

## Chapter 5

# Adaptive Stabilization

In Chap. 4, the plant uncertainty, represented by  $d(t)$ , must be ranged in a compact set  $\mathbb{D}$ . As a result, a high gain feedback control strategy can be used to dominate uncertainty so that the stability of the closed-loop system is maintained regardless of the change of  $d(t)$  as long as  $d(t) \in \mathbb{D}$  for all  $t \geq 0$ . This type of control is also called robust control. In many cases,  $d(t)$  can be unbounded which cannot be handled by the approach in Chap. 4. In this chapter, we will consider another type of uncertainty satisfying the so-called “linear parameterization” property which roughly means that an uncertain function can be represented as a linear function of some unknown constant parameters. We will introduce the so-called adaptive control scheme to handle this type of uncertainty. An adaptive control law is also a type of nonlinear control law that usually includes a mechanism to estimate the unknown parameters. This chapter consists of five sections. In Sect. 5.1, we introduce the adaptive control problem via a simple nonlinear system. In Sect. 5.2, we formalize the method so that it applies to a class of lower triangular uncertain nonlinear systems. In Sect. 5.3, we combine the robust control method studied in Chap. 4 and the adaptive control method studied in Sect. 5.2 leading to the so-called robust adaptive control method. This method can handle nonlinear systems containing both bounded time-varying uncertainty and unbounded constant uncertainty satisfying the linear parameterization property. The case study on the Duffing equation is illustrated in Sect. 5.4. The notes and references are given in Sect. 5.5.

### 5.1 A Motivating Example

To introduce the adaptive control problem, in this section, we first consider a simple nonlinear system of the following form:

$$\dot{x} = f^T(x, t)\mu + u \quad (5.1)$$

where  $x \in \mathbb{R}$  is the state,  $u \in \mathbb{R}$  is the input,  $f : \mathbb{R} \times [t_0, \infty] \mapsto \mathbb{R}^l$  is a known function satisfying locally Lipchitz condition with respect to  $x$  uniformly in  $t$ , and

$\mu \in \mathbb{R}^l$  is an unknown constant parameter vector. We allow each component of  $\mu$  to take any real value. Therefore, if we define

$$g(x, d(t)) = f^\top(x, t)\mu \quad (5.2)$$

with  $d(t) = (t, \mu)$ , then  $d$  is unbounded and the approach in Chap. 4 does not work.

An uncertain function of the form (5.2) is said to satisfy the linear parameterization property since, for any real number  $\gamma$ ,  $f^\top(x, t)(\gamma\mu) = \gamma(f^\top(x, t)\mu)$ . On the other hand, the uncertain function  $\sin(\mu x)$  where  $x \in \mathbb{R}$  and  $\mu$  is a constant real number does not satisfy the linear parameterization property.

In the system (5.1), if  $\mu$  were known, then the control law

$$u = -f^\top(x, t)\mu - \rho x, \quad \rho > 0 \quad (5.3)$$

would lead to a closed-loop system  $\dot{x} = -\rho x$  which is an asymptotically stable linear system. Since  $\mu$  is unknown, the control law (5.3) cannot be implemented. We will instead consider the following control law

$$u = -f^\top(x, t)\hat{\mu} - \rho x, \quad \rho > 0 \quad (5.4)$$

where  $\hat{\mu} \in \mathbb{R}^l$  is a constant vector viewed as an estimation of  $\mu$ . Under this control law, the closed-loop system is

$$\dot{x} = -f^\top(x, t)\tilde{\mu} - \rho x \quad (5.5)$$

where  $\tilde{\mu} = \hat{\mu} - \mu$  is called the parameter estimation error. Now consider the simplest case of (5.5) with  $l = 1$  and  $f(x, t) = x$ . In this case, (5.5) reduces to the following linear system

$$\dot{x} = -(\tilde{\mu} + \rho)x. \quad (5.6)$$

Now since  $\mu$  can take arbitrarily large value, no matter how large  $\rho$  is chosen, one cannot guarantee the stability of (5.6) because  $(\tilde{\mu} + \rho) > 0$  cannot be guaranteed. Thus, a static state feedback control law of the form (5.4) does not work for this case.

Next we turn to the following dynamic state feedback control law

$$\begin{aligned} u &= -f^\top(x, t)\hat{\mu} - \rho x, \quad \rho > 0 \\ \dot{\hat{\mu}} &= \Lambda x f(x, t) \end{aligned} \quad (5.7)$$

where  $\hat{\mu} \in \mathbb{R}^l$  is not a constant vector but a function of time governed by the second equation of (5.7) and  $\Lambda \in \mathbb{R}^{l \times l}$  is a symmetric and positive definite constant matrix. We call  $\hat{\mu}$  as the dynamic estimation of  $\mu$  or simply an estimation of  $\mu$  and call the second equation of (5.7) as the parameter update law or simply update law. The composition of the plant (5.1) and the control law (5.7) is called the closed-loop

system and takes the following form

$$\begin{aligned}\dot{x} &= -f^T(x, t)\tilde{\mu} - \rho x \\ \dot{\hat{\mu}} &= \Lambda x f(x, t).\end{aligned}\tag{5.8}$$

We now show that the closed-loop system has the property that, for any initial condition  $x(0)$  and  $\hat{\mu}(0)$ , the solution of the closed-loop system (5.8) is bounded and  $\lim_{t \rightarrow \infty} x(t) = 0$ . For this purpose, we consider a Lyapunov function candidate

$$W(x, \tilde{\mu}) = x^2/2 + \tilde{\mu}^T \Lambda^{-1} \tilde{\mu}/2 \tag{5.9}$$

for the closed-loop system (5.8). The derivative of  $W(x, \tilde{\mu})$  along the state trajectory of the closed-loop system is

$$\begin{aligned}\dot{W}(x, \tilde{\mu}) &= x\dot{x} + \tilde{\mu}^T \Lambda^{-1} \dot{\tilde{\mu}} \\ &= -\rho x^2 - \tilde{\mu}^T x f(x, t) + \tilde{\mu}^T \Lambda^{-1} \dot{\hat{\mu}} \\ &= -\rho x^2 \leq 0.\end{aligned}\tag{5.10}$$

Thus, the state variables  $x$  and  $\hat{\mu}$  are bounded. Moreover, by Theorem 2.5,  $\lim_{t \rightarrow \infty} x(t) = 0$ .

A few remarks are in order.

*Remark 5.1* (i) The control law (5.7) is a special type of nonlinear dynamic state feedback control law. Since the state  $\hat{\mu}$  is interpreted as a dynamic estimation of the unknown parameter vector  $\mu$  and its value is adapted according to the parameter update law, we call the control law (5.7) a state feedback adaptive control law. The matrix  $\Lambda$  is called the adaptation gain matrix.

(ii) If  $f(0, t) = 0$  for all  $t \geq 0$ , then all such points as  $(x, \hat{\mu}) = (0, \hat{\mu})$  are equilibrium points of the closed-loop system. Thus, the control law (5.7) does not make the origin of the closed-loop system asymptotically stable. Therefore, we call the above problem global adaptive stabilization problem instead of global stabilization problem introduced in Chap. 4. A formal definition of global adaptive stabilization problem for the system of the form (2.7)–(2.9) will be introduced in the next section.

(iii) The control law design philosophy revealed in the above example is called certainty equivalence principle. Indeed, the first equation of the control law (5.7) is obtained as if  $\mu$  were known by mimicking the control law (5.3). The second equation of the control law (5.7) is designed such that the closed-loop system admits a Lyapunov function of the form (5.9) whose derivative along the trajectory of the closed-loop system is negative semi-definite as shown in (5.10). More specifically, suppose the parameter update law is given by  $\dot{\hat{\mu}} = \tau(x, t)$  for some function  $\tau$ . Then the derivative of the Lyapunov function (5.9) along the trajectory of the closed-loop system is

$$\begin{aligned}
\dot{W}(x, \tilde{\mu}) &= x\dot{x} + \tilde{\mu}^\top \Lambda^{-1} \dot{\tilde{\mu}} \\
&= -\rho x^2 - \tilde{\mu}^\top x f(x, t) + \tilde{\mu}^\top \Lambda^{-1} \dot{\tilde{\mu}} \\
&= -\rho x^2 + \tilde{\mu}^\top \left( -x f(x, t) + \Lambda^{-1} \tau(x, t) \right). \tag{5.11}
\end{aligned}$$

Letting  $\dot{W}(x, \tilde{\mu}) = -\rho x^2$  yields  $\tau(x, t) = \Lambda x f(x, t)$ .

(iv) The success of the certainty equivalence principle relies on the linear parameterization property of the uncertain function which leads to two properties, i.e.,  $f(x, t)^\top \hat{\mu} - f(x, t)^\top \mu = f(x, t)^\top \tilde{\mu}$  and  $\dot{\hat{\mu}} = \dot{\tilde{\mu}}$ .

(v) Even though  $\hat{\mu}$  is interpreted as the dynamic estimation of the unknown parameter vector  $\mu$ , the solvability of the adaptive stabilization problem does not have to imply the convergence of  $\hat{\mu}(t)$  to  $\mu$  as  $t$  tends to infinity. The quantity  $\hat{\mu}$  may not even converge at all. The limiting behavior of  $\hat{\mu}(t)$  is known as the parameter convergence issue and will be further studied later in Sect. 5.3 and Chap. 9

To close this section, we summarize the result of this section as follows.

**Theorem 5.1** *The global adaptive stabilization problem for the system (5.1) is solved by the controller*

$$\begin{aligned}
u &= -f^\top(x, t) \hat{\mu} - \rho x, \quad \rho > 0 \\
\dot{\hat{\mu}} &= \Lambda x f(x, t), \quad \Lambda = \Lambda^\top > 0. \tag{5.12}
\end{aligned}$$

## 5.2 Adaptive Stabilization: Tuning Functions Design

In the previous section, we illustrated the adaptive stabilization problem for a simple scalar system. In this section, we will further study the adaptive stabilization problem for nonlinear systems of the form (2.7)–(2.9). Let us first give a formal description of the adaptive stabilization problem as follows.

**Global Adaptive Regulation Problem (GARP):** *Given a nonlinear control system of the form (2.7)–(2.9), design a controller of the form (1.11), such that, for any initial condition  $x_c(0)$  of the closed-loop system (2.10), the state trajectory  $x_c(t)$  of the closed-loop system is bounded and  $\lim_{t \rightarrow \infty} y(t) = 0$ .*

In the description of GARP, we only require the performance output  $y$  to approach the origin asymptotically. If  $y = x$ , then we further call GARP as a global adaptive stabilization problem (GASP). Clearly, the system (5.1) is a special case of (2.7)–(2.9) with  $y_m = y = x$ .

