Representations of Hecke Algebras and Markov Dualities for Interacting Particle Systems

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Lecture 1

Set-up: continuous time Markov processes

- Configuration space: $\Omega := \{0, 1\}^{\mathbb{Z}}$
- Configuration vector: $\eta \in \Omega$; $\eta_i = 1$ indicates a particle at $i \in \mathbb{Z}$; $\eta_i = 0$ indicates a hole at $i \in \mathbb{Z}$
- The space of test functions: $T(\Omega)$ - functions depending on finitely many components of η (cylinder functions)
- The Markov process $(\eta_t)_{t \geq 0}$ is characterised by its infinitesimal generator L
- Defining property of L: for any $f \in T(\Omega)$

$$\frac{d}{dt}\mathbb{E}_{\eta_0}\left[f(\eta_t)\right] = \mathbb{E}_{\eta_0}\left[Lf(\eta_t)\right]$$

The generator and transition rates. Examples

- Let $R(\eta \to \eta')$ be the transition rate from $\eta \in \Omega$ to $\eta' \in \Omega$
- **Claim.** $Lf(\eta) = \sum_{\eta' \in \Omega} R(\eta \to \eta') \, [f(\eta') - f(\eta)]$
- Asymmetric RW on $\mathbb{Z}$:

$$Lf(x) = r(f(x+1) + \ell f(x-1) - f(x) := \Delta^{(l,r)} f(x)$$

- Annihilating random walks on $\mathbb{Z}$ ([ARW]):

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \Big[rf(\ldots, \eta_{x-1}, 0, \eta_x \oplus \eta_{x+1}, \eta_{x+2}, \ldots)$$

$$+\ell f(\ldots, \eta_{x-2}, \eta_x \oplus \eta_{x-1}, 0, \eta_{x+1}, \ldots) - f(\eta) \Big]$$

Markov duality

- Set up: $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ - Markov processes on Ω^X, Ω^Y with infinitesimal generators $L^X,\ L^Y$; $H : \Omega^x \times \Omega^Y \to \mathbb{R}$ - bounded measurable function
- Definition: $X,\ Y$ are called Markov dual w. r. t. H if $\forall (x,y) \in \Omega^X \times \Omega^Y, \forall t > 0,$

$$\mathbb{E}_x^X \left[H(X_t, y) \right] = \mathbb{E}_y^Y \left[H(x, Y_t) \right]$$

- Infinitesimal form: $L^X H(x,y) = L^Y H(x,y)$
- **Claim.** $\frac{d}{dt}\mathbb{E}_x^X \left[H(X_t, y) \right] = L^Y \mathbb{E}_x^X \left[H(X_t, y) \right]$

Matrix representation for the generator

- Relevant generators: $L = \sum_{n\in\mathbb{Z}} q_x$, where q_x acts non-trivially on (η_x, η_{x+1}) only
- $T := \otimes_{n\in\mathbb{Z}} V_n$ - infinite tensor product, $v_n = (1,1)^T := v$ for almost all n's
- $T \cong T(\Omega)$: $\prod_{x=m}^{n} \mathbb{1}_{\alpha_x} \leftrightarrow \dots v \otimes \left(\otimes_{x=m}^{n} e_x^{(\alpha_x)} \right) \otimes v \dots$
- $L \leftrightarrow \sum_{n\in Z} \hat{q}_n$, $\hat{q}_n \in \mathrm{End}(V_n \otimes V_{n+1})$ (hats will be dropped)
- Standard basis: $(e^{(1)} \otimes e^{(1)}, e^{(1)} \otimes e^{(0)}, e^{(0)} \otimes e^{(1)}, e^{(0)} \otimes e^{(0)})$

$$q = \begin{pmatrix} * & R(11\to 10) & R(11\to 01) & R(11\to 00) \\ R(10\to 11) & * & R(10\to 01) & R(10\to 00) \\ R(01\to 11) & R(01\to 10) & * & R(01\to 00) \\ R(00\to 11) & R(00\to 10) & R(00\to 01) & * \end{pmatrix}$$

