Representations of Hecke Algebras and Markov Dualities for Interacting Particle Systems

Oleg Zaboronski and Roger Tribe

University of Warwick

o.v.zaboronski@warwick.ac.uk, r.p.tribe@warwick.ac.uk

Probability and Algebra: New Expressions in Mathematics
10-14.07.2023

Lecture 1

Set-up: continuous time Markov processes

- ▶ Configuration space: $\Omega := \{0, 1\}^{\mathbb{Z}}$

- ▶ Configuration vector: $\eta \in \Omega$; $\eta_i = 1$ indicates a particle at $i \in \mathbb{Z}$; $\eta_i = 0$ indicates a hole at $i \in \mathbb{Z}$

- ▶ The space of test functions: $T(\Omega)$ - functions depending on finitely many components of η (cylinder functions)

- ▶ The Markov process $(\eta_t)_{t \geq 0}$ is characterised by its infinitesimal generator L

- ▶ Defining property of L: for any $f \in T(\Omega)$

$$\frac{d}{dt} \mathbb{E}_{\eta_0} [f(\eta_t)] = \mathbb{E}_{\eta_0} [Lf(\eta_t)]$$

The generator and transition rates. Examples

▶ Let $R(\eta \to \eta')$ be the transition rate from $\eta \in \Omega$ to $\eta' \in \Omega$

▶ **Claim.** $Lf(\eta) = \sum_{\eta' \in \Omega} R(\eta \to \eta') \left[f(\eta') - f(\eta) \right]$

▶ Asymmetric RW on $\mathbb{Z}$:

$$Lf(x) = r(f(x+1) + \ell f(x-1) - f(x) := \Delta^{(l,r)} f(x)$$

▶ Annihilating random walks on $\mathbb{Z}$ ([ARW]):

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \left[rf(\ldots, \eta_{x-1}, 0, \eta_x \oplus \eta_{x+1}, \eta_{x+2}, \ldots) \right.$$

$$\left. + \ell f(\ldots, \eta_{x-2}, \eta_x \oplus \eta_{x-1}, 0, \eta_{x+1}, \ldots) - f(\eta) \right]$$

Markov duality

- ▶ Set up: $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$ - Markov processes on Ω^X, Ω^Y with infinitesimal generators L^X, L^Y; $H : \Omega^x \times \Omega^Y \to \mathbb{R}$ - bounded measurable function

- ▶ Definition: X, Y are called Markov dual w. r. t. H if $\forall (x, y) \in \Omega^X \times \Omega^Y$, $\forall t > 0$,

$$\mathbb{E}_x^X [H(X_t, y)] = \mathbb{E}_y^Y [H(x, Y_t)]$$

- ▶ Infinitesimal form: $L^X H(x, y) = L^Y H(x, y)$

- ▶ **Claim.** $\frac{d}{dt} \mathbb{E}_x^X [H(X_t, y)] = L^Y \mathbb{E}_x^X [H(X_t, y)]$

Matrix representation for the generator

- Relevant generators: $L = \sum_{n \in \mathbb{Z}} q_x$, where q_x acts non-trivially on (η_x, η_{x+1}) only

- $T := \otimes_{n \in \mathbb{Z}} V_n$ - infinite tensor product, $v_n = (1,1)^T := v$ for almost all n's

- $T \cong T(\Omega)$: $\prod_{x=m}^{n} \mathbb{1}_{\alpha_x} \leftrightarrow \ldots v \otimes \left(\otimes_{x=m}^{n} e_x^{(\alpha_x)} \right) \otimes v \ldots$

- $L \leftrightarrow \sum_{n \in Z} \hat{q}_n$, $\hat{q}_n \in \mathsf{End}(V_n \otimes V_{n+1})$ (hats will be dropped)

- Standard basis:
  $(e^{(1)} \otimes e^{(1)}, e^{(1)} \otimes e^{(0)}, e^{(0)} \otimes e^{(1)}, e^{(0)} \otimes e^{(0)})$

$$
q = \begin{pmatrix}
* & R(11 \to 10) & R(11 \to 01) & R(11 \to 00) \\
R(10 \to 11) & * & R(10 \to 01) & R(10 \to 00) \\
R(01 \to 11) & R(01 \to 10) & * & R(01 \to 00) \\
R(00 \to 11) & R(00 \to 10) & R(00 \to 01) & *
\end{pmatrix}
$$