In this section, we will first study the adaptive stabilization problem for a class of lower triangular nonlinear systems containing linearly parameterized uncertainties described as follows:

$$\dot{x}_i = f_i^\top \left( \begin{smallmatrix} \rightarrow \\ x_i, t \end{smallmatrix} \right) \mu + x_{i+1}, \quad i = 1, \dots, r \tag{5.13}$$

where  $\vec{x}_i = \text{col}(x_1, \dots, x_i)$  with  $x_i \in \mathbb{R}$  is the state,  $u := x_{r+1} \in \mathbb{R}$  the input, and  $\mu \in \mathbb{R}^l$  the unknown constant parameter vector. It is also assumed that all functions in the system (5.13) are sufficiently smooth. The system (5.13) can be viewed as a special case of (2.7)–(2.9) with  $y_m = y = x$ . On the other hand, the system (5.1) is a special case of (5.13) with  $r = 1$ . Like the special case with  $r = 1$  explained in the previous section, the basic idea for solving the GASP for (5.13) is to find a controller and an update law such that the derivative of a suitable Lyapunov function along the state trajectory of the closed-loop system is negative semi-definite. However, when  $r > 1$ , the construction of the control law is recursive and we will detail this procedure as follows.

The design procedure consists of  $r$  steps. At each step  $i$ ,  $i = 1, \dots, r$ , two functions  $S_i$  and  $\tau_i$  will be designed based on previous steps. If the relative degree of the system were equal to  $i$ , then these two functions  $S_i$  and  $\tau_i$  would constitute the control law and the parameter update law, respectively. Otherwise, the process will continue recursively until  $i = r$  so that we obtain the actual control law  $x_{r+1} = S_r(\cdot)$  and the actual parameter update law  $\dot{\hat{\mu}} = \tau_r(\cdot)$ . Since the functions  $\tau_i$  are called tuning functions in the literature, this design approach is called the tuning functions approach which can be viewed as a generalization of Proposition 4.1.

More specifically, define the following dynamic coordinate transformation:

$$\begin{aligned} \mathcal{X}_1 &= x_1 \\ \mathcal{X}_{i+1} &= x_{i+1} - S_i(\vec{x}_i, \hat{\mu}, t), \quad i = 1, \dots, r \\ \dot{\hat{\mu}} &= \tau_r(\vec{x}_r, \hat{\mu}, t) \end{aligned} \quad (5.14)$$

for some sufficiently smooth functions  $S_i$ ,  $i = 1, \dots, r$ , and  $\tau_r$ . Let  $\zeta_i = \text{col}(\vec{x}_i, \hat{\mu})$  and  $\vec{\mathcal{X}}_i = \text{col}(\mathcal{X}_1, \dots, \mathcal{X}_i)$ . Denote the system composed of the plant (5.13) and the transformation (5.14) by  $\dot{\zeta}_r = \varphi_r(\zeta_r, \mathcal{X}_{r+1}, t)$ . Then, Proposition 4.1 can be generalized to the following version.

**Proposition 5.1** *If there exist sufficiently smooth functions  $S_i$ ,  $i = 1, \dots, r$ , and  $\tau_r$ , such that, for any initial condition  $\zeta_r(0)$ , the state  $\zeta_r$  of the system  $\dot{\zeta}_r = \varphi_r(\zeta_r, 0, t)$  is bounded and  $\lim_{t \rightarrow \infty} \vec{\mathcal{X}}_r = 0$ , then the GARP with  $y = x_1$  for the system (5.13) is solved by the following dynamic state feedback controller*

$$\begin{aligned} u &= S_r(\vec{x}_r, \hat{\mu}, t) \\ \dot{\hat{\mu}} &= \tau_r(\vec{x}_r, \hat{\mu}, t). \end{aligned} \quad (5.15)$$

*Moreover, if  $S_i(0, \hat{\mu}, t) = 0$ ,  $i = 1, \dots, r - 1$ , then the GASP with  $y = \vec{x}_r$  for the system (5.13) is solved by the same controller.*

We now introduce a recursive procedure to find the functions  $\mathcal{S}_i$  and  $\tau_i$  for  $i = 1, \dots, r$ . The construction of  $\mathcal{S}_1$  and  $\tau_1$  is motivated from the previous section, i.e.,

$$\begin{aligned}\mathcal{S}_1(x_1, \hat{\mu}, t) &= -f_1^\top(x_1, t)\hat{\mu} - \rho_1 x_1, \quad \rho_1 \geq 5/4 \\ \tau_1(x_1, \hat{\mu}, t) &= \Lambda x_1 f_1(x_1, t), \quad \Lambda = \Lambda^\top > 0.\end{aligned}\quad (5.16)$$

The remaining functions are recursively constructed as follows, for  $i = 2, \dots, r$ ,

$$\mathcal{S}_i(\vec{x}_i, \hat{\mu}, t) = -\varrho_i^\top(\vec{x}_i, \hat{\mu}, t)\hat{\mu} - \rho_i \mathcal{X}_i + v_i(\vec{x}_i, \hat{\mu}, t), \quad \rho_i \geq 9/4 \quad (5.17)$$

$$\tau_i(\vec{x}_i, \hat{\mu}, t) = \tau_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t) + \Lambda \mathcal{X}_i \varrho_i(\vec{x}_i, \hat{\mu}, t) \quad (5.18)$$

where the functions  $v_i$  and  $\varrho_i$  are defined as follows

$$\begin{aligned}v_i(\vec{x}_i, \hat{\mu}, t) &= \sum_{j=1}^{i-1} \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial x_j} x_{j+1} + \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial t} \\ &\quad + \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \tau_i(\vec{x}_i, \hat{\mu}, t) \\ &\quad + \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \Lambda \varrho_i(\vec{x}_i, \hat{\mu}, t)\end{aligned}\quad (5.19)$$

$$\varrho_i(\vec{x}_i, \hat{\mu}, t) = f_i(\vec{x}_i, t) - \sum_{j=1}^{i-1} \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial x_j} f_j(\vec{x}_j, t). \quad (5.20)$$

We now state the main result of this section as follows.

**Theorem 5.2** *The GARP for the system (5.13) with the performance output  $y = x_1$  is solved by the controller (5.15) where the functions  $\mathcal{S}_r$  and  $\tau_r$  are defined in (5.16)–(5.18) and also summarized in Algorithm 5.1. Moreover, if  $f_i(0, t) = 0$ ,  $i = 1, \dots, r-1$ , then the GAS for the system (5.13) with the performance output  $y = \vec{x}_r$  is solved by the same controller.*

*Proof* Consider the system (5.13) under the coordinate transformation (5.14). We will show that, for  $1 \leq i \leq r$ , along the state trajectory of the  $\vec{x}_i$ -subsystem, the derivative of the following function

$$W_i(\vec{x}_i, \hat{\mu}) = \sum_{j=1}^i x_j^2/2 + \tilde{\mu}^\top \Lambda^{-1} \tilde{\mu}/2 \quad (5.21)$$

satisfies

$$\begin{aligned} \dot{W}_i(\vec{x}_i, \hat{\mu}) &\leq -\sum_{j=1}^i x_j^2 + x_{i+1}^2 \\ &+ \left( \tilde{\mu}^\top \Lambda^{-1} - \sum_{j=1}^{i-1} x_{j+1} \frac{\partial S_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \right) \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, t) \right] \end{aligned} \quad (5.22)$$

where  $\tilde{\mu} = \hat{\mu} - \mu$  is the parameter estimation error.

For  $i = 1$ , direct calculation shows that the derivative of  $W_1(x_1, \hat{\mu}) = x_1^2/2 + \tilde{\mu}^\top \Lambda^{-1} \tilde{\mu}/2$  along the state trajectory of the  $x_1$ -subsystem satisfies

$$\begin{aligned} \dot{W}_1(x_1, \hat{\mu}) &= x_1 (f_1(x_1, t)^\top \mu + S_1(x_1, \hat{\mu}, t) + x_2) + \tilde{\mu}^\top \Lambda^{-1} \dot{\tilde{\mu}} \\ &= x_1 (-f_1(x_1, t)^\top \tilde{\mu} - \rho_1 x_1 + x_2) + \tilde{\mu}^\top \Lambda^{-1} \dot{\tilde{\mu}} \\ &\leq -x_1^2 + x_2^2 + \tilde{\mu}^\top \Lambda^{-1} \left[ \dot{\hat{\mu}} - \tau_1(x_1, \hat{\mu}, t) \right]. \end{aligned} \quad (5.23)$$

If  $r = 1$  as in the case of Theorem 5.1, then letting  $x_2 = 0$  gives the actual control  $x_2 = S_1(x_1, \hat{\mu}, t)$ . And the update function is chosen as  $\dot{\hat{\mu}} = \tau_1(x_1, \hat{\mu}, t)$  to make  $\dot{W}_1(x_1, \hat{\mu}) \leq -x_1^2$ . But, when  $r > 1$ ,  $x_2$  is not the actual control in this theorem, so we should retain  $\tau_1$  as the first tuning function and allow the presence of  $\dot{\hat{\mu}}$  in  $\dot{W}_1(x_1, \hat{\mu})$  at this stage.