(Type-A) Hecke algebras

- Let $q \in (0,1]$, $Q = (q + q^{-1})^{-2}$
- $\mathbb{H}_n(q)$ - a unital associative algebra over $\mathbb{R}$ generated by $(s_i)_{i=1}^{n-1}$ subject to

$$\begin{cases} s_i s_j = s_j s_i & |i-j| \geq 2 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & 1 \leq i \leq n-2 \\ s_i^2 = 1 + (q - q^{-1}) s_i & 1 \leq i \leq n-1 \end{cases}$$

- Stochastic generators: $\sigma_i = \frac{q - s_i}{q + q^{-1}}$, $1 \leq i \leq n-1$
- $\sigma_i^2 = \sigma_i$, $\sigma_i \sigma_{i+1} \sigma_i - Q\sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} - Q\sigma_{i+1}$
- Markov generators: $q_i = \sigma_i - 1$, $1 \leq i \leq n-1$
- $q_i^2 = -q_i$, $q_i q_{i+1} q_i - Qq_i = q_{i+1} q_i q_{i+1} - Qq_{i+1}$
- $\mathbb{H}_\infty(q) := \varinjlim \mathbb{H}_n(q)$ $(= \amalg_{n \in \mathbb{N}} \mathbb{H}_n(q) / \sim)$

Lecture 2

The model

- The rates: $(\eta_1,\eta_2)\xrightarrow{r}(0,\eta_1\oplus\eta_2)$, $(\eta_1,\eta_2)\xrightarrow{\ell}(\eta_1\oplus\eta_2,0)$
- Timescale: $\ell+r=1$
- $L=\sum_{n\in\mathbb{Z}}q_n=\sum_{n\in\mathbb{Z}}(\sigma_n+I)$, $\sigma,q\in\mathrm{End}(V\otimes V)$:

$$q=\begin{pmatrix}-1&0&0&1\\0&-r&r&0\\0&\ell&-\ell&0\\0&0&0&0\end{pmatrix},\sigma=\begin{pmatrix}0&0&0&1\\0&\ell&r&0\\0&\ell&r&0\\0&0&0&1\end{pmatrix}$$

- $\partial_t\mathbb{E}[\eta_t(0)]=-2\mathbb{E}[\eta_t(0)\eta_t(1)],\ \partial_t\mathbb{E}[\eta_t(0)\eta_t(1)]=$ $3-$point function, ... (T-invariant BBGKY chain)
- Issue: $\mathbb{E}[\eta_t(0)\eta_t(1)]\not\approx\mathbb{E}[\eta_t(0)^2]$ (Otherwise, BBGKY $\implies\mathbb{E}[\eta_t]\sim C/t,\ t\to\infty$ whereas the true answer is $\mathbb{E}[\eta_t]\sim C/\sqrt{t},\ t\to\infty$)

Representation of σ in $V \otimes V$

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \ell & r & 0 \\ 0 & \ell & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathrm{rank}(\sigma) = 2, \sigma^2 = \sigma(\text{ check!}), \sum_{j=1}^{4} \sigma_{ij} = 1$$

- $v := (1,1)^T \leftrightarrow 1$ ('vacuum vector'), $w := (-1,1)^T \leftrightarrow (-1)^\eta$

- $\sigma v \otimes v = v$, ($\Leftarrow$stochasticity); $\sigma w \otimes w = w$ (check!)

