Ch. 4: Fuzzy Inference Systems

- Based on fuzzy set theory, fuzzy IF-THEN rules and fuzzy reasoning.
- It is used for automatic control, data classification, decision analysis, expert systems, time series prediction, robotics, and pattern recognition.
- Due to multidisciplinary nature, the fuzzy inference system is also known as , fuzzy rule-based system, fuzzy expert system, fuzzy model, fuzzy associative memory (FAM) , Fuzzy logic controller and Fuzzy system
- System can take either fuzzy inputs or crisp inputs, but the outputs it produces are almost always fuzzy sets.
- Sometimes, we need to have a crisp output, especially where system used as a controller.

Three components:

- Rule base (fuzzy rules)
- Database (dictionary; membership functions)
- Reasoning/inference mechanism

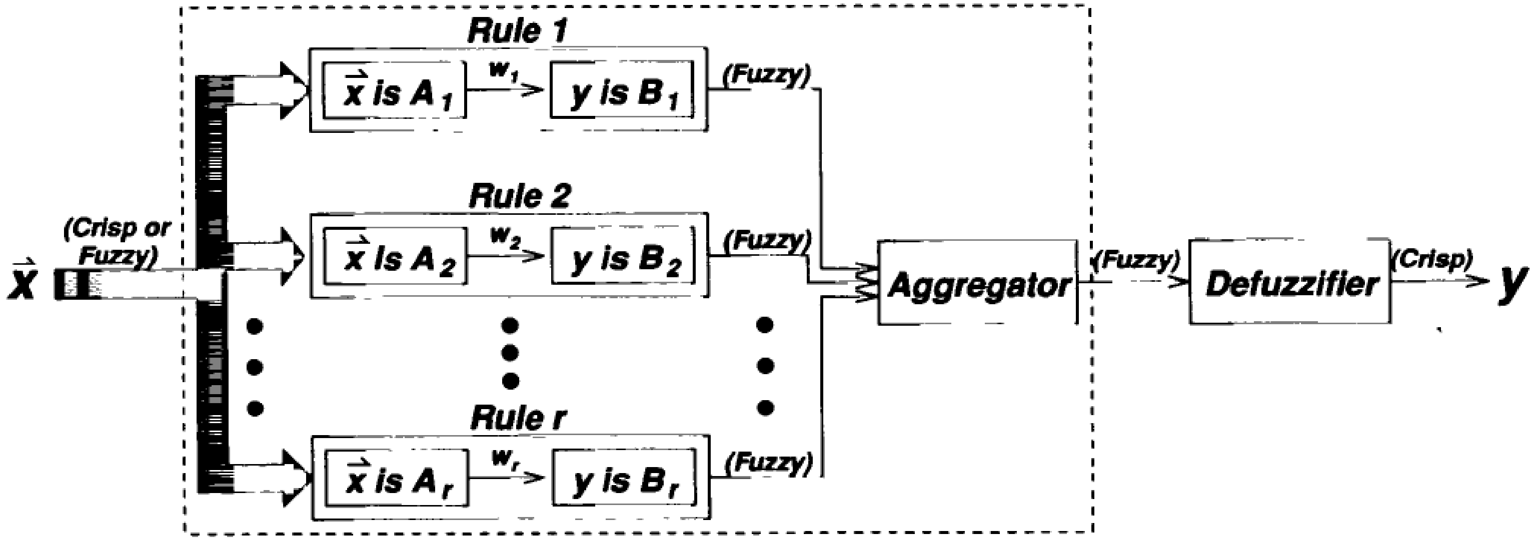
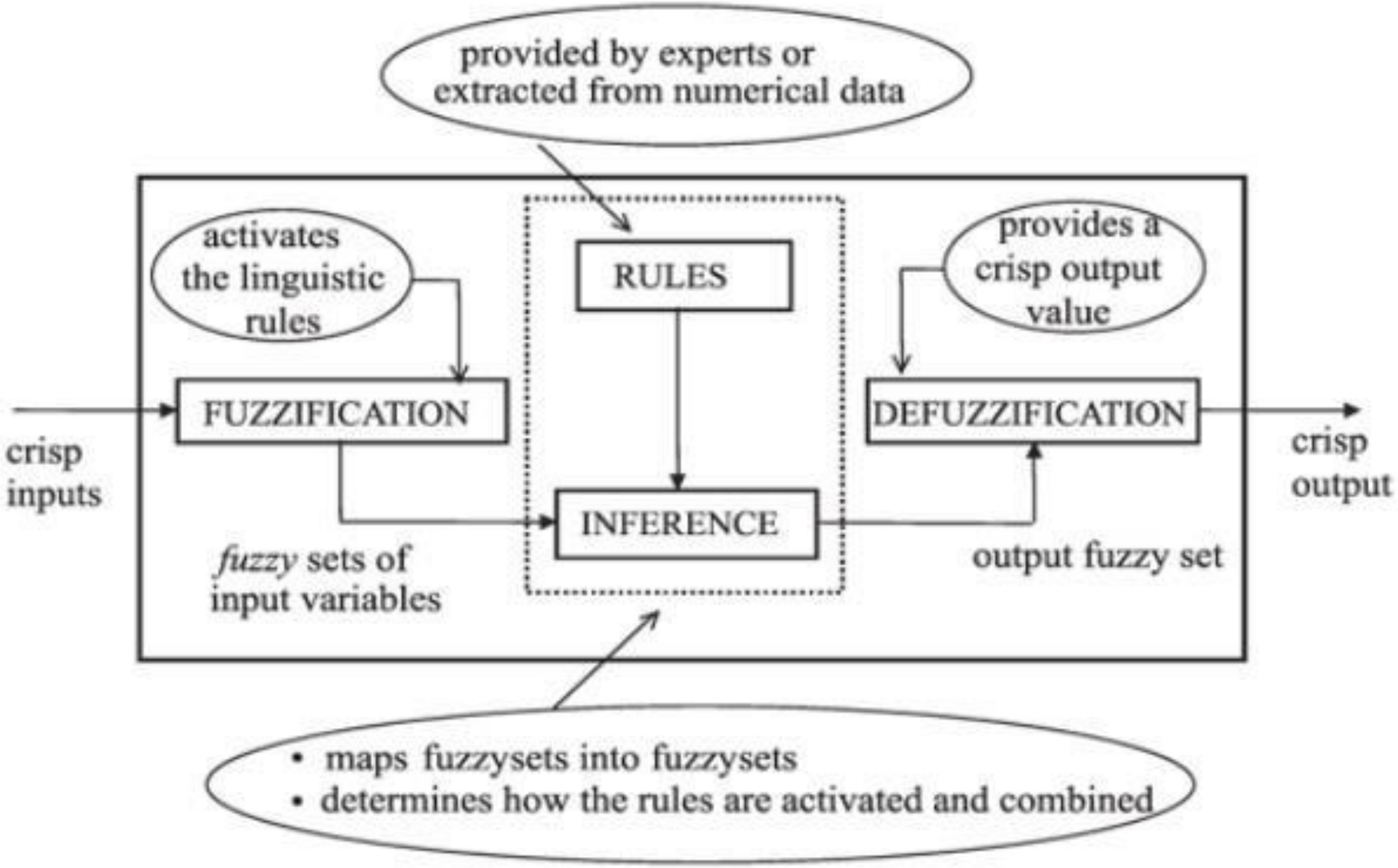


Figure 4.1. *Block diagram for a fuzzy inference system.*


provided by experts or extracted from numerical data
activates the linguistic rules
RULES
provides a crisp output value
FUZZIFICATION
DEFUZZIFICATION
crisp inputs
crisp output
INFERENCE
fuzzy sets of input variables
output fuzzy set
• maps fuzzysets into fuzzysets
• determines how the rules are activated and combined

Outline

1. Fuzzy Inference Systems
   a) Ebrahim Mamdani (University of London)
   b) Sugeno, or TSK (Takagi/Sugeno/Kang)
   c) Tsukamoto
2. Input Space Partitioning
3. • Fuzzy modeling: Required tasks
   a) Surface structure identification
   b) Deep structure identification

Mamdani Fuzzy Models

- It was the first attempt to control the steam engine and boiler combination by set of linguistic control rules.

- Fig.4.2 illustrate how a two-rule Mamdani fuzzy inference system derived the overall output z when subjected to two crisp inputs x and y.

- If we adopt max and algebraic product instead of T-norm and T-conorm operators and use max-product composition instead of max-min composition, then the resulting fuzzy reasoning is shown in figure 4.3. where the inferred output of each rule is a fuzzy set scaled down by its firing strength via algebraic product.

- In Mamdani's application , two fuzzy inference systems were used as two controllers to generate the heat input to the boiler and throttle opening of the engine tht regulate the steam pressure in the boiler and speed of the engine.

- Since the plant takes only crisp values as inputs, we have use defuzzifier to convert a fuzzy set to a crisp value.