(Type-A) Hecke algebras

- ► Let $q \in (0, 1]$, $Q = (q + q^{-1})^{-2}$

- ► $\mathbb{H}_n(q)$ - a unital associative algebra over $\mathbb{R}$ generated by $(s_i)_{i=1}^{n-1}$ subject to

$$\begin{cases} s_i s_j = s_j s_i & |i - j| \geq 2 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & 1 \leq i \leq n - 2 \\ s_i^2 = 1 + (q - q^{-1}) s_i & 1 \leq i \leq n - 1 \end{cases}$$

- ► Stochastic generators: $\sigma_i = \frac{q - s_i}{q + q^{-1}}$, $1 \leq i \leq n - 1$

- ► $\sigma_i^2 = \sigma_i$, $\sigma_i \sigma_{i+1} \sigma_i - Q\sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} - Q\sigma_{i+1}$

- ► Markov generators: $q_i = \sigma_i - 1$, $1 \leq i \leq n - 1$

- ► $q_i^2 = -q_i$, $q_i q_{i+1} q_i - Q q_i = q_{i+1} q_i q_{i+1} - Q q_{i+1}$

- ► $\mathbb{H}_\infty(q) := \varinjlim \mathbb{H}_n(q)$ $(= \amalg_{n \in \mathbb{N}} \mathbb{H}_n(q)/ \sim)$

Lecture 2

The model

- ▶ The rates: $(\eta_1, \eta_2) \xrightarrow{r} (0, \eta_1 \oplus \eta_2)$, $(\eta_1, \eta_2) \xrightarrow{\ell} (\eta_1 \oplus \eta_2, 0)$

- ▶ Timescale: $\ell + r = 1$

- ▶ $L = \sum_{n \in \mathbb{Z}} q_n = \sum_{n \in \mathbb{Z}} (\sigma_n - I)$, $\sigma, q \in \mathsf{End}(V \otimes V)$:

$$
q = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -r & r & 0 \\ 0 & \ell & -\ell & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \ell & r & 0 \\ 0 & \ell & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

- ▶ $\partial_t \mathbb{E}[\eta_t(0)] = -2\mathbb{E}[\eta_t(0)\eta_t(1)]$, $\partial_t \mathbb{E}[\eta_t(0)\eta_t(1)] = F(3, 2 - \text{point fnctns}), \dots$ (T-invariant BBGKY chain)

- ▶ Issue: $\mathbb{E}[\eta_t(0)\eta_t(1)] \not\approx \mathbb{E}[\eta_t(0)^2]$ (Otherwise, BBGKY $\implies \mathbb{E}[\eta_t] \sim C/t$, $t \to \infty$, whereas the true answer is $\mathbb{E}[\eta_t] \sim C/\sqrt{t}$, $t \to \infty$)

Representation of σ in $V \otimes V$

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \ell & r & 0 \\ 0 & \ell & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{rank}(\sigma) = 2, \sigma^2 = \sigma(\text{ check!}), \sum_{j=1}^{4} \sigma_{ij} = 1$$

▶ $v := (1,1)^T \leftrightarrow 1$ ('vacuum vector'), $w := (-1,1)^T \leftrightarrow (-1)^\eta$

▶ $\sigma v \otimes v = v$, ($\Leftarrow$stochasticity); $\sigma w \otimes w = w$ (check!)