Next, we will prove that the statement is true for any  $1 < i \leq r$  if it is true for  $i - 1$ . To this end, we note the relationship

$$W_i(\vec{x}_i, \hat{\mu}) = W_{i-1}(\vec{x}_{i-1}, \hat{\mu}) + x_i^2/2.$$

The derivative of  $x_i^2/2$  along the state trajectory of the  $x_i$ -subsystem satisfies

$$\begin{aligned} x_i \dot{x}_i &= x_i \dot{x}_i - x_i \dot{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t) \\ &= x_i f_i(\vec{x}_i, t)^\top \mu + x_i S_i(\vec{x}_i, \hat{\mu}, t) + x_i x_{i+1} \\ &\quad - x_i \sum_{j=1}^{i-1} \frac{\partial S_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial x_j} \left( f_j(\vec{x}_j, t)^\top \mu + x_{j+1} \right) \\ &\quad - x_i \frac{\partial S_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial t} - x_i \frac{\partial S_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \dot{\hat{\mu}}. \end{aligned}$$

By using the definition of  $\varrho_i^\top(\vec{x}_i, \hat{\mu}, t)$  and  $\mathcal{S}_i(\vec{x}_i, \hat{\mu}, t)$ , we have

$$\begin{aligned}
\mathcal{X}_i \dot{\mathcal{X}}_i &\leq \mathcal{X}_i \varrho_i^\top(\vec{x}_i, \hat{\mu}, t) \mu + \mathcal{X}_i \left[ -\varrho_i^\top(\vec{x}_i, \hat{\mu}, t) \hat{\mu} - \rho_i \mathcal{X}_i + v_i(\vec{x}_i, \hat{\mu}, t) \right] \\
&\quad + \mathcal{X}_i^2/4 + \mathcal{X}_{i+1}^2 - \mathcal{X}_i \sum_{j=1}^{i-1} \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial x_j} x_{j+1} \\
&\quad - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial t} - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \dot{\hat{\mu}} \\
&\leq -\mathcal{X}_i \varrho_i^\top(\vec{x}_i, \hat{\mu}, t) \tilde{\mu} - 2\mathcal{X}_i^2 + \mathcal{X}_{i+1}^2 + \mathcal{X}_i v_i(\vec{x}_i, \hat{\mu}, t) \\
&\quad - \mathcal{X}_i \sum_{j=1}^{i-1} \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial x_j} x_{j+1} - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial t} \\
&\quad - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \dot{\hat{\mu}}.
\end{aligned}$$

Then, using the definition of  $v_i(\vec{x}_i, \hat{\mu}, t)$  and  $\tau_i(\vec{x}_i, \hat{\mu}, t)$  gives

$$\begin{aligned}
\mathcal{X}_i \dot{\mathcal{X}}_i &\leq -\mathcal{X}_i \varrho_i^\top(\vec{x}_i, \hat{\mu}, t) \tilde{\mu} - 2\mathcal{X}_i^2 + \mathcal{X}_{i+1}^2 \\
&\quad + \mathcal{X}_i \sum_{j=1}^{i-2} x_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \Lambda \varrho_i(\vec{x}_i, \hat{\mu}, t) \\
&\quad - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, t) \right] \\
&= -2\mathcal{X}_i^2 + \mathcal{X}_{i+1}^2 - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, t) \right] \\
&\quad + \left( \tilde{\mu}^\top \Lambda^{-1} - \sum_{j=1}^{i-2} x_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \right) \\
&\quad \times \left[ \tau_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t) - \tau_i(\vec{x}_i, \hat{\mu}, t) \right].
\end{aligned}$$

Now, we are ready to show

$$\dot{W}_i(\vec{x}_i, \hat{\mu}) = \dot{W}_{i-1}(\vec{x}_{i-1}, \hat{\mu}) + \mathcal{X}_i \dot{\mathcal{X}}_i$$

$$\begin{aligned}
&\leq - \sum_{j=1}^{i-1} \mathcal{X}_j^2 + \mathcal{X}_i^2 - 2\mathcal{X}_i^2 + \mathcal{X}_{i+1}^2 \\
&\quad - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t)}{\partial \hat{\mu}} \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, t) \right] \\
&\quad + \left( \tilde{\mu}^\top \Lambda^{-1} - \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \right) \left[ \dot{\hat{\mu}} - \tau_{i-1}(\vec{x}_i, \hat{\mu}, t) \right] \\
&\quad + \left( \tilde{\mu}^\top \Lambda^{-1} - \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \right) \\
&\quad \times \left[ \tau_{i-1}(\vec{x}_{i-1}, \hat{\mu}, t) - \tau_i(\vec{x}_i, \hat{\mu}, t) \right] \\
&= - \sum_{j=1}^i \mathcal{X}_j^2 + \mathcal{X}_{i+1}^2 \\
&\quad + \left( \tilde{\mu}^\top \Lambda^{-1} - \sum_{j=1}^{i-1} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, t)}{\partial \hat{\mu}} \right) \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, t) \right].
\end{aligned}$$

The proof of the statement (5.23) is thus completed by mathematical induction.

Finally, consider the closed-loop system composed of the system (5.13) and the controller (5.15), which results from letting  $\mathcal{X}_{r+1} = 0$  and  $\dot{\hat{\mu}} = \tau_r(\vec{x}_r, \hat{\mu}, t)$ . Then, the statement (5.22) with  $i = r$  implies

$$\dot{W}_r(\vec{x}_r, \hat{\mu}) \leq - \sum_{j=1}^r \mathcal{X}_j^2.$$

With  $W_r(\vec{x}_r, \hat{\mu})$  as the Lyapunov function for the closed-loop system, the stability of the closed-loop system can be established by invoking Theorem 2.5. In particular, one has  $\lim_{t \rightarrow \infty} \vec{x}_r(t) = 0$ . That is, the GARP for the system (5.13) with the performance output  $y = x_1$  is solved by Proposition 5.1. To show the moreover part, it suffices to note that  $\mathcal{S}_i(0, t, \hat{\mu}) = 0$  if  $f_i(0, t) = 0$ .  $\square$

### Algorithm 5.1

INPUT:  $f_i, i = 1, \dots, r$

OUTPUT:  $\mathcal{S}_r$  and  $\tau_r$

STEP 1: Let  $i = 1$ ; find the functions  $\mathcal{S}_1$  and  $\tau_1$  from (5.16).

STEP 2: IF  $i = r$ , GO TO STEP 5, ELSE GO TO STEP 3.

STEP 3: Let  $i = i + 1$ ; calculate the functions  $\varrho_i$ ,  $\tau_i$ ,  $v_i$ , and  $\mathcal{S}_i$  in order, from (5.17)–(5.20).

STEP 4: GO TO STEP 2.

STEP 5: END

*Example 5.1* Consider a second order nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_1^2 \mu_1 + x_2 \\ \dot{x}_2 &= \sin(x_1 x_2) \mu_2 + u\end{aligned}\tag{5.24}$$

where  $\mu = [\mu_1 \ \mu_2]^\top$  is an unknown constant parameter vector. The objective is to find the controller  $u$  to solve the GASP with the performance output  $y = [x_1, x_2]^\top$ .

We follow Algorithm 5.1 to explicitly construct the controller. The first step is to pick the following functions based on (5.16),

$$\begin{aligned}\mathcal{S}_1(x_1, \hat{\mu}) &= -x_1^2 \hat{\mu}_1 - 2x_1 \\ \tau_1(x_1, \hat{\mu}) &= [x_1^3 \ 0]^\top.\end{aligned}$$

Next we will calculate  $\varrho_2$ ,  $\tau_2$ ,  $v_2$ , and  $\mathcal{S}_2$  in order using the formulae (5.17)–(5.20). First, we have

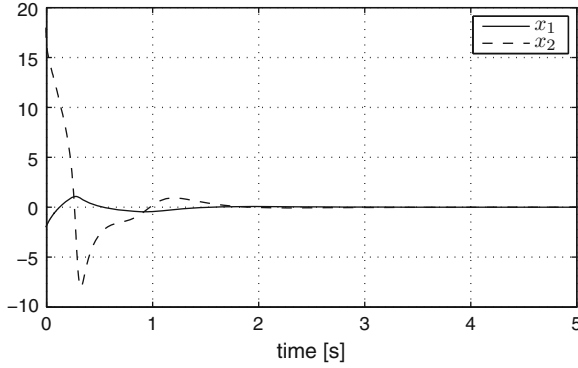
$$\varrho_2(\vec{x}_2, \hat{\mu}) = f_2(\vec{x}_2) - \frac{\partial \mathcal{S}_1(\vec{x}_1, \hat{\mu})}{\partial x_1} f_1(\vec{x}_1) = \begin{bmatrix} (2x_1 \hat{\mu}_1 + 2) x_1^2 & \sin(x_1 x_2) \end{bmatrix}^\top$$

and then,

$$\begin{aligned}\tau_2(\vec{x}_2, \hat{\mu}) &= \tau_1(x_1, \hat{\mu}) + [x_2 - \mathcal{S}_1(x_1, \hat{\mu})] \varrho_2(\vec{x}_2, \hat{\mu}) \\ &= \begin{bmatrix} x_1^3 + (2x_1 \hat{\mu}_1 + 2) x_1^2 (x_2 + x_1^2 \hat{\mu}_1 + 2x_1) \\ \sin(x_1 x_2) (x_2 + x_1^2 \hat{\mu}_1 + 2x_1) \end{bmatrix}.\end{aligned}$$

With the available functions, we are ready to obtain

$$\begin{aligned}v_2(\vec{x}_2, \hat{\mu}) &= \frac{\partial \mathcal{S}_1(\vec{x}_1, \hat{\mu})}{\partial x_1} x_2 + \frac{\partial \mathcal{S}_1(\vec{x}_1, \hat{\mu})}{\partial \hat{\mu}} \tau_2(\vec{x}_2, \hat{\mu}) \\ &= -(2x_1 \hat{\mu}_1 + 2) x_2 - \begin{bmatrix} x_1^2 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} x_1^3 + (2x_1 \hat{\mu}_1 + 2) x_1^2 (x_2 + x_1^2 \hat{\mu}_1 + 2x_1) \\ \sin(x_1 x_2) (x_2 + x_1^2 \hat{\mu}_1 + 2x_1) \end{bmatrix} \\ &= -(2x_1 \hat{\mu}_1 + 2) x_2 - x_1^4 \left( x_1 + (2x_1 \hat{\mu}_1 + 2) (x_2 + x_1^2 \hat{\mu}_1 + 2x_1) \right)\end{aligned}$$



**Fig. 5.1** Profile of state trajectories of the closed-loop system in Example 5.1

and

$$\begin{aligned}
 s_2(\vec{x}_2, \hat{\mu}) &= -\varrho_2^T(\vec{x}_2, \hat{\mu}) \hat{\mu} - \rho_2 \left[ x_2 - s_1(\vec{x}_1, \hat{\mu}) \right] + v_2(\vec{x}_2, \hat{\mu}) \\
 &= - \left[ (2x_1\hat{\mu}_1 + 2) x_1^2 \quad \sin(x_1 x_2) \right] \hat{\mu} - 9/4 \left( x_2 + x_1^2 \hat{\mu}_1 + 2x_1 \right) \\
 &\quad - (2x_1\hat{\mu}_1 + 2) x_2 - x_1^4 \left( x_1 + (2x_1\hat{\mu}_1 + 2) \left( x_2 + x_1^2 \hat{\mu}_1 + 2x_1 \right) \right).
 \end{aligned}$$

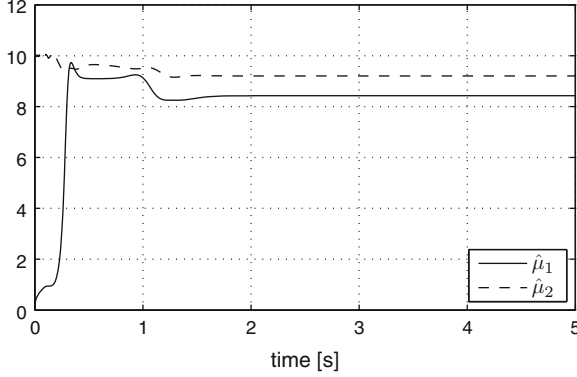
Now, the controller is of the form

$$\begin{aligned}
 u &= s_2(\vec{x}_2, \hat{\mu}) \\
 \dot{\hat{\mu}} &= \tau_2(\vec{x}_2, \hat{\mu}).
 \end{aligned}$$

The performance of the controller is shown in Figs. 5.1 and 5.2 with  $\mu = [2, 4]^T$ . The initial state values are  $x(0) = [-2, 18]^T$  and  $\hat{\mu}(0) = [0, 10]^T$ . It is seen that the state  $x$  of the closed-loop system asymptotically converges to the equilibrium point at the origin and the estimated parameter  $\hat{\mu}$  is bounded, but not convergent to the real value of the parameter  $\mu$ . The condition under which the estimated parameter will converge to its real value will be studied in the next section.

