How to Solve the Stochastic Six Vertex Model

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Introduction

These are lecture notes for PANEM-2023 at Texas A&M on the integrability and asymptotics of the stochastic six vertex model.

1 Six vertex model through different lenses

In the first lecture, we describe the stochastic six vertex model from two diverse perspectives — as a model of statistical mechanics, and as a stochastic particle system.

1.1 Gibbs measures and the six vertex model

1.1.1 Finite-volume Gibbs measures

We begin with describing the useful framework of *Gibbs measures*. For simplicity, we work on the two-dimensional lattice $\mathbb{Z}^2$. Let $\Lambda \subset \mathbb{Z}^2$ be a finite subset (for example, a rectangle). We are interested in *spin configurations* inside Λ which are encoded as $\omega = \{\sigma_e \colon e \text{ is an edge in } \Lambda\}$, where $\sigma_e \in \{0, 1\}$. By an "edge in Λ" we mean that both endpoints of this edge must be inside Λ. Each spin configuration is equipped with boundary conditions, which are fixed spins on all the boundary edges of Λ (an edge is called boundary if it connects Λ to $\mathbb{Z}^2 \setminus \Lambda$).

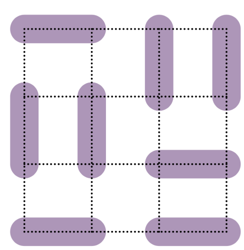
With each spin configuration ω, we associate its energy $H(\omega) \in \mathbb{R}$. This energy may depend on global parameters (e.g., inverse temperature) and local parameters (e.g., edge capacities or vertex rapidities). If a particular spin configuration ω is forbidden, we have $H(\omega) = +\infty$.

Definition 1.1. A (finite-volume) *Gibbs measure* in Λ with fixed boundary conditions and the energy function $H(\cdot)$ is the probability distribution on spin configurations whose probability weights have the form

$$\operatorname{Prob}(\omega) = \frac{1}{Z} \exp\{-H(\omega)\}.$$

Here Z is the *partition function*, which is simply the probability normalizing constant.

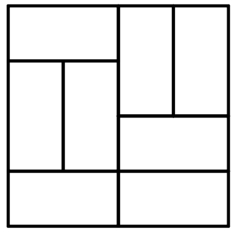
Example 1.2 (Domino tilings on the square grid)**.** A *perfect matching* on Λ is any subset M of its edges such that every vertex is covered by exactly one edge from M. For example, here is a perfect matching on the four by four rectangle:



If the set of allowed spin configurations is the set of perfect matchings, and

$$H(\omega) = \begin{cases} 0, & \omega \text{ is a perfect matching;} \\ +\infty, & \omega \text{ is not a perfect matching,} \end{cases}$$

then the corresponding Gibbs measure is the uniform distribution on the space of *domino tilings*. That is, we identify each covered edge with a 2×1 domino. The domino tiling corresponding to the above perfect matching is



Computing partition functions of various Gibbs measures may be challenging. For example, the number of domino tilings of the 8×8 chessboard is 12,988,816, but its theoretical computation (not via a computer program) requires several nontrivial steps [Kas61], [TF61].

Parameter-dependent partition functions represent many important quantities across all of mathematics, including various families of symmetric functions (such as Schur or Hall-Littlewood functions), and related objects.

1.1.2 Infinite-volume Gibbs measures

1.1.3 Six vertex model

1.2 Stochastic six vertex model and its particle system limits

1.3 Gibbs properties of the stochastic six vertex model

1.4 Basic coupling and colored (multispecies) models

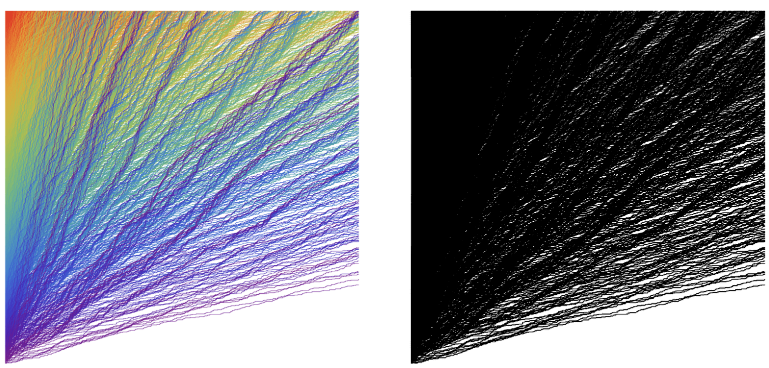


Figure 1: Colored stochastic six vertex model and its monochrome version.

1.5 Stationary distributions and hydrodynamics

Bernoulli is Stationary; also for all the limits we had.

1.6 Limit shape and fluctuation problem

2 Integrability

3 Asymptotics

References

[Kas61] P. W. Kasteleyn, *The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice*, Physica **27** (1961), no. 12, 1209–1225. ↑3

[TF61] H. Temperley and M. Fisher, *Dimer problem in statistical mechanics - an exact result*, Philos. Mag. **6** (1961), no. 68, 1061–1063. ↑3

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