▶ Exercise:
$$\begin{cases} \sigma v \otimes w = \ell v \otimes v + r w \otimes w \\ \sigma w \otimes v = \ell w \otimes w + r v \otimes v \end{cases}$$

▶ Conclusion: the action of q, σ preserves the order of v, w in the tensor product

Representation of $\mathbb{A}$ in T

$$\begin{cases} \sigma v \otimes w = \ell v \otimes v + r w \otimes w \\ \sigma w \otimes v = \ell w \otimes w + r v \otimes v \end{cases}$$

- $f_x := (\otimes_{i \leq x} w_i) \otimes (\otimes_{i > x} v_j)$ $(\notin T\,!)$

- $\sigma_x f_x = \ell f_{x+1} + r f_{x-1}$, $\sigma_y f_x = f_x$, $\forall y \neq x \in \mathbb{Z}$

- $\mathbb{A}$ acts in $T_1 := \operatorname{span}_{\mathbb{R}}(f_x, x \in \mathbb{Z})$; $q_y f_x = \mathbb{1}_x(y) \Delta^{(r,l)} f_x$

- $f_{x_1, x_2, \ldots, x_{2n}}^{(2n)} = \ldots (\otimes_{x_1 + 1 \leq j_1 \leq x_2} w_{j_1}) \cdot (\otimes_{x_{2n-1} + 1 \leq j_n \leq x_{2n}} w_{j_n}) \ldots \in T$

- $T_{2n} := \operatorname{span}_{\mathbb{R}}(f_{x_1, \ldots, x_{2n}}^{2n}, n \in \mathbb{N}_0, x_1 \leq x_2 \leq \ldots x_{2n})$

- **Claim.** $\mathbb{A} T_{2n} \subset T_{2n}$, $n \in \mathbb{N}$ (Lower-Δ representations)

- Exercise: $\sigma_3 f_{34}^{(2)} = \ell f^{(0)} + r f_{24}^{(2)}$

Markov duality

- **Claim.**

$$\begin{cases} L f^{(2n)}_{x_1,\ldots,x_{2n}} = \sum_{k=1}^{2n} \Delta_k^{(r,l)} f^{(2n)}_{x_1,\ldots,x_{2n}}, \ x_1 < x_2 < \ldots < x_n, \ n \in \mathbb{N} \\ f^{(2n)}_{x_1,\ldots,x_i=x_{i+1},\ldots,x_{2n}} = f^{(2n-2)}_{x_1,\ldots,x_{i-1},x_{i+2},\ldots,x_{2n}}, \ 1 \le i \le 2n-1 \end{cases}$$

- Conclusion: For each $n \in \mathbb{N}$, [ARW] is $f^{(2n)}$-dual to ARW with at most $2n$ particles with right hopping rate ℓ and left hopping rate r

- $w \leftrightarrow (-1)^\eta \implies f^{(2n)}_{x_1,\ldots,x_{2n}} \leftrightarrow \prod_{k=1}^n (-1)^{\sum_{j_k=x_{2k-1}+1}^{x_{2k}} \eta(j_k)}$

- $\eta = \frac{1-(-1)^\eta}{2} \implies \mathbb{E}[f^{(2n)}]$'s determine the law of η_t

- Define $\Phi_t^{(2n)}(x_1,\ldots,x_{2n}) := \mathbb{E}[f^{(2n)}_{x_1,\ldots,x_{2n}}(\eta_t)]$

From dualities to Pfaffians

$$\begin{cases} \left(\partial_t - \sum_{k=1}^{2n} \Delta_k^{(r,l)}\right) \Phi_t^{(2n)}(x_1, \ldots, x_{2n}) = 0, t > 0, x_1 < \ldots < x_{2n}, n \in \mathbb{N} \\ \Phi_t^{(2n)}(\ldots x_i = x_{i+1} \ldots) = \Phi_t^{(2n-2)}(\ldots \hat{x}_{i-1}, \hat{x}_{i+2} \ldots), \ 1 \leq i \leq 2n-1 \\ \Phi_0^{(2n)} = \text{det. initial condition} \end{cases}$$

- ▶ **(A non-trivial) exercise**: The above system has a unique solution

  $$\Phi_t^{(2n)}(x_1, \ldots, x_{2n}) = \text{pfaff}[\Phi_t^{(2)}(x_i, x_j), 1 \leq i < j \leq 2n],$$

  $t \geq 0, x_1 \leq x_2 \leq \ldots \leq x_{2n}, n \in \mathbb{N}$. (F-Wick's theorem)

