0.6
y=δ	0.7	0.2

How relevant is x=2 to z=a?

Linguistic Variables

Precision vs. significance:

A linguistic value is characterized by a quintuple $(x, T(x), X, G, M)$ the variable name (age), the term set, the **universe of discourse**, a syntactic rule, and a semantic rule.
The **variable name** is just that : *age*
The **term set** is the set of its linguistic values:

T(age) = {young, not young, very young, ...
middle aged, not middle aged, ...
old, not old, very old, more or less old, ...
not very young and not very old, ...}

The **syntactic rule** refers to how the linguistic values are generated.
The **semantic rule** defines the membership value of each linguistic variable.

Linguistic Variables

A numerical variable takes numerical values:

Age = 65

A linguistic variables takes linguistic values:

Age is old

A linguistic value is a fuzzy set.

All linguistic values form a <u>term set</u> (set of terms):

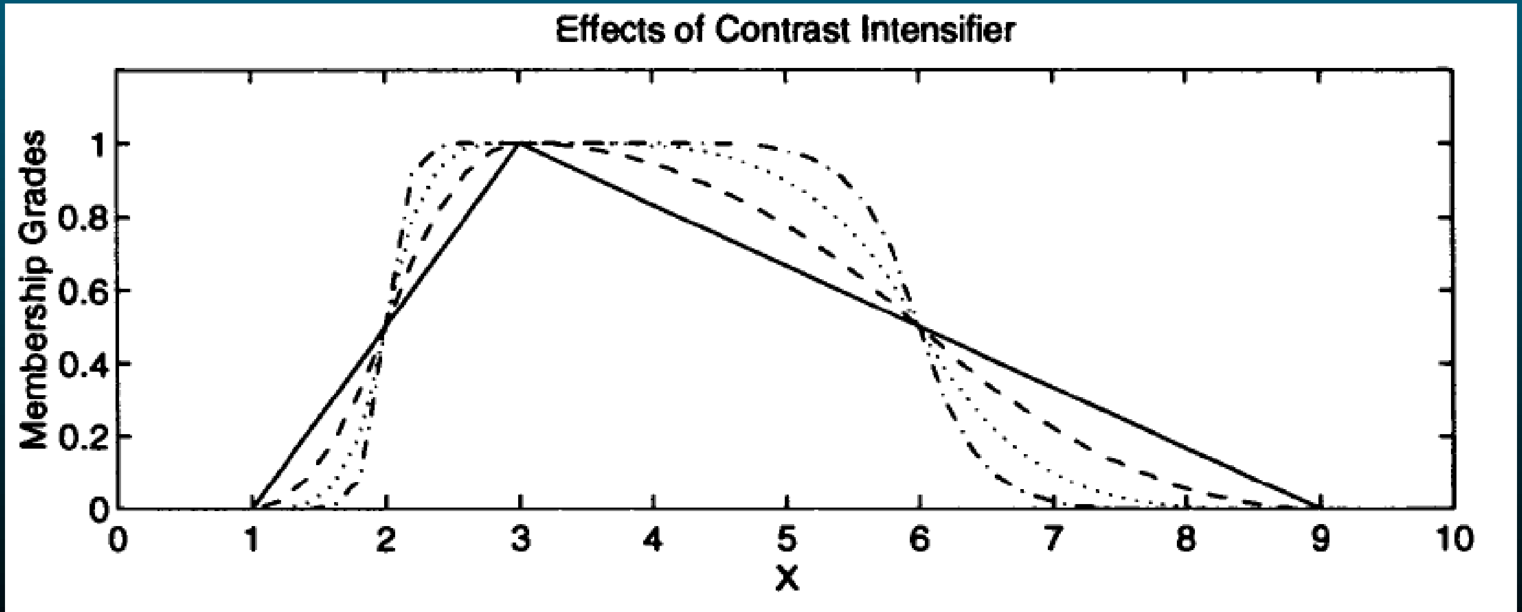
T(age) = {young, not young, very young, ...
middle aged, not middle aged, ...
old, not old, very old, more or less old, ...
not very young and not very old, ...}