### 5.3 Robust Adaptive Stabilization

In this section, we combine the robust and adaptive techniques to deal with a class of uncertain nonlinear systems containing both disturbances and unknown parameters. The systems are described as follows:



**Fig. 5.2** Profile of estimated parameters of the closed-loop system in Example 5.1

$$\begin{aligned}
 \dot{z} &= q(z, x_1, d(t)) \\
 \dot{x}_1 &= f_1(z, x_1, d(t)) + b f_a^T(x_1, t) \mu + b x_2 \\
 \dot{x}_i &= f_i(\vec{x}_i) + x_{i+1}, i = 2, \dots, r
 \end{aligned} \tag{5.25}$$

where  $z \in \mathbb{R}^n$  and  $\vec{x}_i = \text{col}(x_1, \dots, x_i)$  with  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, r$ , are the state variables,  $u := x_{r+1}$  is the input,  $\mu \in \mathbb{R}^{l_1}$  is the unknown constant parameter, and  $d : [0, \infty) \mapsto \mathbb{D} \subset \mathbb{R}^{l_2}$  with  $\mathbb{D}$  a compact set represents unknown parameters and/or disturbances. The functions  $q$ ,  $f_a$  and  $f_i$ ,  $i = 1, \dots, r$ , are sufficiently smooth with  $q(0, 0, d) = 0$ ,  $f_1(0, 0, d) = 0$ , for all  $d \in \mathbb{D}$  and  $f_i(0) = 0$ ,  $i = 2, \dots, r$ . The subscript  $a$  means the term  $f_a^T(x_1, t) \mu$  will be dealt with by adaptive control. In (5.25), the function  $q$  may not be known precisely, and/or the state  $z$  may not be used for feedback control. Thus, the dynamics governing  $z$  are regarded as the dynamic uncertainty. When  $\mu = 0$ , the system (5.25) reduces to the filter extended form studied in the previous chapter. The parameter  $b$  in the system is an unknown positive number satisfying the following assumption.

**Assumption 5.1** There exist two known positive constants  $\bar{b}$  and  $\underline{b}$  such that  $\bar{b} \geq b \geq \underline{b}$ .

### 5.3.1 Systems with Relative Degree One

We first consider the system (5.25) with the relative degree  $r = 1$ , i.e.,

$$\begin{aligned}
 \dot{z} &= q(z, x, d(t)) \\
 \dot{x} &= f(z, x, d(t)) + b f_a^T(x, t) \mu + b u.
 \end{aligned} \tag{5.26}$$

In the system (5.26), the stabilization problem with  $\mu = 0$  has been studied in Chap. 4 by the robust control technique. The special case of (5.26) with  $z \in \mathbb{R}^0$  and  $b = 1$  has been studied in Sect. 5.2. To handle the nontrivial inverse dynamics with  $y = x$ , we need the following assumption which plays the same role as Assumption 4.1.

**Assumption 5.2** The subsystem  $\dot{z} = q(z, x, d)$  has an ISS Lyapunov function  $V(z)$ , i.e.,

$$V(z) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{z} = q(z, x, d)\}$$

and

$$\lim_{s \rightarrow 0^+} \sup \frac{s^2}{\alpha(s)} < \infty, \quad \lim_{s \rightarrow 0^+} \sup \frac{\sigma(s)}{s^2} < \infty.$$

The first step is to repeat the procedure in Sect. 4.2. Specifically, by (11.13) of the Appendix, we can find two sufficiently smooth and non-negative functions  $m_1$  and  $m_2$  such that

$$|f(z, x, d)| \leq m_1(z)\|z\| + m_2(x)|x|, \quad \forall d \in \mathbb{D}. \quad (5.27)$$

Pick a sufficiently smooth function

$$\Delta(z) \geq 1 + m_1^2(z). \quad (5.28)$$

By Corollary 2.2, there exists a continuously differentiable function  $V'(z)$  satisfying  $\underline{\alpha}'(\|z\|) \leq V'(z) \leq \bar{\alpha}'(\|z\|)$  for some class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}'$  and  $\bar{\alpha}'$ , such that, along the state trajectory of  $\dot{z} = q(z, x, d)$ ,

$$\dot{V}'(z) \leq -\Delta(z)\|z\|^2 + \varkappa(x)x^2 \quad (5.29)$$

for a sufficiently smooth function  $\varkappa$ . Then, we define a sufficiently smooth function  $\rho(x)$  such that

$$\rho(x) \geq [\varkappa(x) + m_2(x) + 5/4]/b \quad (5.30)$$

with which the controller can be constructed in the following theorem.

**Theorem 5.3** Consider the system (5.26) with  $\mathbb{D}$  a prescribed compact set. Under Assumptions 5.1 and 5.2, the GASP of (5.26) with the performance output  $y = \text{col}(z, x)$  is solved by the following controller:

$$\begin{aligned} u &= -f_a^\top(x, t)\hat{\mu} - \rho(x)x \\ \dot{\hat{\mu}} &= \Lambda x f_a(x, t), \quad \Lambda = \Lambda^\top > 0 \end{aligned} \quad (5.31)$$

where the function  $\rho$  is given in (5.30).

*Proof* Define a Lyapunov function candidate for the closed-loop system as follows:

$$W(z, x, \hat{\mu}) = V'(z) + x^2/2 + b\tilde{\mu}^\top \Lambda^{-1} \tilde{\mu}/2$$

where  $\tilde{\mu} = \hat{\mu} - \mu$  is the parameter estimation error. Direct calculation shows that the derivative of  $W(z, x, \hat{\mu})$  along the state trajectory of the closed-loop system satisfies

$$\begin{aligned} \dot{W}(z, x, \hat{\mu}) &\leq -\Delta(z)\|z\|^2 + \varkappa(x)x^2 + x[f(z, x, d) + bf_a^\top(x, t)\mu + bu] + b\tilde{\mu}^\top \Lambda^{-1} \dot{\hat{\mu}} \\ &\leq -\Delta(z)\|z\|^2 + \varkappa(x)x^2 + x[f(z, x, d) - b\rho(x)x] \\ &\quad + bx[f_a^\top(x, t)\mu - f_a^\top(x, t)\hat{\mu}] + b\tilde{\mu}^\top \Lambda^{-1} \dot{\hat{\mu}} \\ &\leq -\Delta(z)\|z\|^2 + m_1^2(z)\|z\|^2 + x^2[\varkappa(x) + 1/4 + m_2(x) - b\rho(x)] \\ &\quad + b\tilde{\mu}^\top (-xf_a(x, t) + \Lambda^{-1} \dot{\hat{\mu}}) \\ &\leq -\|z\|^2 - x^2. \end{aligned} \tag{5.32}$$

Now the stability of the closed-loop system can be established by invoking Theorem 2.5. In particular, one has  $\lim_{t \rightarrow \infty} \text{col}(z(t), x(t)) = 0$ . The GASP is thus solved.  $\square$

Finally, we will consider the convergence issue of the parameter  $\hat{\mu}$  as illustrated by the following corollary.

**Corollary 5.1** *In Theorem 5.3, the closed-loop system composed of the given system (5.26) and the controller (5.31) has the properties that, for any initial state,*

$$\lim_{t \rightarrow \infty} \dot{\hat{\mu}}(t) = 0, \tag{5.33}$$

*and, if  $d(t)$ ,  $\dot{d}(t)$ ,  $f_a(x, t)$ ,  $\frac{\partial f_a(x, t)}{\partial t}$ , and  $\frac{\partial f_a(x, t)}{\partial x}$  are bounded for all  $t \geq 0$ , then*

$$\lim_{t \rightarrow \infty} f_a^\top(0, t) (\hat{\mu}(t) - \mu) = 0. \tag{5.34}$$

*Moreover, if  $f_a(0, t)$  is PE, then*

$$\lim_{t \rightarrow \infty} \hat{\mu}(t) = \mu. \tag{5.35}$$

*Proof* In Theorem 5.3, it has been proved that  $\lim_{t \rightarrow \infty} \text{col}(z(t), x(t)) = 0$  which implies (5.33). Since  $d(t)$ ,  $\dot{d}(t)$ ,  $f_a(x, t)$ ,  $\frac{\partial f_a(x, t)}{\partial t}$ , and  $\frac{\partial f_a(x, t)}{\partial x}$  are bounded,  $\ddot{x}$  is bounded and hence  $\dot{x}$  is uniformly continuous. By Barbalat's Lemma,  $\dot{x}(t)$  approaches zero as  $t \rightarrow \infty$ , which implies (5.34). Finally, by Lemma 2.4, (5.35) holds if  $f_a(0, t)$  is PE.  $\square$

### 5.3.2 Systems with High Relative Degree

In this section, we consider the general system (5.25) with  $r > 1$ . When  $r > 1$ ,  $x_2$  in (5.25) is not a real control, but the controller (5.31) in Theorem 5.3 motivates a coordinate transformation as follows:

$$\mathcal{X}_2 = x_2 - \mathcal{S}_1(x_1, \hat{\mu}, t), \quad \mathcal{S}_1(x_1, \hat{\mu}, t) = -f_a^T(x_1, t)\hat{\mu} - \rho_1(x_1)x_1$$

where  $\mathcal{X}_2$  is governed by the following equation:

$$\begin{aligned} \dot{\mathcal{X}}_2 &= f_2(\vec{x}_2) - \dot{\mathcal{S}}_1(x_1, \hat{\mu}, t) + x_3 \\ &= f_2(\vec{x}_2) - \frac{\partial \mathcal{S}_1(x_1, \hat{\mu}, t)}{\partial x_1} [f_1(z, x_1, d) + b f_a^T(x_1, t)\mu + b x_2] \\ &\quad - \frac{\partial \mathcal{S}_1(x_1, \hat{\mu}, t)}{\partial \hat{\mu}} \dot{\hat{\mu}} - \frac{\partial \mathcal{S}_1(x_1, \hat{\mu}, t)}{\partial t} + x_3. \end{aligned} \quad (5.36)$$

In (5.36), besides  $\mu$ , the parameter  $b$  is also unknown and should be estimated. Let  $\hat{b}$  be the estimation of  $b$ , and  $\tilde{b} = \hat{b} - b$  the estimation error.