- ▶ Conclusion: η_t is a Pfaffian PP for any fixed $t > 0$ and all deterministic IC's

- ▶ Benefits: $\rho_t^{(n)}(x) \sim C_n|\Delta(x)|t^{-n/2-n(n-1)/4}$; persistence exponent $(1/4)$; the distribution of the rightmost particle (Rider-Sinclair's distribution)

Duality and Hecke relations

- ▶ Basic ingredients of the solution:
    1. $\exists (v, w)$ - basis of V: $\sigma v \otimes v = v \otimes v$, $\sigma w \otimes w = w \otimes w$
    2. σ-action preserves the order of v and w in T

- ▶ **Check:** $q_i q_{i+1} q_i - Q q_i = q_{i+1} q_i q_{i+1} - Q q_{i+1}$, $i \in \mathbb{Z}$, $Q = r\ell$

- ▶ Conclusion: $\mathbb{A}$ is a quotient of $\mathbb{H}_\infty$

- ▶ **Theorem.** Let $(\sigma_n)_{n \in \mathbb{Z}}$ be stochastic generators of $\mathbb{H}_\infty$: rank$(\sigma - I) = 2$ and the above condition 1. is satisfied. Then EITHER

$$
\left\{
\begin{array}{l}
\sigma v \otimes w = 0 w \otimes v + \dots \\
\sigma w \otimes v = 0 v \otimes w \dots
\end{array}
\right.
\text{ OR }
\left\{
\begin{array}{l}
\sigma v \otimes w = \alpha w \otimes v + \beta v \otimes w \\
\sigma w \otimes v = \gamma v \otimes w + \delta w \otimes v
\end{array}
\right.
$$

- ▶ The second possiiblity corresponds to duality functions of product moment type ([SEP], voter models, etc.)

Lecture 3

The classification theorem

- ▶ $\rho \in \mathrm{End}(V \otimes V)$: $\rho(a \otimes b) := b \otimes a$ (reflection)
- ▶ $\tau \in \mathrm{End}(V \otimes V)$: $\tau(e^{(\alpha)} \otimes e^{\beta}) := e^{(1-\alpha)} \otimes e^{(1-\beta)}$ $(0 \leftrightarrow 1)$
- ▶ Consider a continuous time Markov process on $\Omega = \{0, 1\}^{\mathbb{Z}}$, characterised by the generator

$$L = \sum_{n \in \mathbb{Z}} (\sigma_n - I), \ \sigma_n \in \mathrm{End}(V_n \otimes V_{n+1}), \sigma_n^2 = \sigma_n, \ n \in \mathbb{Z}$$

- ▶ Assume the reflective symmetry of the chain,

$$\rho_n \circ \sigma_n = \sigma_n \circ \rho_n, n \in \mathbb{Z}.$$

Up to the particle-hole conjugation $(\sigma \to \tau\sigma\tau)$ there are **nine** Markov processes with generator L s. t. $\sigma \neq I$.

"The nine-fold way"

1. Hecke class
   1.1 Symmetric exclusion process [SSEP]
   1.2 Biased voter model $[\text{BVM}]_\theta$
   1.3 Symmetric anti-voter model [SAVM]
   1.4 Annihilating coalescing random walk $[\text{ACSRW}]_\theta$
   1.5 Coalescing symmetric random walk with branching
       $[\text{CSRWB}]_\theta$
   1.6 Annihilating random walk with pairwise immigration
       $[\text{ASRWPI}]_\theta$

2. Exceptional models
   2.1 Stationary annihilation coalescence model $[\text{SCAM}]_\theta$
   2.2 Dimer model $[\text{DM}]_\theta$
   2.3 Reshuffle model $[\text{RM}]_{\theta_1, \theta_2, \theta_3}$

Lemma. For each model of Hecke class, the two-site
generators obey the braid relation

$\sigma_i \sigma_{i+1} \sigma_i - Q\sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} - Q\sigma_{i+1}$, where Q can depend on
the parameter θ of the model

Steps of the proof of the classification theorem.

