Lecture 1.

A. Interacting particle systems as Markov processes.

Extra assumption: T-invariance

Class of IPS considered: particles live on $\mathbb{Z}$, there can be at most one particle per site, the interactions are with the nearest neighbours only, the evolution occurs in continuous time and processes prior to t and after t are independent conditionally on the state of the process at t.

I.e. configurations of particles

Mathematically, such IPS's are described as continuous time Markov processes on $\Omega = \{0,1\}^{\mathbb{Z}}$, where $\eta \in \Omega$ is the configuration vector: $\eta_i = 1$ indicates that there is a particle at $i \in \mathbb{Z}$, $\eta_i = 0$ indicates a hole ($\equiv$ empty site). The Ω-valued process $(\eta_t)_{t \geq 0}$ is char. by its generator L defined on the space of test functions $T(\Omega)$, consisting of functions $\eta \mapsto f(\eta)$ depending on finitely many variables (cylinder functions).

Ω-config space, is usually equipped with σ-algebra gen. by finite sets

The generator encodes the interaction rules and has the following defining property: $\forall f \in T(\Omega)$

$$\frac{d}{dt} \mathbb{E}_{\eta_0}\left[f(\eta_t)\right] = \mathbb{E}_{\eta_0}\left[Lf(\eta_t)\right] \qquad (1)$$

Initial condition, will be omitted unless needs emphasizing

Generators are analogous to Q-matrices for Markov chains on finite config. spaces.

Recall that a (time-homogeneous) Markov process can be defined by a (continuous parameter) semigroup $(K_t)_{t\geq 0}$ of transition operators,

We assume that $(\eta_t)_{t\geq 0}$ is adapted to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$

$$\mathbb{E}\left[f(\eta_s) \mid \mathcal{F}_t\right] = (K_{s-t} f)(\eta_t), \qquad (*)$$

$$s \geq t,\ f \in T(\Omega)$$

The generator is defined as

$$Lf = \lim_{h \downarrow 0} \frac{K_h f - f}{h}. \qquad (**)$$

Remarks in between # signs can be omitted

Therefore

$$\frac{d}{dt} \mathbb{E}\left[f(\eta_s / \mathcal{F}_t)\right] = \lim_{h\downarrow 0} \mathbb{E}\left[\frac{f(\eta_{s+h}) - f(\eta_s)}{h} \Big| \mathcal{F}_t\right]$$

$$\overset{(*)}{=} \lim_{h\downarrow 0} \frac{(K_{s+h-t} f)(\eta_t) - (K_{s-t} f)(\eta_t)}{h}$$

$$\overset{SGP}{=} \lim_{h\downarrow 0} \frac{K_{s-t}(K_h f - f)(\eta_t)}{h}$$

$$= K_{s-t} Lf(\eta_t) \overset{(*)}{=} \mathbb{E}\left[Lf(\eta_t) \mid \mathcal{F}_t\right] \#$$

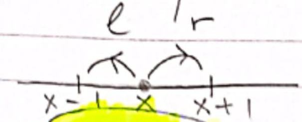
Explicitly,

An IPS is usually defined in terms of rates (2)

if $R(\eta\to\eta')$ is the (exponential) transition rate from state $\eta\in\Omega$ to $\eta'\in\Omega'$, then