With the introduction of the estimation  $\hat{b}$ , various functions that define the control law may also rely on  $\hat{b}$ . Thus, we define the dynamic coordinate transformation as follows:

$$\begin{aligned} \mathcal{X}_1 &= x_1 \\ \mathcal{X}_{i+1} &= x_{i+1} - \mathcal{S}_i(\vec{x}_i, \hat{\mu}, \hat{b}, t), \quad i = 1, \dots, r \\ \dot{\hat{\mu}} &= \tau_r(\vec{x}_r, \hat{\mu}, \hat{b}, t) \\ \dot{\hat{b}} &= \varpi_r(\vec{x}_r, \hat{\mu}, \hat{b}, t) \end{aligned} \quad (5.37)$$

for some sufficiently smooth functions  $\mathcal{S}_i, i = 1, \dots, r, \tau_r$  and  $\varpi_r$ , to be specified. Let  $\zeta_i = \text{col}(\vec{x}_i, \hat{\mu}, \hat{b})$ . Denote the system governing  $\zeta_r$  by  $\dot{\zeta}_r = \varphi_r(\zeta_r, \mathcal{X}_{r+1}, t)$ . Then, Proposition 4.1 can be generalized to the following form.

**Proposition 5.2** *If there exist sufficiently smooth functions  $\mathcal{S}_i, i = 1, \dots, r$ , and  $\tau_r$ , such that, the state  $\zeta_r$  of the system  $\dot{\zeta}_r = \varphi_r(\zeta_r, 0, t)$  is bounded and  $\lim_{t \rightarrow \infty} \vec{x}_r = 0$ , then the GARP with  $y = x_1$  for the system (5.25) is solved by the following dynamic controller*

$$\begin{aligned} u &= \mathcal{S}_r(\vec{x}_r, \hat{\mu}, \hat{b}, t) \\ \dot{\hat{\mu}} &= \tau_r(\vec{x}_r, \hat{\mu}, \hat{b}, t) \\ \dot{\hat{b}} &= \varpi_r(\vec{x}_r, \hat{\mu}, \hat{b}, t). \end{aligned} \quad (5.38)$$

Moreover, if  $\mathcal{S}_i(0, \hat{\mu}, \hat{b}, t) = 0$ ,  $i = 1, \dots, r-1$ , then the GASP with  $y = \vec{x}_r$  for the system (5.25) is solved by the same controller.

Like in Sect. 5.2, we need to find the functions  $\mathcal{S}_r$ ,  $\tau_r$ , and  $\varpi_r$  recursively. The first step is motivated from the previous case with  $r = 1$ . Again we note that, by (11.13) of the Appendix, one has

$$|f_1(z, x_1, d)| \leq m_1(z)\|z\| + m_2(x_1)|x_1|, \quad \forall d \in \mathbb{D} \quad (5.39)$$

for some sufficiently smooth and non-negative functions  $m_1$  and  $m_2$ . Let

$$\Delta(z) \geq 1 + rm_1^2(z). \quad (5.40)$$

Under Assumption 5.2, by using the changing supply function technique (Corollary 2.2), there exists a continuously differentiable function  $V'(z)$  satisfying  $\underline{\alpha}'(\|z\|) \leq V'(z) \leq \bar{\alpha}'(\|z\|)$  for some class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}'$  and  $\bar{\alpha}'$ , such that, along the state trajectory of  $\dot{z} = q(z, x_1, d)$ ,

$$\dot{V}'(z) \leq -\Delta(z)\|z\|^2 + \varkappa(x_1)x_1^2 \quad (5.41)$$

for a sufficiently smooth and non-negative function  $\varkappa$ . Let

$$\rho_1(x_1) \geq [\varkappa(x_1) + m_2(x_1) + 5/4 + (r-1)m_2^2(x_1)]/\underline{b} + \bar{b}/4. \quad (5.42)$$

Then we define

$$\begin{aligned} \mathcal{S}_1(x_1, \hat{\mu}, \hat{b}, t) &= -f_a^\top(x_1, t)\hat{\mu} - \rho_1(x_1)x_1 \\ \tau_1(x_1, \hat{\mu}, \hat{b}, t) &= \Lambda_\mu x_1 f_a(x_1, t) \\ \varpi_1(x_1, \hat{\mu}, \hat{b}, t) &= 0. \end{aligned} \quad (5.43)$$

The remaining functions are constructed as follows, for  $i = 2, \dots, r$ ,

$$\begin{aligned} \mathcal{S}_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) &= \mathcal{Q}_i(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) - \rho_i(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t)x_i + v_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) \\ \tau_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) &= \tau_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) \\ &\quad - \Lambda_\mu x_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t)}{\partial x_1} f_a(x_1, t) \\ \varpi_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) &= \varpi_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) \\ &\quad - \Lambda_b x_i \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t)}{\partial x_1} [f_a^\top(x_1, t)\hat{\mu} + x_2] \end{aligned} \quad (5.44)$$

where

$$\begin{aligned}
 \varrho_i \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) &= \left( - \sum_{j=1}^{i-2} x_{j+1} \frac{\partial S_j \left( \vec{x}_j, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \Lambda_{\mu} \right) \\
 &\quad \times \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} f_a(x_1, t) \\
 &\quad + \left( \hat{b} - \sum_{j=1}^{i-2} x_{j+1} \frac{\partial S_j \left( \vec{x}_j, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \Lambda_b \right) \\
 &\quad \times \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} [f_a^T(x_1, t) \hat{\mu} + x_2] \\
 \rho_i \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) &\geq \left[ \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} \right]^2 / 2 + 9/4,
 \end{aligned}$$

and

$$\begin{aligned}
 v_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) &= -f_i \left( \vec{x}_i \right) + \sum_{j=2}^{i-1} \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_j} [f_j \left( \vec{x}_j \right) + x_{j+1}] \\
 &\quad + \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \tau_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \\
 &\quad + \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \varpi_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \\
 &\quad + \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial t}.
 \end{aligned}$$

In the above equations, the matrix  $\Lambda_{\mu} = \Lambda_{\mu}^T > 0$  and the scalar  $\Lambda_b > 0$  can be arbitrarily selected. It is noted that, for  $i \geq 2$ , the function  $S_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right)$  is split into three terms for the convenience of proving the following theorem.

**Theorem 5.4** Consider the system (5.25) with  $\mathbb{D}$  a prescribed compact set. Under Assumptions 5.1 and 5.2, the GARP of (5.25) with the performance output  $y = \text{col}(z, x_1)$  is solved by the controller (5.38) where the functions  $S_r$  and  $\tau_r$  are defined in (5.43)–(5.44) and also summarized in Algorithm 5.2. Moreover, if  $f_a(0, t) = 0$ , then the GASP of (5.25) with the performance output  $y = \text{col}(z, \vec{x}_r)$  is solved by the same controller.

*Proof* Consider the system (5.25) under the coordinate transformation (5.37). Let

$$W_1(z, x_1, \hat{\mu}, \hat{b}) = V'(z) + x_1^2/2 + b\tilde{\mu}^\top \Lambda_\mu^{-1} \tilde{\mu}/2 + \Lambda_b^{-1} \tilde{b}^2/2$$

and, for  $2 \leq i \leq r$ ,

$$W_i(z, \vec{x}_i, \hat{\mu}, \hat{b}) = W_{i-1}(z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}) + x_i^2/2. \quad (5.45)$$

Then we will show that the derivative of the function  $W_i(z, \vec{x}_i, \hat{\mu}, \hat{b})$  along the state trajectory of the closed-loop system satisfies

$$\begin{aligned} \dot{W}_i(z, \vec{x}_i, \hat{\mu}, \hat{b}) &\leq -(r-i)m_1^2(z)\|z\|^2 - (r-i)m_2^2(x_1)\|x_1\|^2 \\ &\quad - \|z\|^2 - \sum_{j=1}^i x_j^2 + x_{i+1}^2 + \nabla_i(z, \vec{x}_i, \hat{\mu}, \hat{b}, t), \end{aligned} \quad (5.46)$$

where

$$\begin{aligned} \nabla_i(z, \vec{x}_i, \hat{\mu}, \hat{b}, t) &= \left( b\tilde{\mu}^\top \Lambda_\mu^{-1} - \sum_{j=1}^{i-1} x_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, \hat{b}, t)}{\partial \hat{\mu}} \right) \\ &\quad \times \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) \right] \\ &\quad + \left( \Lambda_b^{-1} \tilde{b} - \sum_{j=1}^{i-1} x_{j+1} \frac{\partial \mathcal{S}_j(\vec{x}_j, \hat{\mu}, \hat{b}, t)}{\partial \hat{b}} \right) \\ &\quad \times \left[ \dot{\hat{b}} - \varpi_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) \right]. \end{aligned}$$

The first step is to verify (5.46) for  $i = 1$ . Direct calculation shows that the derivative of  $W_1(z, x_1, \hat{\mu}, \hat{b})$  along the state trajectory satisfies

$$\begin{aligned} \dot{W}_1(z, x_1, \hat{\mu}, \hat{b}) &\leq -\Delta(z)\|z\|^2 + \kappa(x_1)x_1^2 + x_1 \left[ f_1(z, x_1, d(t)) + b f_a^\top(x_1, t) \mu \right. \\ &\quad \left. + b \mathcal{S}_1(x_1, t, \hat{\mu}) + b x_2 \right] + b\tilde{\mu}^\top \Lambda_\mu^{-1} \dot{\hat{\mu}} + \Lambda_b^{-1} \tilde{b} \dot{\hat{b}} \\ &\leq -(r-1)m_1^2(z)\|z\|^2 - (r-1)m_2^2(x_1)\|x_1\|^2 - \|z\|^2 \\ &\quad - x_1^2 + x_2^2 + b\tilde{\mu}^\top \Lambda_\mu^{-1} \left[ \dot{\hat{\mu}} - \tau_1(x_1, \hat{\mu}, \hat{b}, t) \right] + \Lambda_b^{-1} \tilde{b} \dot{\hat{b}}. \end{aligned}$$

Thus (5.46) is verified with  $i = 1$ .