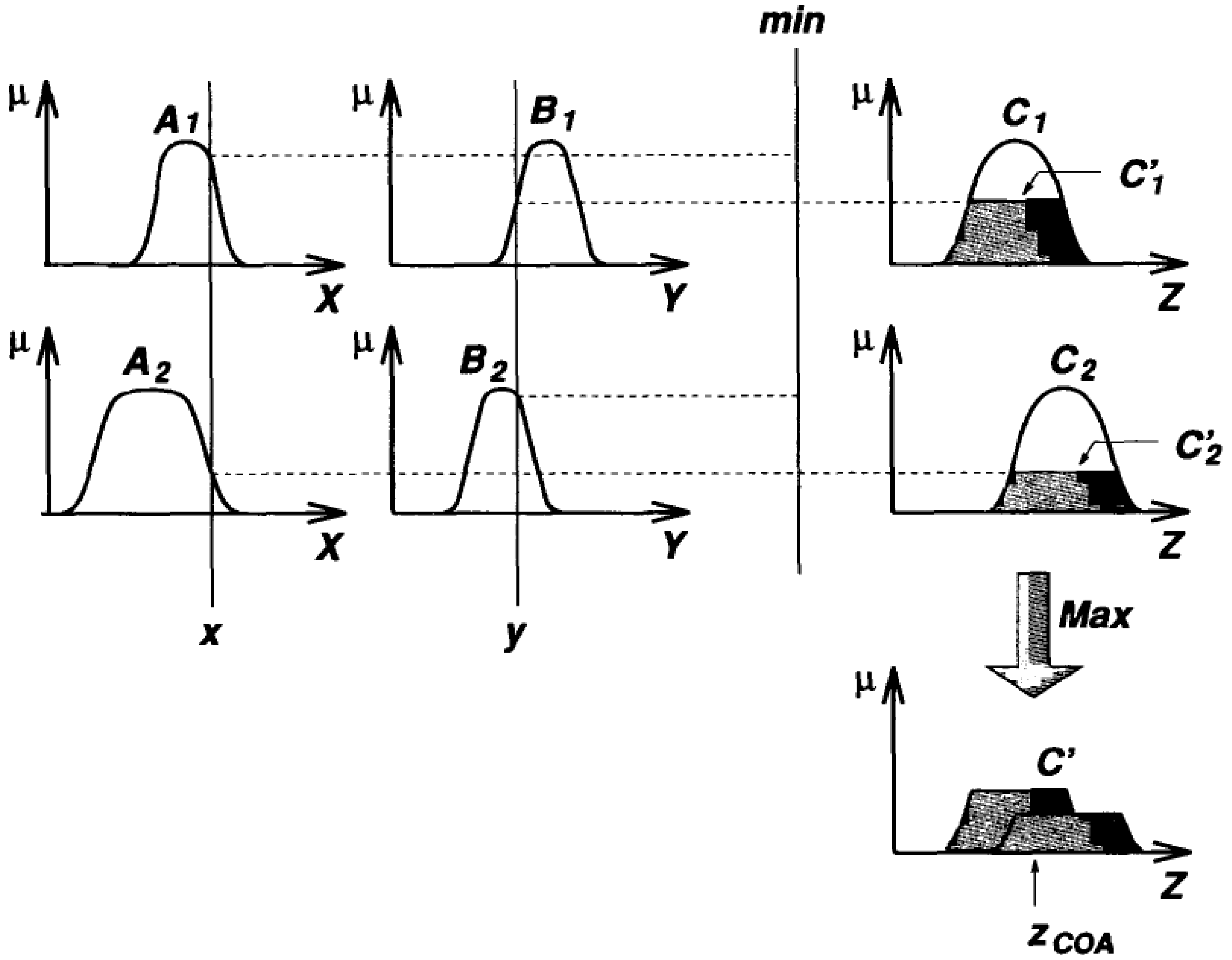


Figure 4.2. *The Mamdani fuzzy inference system using* min *and* max *for T-norm and T-conorm operators, respectively.*

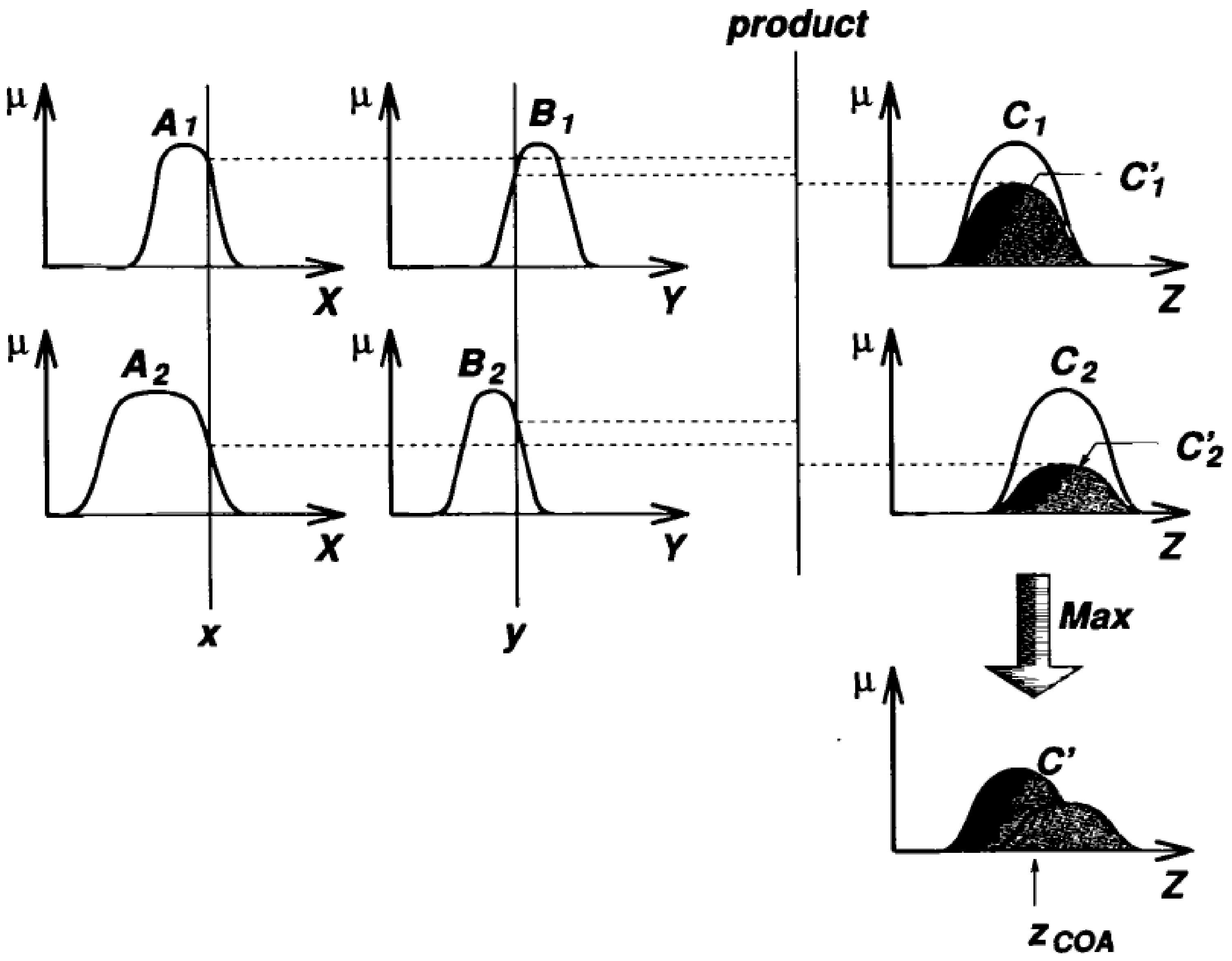


Figure 4.3. *The Mamdani fuzzy inference system using product and* max *for T-norm and T-conorm operators, respectively.*

Defuzzification

- It is the way to extract crisp value from fuzzy set.
- In general there are five methods for defuzzifying a fuzzy set A of the universe of discourse Z

- Centroid of area z_{COA}:

$$z_{\mathrm{COA}} = \frac{\int_Z \mu_A(z) z \, dz}{\int_Z \mu_A(z) \, dz}, \qquad (4.1)$$

where $\mu_A(z)$ is the aggregated output MF. This is the most widely adopted defuzzification strategy, which is reminiscent of the calculation of expected values of probability distributions.

- Bisector of area z_{BOA}: z_{BOA} satisfies

$$\int_{\alpha}^{z_{\mathrm{BOA}}} \mu_A(z)\,dz = \int_{z_{\mathrm{BOA}}}^{\beta} \mu_A(z)\,dz, \tag{4.2}$$

  where $\alpha = \min\{z|z \in Z\}$ and $\beta = \max\{z|z \in Z\}$. That is, the vertical line $z = z_{\mathrm{BOA}}$ partitions the region between $z = \alpha$, $z = \beta$, $y = 0$ and $y = \mu_A(z)$ into two regions with the same area.