Lemma (1)

Let $\sigma \in End(V \otimes V)$ be a stochastic matrix: $\sigma^2 = \sigma$, $\rho \circ \sigma = \sigma \circ \rho$. Then either (i) there is $w \in V$, linearly independent of v: $\sigma w \otimes w = w \otimes w$ or (ii) σ corresponds to [SAVM], [BVM]$_{\theta=1/2}$ or [RM]

Lemma (2)

Let $\sigma \in End(V \otimes V)$ be a stochastic idempotent matrix s. t. $\rho\sigma = \sigma\rho$. Assume $\exists w \in V$ lin. indep. of v s. t. $\sigma w \otimes w = w \otimes w$. Then, after possibly a particle hole conjugation there are seven non-trivial matrices $\sigma \neq I$ corresponding to [SSEP], [BVM]$_{\theta \neq 1/2}$, [ACSRW], [CSRWB], [ASRWPI], [SCAM], [DM]

Probabilistic discussion

BVM and [SAVM] models are equivalent to ARW's via the domain-walls map

ACSRW , [CSRWB] and [ASRWPI] are Pfaffian point processes for any deterministic initial condition. Their analysis is identical to [ARW] using the coresp. (v, w) system

SCAM is solvable via dualites. If $\eta_0 \equiv 1$, η_t is a renewal process with explicit kernel for any $t > 0$

RM is exactly solvable for any one-dependent η_0. η_t is a determinantal point process for any $t > 0$.

DM is equivalent to an inhomogeneous [SEP]

Algebraic discussion

- ▶ Consider [ARW] on $\mathbb{Z}_N$ with $L_N = \sum_{k=1}^{N-1} q_k$. Then
  $\mathbb{A}_N := <1, \sigma_i, 1 \leq i \leq N-1>$ has irreps of dimensions
  $\binom{N-1}{k}^{(2)}$, $0 \leq k \leq N-1$. The spaces of these irreps are

  $$\frac{\text{span}_\mathbb{R}(\text{Duality functions with at most } k \text{ jumps})}{\text{span}_\mathbb{R}(\text{Duality functions with at most}(k-1) \text{ jumps})}$$

- ▶ Interpreting duality functions as intertwiners, can construct
  the following coordinate representations of $\mathbb{H}$: for $Q = r\ell$,
    1. $(\mathbb{1}_x \Delta^{(r,\ell)})_{x \in \mathbb{Z}}$ is a representation of Temperley-Lieb algebra
       in $L_2(\mathbb{Z})$ with param. Q
    2. $(\sum_{k=1}^n \mathbb{1}_x^{(k)} \Delta^{(r,\ell)})_{x \in \mathbb{Z}}$ is a representation of $\mathbb{H}_\infty(Q)$ in
       $L_2(W^n)$; $\mathbb{1}_x^{(k)}(y) = \mathbb{1}_x(y_k)$

Baxterisation

Theorem (V. Jones, 1990)
If $(s_n)_{n=1}^M$ are canonical generators of $\mathbb{H}_{M+1}(q)$, then
$R_n(x) := s_n - x s_n^{-1}$, $1 \le n \le M$, $x \in \mathbb{C}$, solves the YBE,

$$R_{n-1}(x) R_n(xy) R_n(y) = R_{n-1}(y) R_n(xy) R_{n-1}(x)$$

Example. The R-matrix for [CSRWB]:

$$
R(x) =
\begin{pmatrix}
q^{-1} + x(q - 2q^{-1}) & -(1-x)q^{-1} & -(1-x)q^{-1} & 0 \\
-(1-x)(q-q^{-1}) & q - q^{-1} & -(1-x)q^{-1} & 0 \\
-(1-x)(q-q^{-1}) & -(1-x)q^{-1} & q - q^{-1} & 0 \\
0 & 0 & 0 & -q^{-1} + qx
\end{pmatrix},
$$

where $x \in C$, $Q = \theta(1 - \theta)$, $q + q^{-1} = \frac{1}{\sqrt{Q}}$

Thank you

For more details, see Povolotsky et al., *Representations of Hecke algebras and Markov dualities for interacting particle systems*, arXiv:2212.12944