$$Lf(\eta)=\sum_{\eta'}R(\eta\to\eta')\big(f(\eta')-f(\eta)\big)$$

Examples. (i) A single RW on $\mathbb{Z}$



Rates

$\boxed{r+l=1}$

$$\eta=(\dots 0\ 1\ 0\dots)$$
$\uparrow$ x-component, $x\in\mathbb{Z}$

So, it is conv. to identify Ω with $\mathbb{Z}$. Then

$$Lf(x)=r\big(f(x+1)-f(x)\big)+l\big(f(x-1)-f(x)\big)$$

$\sum$ over all possible transitions

$$\equiv \Delta^{(l,r)}f(x),$$

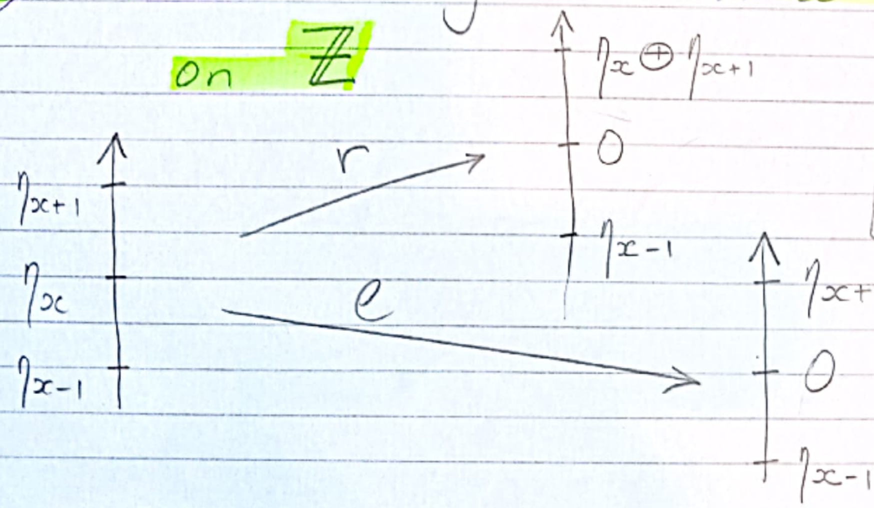
where $\Delta^{(l,r)}=rD+lD^{-1}-(r+l)\cdot I$ is the discrete Laplacian. (3)

D: Forward shift operator.

Remark. By time rescaling can always ensure that $r+l=1$.

(ii) Annihilating random walks on $\mathbb{Z}$

Rates



$\oplus$ - addition mod 2

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \Big[r\big(f(\ldots \eta_{x-1} \underset{x}{0} \eta_x \oplus \eta_{x+1} \eta_{x+2} \ldots)$$

$$- f(\eta)\big) + \ell\big(f(\ldots \eta_{x-2} \eta_x \oplus \eta_{x-1} \underset{x}{0} \eta_{x+1} \ldots)$$

$$- f(\eta)\big)\Big] \tag{4}$$

B. Markov duality

Let $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$ be Markov processes on Ω^X, Ω^Y respectively. Let $H: \Omega^X \times \Omega^Y \to \mathbb{R}$ be bnded measurable. Let L^X, L^Y be the generators of X, Y. The processes X, Y are called (Markov) dual w.r.t. H if $\forall (x,y) \in \Omega^X \times \Omega^Y$, $\forall t > 0$,

$$\underbrace{\mathbb{E}^X_x}_{\substack{\text{Expectation} \\ \text{w.r.t. prob.} \\ \text{measure of } X \\ \text{started at } x}} \left[H(X_t, y) \right] = \mathbb{E}^Y_y \left[H(x, Y_t) \right] \tag{5}$$

Can have a proportionality constant

(5) has an infinitesimal form: differentiating (5) w.r.t. t,

Recall: $\mathbb{E}^X_x[H(X_t,y)] = K_t H(\cdot,y)(x)$

$$\underbrace{\frac{d}{dt} K^X_t}_{L^X K^X_t} H(\cdot, y)(x) = \underbrace{\frac{d}{dt} K^Y_t}_{L^Y K^Y_t} H(x, \cdot)(y)$$

$$\Downarrow$$

$$L^X \mathbb{E}^X_x \left[H(X_t, y) \right] = L^Y \mathbb{E}^Y_y \left[H(x, Y_t) \right] \quad \forall t \geq 0$$

This is just the BKE

Setting $t=0$,

$$L^X H(x,y) = L^Y H(x,y) \qquad (6)$$

(6) can be used in the following interesting way:

$$\frac{d}{dt}\mathbb{E}_x^X\left[H(X_t,y)\right]$$

$$= \frac{d}{dt} K_t^X H(x,y) = K_t^X L^X H(x,y)$$

$$= \mathbb{E}_x^X\left[L^X H(X_t,y)\right]$$

$$\overset{(6)}{=} \mathbb{E}_x^X\left[L^Y H(X_t,y)\right]$$

$$= L^Y \mathbb{E}_x^X\left[H(X_t,y)\right] \qquad (7)$$

– a closed eq-n for $\mathbb{E}_x^X\left[H(X_t,y)\right]$

(i) If the process Y is "simple" (e.g. has finitely many particles), (7) can be solvable

(ii) If one can find a set $(H_i)_{i\in I}$ of duality functions s.t. $\left(\mathbb{E}_x^X\left[H_i(X_t,y)\right]\right)_{i\in I}$ determines the law of X_t, we call such a set complete. Existence of a complete set of duality functions can be a useful definition of integrability in probability theory.

(iii) Solving (7) often gives observables for the X process. Contrast this with methods of studying statistics of X by diagonalizing $\mathcal{L}^X$.