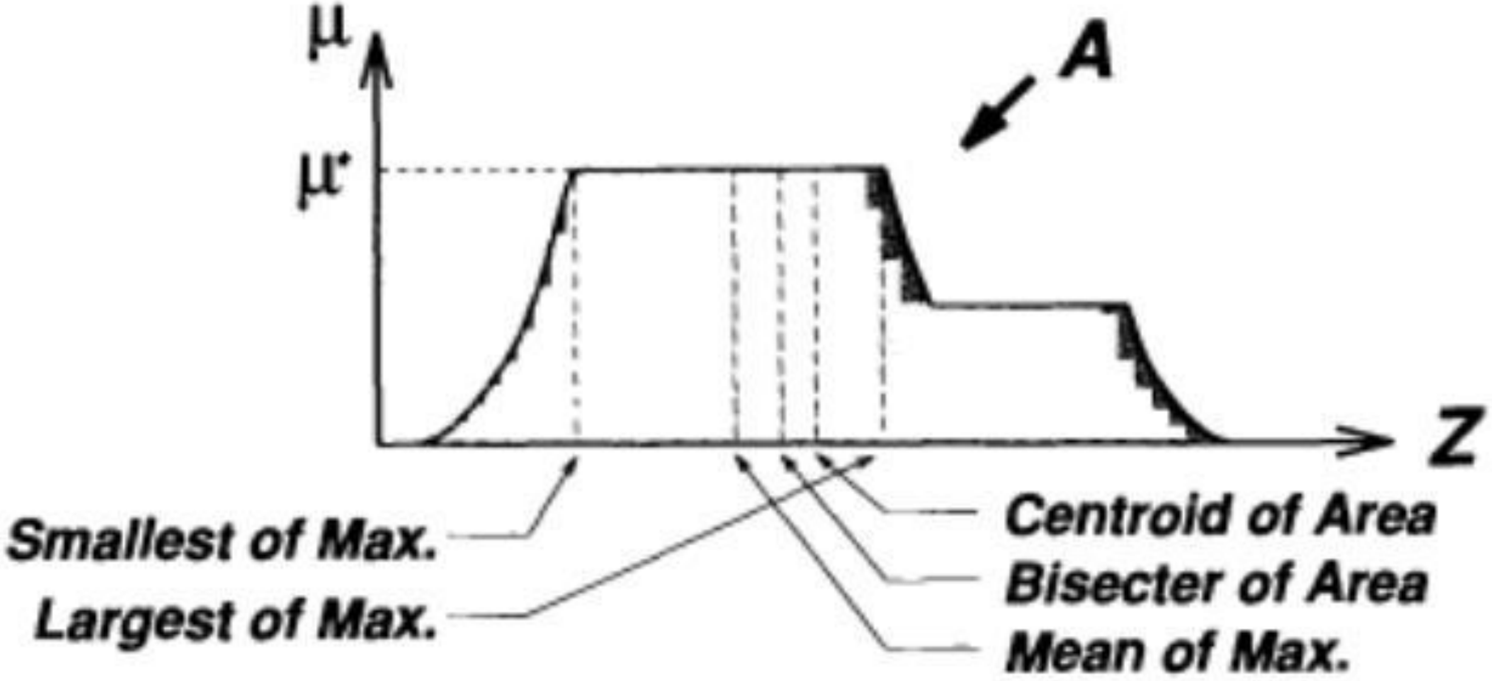
- Mean of maximum z_{MOM}: z_{MOM} is the average of the maximizing z at which the MF reach a maximum μ^*. In symbols,

$$z_{\mathrm{MOM}} = \frac{\int_{Z'} z\,dz}{\int_{Z'} dz}, \tag{4.3}$$

- Smallest of maximum z_{SOM}: z_{SOM} is the minimum (in terms of magnitude) of the maximizing z.

- Largest of maximum z_{LOM}: z_{LOM} is the maximum (in terms of magnitude) of the maximizing z. Because of their obvious bias, z_{SOM} and $_{\mathrm{LOM}}$ are not used as often as the other three defuzzification methods.

Defuzzification



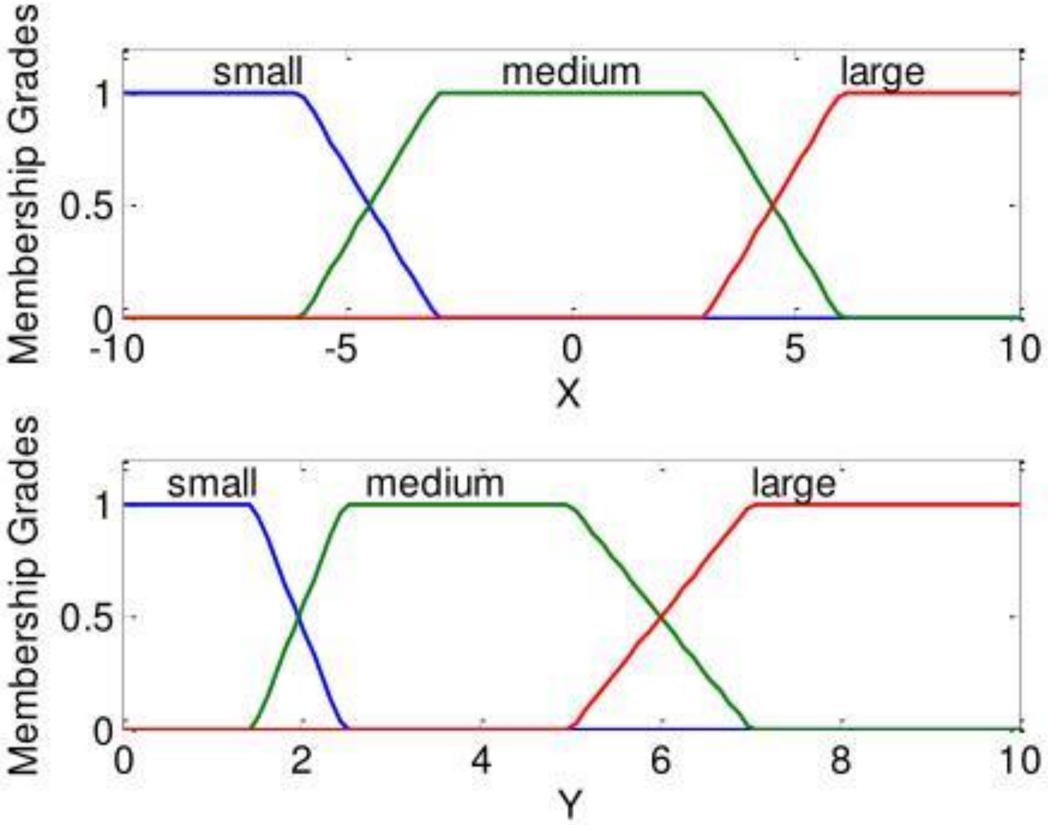
Jang, page 77. See pages 75–76 for formulas.

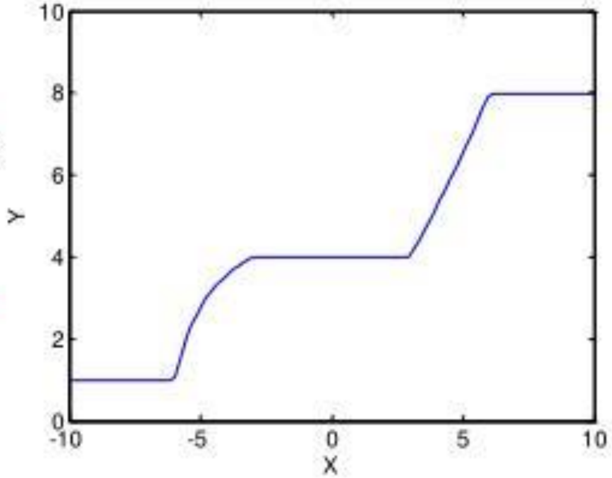
Example 4.1 *Single-input single-output Mamdani fuzzy model*

An example of a single-input single-output Mamdani fuzzy model with three rules can be expressed as

$$\begin{cases} \text{If } X \text{ is small then } Y \text{ is small.} \\ \text{If } X \text{ is medium then } Y \text{ is medium.} \\ \text{If } X \text{ is large then } Y \text{ is large.} \end{cases}$$

Figure 4.5(a) plots the membership functions of input X and output Y, where the input and output universe are $[-10,\ 10]$ and $[0,\ 10]$, respectively. With max-min composition and centroid defuzzification, we can find the overall input-output curve, as shown in Figure 4.5(b). Note that the output variable never reaches the maximum (10) and minimum (0) of the output universe. Instead, the reachable minimum and maximum of the output variable are determined by the centroids of the leftmost and rightmost consequent MFs, respectively.