Operations on Linguistic Values

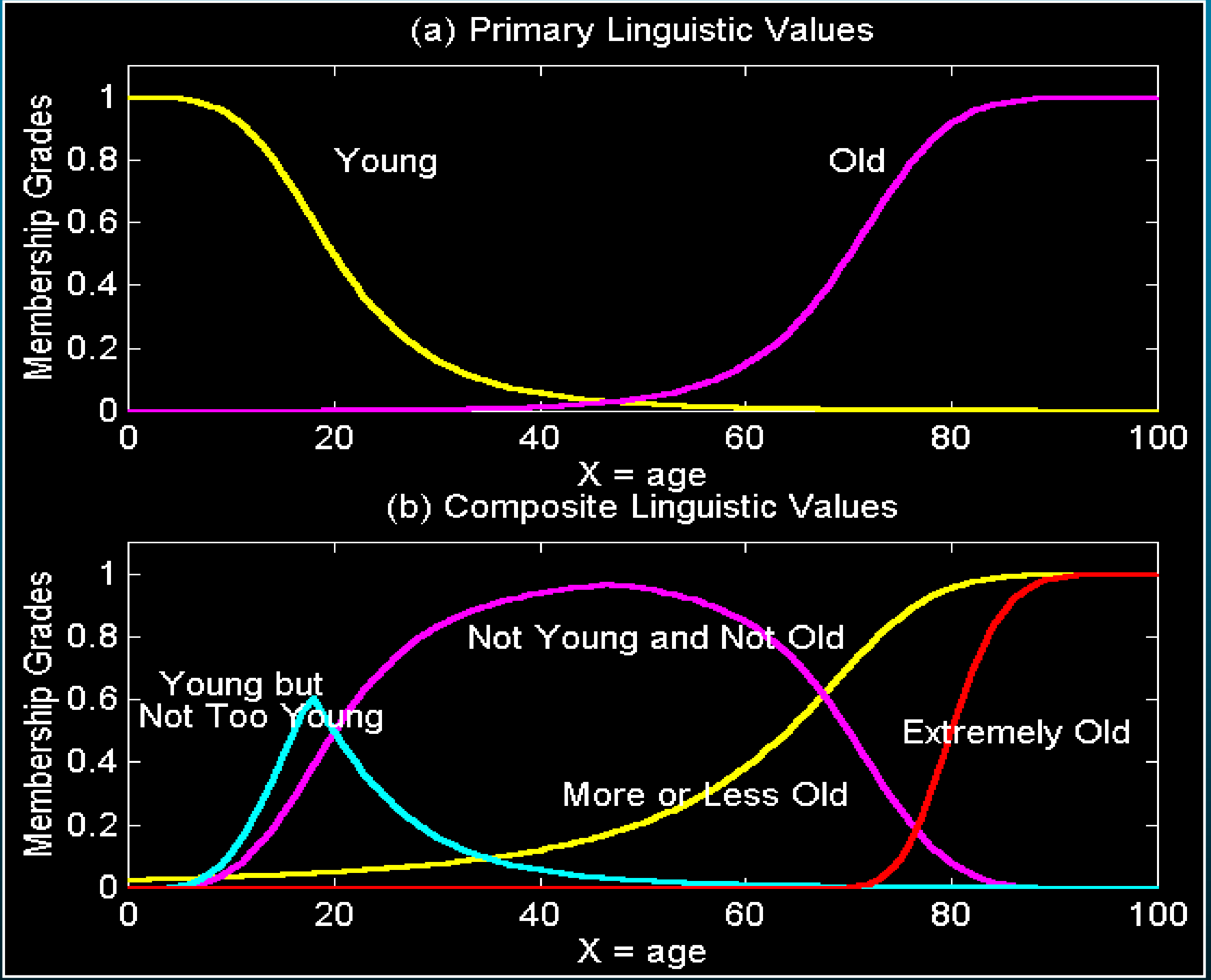
Concentration: ⇒ $CON(A) = A^2$ (very)

Dilation: ⇒ $DIL(A) = A^{0.5}$ (more or less)

Contrast intensification: ⇒ $$INT(A) = \begin{cases} 2A^2, & 0 \le \mu_A(x) \le 0.5 \\ \neg 2(\neg A)^2, & 0.5 \le \mu_A(x) \le 1 \end{cases}$$



Linguistic Values (Terms)



How are these derived from the above MFs?

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Fuzzy If-Then Rules

A fuzzy if-then rule(also known as fuzzy rule, fuzzy implication, or fuzzy conditional statement) assume the form:

“If x is A then y is B”

abbreviated as A->B

Often “x is A” is called the antecedent or premise, while “y is B” is called consequence or conclusion.

This is interpreted as a fuzzy set

Examples:

- If pressure is high, then volume is small.
- If the road is slippery, then driving is dangerous.
- If a tomato is red, then it is ripe.
- If the speed is high, then apply the brake a little.

Fuzzy If-Then Rules

If we interpret A->B as A coupled with B, then

$$R = A \to B = A \times B = \int_{X \times Y} \mu_A(x) \,\tilde{*}\, \mu_B(y)/(x, y),$$

where $\tilde{*}$ is a T-norm operator and $A \to B$ is used again to represent the fuzzy relation R. On the other hand, if $A \to B$ is interpreted as A **entails** B, then it can be written as four different formulas:

- Material implication:

$$R = A \to B = \neg A \cup B. \tag{3.19}$$

- Propositional calculus:

$$R = A \to B = \neg A \cup (A \cap B). \tag{3.20}$$

- Extended propositional calculus:

$$R = A \to B = (\neg A \cap \neg B) \cup B. \tag{3.21}$$

- Generalization of modus ponens:

$$\mu_R(x, y) = \sup\{c \mid \mu_A(x) \,\tilde{*}\, c \leq \mu_B(y) \text{ and } 0 \leq c \leq 1\}, \tag{3.22}$$

where $R = A \to B$ and $\tilde{*}$ is a T-norm operator.

Fuzzy If-Then Rules

Example:

if (profession is athlete) then (fitness is high)

Coupling: Athletes, and only athletes, have high fitness.

The "if" statement (antecedent) is a necessary and sufficient condition.

Entailing: Athletes have high fitness, and non-athletes may or may not have high fitness.

The "if" statement (antecedent) is a sufficient but not necessary condition.

Fuzzy If-Then Rules

Two ways to interpret "If x is A then y is B":

- **A coupled with B: (*A and B* – T-norm)**

$$R = A \to B = A \times B = \int \mu_A(x) \tilde{*} \mu_B(y) | (x, y)$$

- **A entails B: (*not A or B*)**
  - Material implication $\neg A \cup B$
  - Propositional calculus $\neg A \cup (A \cap B)$
  - Extended propositional calculus $(\neg A \cap \neg B) \cup B$
  - Generalization of modus ponens $\mu_R(x, y) = \sup\{c \mid \mu_A(x) \tilde{*} c \leq \mu_B(y) \text{ and } 0 \leq c \leq 1\}$

Fuzzy If-Then Rules

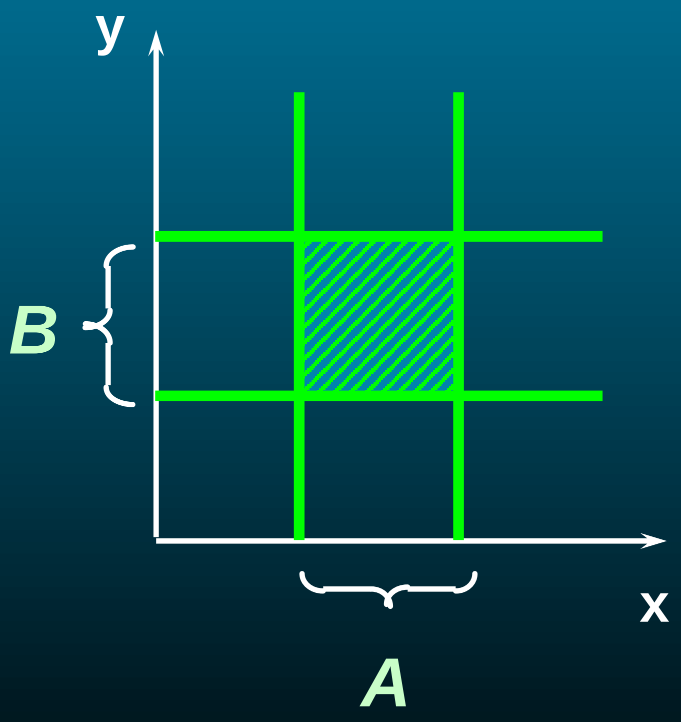
Two ways to interpret "If x is A then y is B"