Next, we will prove (5.46) for any  $1 < i \leq r$  assuming it is true for  $i - 1$ . To this end, note that the derivative of  $x_i^2/2$  along the state trajectory of the  $x_i$ -dynamics

satisfies

$$\begin{aligned}
\mathcal{X}_i \dot{\mathcal{X}}_i &= \mathcal{X}_i \dot{x}_i - \mathcal{X}_i \dot{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) \\
&\leq \mathcal{X}_i f_i \left( \vec{x}_i \right) + \mathcal{X}_i S_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) + \mathcal{X}_i^2/4 + \mathcal{X}_{i+1}^2 \\
&\quad + \left[ \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} \right]^2 / 2 + m_1^2(z) \|z\|^2 + m_2^2(x_1) x_1^2 \\
&\quad - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} \left[ b f_a^\top(x_1, t) \mu + b x_2 \right] \\
&\quad - \mathcal{X}_i \sum_{j=2}^{i-1} \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_j} \left[ f_j \left( \vec{x}_j \right) + x_{j+1} \right] \\
&\quad - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \dot{\hat{\mu}} - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \dot{\hat{b}} \\
&\quad - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial t} \\
&\leq \mathcal{X}_i f_i \left( z, \vec{x}_i \right) + \mathcal{X}_i S_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) + \mathcal{X}_i^2/4 + \mathcal{X}_{i+1}^2 \\
&\quad + \left[ \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} \right]^2 / 2 + m_1^2(z) \|z\|^2 + m_2^2(x_1) x_1^2 \\
&\quad - \mathcal{X}_i \sum_{j=2}^{i-1} \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_j} \left[ f_j \left( \vec{x}_j \right) + x_{j+1} \right] \\
&\quad - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \tau_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \\
&\quad - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \varpi_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \\
&\quad - \mathcal{X}_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial t} - \mathcal{X}_i \Gamma_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right)
\end{aligned}$$

where

$$\Gamma_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) = \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} b \left[ f_a^\top(x_1, t) \hat{\mu} + x_2 \right]$$

$$\begin{aligned}
& + \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t)}{\partial x_1} [-bf_a^\top(x_1, t)\tilde{\mu}] \\
& + \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t)}{\partial \hat{\mu}} \left[ \dot{\hat{\mu}} - \tau_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) \right] \\
& + \frac{\partial \mathcal{S}_{i-1}(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t)}{\partial \hat{b}} \left[ \dot{\hat{b}} - \varpi_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) \right].
\end{aligned}$$

By using the definition of  $\mathcal{S}_i(\vec{x}_i, \hat{\mu}, \hat{b}, t)$ , we have

$$\begin{aligned}
\mathcal{X}_i \dot{\mathcal{X}}_i & \leq \mathcal{X}_i \mathcal{Q}_i(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) - 2\mathcal{X}_i^2 + \mathcal{X}_{i+1}^2 + m_1^2(z)\|z\|^2 \\
& + m_2^2(x_1)x_1^2 - \mathcal{X}_i \Gamma_i(\vec{x}_i, \hat{\mu}, \hat{b}, t).
\end{aligned}$$

Now we are ready to verify that

$$\begin{aligned}
\dot{W}_i(z, \vec{x}_i, \hat{\mu}, \hat{b}) & = \dot{W}_{i-1}(z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}) + \mathcal{X}_i \dot{\mathcal{X}}_i \\
& \leq -(r-i+1)m_1^2(z)\|z\|^2 - (r-i+1)m_2^2(x_1)\|x_1\|^2 \\
& \quad - \|z\|^2 - \sum_{j=1}^{i-1} \mathcal{X}_j^2 + \mathcal{X}_i^2 + \nabla_{i-1}(z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) \\
& \quad + \mathcal{X}_i \mathcal{Q}_i(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) - 2\mathcal{X}_i^2 + \mathcal{X}_{i+1}^2 + m_1^2(z)\|z\|^2 \\
& \quad + m_2^2(x_1)x_1^2 - \mathcal{X}_i \Gamma_i(\vec{x}_i, \hat{\mu}, \hat{b}, t) \\
& \leq -(r-i)m_1^2(z)\|z\|^2 - (r-i)m_2^2(x_1)\|x_1\|^2 \\
& \quad - \|z\|^2 - \sum_{j=1}^i \mathcal{X}_j^2 + \mathcal{X}_{i+1}^2 + \nabla_{i-1}(z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) \\
& \quad + \mathcal{X}_i \mathcal{Q}_i(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) - \mathcal{X}_i \Gamma_i(\vec{x}_i, \hat{\mu}, \hat{b}, t).
\end{aligned}$$

To complete the proof of (5.46), it suffices to verify that

$$\begin{aligned}
\mathcal{X}_i \mathcal{Q}_i(\vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) & = \nabla_i(z, \vec{x}_i, \hat{\mu}, \hat{b}, t) - \nabla_{i-1}(z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t) \\
& + \mathcal{X}_i \Gamma_i(\vec{x}_i, \hat{\mu}, \hat{b}, t).
\end{aligned}$$

In fact, direct calculation shows

$$\begin{aligned}
& \nabla_i \left( z, \vec{x}_i, \hat{\mu}, \hat{b}, t \right) - \nabla_{i-1} \left( z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) \\
&= -\mathcal{X}_i \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \left[ \dot{\hat{\mu}} - \tau_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \right] \\
&\quad + \left( b \tilde{\mu}^\top \Lambda_\mu^{-1} - \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j \left( \vec{x}_j, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \right) \\
&\quad \times \left[ -\tau_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) + \tau_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) \right] \\
&\quad - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \left[ \dot{\hat{b}} - \varpi_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \right] \\
&\quad + \left( \Lambda_b^{-1} \tilde{b} - \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j \left( \vec{x}_j, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \right) \\
&\quad \times \left[ -\varpi_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) + \varpi_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) \right].
\end{aligned}$$

Using the equations in (5.44), one has

$$\begin{aligned}
& \nabla_i \left( z, \vec{x}_i, \hat{\mu}, \hat{b}, t \right) - \nabla_{i-1} \left( z, \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) \\
&= -\mathcal{X}_i \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \left[ \dot{\hat{\mu}} - \tau_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \right] \\
&\quad + \left( b \tilde{\mu}^\top - \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j \left( \vec{x}_j, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{\mu}} \Lambda_\mu \right) \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} f_a(x_1, t) \\
&\quad - \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \left[ \dot{\hat{b}} - \varpi_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right) \right] \\
&\quad + \left( \tilde{b} - \sum_{j=1}^{i-2} \mathcal{X}_{j+1} \frac{\partial \mathcal{S}_j \left( \vec{x}_j, \hat{\mu}, \hat{b}, t \right)}{\partial \hat{b}} \Lambda_b \right) \\
&\quad \times \mathcal{X}_i \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right)}{\partial x_1} \left[ f_a^\top(x_1, t) \hat{\mu} + x_2 \right] \\
&= \mathcal{X}_i \varrho_i \left( \vec{x}_{i-1}, \hat{\mu}, \hat{b}, t \right) - \mathcal{X}_i \Gamma_i \left( \vec{x}_i, \hat{\mu}, \hat{b}, t \right).
\end{aligned}$$

By mathematical induction, (5.46) holds for  $i = 1, \dots, r$ .

Finally, we consider the closed-loop system composed of (5.25) and the controller (5.38), i.e.  $\mathcal{X}_{r+1} = 0$ ,  $\dot{\hat{\mu}} = \tau_r \left( \vec{x}_r, \hat{\mu}, \hat{b}, t \right)$ , and  $\dot{\hat{b}} = \varpi_r \left( \vec{x}_r, \hat{\mu}, \hat{b}, t \right)$ . Then, the

inequality (5.46) with  $i = r$  becomes

$$\dot{W}_r \left( z, \vec{x}_r, \hat{\mu}, \hat{b} \right) \leq -\|z\|^2 - \sum_{j=1}^r x_j^2.$$

With  $W_r \left( z, \vec{x}_r, \hat{\mu}, \hat{b} \right)$  as the Lyapunov function for the closed-loop system, the stability of the closed-loop system is established by invoking Theorem 2.5 again. In particular,  $\lim_{t \rightarrow \infty} \text{col}(z(t), \vec{x}_r(t)) = 0$ . That is, the GARP for the system (5.25) with the performance output  $y = \text{col}(z, x_1)$  is solved by Proposition 5.2. To show the moreover part, it suffices to note  $S_i \left( 0, \hat{\mu}, \hat{b}, t \right) = 0$  if  $f_1(0, 0, d) = 0$ ,  $f_a(0, t) = 0$ , and  $f_i(0) = 0$ .  $\square$

### Algorithm 5.2

INPUT:  $f_i, i = 1, \dots, r, f_a, \bar{b}, \underline{b}, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \mathbb{D}$

OUTPUT:  $S_r, \tau_r$ , and  $\varpi_r$

STEP 1: For the given  $f_1(z, x_1, d)$ , find  $m_1$  and  $m_2$  from (5.39).

STEP 2: Pick the function  $\Delta$  from (5.40) and call

$$(\underline{\alpha}', \bar{\alpha}', \varkappa) = \text{ALGORITHM 2.3}(\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \Delta).$$

STEP 3: Let  $i = 1$ ; find the functions  $S_1$  and  $\tau_1$  from (5.43).

STEP 4: IF  $i = r$ , GO TO STEP 7, ELSE GO TO STEP 5.

STEP 5: Let  $i = i + 1$ ; calculate the functions  $S_i, \tau_i$ , and  $\varpi_i$  from (5.44).

STEP 6: GO TO STEP 4.

STEP 7: END

*Remark 5.2* When the parameter  $b$  is known, e.g.,  $b = 1$ , the controller (5.38) can be simplified. In particular, the functions in (5.44) reduce to

$$\begin{aligned} S_i \left( \vec{x}_i, \hat{\mu}, t \right) &= Q_i \left( \vec{x}_{i-1}, \hat{\mu}, t \right) - \rho_i \left( \vec{x}_{i-1}, \hat{\mu}, t \right) x_i + v_i \left( \vec{x}_i, \hat{\mu}, t \right) \\ \tau_i \left( \vec{x}_i, \hat{\mu}, t \right) &= \tau_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right) - \Lambda x_i \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial x_1} f_a(x_1, t) \end{aligned}$$

where

$$\begin{aligned} Q_i \left( \vec{x}_{i-1}, \hat{\mu}, t \right) &= \left( - \sum_{j=1}^{i-2} x_{j+1} \frac{\partial S_j \left( \vec{x}_j, \hat{\mu}, t \right)}{\partial \hat{\mu}} \Lambda \right) \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial x_1} f_a(x_1, t) \\ &\quad + \frac{\partial S_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial x_1} \left[ f_a^\top(x_1, t) \hat{\mu} + x_2 \right] \end{aligned}$$

$$\begin{aligned}
\rho_i \left( \vec{x}_{i-1}, \hat{\mu}, t \right) &\geq \left[ \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial x_1} \right]^2 / 2 + 9/4 \\
v_i \left( \vec{x}_i, \hat{\mu}, t \right) &= -f_i \left( \vec{x}_i \right) + \sum_{j=2}^{i-1} \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial x_j} \left[ f_j \left( \vec{x}_j \right) + x_{j+1} \right] \\
&\quad + \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial \hat{\mu}} \tau_i \left( \vec{x}_i, \hat{\mu}, t \right) \\
&\quad + \frac{\partial \mathcal{S}_{i-1} \left( \vec{x}_{i-1}, \hat{\mu}, t \right)}{\partial t}.
\end{aligned}$$

In the above equations, the matrix  $\Lambda = \Lambda^\top > 0$  can be arbitrarily selected.