Results depend on T-norm, S-norm, and defuzzification method

Example 4.1 – mam1.m

- if x is small then y is small
- if x is medium then y is medium
- if x is large then y is large

Review Mam1.m software

Other Variants

Figure 4.2 and 4.3 confirm to the fuzzy reasoning. However, in consideration of computational efficiency or mathematical tractability, a fuzzy inference system may have a certain reasoning mechanism that doesn't follow the strict definition of compositional rule of inference. For instance, one might use product for computing firing strengths(for rules with AND'ed antecedent), min for computing qualified consequent MFs, and max for aggregating them into an overall output MF. Therefore, to completely specify the operation of a Mamdani fuzzy inference system, we need to assign a function for each of the following operators:

- AND operator (usually T-norm) for calculating the firing strength of a rule with AND'ed antecedents.
- OR operator (Usually T-conorm) for calculating the firing strength of a rule with OR'ed antecedents.
- Implication operator (usually T-norm) for calculating qualified consequent MFs based on given firing strength.
- Aggregate operator for transforming an output MF to a crisp single output value.
- Defuzzification operator for transforming an output MF to a crisp single output value.

Theorem: Computational shortcut to Mamdani fuzzy inference system.

Under sum-product composition the output of Mamdani fuzzy inference system with centroid defuzzification is equal to the weighted average of centroids of consequent MFs where each of the weighting factors is equal to the product of a firing strength and consequent MFs area.

Proof: We can proof this theorem by using two rules. By using product and sum for implication and aggregate operators respectively, we have

$$\mu_{C'}(z) = w_1 \mu_{C_1}(z) + w_2 \mu_{C_2}(z).$$

The MF could have value greater than 1 at certain points.

The crisp output under centroid defuzzification is defined

$$\begin{aligned} z_{\mathrm{COA}} &= \frac{\int_Z \mu_{C'}(z) z dz}{\int_Z \mu_{C'}(z) dz} \\ &= \frac{w_1 \int \mu_{C_1}(z) z dz + w_2 \int \mu_{C_2}(z) z dz}{w_1 \int \mu_{C_1}(z) dz + w_2 \int \mu_{C_2}(z) dz} \\ &= \frac{w_1 a_1 z_1 + w_2 a_2 z_2}{w_1 a_1 + w_2 a_2}, \end{aligned}$$

where a_i $(= \int_Z \mu_{C_i}(z) dz)$ and z_i $\left(= \frac{\int_Z \mu_{C_i}(z) z dz}{\int_Z \mu_{C_i}(z) dz}\right)$ are the area and centroid of the consequent MF $\mu_{C_i}(z)$, respectively.

Mamdani Fuzzy Models

y

x

	B_1	B_2	B_3	B_4
A_1	C_1	C_2	C_3	C_4
A_2	C_5	C_6	C_7	C_8
A_3	C_9	C_{10}	C_{11}	C_{12}

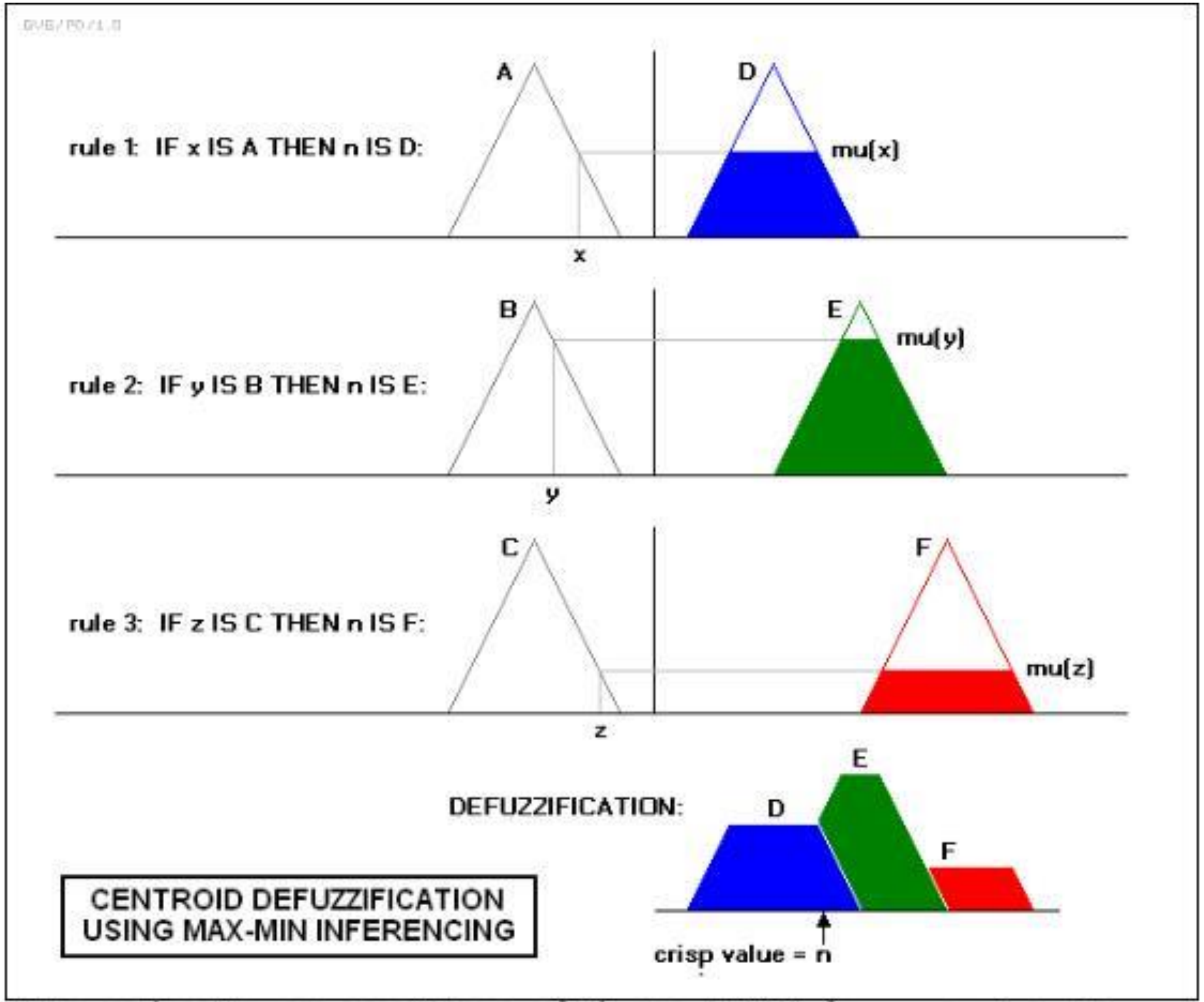
z

Batting average

Fielding percentage

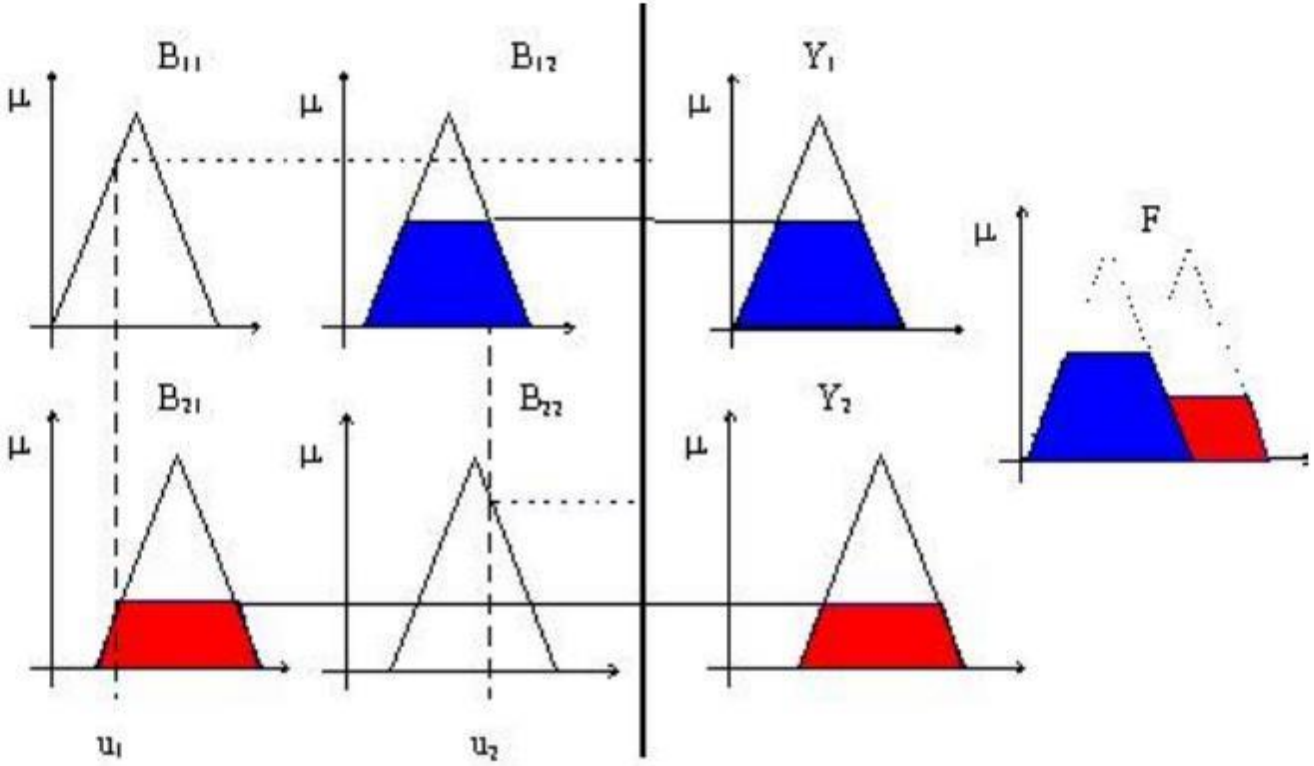
	L	M	H	VH
L	Z	L	M	H
M	L	M	H	VH
H	M	H	VH	VH