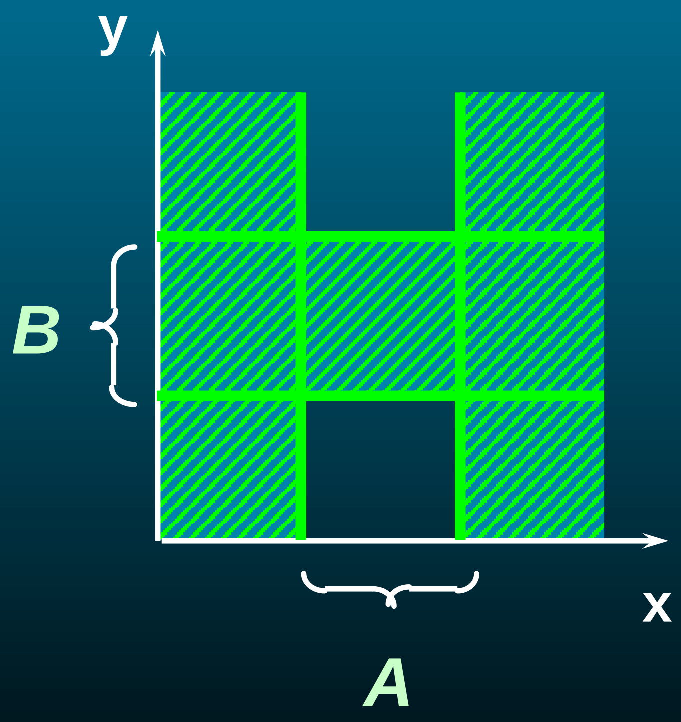
A is coupled with B:

(x is A) ∩ (y is B)

A entails B:

(x is not A) ∪ (y is B)





"A coupled with B" as a meaning of A->B

- $R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y)/(x,y)$, or $f_c(a,b) = a \wedge b$. This relation, which was proposed by Mamdani [3], results from using the min operator for conjunction.
- $R_p = A \times B = \int_{X \times Y} \mu_A(x)\mu_B(y)/(x,y)$, or $f_p(a,b) = ab$. Proposed by Larsen [2], this relation is based on using the algebraic product operator for conjunction.
- $R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y)/(x,y) = \int_{X \times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1)/(x,y)$, or $f_{bp}(a,b) = 0 \vee (a+b-1)$. This formula employs the bounded product operator for conjunction.
- $R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\cdot} \mu_B(y)/(x,y)$, or

$$f(a,b) = a \hat{\cdot} b = \begin{cases} a & \text{if } b = 1. \\ b & \text{if } a = 1. \\ 0 & \text{otherwise.} \end{cases}$$

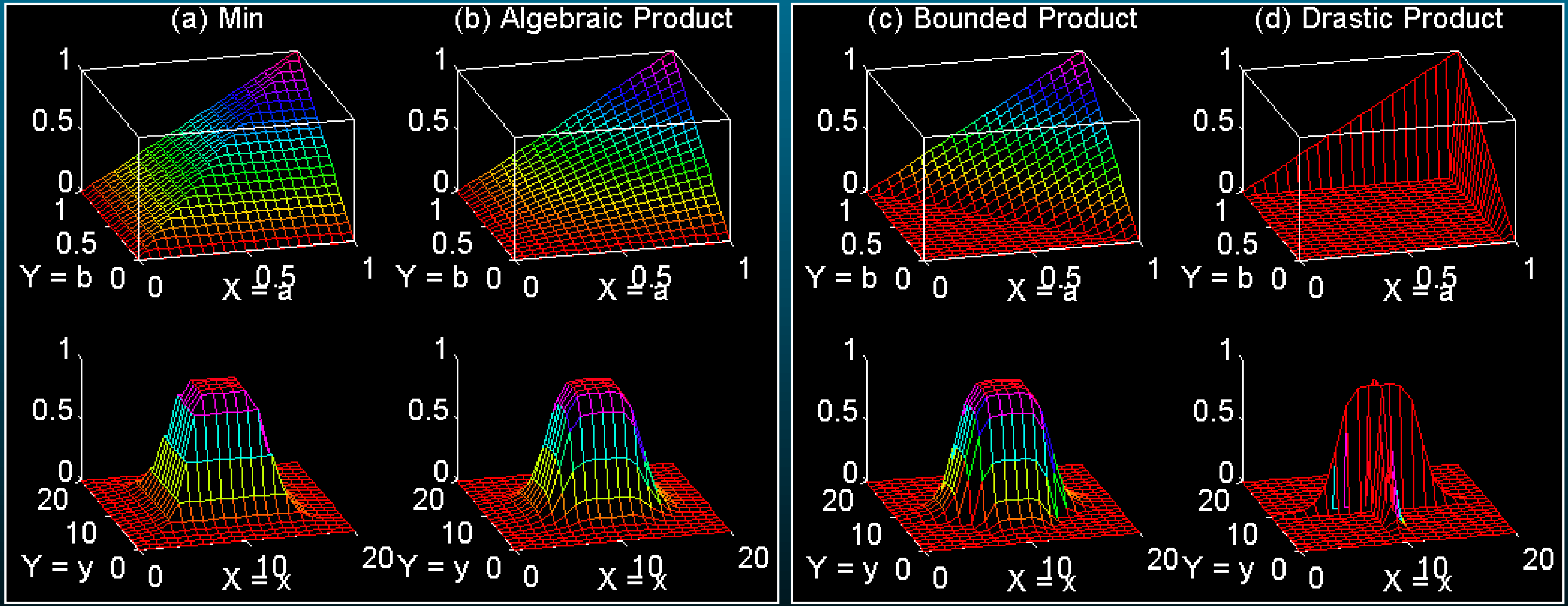
This formula uses the drastic product operator for conjunction.