Now we will consider the convergence issue of the parameter  $\hat{\mu}$  by the following corollary.

**Corollary 5.2** *In Theorem 5.4, the closed-loop system composed of the given system (5.25) and the controller (5.38) has the properties that, for any initial state,*

$$\lim_{t \rightarrow \infty} \hat{\mu}(t) = 0, \quad (5.47)$$

and, if  $d(t)$ ,  $\dot{d}(t)$ ,  $f_a(x_1, t)$ ,  $\frac{\partial f_a(x_1, t)}{\partial t}$ , and  $\frac{\partial f_a(x_1, t)}{\partial x_1}$  are bounded for all  $t \geq 0$ , then

$$\lim_{t \rightarrow \infty} f_a^\top(0, t) (\hat{\mu}(t) - \mu) = 0. \quad (5.48)$$

Moreover, if  $f_a(0, t)$  is PE, then

$$\lim_{t \rightarrow \infty} \hat{\mu}(t) = \mu. \quad (5.49)$$

*Proof* In Theorem 5.4, it has been proved that  $\lim_{t \rightarrow \infty} \text{col} \left( z, \vec{x}_r \right) = 0$  which implies  $\lim_{t \rightarrow \infty} \tau_r \left( \vec{x}_r(t), \hat{\mu}(t), \hat{b}(t), t \right) = 0$ , i.e., (5.47) is satisfied.

To show (5.48), we observe that since both  $x_1$  and  $\mathcal{X}_2 = x_2 + f_a^\top(x_1, t)\hat{\mu} + x_1\rho_1(x_1)$  approach 0 as  $t \rightarrow \infty$ , so does  $x_2 + f_a^\top(0, t)\hat{\mu}$ , i.e.,

$$\lim_{t \rightarrow \infty} (x_2(t) + f_a^\top(0, t)\hat{\mu}(t)) = 0. \quad (5.50)$$

Next, we will show

$$\lim_{t \rightarrow \infty} (x_2(t) + f_a^\top(0, t)\mu) = 0. \quad (5.51)$$

In fact, on one hand,  $x_1$  approaches zero as  $t \rightarrow \infty$ . On the other hand, since  $d(t)$ ,  $\dot{d}(t)$ ,  $f_a(x_1, t)$ ,  $\frac{\partial f_a(x_1, t)}{\partial t}$ , and  $\frac{\partial f_a(x_1, t)}{\partial x_1}$  are bounded,  $\ddot{x}_1$  is bounded and hence  $\dot{x}_1$  is uniformly continuous. By Barbalat's Lemma,  $\dot{x}_1$  approaches zero as  $t \rightarrow \infty$ , which implies (5.51). Comparing (5.50) and (5.51), we have  $\lim_{t \rightarrow \infty} f_a^T(0, t) (\hat{\mu}(t) - \mu) = 0$ , i.e., (5.48). Finally, by Lemma 2.4, (5.49) holds if  $f_a(0, t)$  is PE.  $\square$

*Example 5.2* Consider a nonlinear system

$$\begin{aligned}\dot{z} &= -z + x_1/2 \\ \dot{x}_1 &= d(t)(\sin x_1)z^2 + \mu[\cos(x_1) + a] + x_2 \\ \dot{x}_2 &= u\end{aligned}\tag{5.52}$$

where  $d(t) \in [-2, 2]$  is an external disturbance and  $\mu$  is an unknown constant parameter. The parameter  $a$  is a known constant. Design a controller  $u$  to solve the GASP with the performance output  $y = \text{col}(z, x_1, x_2)$ .

It is easy to verify the satisfaction of Assumption 5.2. So, the controller can be given in the form (5.38) where the functions  $S_2$  and  $\tau_2$  are given below.

First, we note  $|d(\sin x_1)z^2| \leq 2|z||z|$ . Let  $\Delta(z) \geq 1 + 8z^2$ . Consider the function  $V(z) = z^2 + 4z^4$  whose derivative along the state trajectory of  $\dot{z} = -z + x_1/2$  satisfies

$$\begin{aligned}\dot{V}(z) &= 2z(-z + x_1/2) + 16z^3(-z + x_1/2) \\ &\leq -z^2 + x_1^2/4 - 8z^4 + 8x_1^4 \\ &= -\left(1 + 8z^2\right)z^2 + \left(1/4 + 8x_1^2\right)x_1^2.\end{aligned}$$

Then, we can pick a function  $\rho_1(x_1) = 2 + 8x_1^2$  and hence

$$\begin{aligned}S_1(x_1, \hat{\mu}) &= -[\cos(x_1) + a]\hat{\mu} - x_1\rho_1(x_1) \\ \tau_1(x_1) &= x_1[\cos(x_1) + a].\end{aligned}$$

Define

$$S_{1x}(x_1, \hat{\mu}) = \frac{\partial S_1(x_1, \hat{\mu})}{\partial x_1} = \sin(x_1)\hat{\mu} - 2 - 24x_1^2.$$

The remaining functions are given in order:

$$\begin{aligned}\rho_2(x_1, \hat{\mu}) &= S_{1x}^2(x_1, \hat{\mu})/2 + 9/4 \\ \varrho_2(x_1, \hat{\mu}) &= S_{1x}(x_1, \hat{\mu})[\cos(x_1) + a]\hat{\mu} + S_{1x}(x_1, \hat{\mu})x_2\end{aligned}$$

and

$$\begin{aligned}\tau_2 \left( \vec{x}_2, \hat{\mu} \right) &= \tau_1(x_1) - \left[ x_2 - \mathcal{S}_1 \left( \vec{x}_1, \hat{\mu} \right) \right] \mathcal{S}_{1x} \left( x_1, \hat{\mu} \right) [\cos(x_1) + a] \\ v_2 \left( \vec{x}_2, \hat{\mu} \right) &= -[\cos(x_1) + a] \tau_2 \left( \vec{x}_2, \hat{\mu} \right).\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathcal{S}_2 \left( \vec{x}_2, \hat{\mu} \right) &= \mathcal{S}_{1x} \left( x_1, \hat{\mu} \right) [\cos(x_1) + a] \hat{\mu} + \mathcal{S}_{1x} \left( x_1, \hat{\mu} \right) x_2 \\ &\quad - \left[ x_2 - \mathcal{S}_1 \left( x_1, \hat{\mu} \right) \right] \left[ \mathcal{S}_{1x}^2 \left( x_1, \hat{\mu} \right) / 2 + 9/4 \right] \\ &\quad - [\cos(x_1) + a] \tau_2 \left( \vec{x}_2, \hat{\mu} \right).\end{aligned}$$

By Theorem 5.4, the GASP is solved by the controller

$$\begin{aligned}u &= \mathcal{S}_2 \left( \vec{x}_2, \hat{\mu} \right) \\ \dot{\hat{\mu}} &= \tau_2 \left( \vec{x}_2, \hat{\mu} \right).\end{aligned}$$

The performance of the controller is simulated with  $\mu = 2$ ,  $z(0) = 5$ ,  $x(0) = [1, 1]^T$  and  $\hat{\mu}(0) = 0$ , and some results are shown in Figs. 5.3, 5.4, 5.5 and 5.6. Two cases are observed as follows.

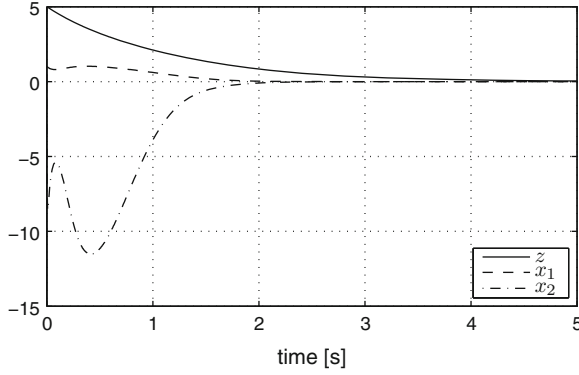
*Case 1:*  $a = -1$ . In this case, the uncontrolled system has an equilibrium point  $\text{col}(z, x) = [0, 0, 0]$ . The simulation result is shown in Figs. 5.3 and 5.4. In particular, the state variables  $z$  and  $x$  of the closed-loop system converge to the origin. However, the function  $f_a(0, t) = \cos(0) + a = 0$  is obviously not PE. Therefore, the estimated parameter  $\hat{\mu}$  does not converge to its real value of  $\mu = 2$ .

*Case 2:*  $a = 0$ . In this case, the origin  $\text{col}(z, x) = [0, 0, 0]$  is not an equilibrium point of the uncontrolled system. The simulation result is shown in Figs. 5.5 and 5.6. We can see that the state variables  $z$  and  $x_1$  of the closed-loop system converge to 0 but the state  $x_2$  does not. The function  $f_a(0, t) = \cos(0) + a = 1$  is obviously PE. Therefore, the estimated parameter  $\hat{\mu}$  converges to its real value of  $\mu = 2$ .

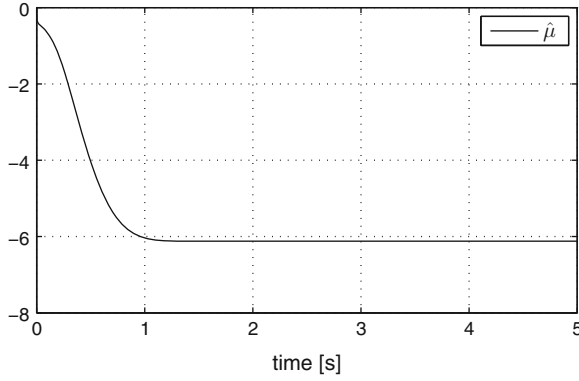
## 5.4 Adaptive Stabilization of the Duffing Equation

In this section, we consider the controlled Duffing equation (3.33). The external disturbance  $v(t) = \gamma \cos(\omega t + \phi)$  can be rewritten as  $v(t) = \gamma_1 \cos(\omega t) + \gamma_2 \sin(\omega t)$  for some constants  $\gamma_1$  and  $\gamma_2$ . As a result, the model is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 - \alpha x_1 - \beta x_1^3 + \gamma_1 \cos(\omega t) + \gamma_2 \sin(\omega t) + u.\end{aligned}\tag{5.53}$$



**Fig. 5.3** Profile of state trajectories of the closed-loop system in Example 5.2: Case 1



**Fig. 5.4** Profile of estimated parameters of the closed-loop system in Example 5.2: Case 1

In (5.53), the parameters  $\delta$ ,  $\alpha$ ,  $\beta$ ,  $\gamma_1$  and  $\gamma_2$  are unknown, but the frequency  $\omega$  of the external signal  $v(t)$  is assumed to be known. We will consider the regulation problem with the performance output and the measurement output being  $y = y_m = x = [x_1, x_2]^T$ .