Playing time



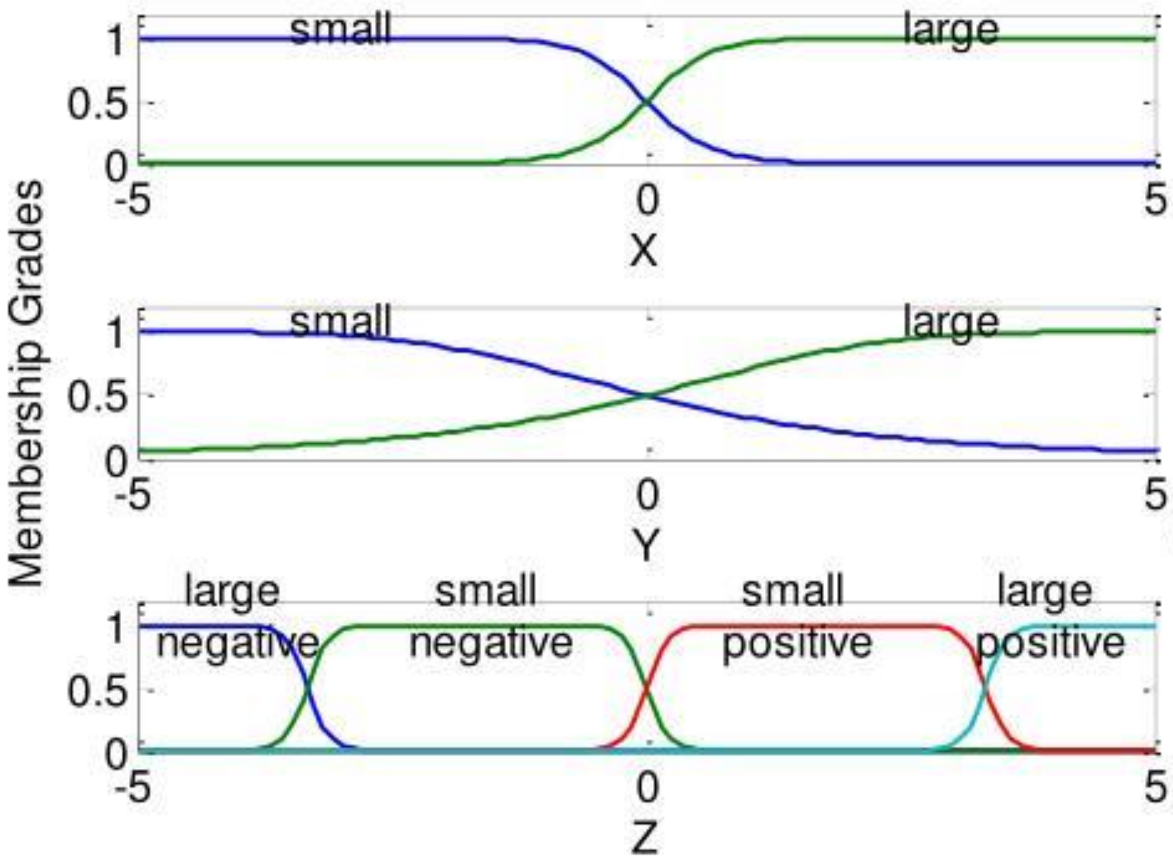
Mamdani composition of three SISO fuzzy outputs

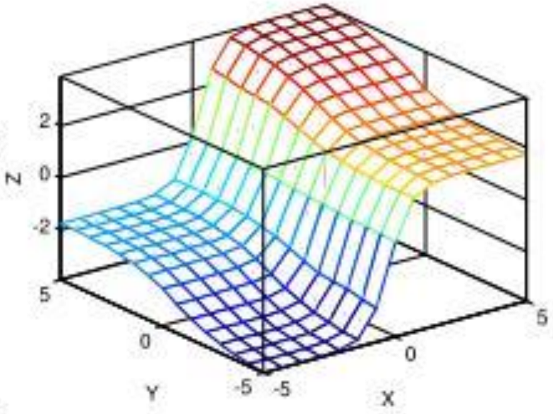
http://en.wikipedia.org/wiki/Fuzzy_control_system



Mamdani composition of two-rule fuzzy system max-min inferencing

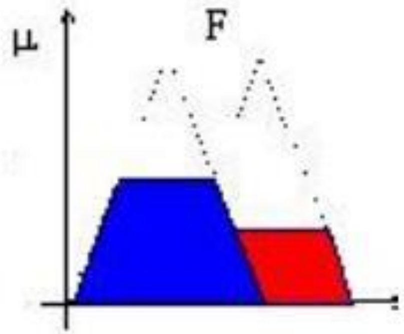
http://aragorn.pb.bialystok.pl/~radev/logic/logrozm.htm





Example 4.2 – mam2.m

- if x is small & y is small then z is large negative
- if x is small & y is large then z is small negative
- if x is large & y is small then z is small positive
- if x is large & y is large then z is large positive



Defuzzification can be computationally expensive. What is the center of area of this fuzzy set?

Theorem 4.1: If we use the product T-norm, and the summation S-norm (which is not really an S-norm), and centroid defuzzification, then the crisp output z is:

$$z = \frac{\sum w_i a_i z_i}{\sum w_i a_i}$$

w_i = firing strength (input MF value)

a_i = consequent MF area

z_i = consequent MF centroid

a_i and z_i can be calculated ahead of time!