- Exercise:
$$\begin{cases} \sigma v \otimes w = \ell v \otimes v + r w \otimes w \\ \sigma w \otimes v = \ell w \otimes w + r v \otimes v \end{cases}$$

- Conclusion: the action of q, σ preserves the order of v, w in the tensor product

Representation of $\mathbb{A}$ in T

$$\begin{cases} \sigma v \otimes w = \ell v \otimes v + r w \otimes w \\ \sigma w \otimes v = \ell w \otimes w + r v \otimes v \end{cases}$$

- $f_x := (\otimes_{i \le x} w_i) \otimes (\otimes_{i > x} v_j)$ ($\notin T$!)
- $\sigma_x f_x = \ell f_{x+1} + r f_{x-1}$, $\sigma_y f_x = f_x$, $\forall y \neq x \in \mathbb{Z}$
- $\mathbb{A}$ acts in $T_1 := \mathrm{span}_{\mathbb{R}}(f_x, x \in \mathbb{Z})$; $q_y f_x = \mathbb{1}_x(y) \Delta^{(r,l)} f_x$
- $f^{(2n)}_{x_1, x_2, \ldots, x_{2n}} = \ldots (\otimes_{x_1+1 \le j_1 \le x_2} w_{j_1}) \ldots (\otimes_{x_{2n-1}+1 \le j_n \le x_{2n}} w_{j_n}) \ldots \in T$
- $T_n := \mathrm{span}_{\mathbb{R}}(f^{2n}_{x_1, \ldots, x_{2n}}, n \in \mathbb{N}_0, x_1 \le x_2 \le \ldots x_{2n})$
- **Claim.** $\mathbb{A} T_{2n} \subset T_{2n}$, $n \in \mathbb{N}$ (Lower-Δ representations)
- Exercise: $\sigma_3 f^{(2)}_{34} = \ell f^{(0)} + r f^{(2)}_{24}$

Markov duality

- **Claim.**

$$\begin{cases} Lf^{(2n)}_{x_1,\dots,x_{2n}} = \sum_{k=1}^{2n} \Delta_k^{(r,l)} f^{(2n)}_{x_1,\dots,x_{2n}}, \ x_1 < x_2 < \ldots < x_n, \ n \in \mathbb{N} \\ f^{(2n)}_{x_1,\dots,x_i=x_{i+1},\dots,x_{2n}} = f^{(2n-2)}_{x_1,\dots,x_{i-1},x_{i+2},\dots,x_{2n}}, \ 1 \le i \le 2n-1 \end{cases}$$

- Conclusion: For each $n \in \mathbb{N}$, [ARW] is $f^{(2n)}$-dual to ARW with at most $2n$ particles with right hopping rate ℓ and left hopping rate r
- $w \leftrightarrow (-1)^{\eta} \implies f^{(2n)}_{x_1,\dots,x_{2n}} \leftrightarrow \prod_{k=1}^{n} (-1)^{\sum_{j_k=x_{2k-1}+1}^{x_{2k}} \eta(j_k)}$
- $\eta(x) = \frac{1-(-1)^{\eta}}{2} \implies \mathbb{E}[f^{(2n)}]$'s determine the law of η_t
- Define $\Phi_t^{(2n)}(x_1,\dots,x_{2n}) := \mathbb{E}[f^{(2n)}_{x_1,\dots,x_{2n}}(\eta_t)]$

From dualities to Pfaffians

$$\begin{cases} \left(\partial_t - \sum_{k=1}^{2n} \Delta_k^{(r,l)}\right) \Phi_t^{(2n)}(x_1, \ldots, x_{2n}) = 0, t > 0, x_1 < \ldots < x_{2n}, n \in \mathbb{N} \\ \Phi_t^{(2n)}(\ldots x_i = x_{i+1} \ldots) = \Phi_t^{(2n-2)}(\ldots \hat{x}_{i-1}, \hat{x}_{i+2} \ldots),\ 1 \leq i \leq 2n-1 \\ \Phi_0^{(2n)} = \text{det. initial condition} \end{cases}$$

- **(A non-trivial) exercise**: The above system has a unique solution

$$\Phi_t^{(2n)}(x_1, \ldots, x_{2n}) = \mathrm{pfaff}[\Phi_t^{(2n)}(x_i, x_j), 1 \leq i < j \leq 2n],$$