Fuzzy If-Then Rules

Fuzzy implication $\mu_R(x,y) = f(\mu_A(x), \mu_B(y)) = f(a,b)$

A coupled with B (bell-shaped MFs, T-norm operators)

Example: only fit athletes satisfy the rule



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" A entails B" as a meaning of A->B

- $R_a = \neg A \cup B = \int_{X\times Y} 1 \wedge (1-\mu_A(x)+\mu_B(y))/(x,y)$, or $f_a(a,b) = 1 \wedge (1-a+b)$. This is Zadeh's arithmetic rule, which follows Equation (3.19) by using the bounded sum operator for $\cup$.

- $R_{mm} = \neg A \cup (A \cap B) = \int_{X\times Y} (1-\mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))/(x,y)$, or $f_m(a,b) = (1-a) \vee (a \wedge b)$. This is Zadeh's max-min rule, which follows Equation (3.20) by using min for $\cap$ and max for $\cup$.

- $R_s = \neg A \cup B = \int_{X\times Y} (1-\mu_A(x)) \vee \mu_B(x)$, or $f_s(a,b) = (1-a) \vee b$. This is Boolean fuzzy implication using max for $\cup$.

- $R_\triangle = \int_{X\times Y} (\mu_A(x) \tilde{<} \mu_B(y))/(x,y)$, where

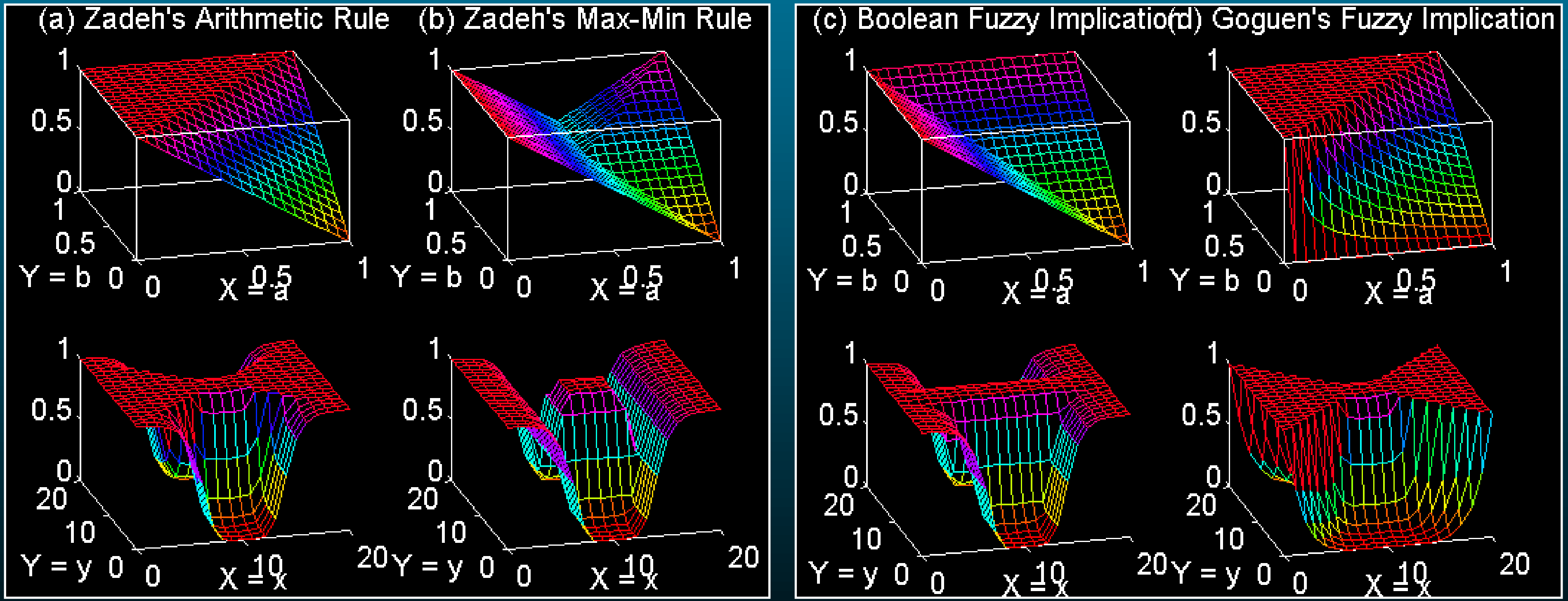
$$a \tilde{<} b = \begin{cases} 1 & \text{if } a \le b. \\ b/a & \text{if } a > b. \end{cases}$$

Fuzzy If-Then Rules

A entails B (bell-shaped MFs)

Arithmetic rule: (x is not A) ∪ (y is B) → (1 – x) + y

Example: everyone except non-fit athletes satisfies the rule



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- Same idea as max-min composition
- Suppose that we have a curve *$y = f(x)$ that regulate the relation between x and y. when we are given $x = a$ then we can infer that $y = b=f(a)$;*
- *a generalization process would allow a to be an interval and f(x) to be an interval valued function.*
- *By cylindrical extension we get the projection of I onto the y-axis yields the interval $y=b$.*
- *For further generalization we assume that F is a fuzzy relation on XxY and A is a fuzzy set of X. To find the resulting fuzzy set B again cylindrical extension c(A) with base A.*
- *The intersection of c(A) and F forms the analog of the region of intersection I. By projecting $c(A) \cap F$ onto the y-axis, we infer y as a fuzzy set B on the y-axis.*