C. Matrix representation for the inf. generator

As we consider nearest neighbour interactions only and assume T-invariance,

$$(8) \qquad \mathcal{L} = \sum_{x \in \mathbb{Z}} \underbrace{q_x}$$

acts non-trivially on functions of (η_i, η_{i+1}) only.

As $T(\Omega)$ is linear can represent $\mathcal{L}$ as an ∞ matrix by choosing an appropriate basis in the space of test functions

Warm up: $\{f: \{0,1\} \to \mathbb{R}\} \cong \mathbb{R}^2$.

Indeed, $f(\eta) = f(\eta)\left(\mathbb{1}_0(\eta) + \mathbb{1}_1(\eta)\right)$

$= f(0)\,\mathbb{1}_0(\eta) + f(1)\,\mathbb{1}_1(\eta)$

Possible isomorphism: $\begin{cases} \mathbb{1}_1 \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} := e^{(1)} \\ \mathbb{1}_0 \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} := e^{(0)} \end{cases}$

So, $f \longleftrightarrow \begin{pmatrix} f(1) \\ f(0) \end{pmatrix}$

Examples. **Constant function** $f(\eta)\equiv 1 \longleftrightarrow \begin{pmatrix}1\\1\end{pmatrix} := \sigma$

σ is the "vacuum vector"

Parity function $f(\eta)=(-1)^{\eta} \longleftrightarrow \begin{pmatrix}-1\\1\end{pmatrix} := \omega$

Full construction: let V be a copy of $\mathbb{R}^2$.

Define: $T := \bigotimes_{n\in\mathbb{Z}} V_n$ – linear space over $\mathbb{R}$

an ∞ tensor product of V's

spanned by $\otimes_{n\in\mathbb{Z}}\sigma_n$, where $\sigma_n \in V_n$ and $\sigma_n = \sigma$ for all but finitely many values of $n\in\mathbb{Z}$, modulo the usual equivalence relations:

$$\begin{cases} \ldots \otimes \alpha a_m \otimes \ldots \otimes \beta_n \otimes \ldots \sim \ldots \otimes \alpha_m \otimes \ldots \otimes \alpha\beta_n \otimes \ldots \quad \forall m<n\in\mathbb{Z},\ \alpha\in\mathbb{R} \\ \ldots \otimes (a_n+b_n)\otimes \ldots \sim \ldots \otimes a_n \otimes \ldots + \ldots \otimes b_n \otimes \ldots \end{cases}$$

Claim $T(\Omega) \cong T$

Check: $T(\Omega)$ is spanned by indicator fnctns

$$\eta \longmapsto \prod_{x=m}^{n} \mathbb{1}_{\alpha_x}(\eta(x)),\quad m\le n,\ \alpha_x\in\{0,1\}$$

The isomorphism in question is given by $\prod_{x=m}^{n}\mathbb{1}_{\alpha_x} \longleftrightarrow \ldots \otimes \sigma \left(\bigotimes_{x=m}^{n} e_x^{(\alpha_x)}\right)\otimes\sigma\otimes\ldots$

As an operator on T,

$$L = \sum_{n\in\mathbb{Z}} \underbrace{q_n}_{\in\, \mathrm{End}(V_n\otimes V_{n+1})},$$ where q_n is a 4×4 matrix.

In the basis $(e^{(1)}\otimes e^{(1)}, e^{(1)}\otimes e^{(0)}, e^{(0)}\otimes e^{(1)}, e^{(0)}\otimes e^{(0)})$, (8)

$$q = \begin{array}{c} \\ 11 \\ 10 \\ 01 \\ 00 \end{array}\begin{array}{c} \begin{array}{cccc} 11 & 10 & 01 & 00 \end{array} \\ \begin{pmatrix} * & R(11\to 10) & R(11\to 01) & R(11\to 00) \\ R(10\to 11) & * & R(10\to 01) & R(10\to 00) \\ R(01\to 11) & R(01\to 10) & * & R(01\to 00) \\ R(00\to 11) & R(00\to 10) & R(00\to 01) & * \end{pmatrix} \end{array},$$

where the diagonal elements are chosen so that each row sum is zero,

$$\sum_J q_{IJ} = 0, \quad I \in (11, 10, 01, 00)$$

Remark. $\mathbb{T}$ can be turned into a ring by introducing Hadamard multiplication:

$$\left(\bigotimes_{i\in\mathbb{Z}} a_i, \bigotimes_{i\in\mathbb{Z}} b_i\right) \overset{*}{\longmapsto} \bigotimes_{i\in\mathbb{Z}} (a_i * b_i),$$

where $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$.