$t \geq 0, x_1 \leq x_2 \leq \ldots \leq x_{2n}, n \in \mathbb{N}$. (F-Wick's theorem)

- Conclusion: η_t is a Pfaffian PP for any fixed $t > 0$ and all deterministic IC's
- Benefits: $\rho_t^{(n)}(x) \sim C_n |\Delta(x)| t^{-n/2-n(n-1)/4}$; persistence exponent $(1/4)$; the distribution of the rightmost particle (Rider-Sinclair's distribution)

Duality and Hecke relations

- Basic ingredients of the solution:
  1. $\exists(v, w)$ - basis of V: $\sigma v \otimes v = v \otimes v$, $\sigma w \otimes w = w \otimes w$
  2. σ-action preserves the order of v and w in T
- **Check:** $q_i q_{i+1} q_i - Q q_i = q_{i+1} q_i q_{i+1} - Q q_{i+1}, i \in \mathbb{Z}, Q = r\ell$
- Conclusion: $\mathbb{A}$ is a quotient of $\mathbb{H}_\infty$
- **Theorem.** Let $(\sigma_n)_{n\in\mathbb{Z}}$ be stochastic generators of $\mathbb{H}_\infty$: rank$(\sigma - I) = 2$ and the above condition 1. is satisfied. Then EITHER

$$\left\{ \begin{array}{c} \sigma v \otimes w = 0 w \otimes v + \ldots \\ \sigma w \otimes v = 0 v \otimes w \ldots \end{array} \right. \text{OR} \left\{ \begin{array}{c} \sigma v \otimes w = \alpha w \otimes v + \beta v \otimes w \\ \sigma w \otimes v = \gamma v \otimes w + \delta w \otimes v \end{array} \right.$$

- The second possiiblity corresponds to duality functions of product moment type ([SEP], voter models, etc.)

Lecture 3

The classification theorem

- $\rho \in \mathrm{End}(V \otimes V)$: $\rho(a \otimes b) := b \otimes a$ (reflection)
- $\tau \in \mathrm{End}(V \otimes V)$: $\tau(e^{(\alpha)} \otimes e^{\beta}) := e^{(1-\alpha)} \otimes e^{(1-\beta)}$ $(0 \leftrightarrow 1)$
- Consider a continuous time Markov process on $\Omega = \{0, 1\}^{\mathbb{Z}}$, characterised by the generator

$$L = \sum_{n \in \mathbb{Z}} (\sigma_n - I), \;\; \sigma_n \in \mathrm{End}(V_n \otimes V_{n+1}), \sigma_n^2 = \sigma_n, \;\; n \in \mathbb{Z}$$

- Assume the reflective symmetry of the chain,

$$\rho_n \circ \sigma_n = \sigma_n \circ \rho_n, n \in \mathbb{Z}. \tag{1}$$

Up to the particle-hole conjugation ($\sigma \to \tau\sigma\tau$) there are **nine** such Markov processes with generator L s. t. $\sigma \neq I$.

“The nine-fold way”

1. Hecke class
   1.1 Symmetric exclusion process [SEP]
   1.2 Biased voter model [BVM]$_\theta$
   1.3 Symmetric anti-voter model [SAVM]
   1.4 Annihilating coalescing random walk [ACSRW]$_\theta$
   1.5 Colaescing symmetric random walk with branching [CSRWB]$_\theta$
   1.6 Annihilating random walk with pairwise immigration [ASRWPI]$_\theta$
2. Exceptional models
   2.1 Stationary annihilation coalescence model [SCAM]$_\theta$
   2.2 Dimer model [DM]$_\theta$
   2.3 Reshuffle model [RM]$_{\theta_1,\theta_2,\theta_3}$

Lemma. For each model of Hecke class, the two-site generators obey the braid relation $\sigma_i\sigma_{i+1}\sigma_i - Q\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} - Q\sigma_{i+1}$, where Q can depend on the parameter θ of the model

Steps of the proof of the classification theorem.