- *Spefically, let μ_A, $\mu_{c(A)}$, μ_B and μ_F be the MFs of A, c(A), B, and F, respectively, where $\mu_{c(A)}$ is related to μ_A through $\mu_{c(A)}(x, y)= \mu_A(x)$.*

Then

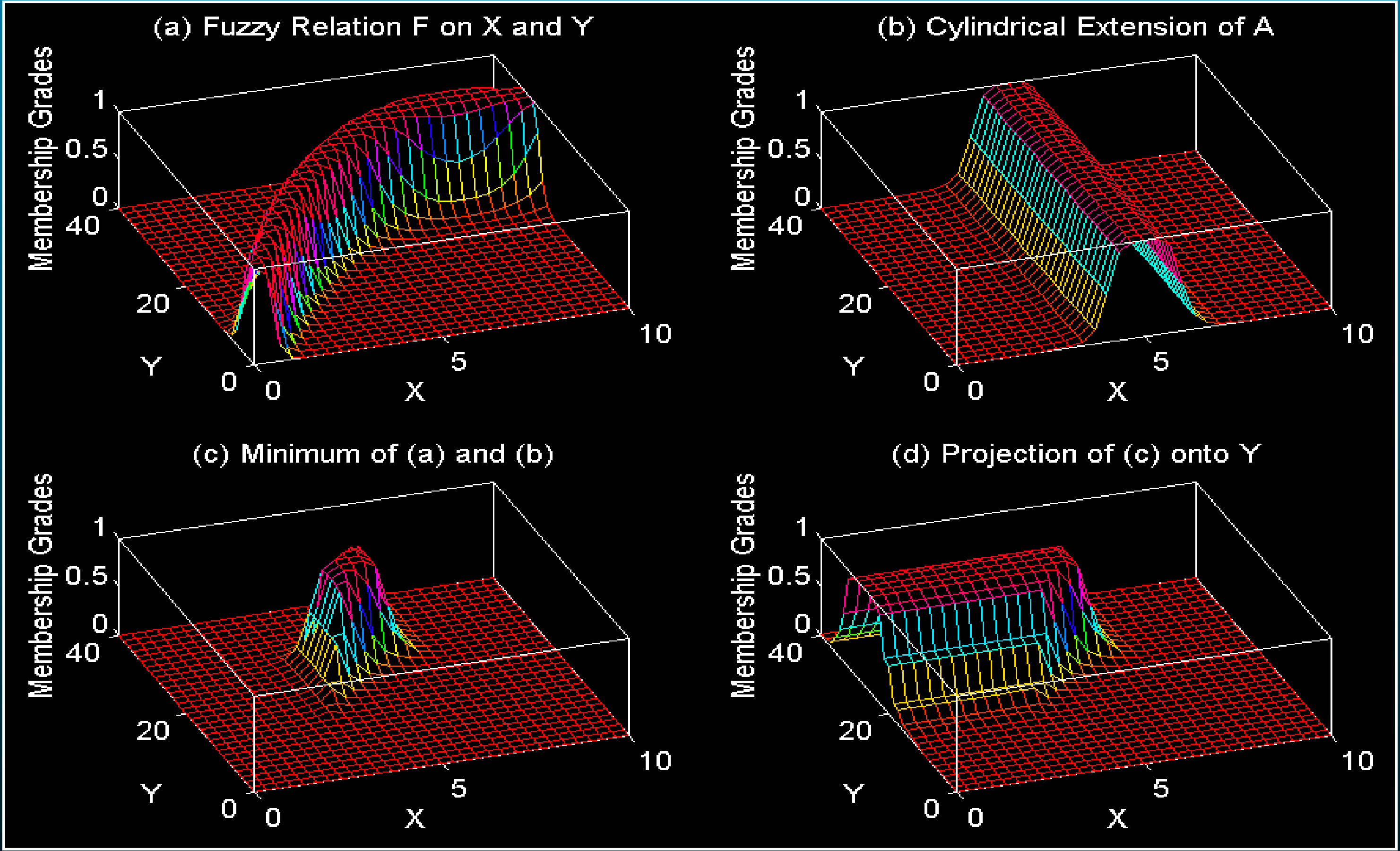
$$\begin{aligned}\mu_{c(A)\cap F}(x,y) &= \min[\mu_{c(A)}(x,y), \mu_F(x,y)] \\ &= \min[\mu_A(x), \mu_F(x,y)].\end{aligned}$$

By projecting $c(A) \cap F$ onto the y-axis, we have

$$\begin{aligned}\mu_B(y) &= \max_x \min[\mu_A(x), \mu_F(x,y)] \\ &= \vee_x[\mu_A(x) \wedge \mu_F(x,y)].\end{aligned}$$

Compositional Rule of Inference

A is a fuzzy set of x and *y* = *f(x)* is a fuzzy relation:



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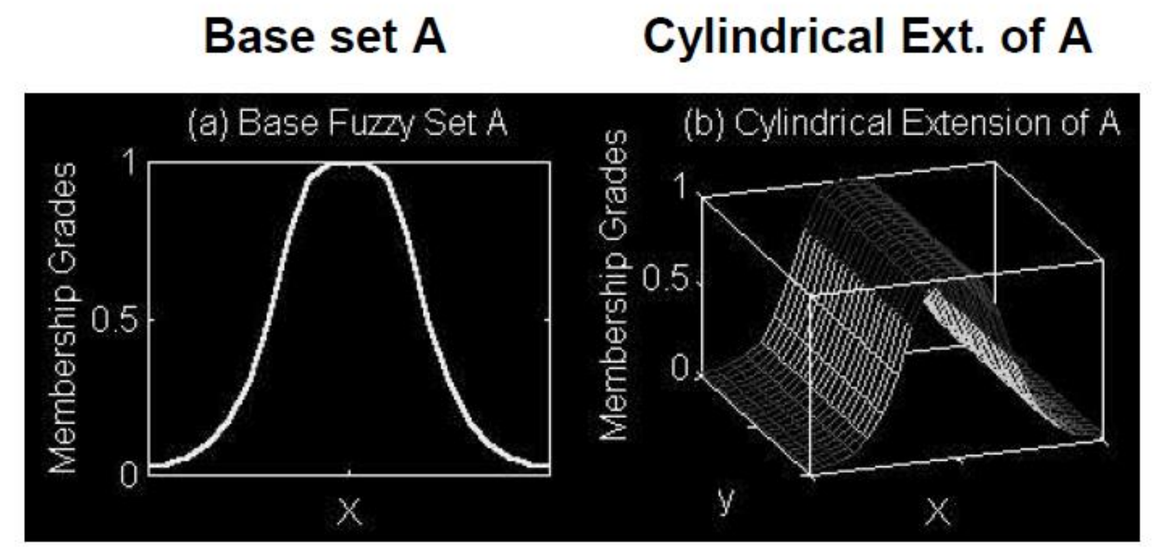
To find the resulting interval y=b (which corresponds to x=a)