First, let

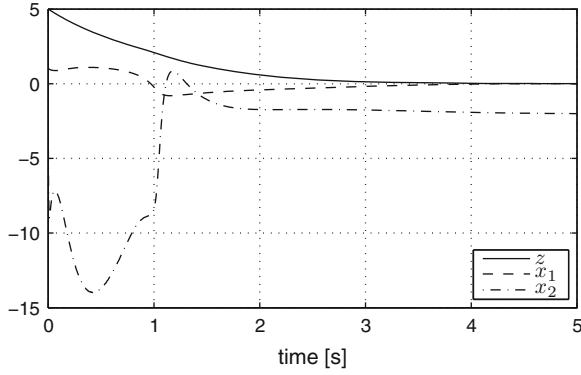
$$\mathcal{X}_2 = x_2 + \rho_1 x_1, \quad \rho_1 > 0.$$

As a result, we have

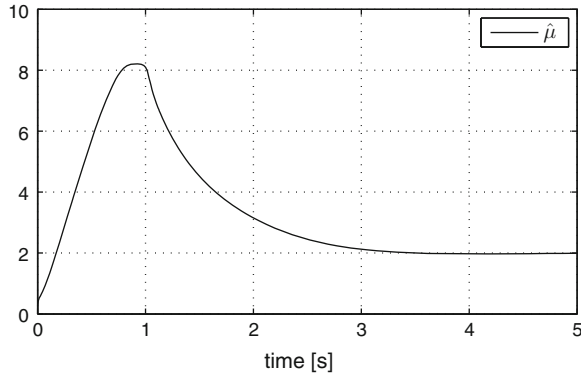
$$\dot{x}_1 = -\rho_1 x_1 + \mathcal{X}_2$$

which is ISS with the state  $x_1$  and the input  $\mathcal{X}_2$ . The  $\mathcal{X}_2$ -dynamics become

$$\dot{\mathcal{X}}_2 = f_a^T(x_1, x_2, t)\mu + u \quad (5.54)$$



**Fig. 5.5** Profile of state trajectories of the closed-loop system in Example 5.2: Case 2



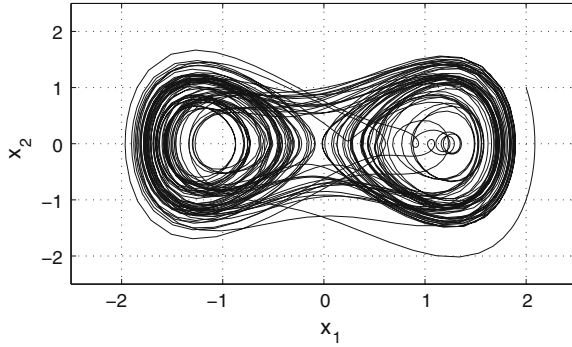
**Fig. 5.6** Profile of estimated parameters of the closed-loop system in Example 5.2: Case 2

where

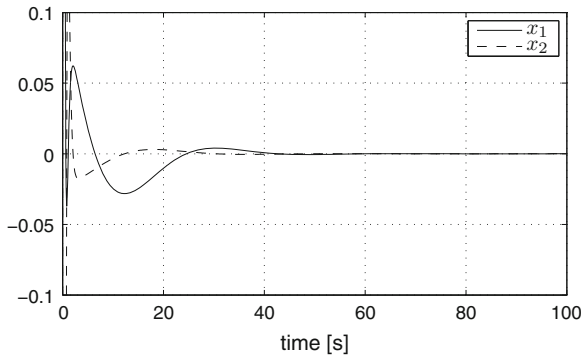
$$f_a(x_1, x_2, t) = \begin{bmatrix} x_2 \\ x_1 \\ x_1^3 \\ \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}, \quad \mu = \begin{bmatrix} -\delta + \rho_1 \\ -\alpha \\ -\beta \\ \gamma_1 \\ \gamma_2 \end{bmatrix}.$$

By Theorem 5.4, an adaptive controller can be designed as follows:

$$\begin{aligned} u &= -\rho_2 x_2 - f_a^\top(x_1, x_2, t) \hat{\mu}, \quad \rho_2 > 0 \\ \dot{\hat{\mu}} &= \Lambda x_2 f_a(x_1, x_2, t). \end{aligned} \quad (5.55)$$



**Fig. 5.7** Profile of state trajectories of the uncontrolled Duffing equation



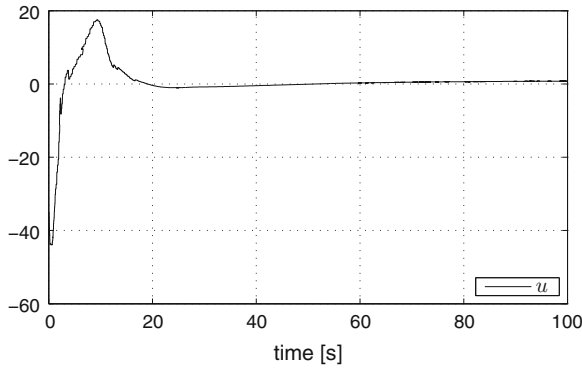
**Fig. 5.8** Profile of state trajectories of the controlled Duffing equation

The performance of the controller is simulated with  $\delta = 0.03$ ,  $\alpha = -1$ ,  $\beta = 1$ , and  $v(t) = 0.8 \cos(0.2t + 0.1)$ . The uncontrolled system trajectories are shown in Fig. 5.7. When the controller is applied with  $\rho_1 = 3$  and  $\rho_2 = 5$ , the results are shown in Figs. 5.8 and 5.9. In particular, it is shown in Fig. 5.8 that the system state asymptotically approaches the equilibrium point  $x = 0$ . The control signal is shown in Fig. 5.9.

It is noted that the adaptive regulation method developed in this chapter cannot handle the unknown  $\omega$  because  $\cos(\omega t)$  and  $\sin(\omega t)$  are not linear in  $\omega$ . However, the internal model approach to be introduced in Chap. 9 is able to handle the unknown  $\omega$ .

## 5.5 Notes and References

The adaptive backstepping design for time-invariant nonlinear systems was first introduced in [1]. It requires multiple estimates of the same parameter. This over-



**Fig. 5.9** Profile of control signal of the controlled Duffing equation

parametrization was eliminated in [2] by using the tuning functions design. The adaptive control of time-invariant nonlinear systems is thoroughly covered in the book [3]. In this chapter, we have extended the approach in [1] to time-varying systems so that this approach can be applied to the adaptive output regulation problem studied in Chaps. 9 and 10 and we have also thoroughly addressed the parameter convergence issue. The materials in this chapter can find the references in [4–6]. The example on the Duffing equation can find the reference in [7].

## 5.6 Problems

**Problem 5.1** Solve the GASP for each of the following systems with state  $x$ , input  $u$ , and an unknown constant parameter  $\mu \in \mathbb{R}^2$ .

- (a)  $\dot{x} = [\sin t, x + 1]\mu + u$ ;
- (b)  $\dot{x} = [x, x^2]\mu + u$ ;
- (c)  $\dot{x} = [\cos x, x/(1+t)]\mu + u$ .

Simulate the closed-loop system for  $\mu = [2, 1]^T$  starting from different initial conditions.

**Problem 5.2** Use tuning functions design approach to solve the GARP for each of the following systems ( $a = 1$ ) with state  $x$ , input  $u$ , and unknown constant parameters  $\mu_1, \mu_2 \in \mathbb{R}$ .

- (a)  $\dot{x}_1 = \mu_1(x_1^2 + a) + x_2, \dot{x}_2 = \mu_2 x_1 x_2 + u$ ;
- (b)  $\dot{x}_1 = \mu_1 x_1^2 + \mu_2 a + x_2, \dot{x}_2 = u$ ;
- (c)  $\dot{x}_1 = \mu_1(x_1 + a)/(t + 1) + x_2, \dot{x}_2 = \mu_2 x_2 + u$ .

Simulate the closed-loop system for  $\mu_1 = 3$  and  $\mu_2 = -4$  starting from different initial conditions.

**Problem 5.3** Repeat Problem 5.2 for  $a = 0$  and show that the controllers indeed solve the GASP for the systems.

**Problem 5.4** Solve the GASP for each of the following systems with state  $x$ , input  $u$ , and an unknown constant parameter  $\mu \in \mathbb{R}^2$ . The systems contain static uncertainty  $-1 \leq d(t) < 1$  and dynamic uncertainty represented by the  $z$ -dynamics.

- (a)  $\dot{z} = -z + 0.5x$ ,  $\dot{x} = d(t)|zx| + [\sin t, x + 1]\mu + u$ ;
- (b)  $\dot{z} = -(2 + d(t))z + x$ ,  $\dot{x} = z^2 + 2d(t)x^2 + [x, x^2]\mu + u$ ;
- (c)  $\dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} x$ ,  $\dot{x} = d(t)\|z\|x + [\cos x, x/(1+t)]\mu + u$ .

Simulate the closed-loop system for  $\mu = [2, 1]^T$  starting from different initial conditions.

**Problem 5.5** Repeat Problem 5.4 for the systems with an input filter added, i.e.,

- (a)  $\dot{z} = -z + 0.5x_1$ ,  $\dot{x}_1 = d(t)|zx_1| + [\sin t, x_1 + 1]\mu + x_2$ ,  $\dot{x}_2 = u$ ;
- (b)  $\dot{z} = -(2 + d(t))z + x_1$ ,  $\dot{x}_1 = z^2 + 2d(t)x_1^2 + [x_1, x_1^2]\mu + x_2$ ,  $\dot{x}_2 = u$ ;
- (c)  $\dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} x_1$ ,  $\dot{x}_1 = d(t)\|z\|x_1 + [\cos x_1, x_1/(1+t)]\mu + x_2$ ,  $\dot{x}_2 = u$ .

**Problem 5.6** Determine if the estimation  $\hat{\mu}$  of the unknown parameter  $\mu$  converges to the real value of  $\mu$  in Problem 5.5. Justify your answer.

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