Lemma (1)

Let $\sigma \in End(V \otimes V)$ *be a stochastic matrix:* $\sigma^2 = \sigma$, $\rho \circ \sigma = \sigma \circ \rho$. *Then either (i) there is* $w \in V$, *linearly independent of* v: $\sigma w \otimes w = w \otimes w$ *or (ii)* σ *corresponds to [SAVM], [BVM]*$_{\theta=1/2}$ *or [RM]*

Lemma (2)

Let $\sigma \in End(V \otimes V)$ *be a stochastic idempotent matrix s. t.* $\rho\sigma = \sigma\rho$. *Assume* $\exists w \in V$ *lin. indep. of* v *s. t.* $\sigma w \otimes w = w \otimes w$. *Then, after possibly a particle hole conjugation there are seven non-trivial matrices* $\sigma \neq I$ *corresponding to [SEP], [BVM]*$_{\theta\neq 1/2}$*, [ACSRW], [CSRWB], [ASRWPI], [SCAM], [DM]*

Probabilistic discussion

BVM and [SAVM] models are equivalent to ARW's via the domain-walls map

ACSRW , [CSRWB] and [ASRWPI] are Pfaffian point processes for any deterministic initial condition. Their analysis is identicall using the corresp. (v, w) system

SCAM is solvable via dualites. If $\eta_0 \equiv= 1$, η_t is a renewal process with explicit kernel for any $t > 0$

RM is exactly solvable for any one-dependent η_0. η_t is a determinantal point process for any $t > 0$.

DM is equivalent to an inhomogeneous [SEP]

Algebraic discussion

- Consider [ARW] on $\mathbb{Z}_N$ with $L_N = \sum_{k=1}^{N-1} q_k$. Then $\mathbb{A}_N :=< 1, \sigma_i, 1 \leq i \leq N-1 >$ has irreps of dimensions $\binom{N-1}{k}^{(2)}$, $0 \leq k \leq N-1$. The spaces of these irreps are

$$\frac{\text{span}_{\mathbb{R}}(\text{Duality functions with at most } k \text{ jumps})}{\text{span}_{\mathbb{R}}(\text{Duality functions with at most}(k-1) \text{ jumps})}$$

- Interpreting duality functions as intertwiners, can construct the following coordinate representations of $\mathbb{H}$: for $Q = r\ell$,
  1. $(\mathbb{1}_x \Delta^{(r,\ell)})_{x \in \mathbb{Z}}$ is a representation of Temperley-Lieb algebra in $L_2(\mathbb{Z})$ with param. Q
  2. $(\sum_{k=1}^{n} \mathbb{1}_x^{(k)} \Delta^{(r,\ell)})_{x \in \mathbb{Z}}$ is a representation of $\mathbb{H}_\infty(Q)$ in $L_2(W^n)$; $\mathbb{1}_x^{(k)}(y) = \mathbb{1}_x(y_k)$

Baxterisation: [CSRWB]

Theorem (V. Jones, 1990)

If $(s_n)_{n=1}^{M}$ are canonical generators of $\mathbb{H}_{M+1}(q)$, then $R_n(x) := s_n - xs_n^{-1}$, $1 \leq n \leq M,\ x \in \mathbb{C}$, solves the YBE,

$$R_{n-1}(x)R_n(xy)R_n(y) = R_{n-1}(y)R_n(xy)R_{n-1}(x)$$

Example.

$$R(x) = \begin{pmatrix} q^{-1} + x(q - 2q^{-1}) & -(1-x)q^{-1} & -(1-x)q^{-1} & 0 \\ -(1-x)(q-q^{-1}) & q - q^{-1} & -(1-x)q^{-1} & 0 \\ -(1-x)(q-q^{-1}) & -(1-x)q^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1} + qx \end{pmatrix},$$

where $x \in C$, $Q = \theta(1-\theta)$, $q + q^{-1} = \frac{1}{\sqrt{Q}}$