- construct a cylindrical extension of a
- find intersection with curve
- project intersection to y-axis

Recall :cylindrical extension



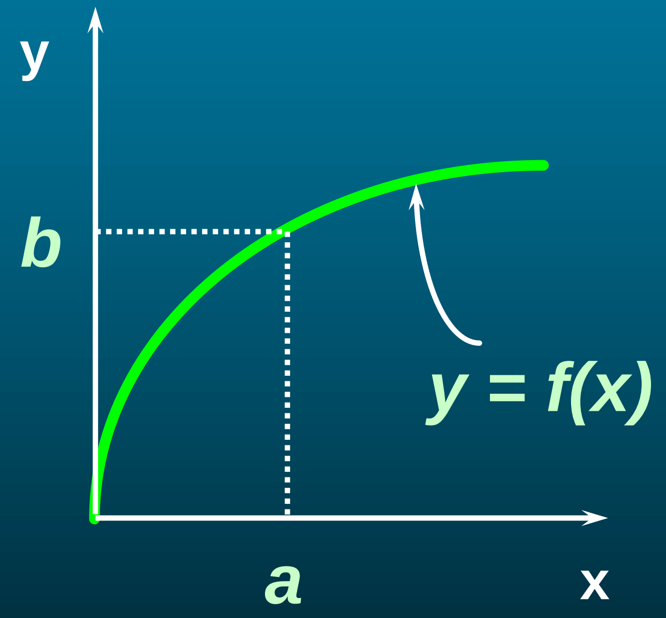
$$\mu_{c(A)}(x,y) = \mu_A(x)$$

Compositional Rule of Inference

Same idea as max-min composition

Derivation of $y = b$ from $x = a$ and $y = f(x)$:

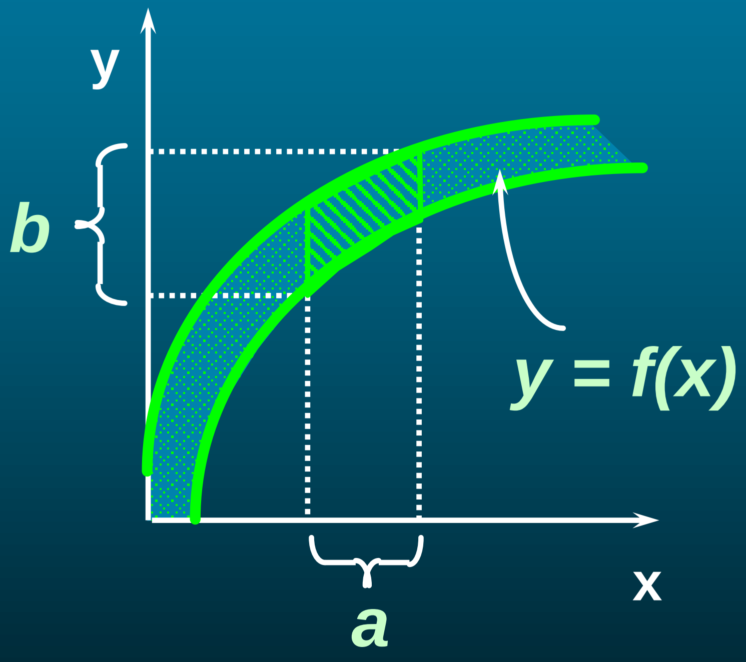
Derivation of $y = b$ from $x = a$ and $y = f(x)$:



a and b : points

$y = f(x)$: a curve

Crisp : if x = a, then y=b



a and b : intervals

$y = f(x)$: interval-valued function

Fuzzy : if (x is a) then (y is b)

Fuzzy Reasoning

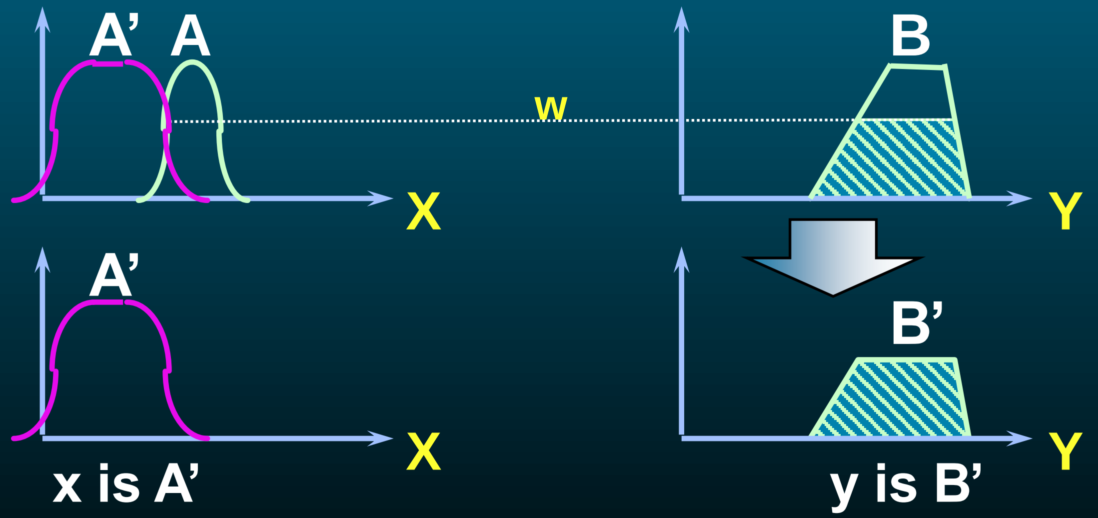
Modus Ponens:

Fact:	x is A
Rule:	IF x is A THEN y is B
Conclusion:	y is B

Single rule with single antecedent

- Rule: if x is A then y is B
- Premise: x is A', where A' is close to A
- Conclusion: y is B'

Use max of intersection between A and A' to get B'

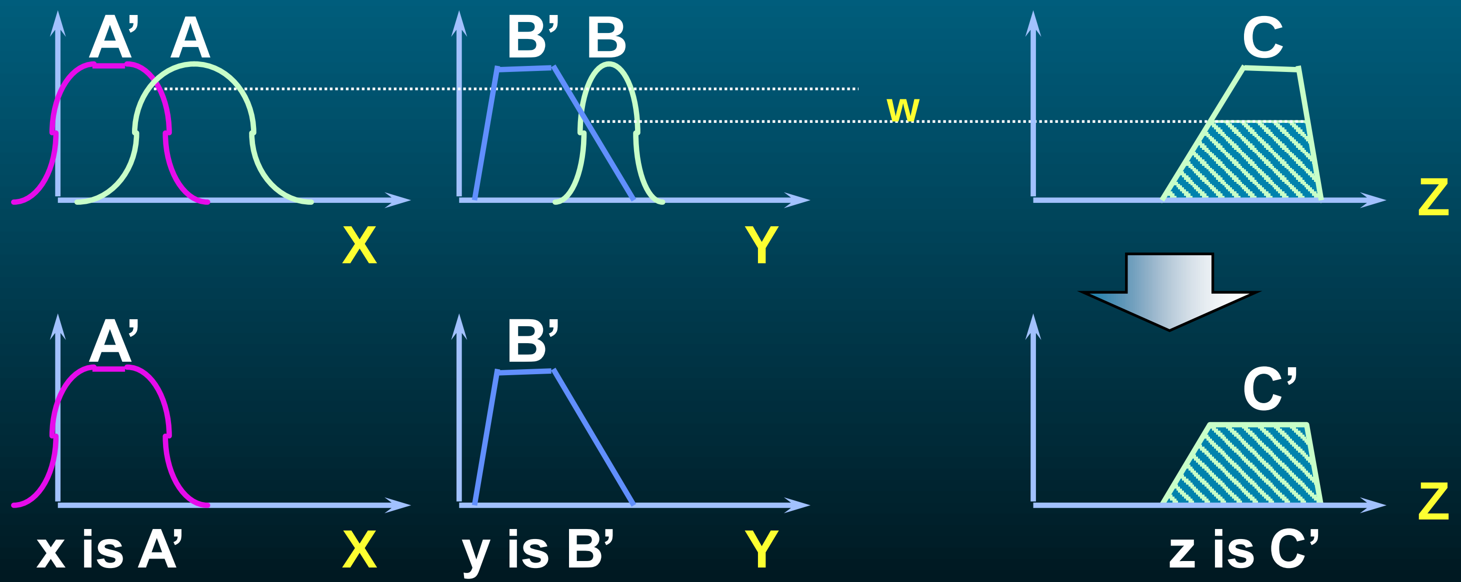


Fuzzy Reasoning

Single rule with multiple antecedents

- Rule: if x is A and y is B then z is C
- Premise: x is A' and y is B'
- Conclusion: z is C'

Use min of (A ∩ A') and (B ∩ B') to get C'



Fuzzy Reasoning

Multiple rules with multiple antecedents

Rule 1: if x is A1 and y is B1 then z is C1

Rule 2: if x is A2 and y is B2 then z is C2

Premise: x is A’ and y is B’

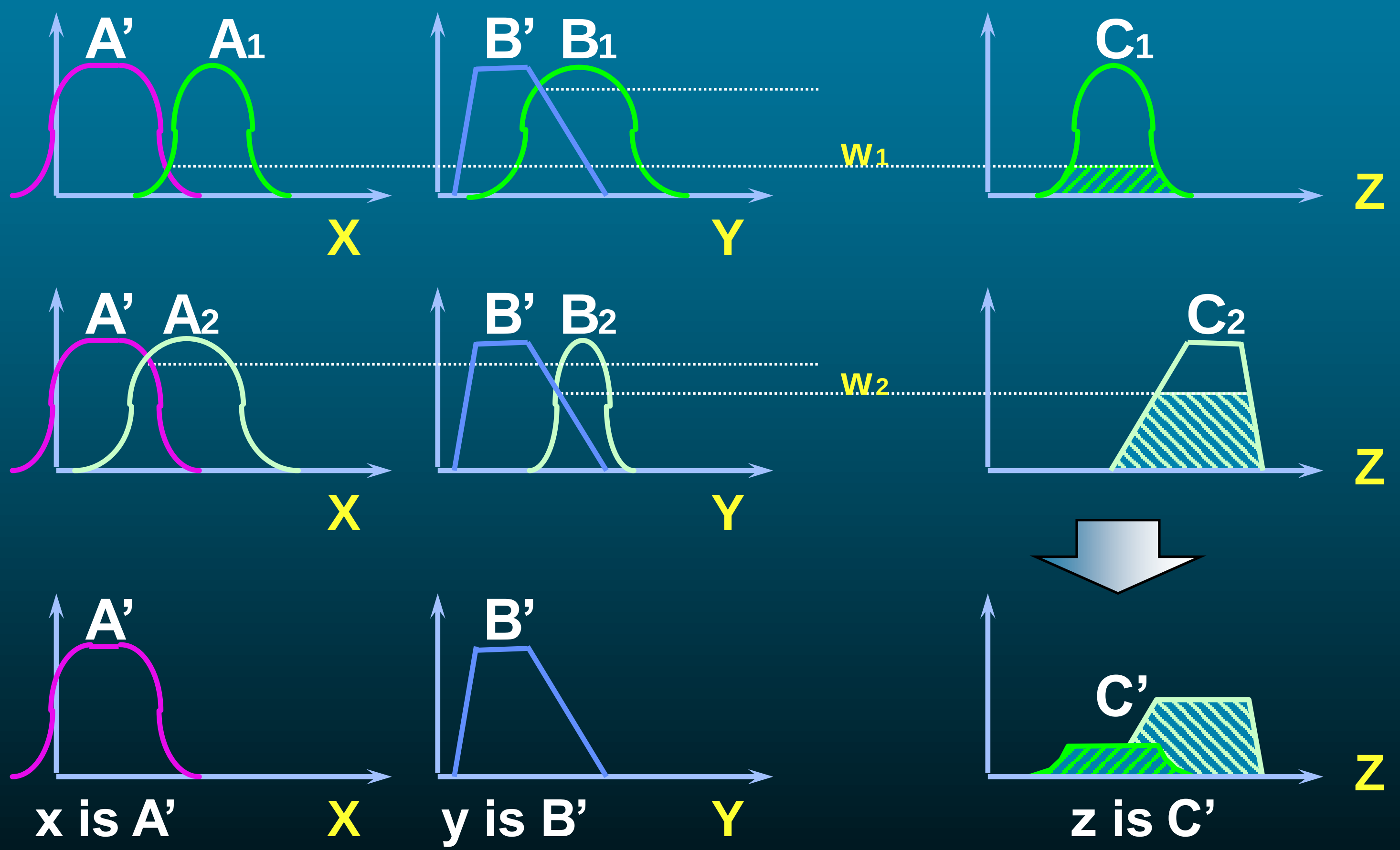
Conclusion: z is C’