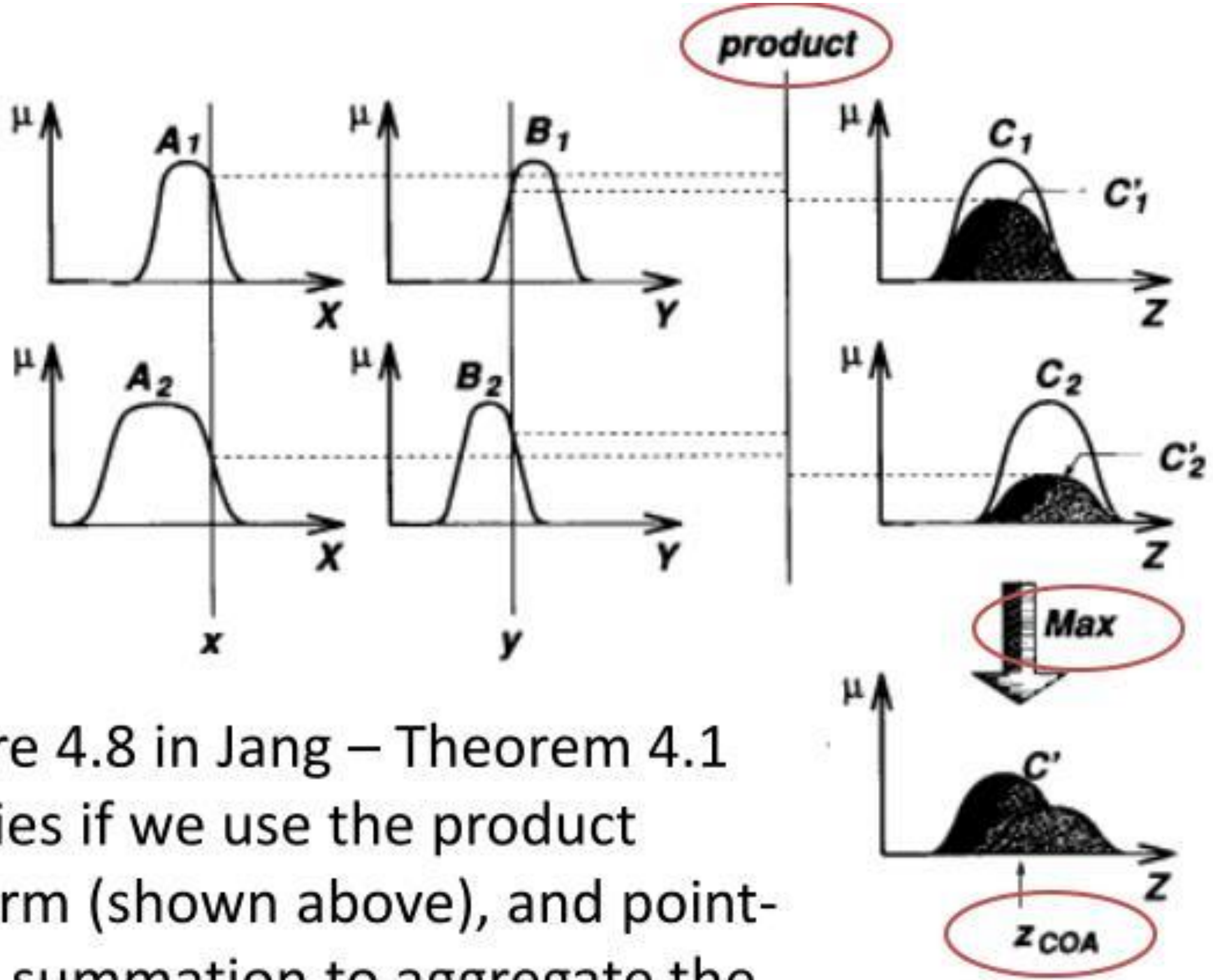


Figure 4.8 in Jang – Theorem 4.1 applies if we use the product T-norm (shown above), and point-wise summation to aggregate the output MFs (not shown above), and centroid defuzzification.

What would point-wise summation look like?

Sugeno (TSK) Fuzzy Models

- Two-input, one-ouput example:

  If x is A_i and y is B_k then $z = f_{m(i,k)}(x, y)$
  Antecedents are fuzzy, consequents are crisp
- Special case of singleton outputs:

  If x is A_i and y is B_k then $z = c_{m(i,k)}$ (constant)
  This is also called a zero-order TSK model, and it is also a special case of a Mamdani model

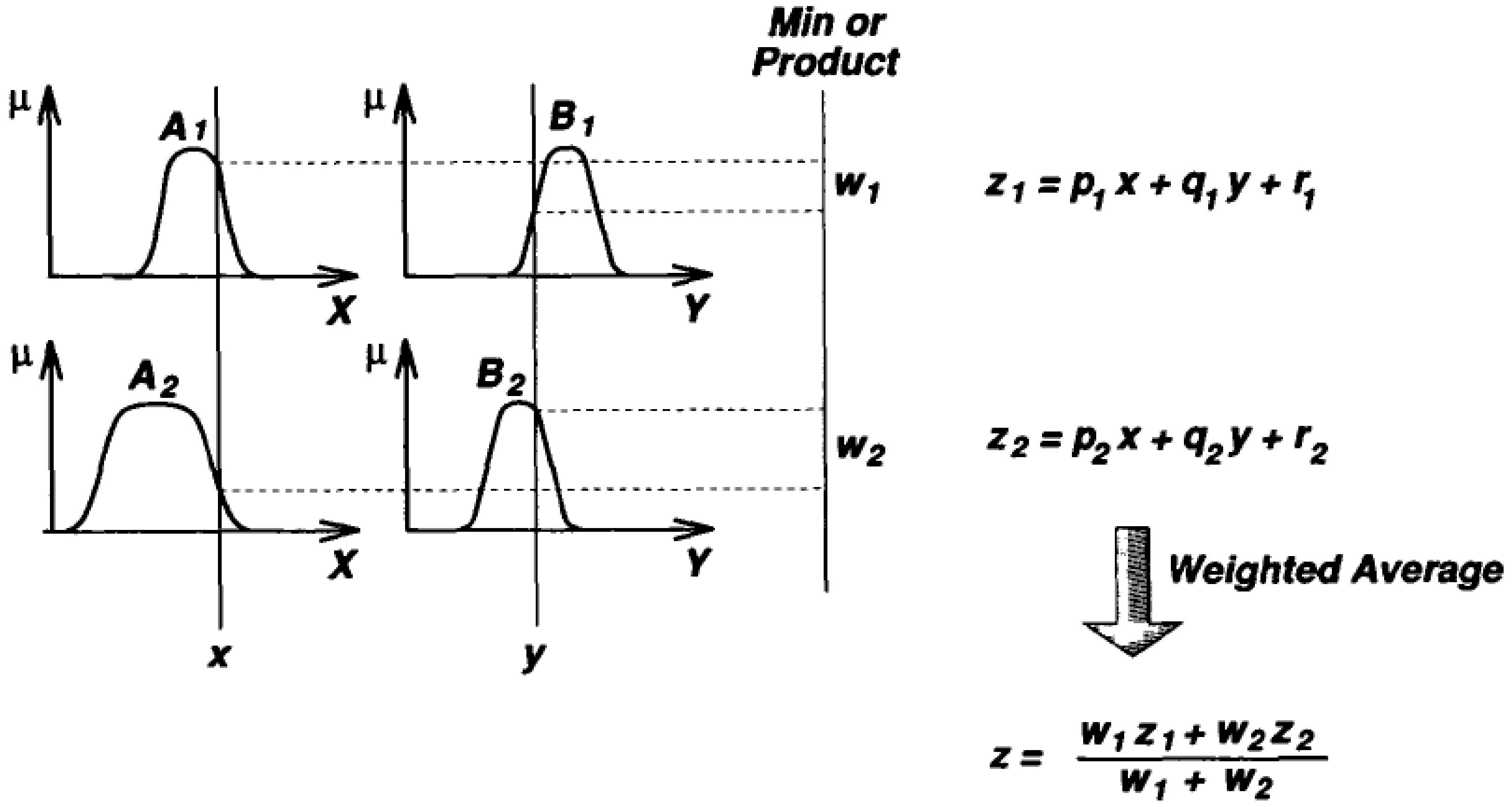


Figure 4.8. *The Sugeno fuzzy model.*

Output function generally polynomial but it can be anyn function as long as it map input to output correctly.
When output function is first order polynomial then resulting FIS is called first-order Sugeno fuzzy model.
When f is a constant it it referred as zero-order Sugeno fuzzy model.

Since each has a crisp output, the overall output is obtained via weighted average, and avoid defuzzification. Weighted average operator is sometimes replaced with the weighted sum operator $z = w_1 z_1 + w_2 z_2$

Sugeno (TSK) Fuzzy Models

The output is a weighted average:

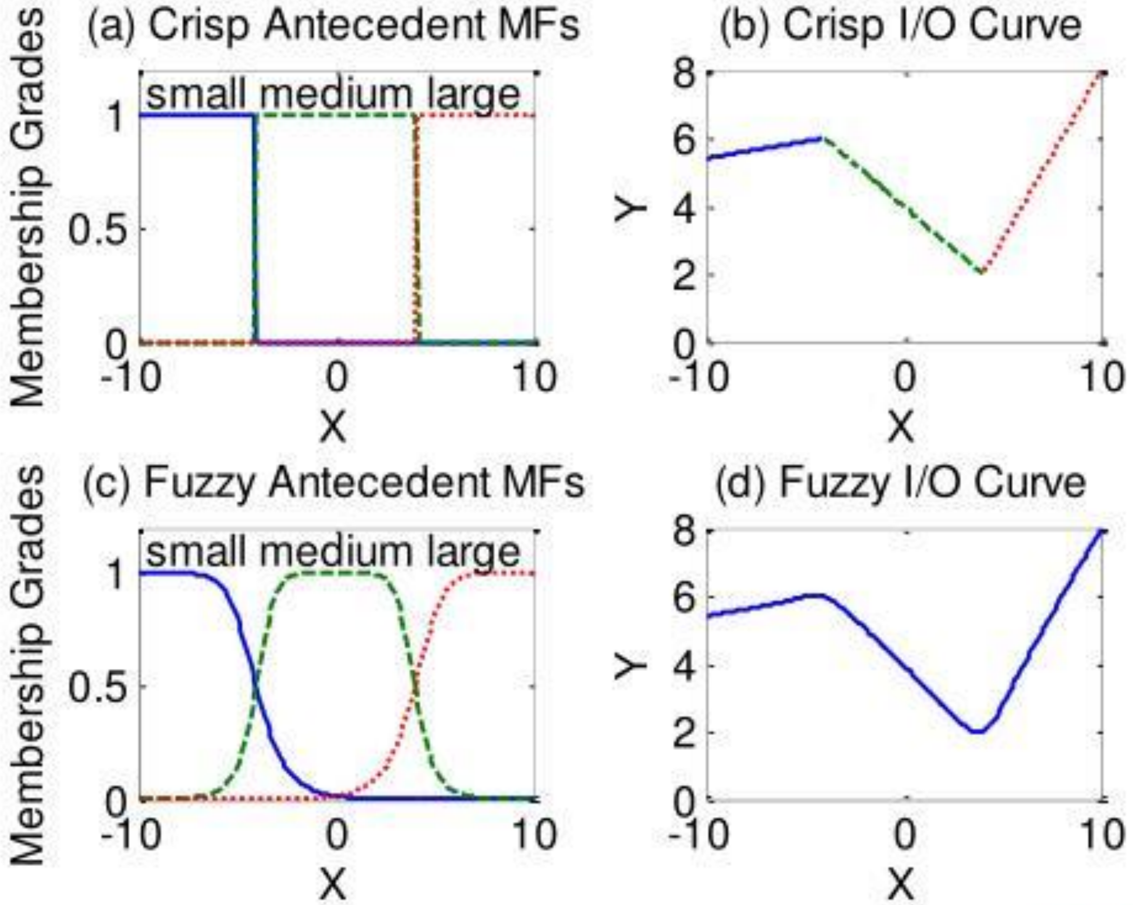
$$z = \frac{\sum \mu_{A_i, B_k}(x, y) f_{m(i,k)}(x, y)}{\sum \mu_{A_i, B_k}(x, y)}$$