The above isomorphism between $T(\Omega)$ and T as linear spaces extends to the isomorphism between rings of functions

(The main step is ~~notice~~ to notice that if $f \leftrightarrow \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$, $g \leftrightarrow \begin{pmatrix} g(1) \\ g(2) \end{pmatrix}$, then $f\cdot g \leftrightarrow \begin{pmatrix} f(1)g(1) \\ f(2)g(2) \end{pmatrix} = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix} * \begin{pmatrix} g(1) \\ g(2) \end{pmatrix}$ #

D Hecke algebras

For us it is sufficient to define HA $\mathbb{H}_n(q)$ as a unital associative algebra over R generated by $(S_i)_{1\le i\le n-1}$ subject to

$$\begin{cases} S_iS_j = S_jS_i & |i-j|\geq 2 \\ S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1} & 1\le i\le n-2 \\ S_i^2 = 1 + (q-q^{-1})S_i & \end{cases}$$

Here $q\in R\setminus\{0\}$ is fixed. Due to the isomorphism between $\mathbb{H}_n(q)$, $\mathbb{H}_n(-q)$ and $\mathbb{H}_n(q^{-1})$, can assume that $q\in(0,1]$.

Remarks. (i) $\mathbb{H}_n(1)\cong RS_n$ - the group algebra of the symmetric group S_n. So $\mathbb{H}_n(q)$ is a deformation of RS_n; (ii) Typically, HA is considered as an algebra over $R[q,q^{-1}]$, where q is a formal parameter. Our definition corresponds to a specialization of HA by assigning a numerical value to q. (iii) More generally, $\mathbb{H}_n(q)$ is defined given a Coxeter system (W, S). Our definition

W — Coxeter group; S — Set of generators

corresponds to a particular choice of Coxeter group - the symmetric group S_n. It's relations are visualized by Dynkin diagram of type A, so we are actually discussing: type-A Hecke algebras. $\boxed{(s_i s_j)^{m_{ij}}=1 \text{ or use } (T_w)_{w\in W}}$

(iv) The quadratic relation can be re-written as $(s_i+q)(s_i-q^{-1})=0$ giving the minimal polynomial for all the generators.

In the context of IPS's it is convient to use two alternative sets of generators:

Stochastic generators

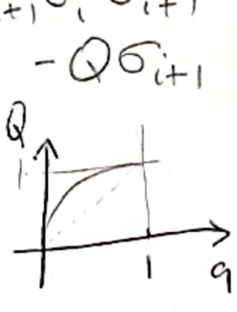
$$\sigma_i = \frac{q-s_i}{q+q^{-1}},$$

$1\leq i\leq n-1.$

Then

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|\geq 2 \\ \sigma_i^2 = \sigma_i & \\ \sigma_i\sigma_{i+1}\sigma_i - Q\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} - Q\sigma_{i+1} & \end{cases}$$

$$Q = \frac{1}{(q+q^{-1})^2}$$



Markov generators

$$g_i = \sigma_i - 1 = \frac{q^{-1}+s_i}{q+q^{-1}},$$

$1\leq i\leq n-1.$

Then

$$\begin{cases} g_i g_j = g_j g_i & |i-j|\geq 2 \\ g_i^2 = -g_i & \\ g_i g_{i+1} g_i - Q g_i = g_{i+1} g_i g_{i+1} - Q g_{i+1} & \end{cases}$$

$$Q = \frac{1}{(q+q^{-1})^2}$$

We will be interested in algebras generated by $(g_i)_{i\in\mathbb{Z}}$. To construct $\mathbb{H}_\infty$, consider $(\mathbb{H}_n)_{n\geq 1}$ and $e_{nm}: \mathbb{H}_m \hookrightarrow \mathbb{H}_n$ - a natural embedding, where $m\leq n$

Then $\mathbb{H}_\infty := \varinjlim \mathbb{H}_n = \bigsqcup_{i \in \mathbb{N}} \mathbb{H}_n / \sim$,

where $a_i \in \mathbb{H}_i \sim a_j \in \mathbb{H}_j$ if $\exists k$:

$$e_{kj}(a_j) = e_{ki}(a_i).$$

Elements of $\mathbb{H}_\infty$ are finite linear combinations of words of finite length with letters from the alphabet $\{a_i\}_{i \in \mathbb{Z}}$

Annoyingly, the generator itself doesn't belong to $\mathbb{H}_\infty$. Fortunately, this fact is irrelevant for the following construction.