Use previous slide to get C_1’ and C_2’

Use max of C_1’ and C_2’ to get C’ (next slide)

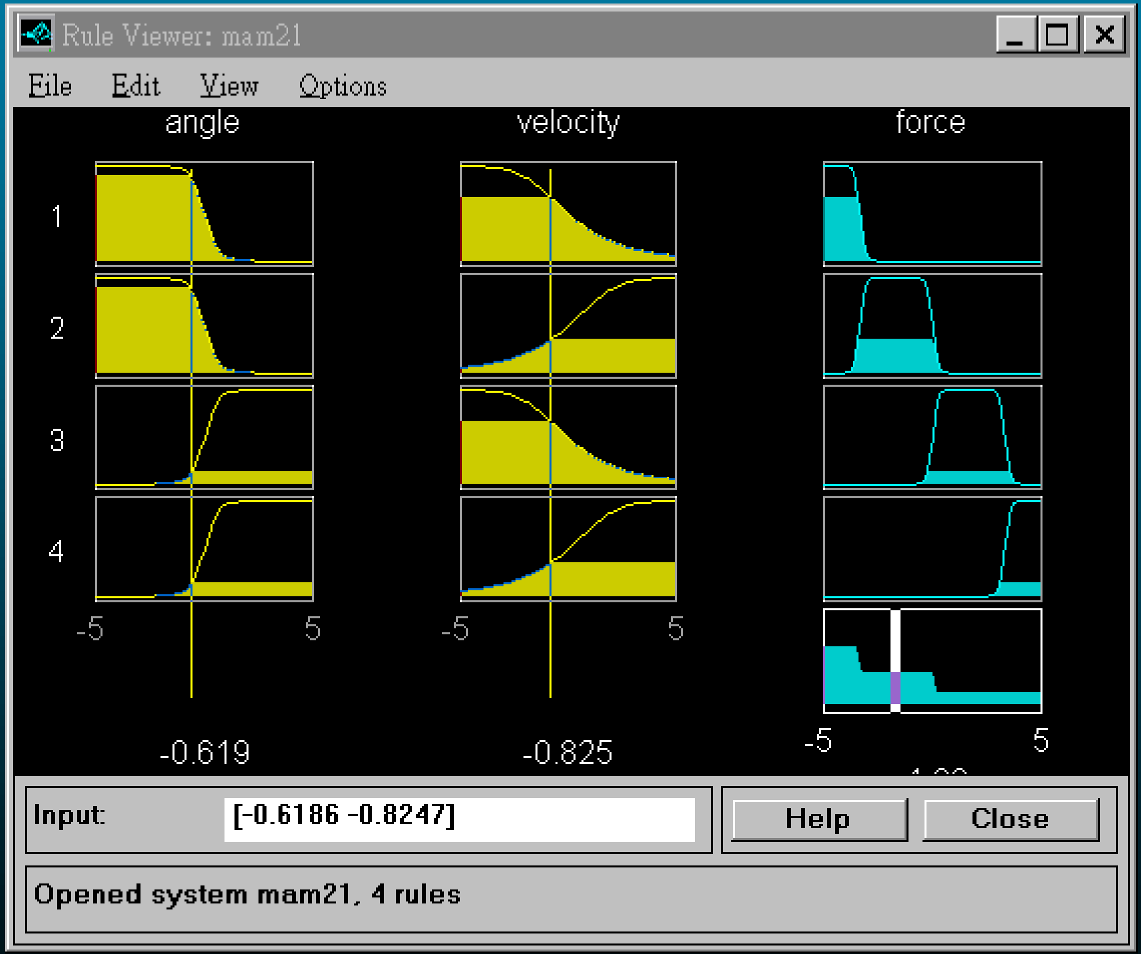
Fuzzy Reasoning

Multiple rules with multiple antecedents



Fuzzy Reasoning: MATLAB Demo

>> ruleview mam21 (Matlab Fuzzy Logic Toolbox)



Other Variants

Some terminology:

- Degrees of compatibility (match between input variables and fuzzy input MFs)
- Firing strength calculation (we used MIN)
- Qualified (induced) MFs (combine firing strength with fuzzy outputs)
- Overall output MF (we used MAX)