Double summation over all *i* (x MFs) and all *k* (y MFs)

$$= \frac{\sum w_i f_i(x, y)}{\sum w_i}$$

Summation over all *i* (fuzzy rules)

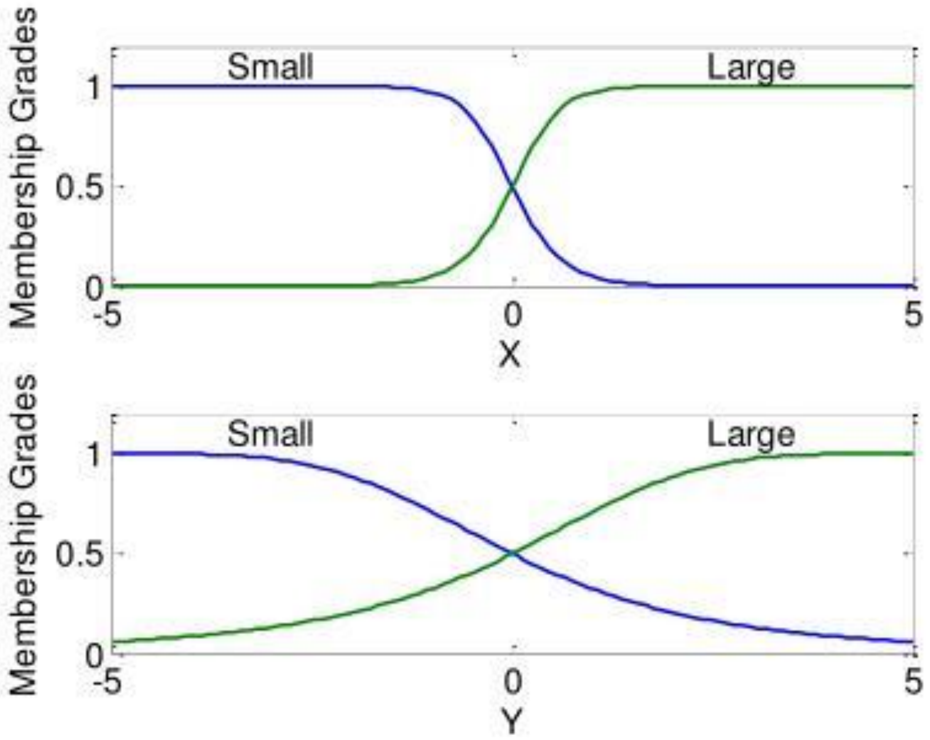
where w_i is the firing strength of the *i*-th output

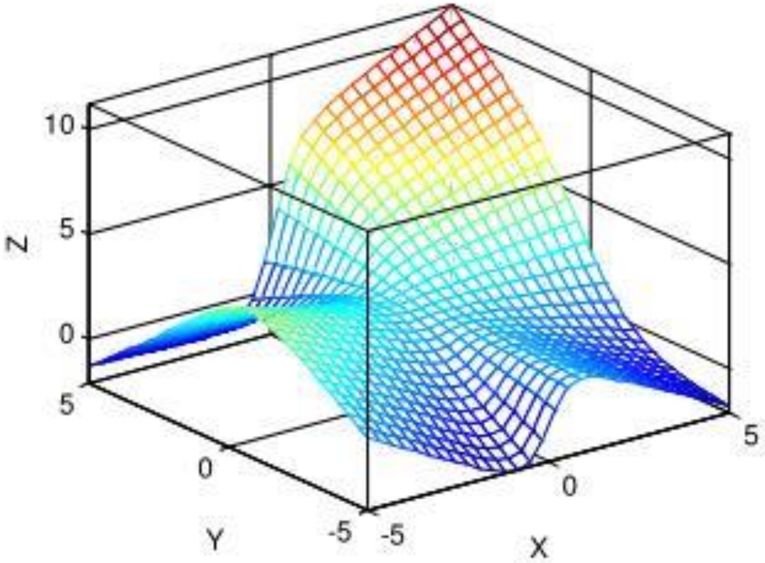


Example 4.3 – sug1.m

- If x is small then y = 0.1x + 6.4
- If x is medium then y = –0.5x + 5
- If x is large then y = x – 2

Review sug1.m software



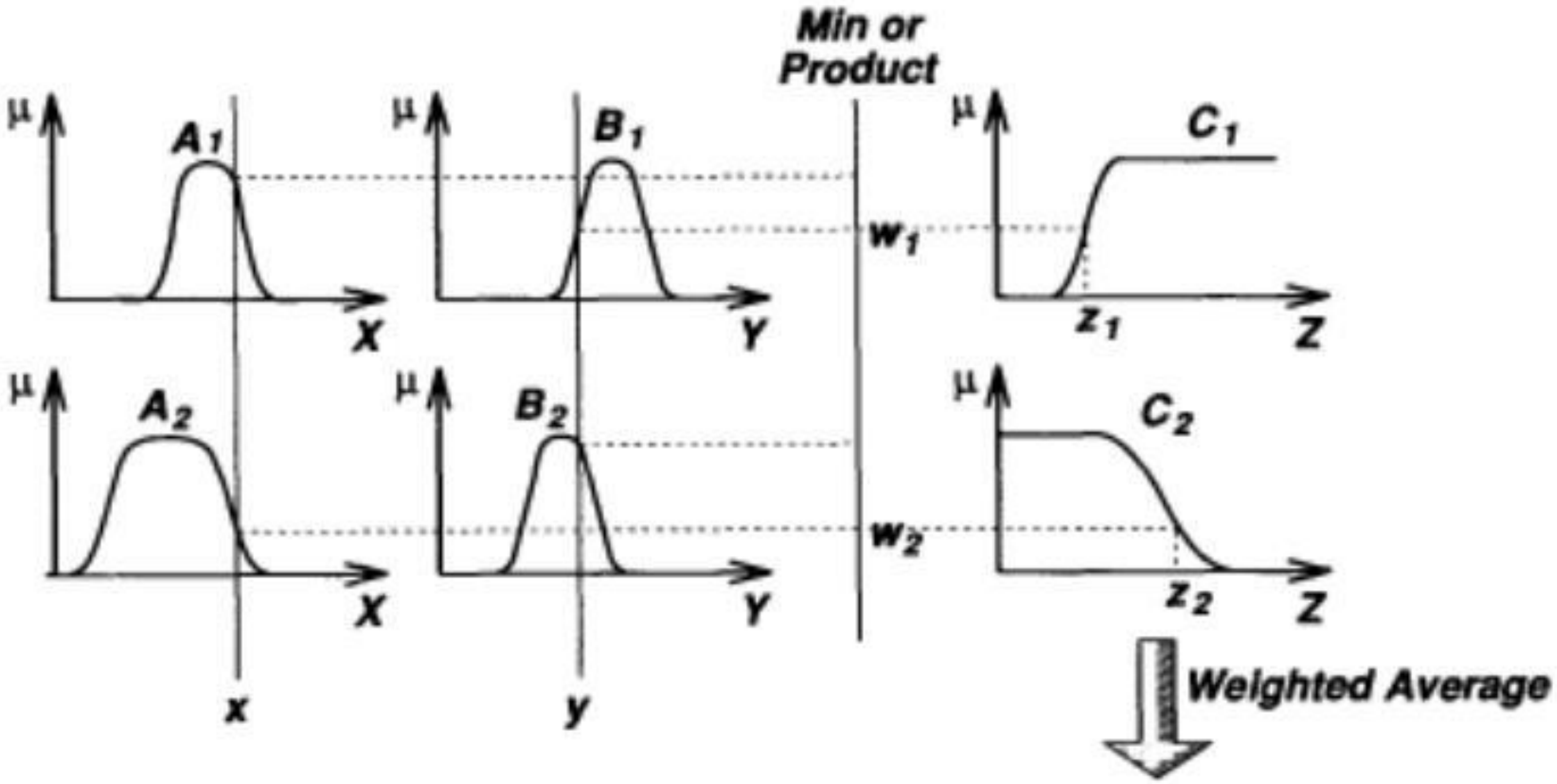


Example 4.4 – sug2.m

- If x is small and y is small then z = –x + y + 1
- If x is small and y is large then z = –y + 5
- If x is large and y is small then z = –x + 3
- If x is large and y is large then z = x + y + 2

Tsukamoto Fuzzy Models

- Section 4.4 in the text
- Special type of Mamdani model
  - Output MFs are open
  - Crisp output is weighted average of fuzzy outputs

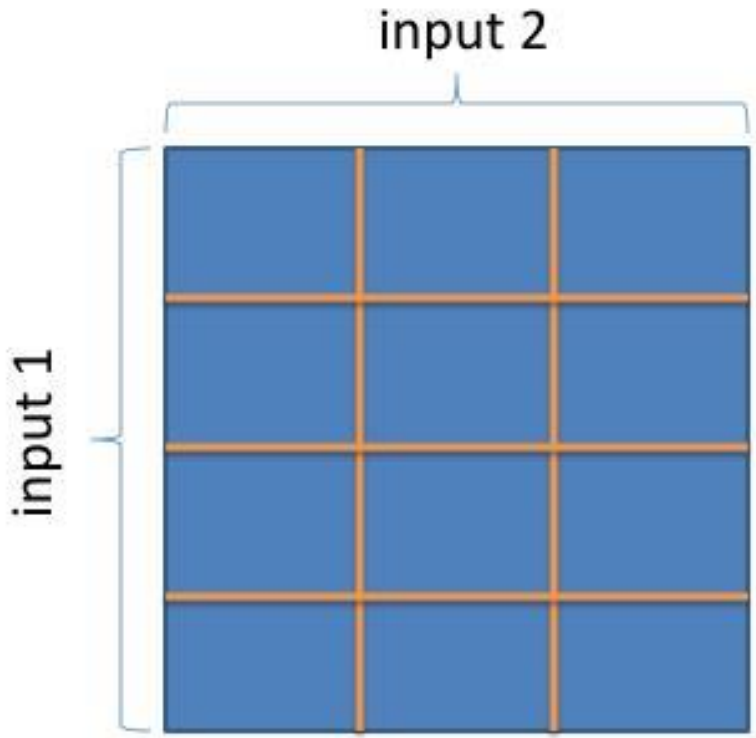


$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

Tsukamoto fuzzy model

Jang, Figure 4.11

Input Space Partitioning



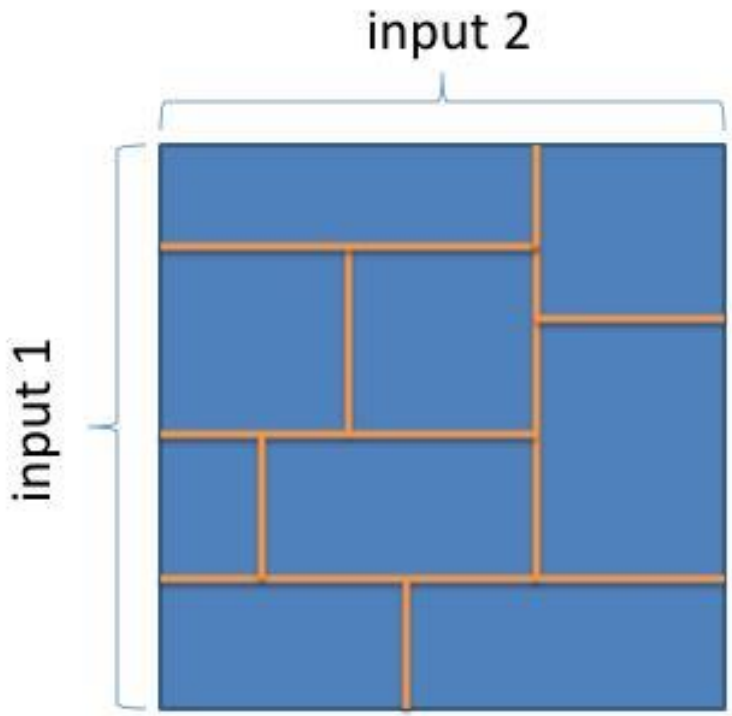
Grid Partition

<u>The curse of dimensionality:</u>

If we have *n* inputs and *m* FMs per input, then we have m^n if-then rules.

For example, 6 inputs and 5 memberships per input $\rightarrow 5^6 = 15{,}625$ rules!

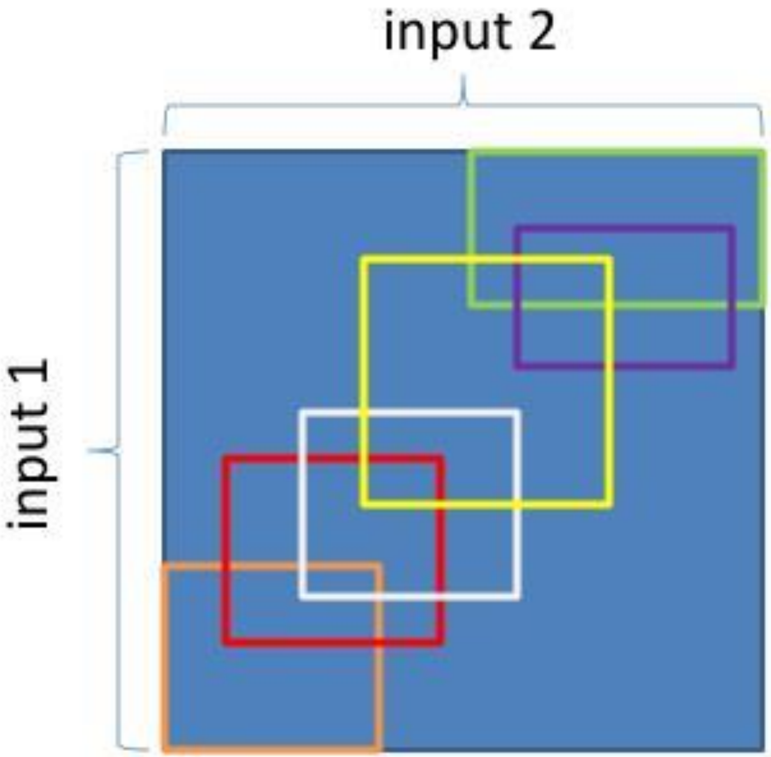
Input Space Partitioning



Tree Partition

- Reduces the number of rules
- Requires more MFs per input
- MFs do not have clear linguistic meanings

Input Space Partitioning



Scatter Partition

- In many systems, extremes occur rarely
- Number of active rules depends on input values

Fuzzy Modeling

- *Fuzzy modeling* consists of constructing a fuzzy inference system. This can be done using:
  - Domain (expert) knowledge
  - Numerical training data

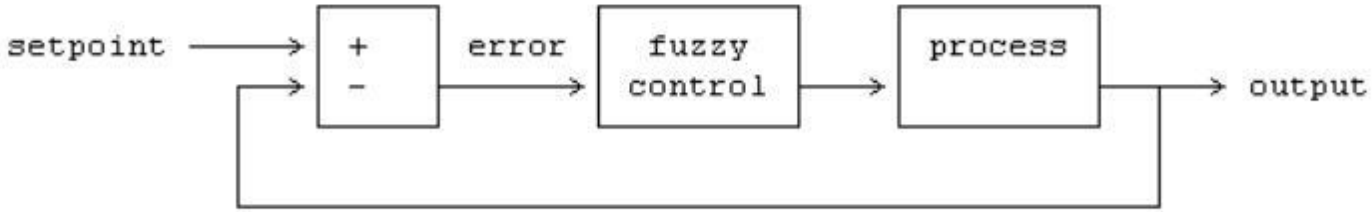
Fuzzy Modeling

Fuzzy modeling includes two stages:

- Surface structure identification
  - Specify input and output variables
  - Specify the type of fuzzy inference system
  - Specify the number of MFs for inputs and outputs
  - Specify the fuzzy if-then rules
- Deep structure identification
  - Specify the type of MFs
  - Specify the MF parameters using human expertise and numerical optimization

Fuzzy Modeling

How could you construct a fuzzy control system?



http://en.wikipedia.org/wiki/File:Fuzzy_control_system-feedback_controller.png