Exactly solvable interacting particle systems described by determinantal measures

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Probability and Algebra: New Expressions In Mathematics
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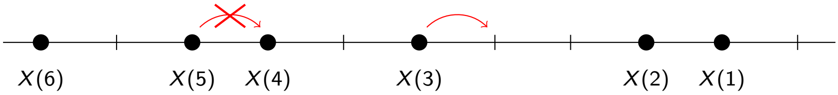
Plan of my talk

1. Motivation and examples of interacting particle systems
2. Method of exact solution
3. Some interesting findings and questions

Motivation — TASEP

[M., Quastel, Remenik, 2021. The KPZ fixed point]

◇ Developed a method of exact solution for continuous-time TASEP (totally asymmetric simple exclusion process)



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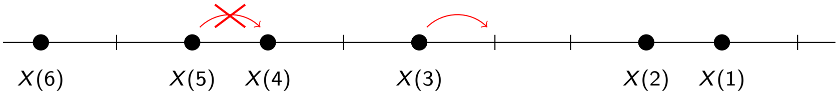

◇ The method gives formulas in the form

$$\mathbb{P}_{X_0}\Big(X_t(n_i) > a_i,\ i = 1, \ldots, m\Big) = \det\big(I - K\big)_{L^2(\{n_1,\cdots,n_m\}\times\mathbb{Z})}$$

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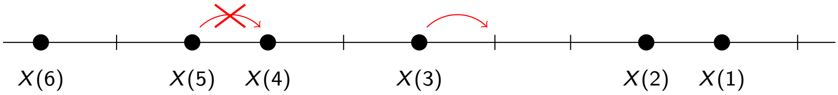

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with a trace class operator K depending on t, a_i, X_0

⋄ The formula is amenable to asymptotic analysis (the KPZ scaling limit)

Motivation — KPZ scaling limit

If the initial states satisfy

$$-\gamma\varepsilon^{1/2}\Big(X_0^{\varepsilon}(-\sigma\varepsilon^{-1}\boldsymbol{x}) - \delta\varepsilon^{-1}\boldsymbol{x}\Big) \quad \xrightarrow[\varepsilon\to 0]{} \quad \mathfrak{h}_0(\boldsymbol{x})$$

then for every $\boldsymbol{t} > 0$

$$-\gamma\varepsilon^{1/2}\Big(X_{\varepsilon^{-3/2}\boldsymbol{t}}(\alpha\varepsilon^{-3/2}\boldsymbol{t} - \sigma\varepsilon^{-1}\boldsymbol{x}) - \beta\varepsilon^{-3/2}\boldsymbol{t} - \delta\varepsilon^{-1}\boldsymbol{x}\Big) \quad \xrightarrow[\varepsilon\to 0]{} \quad \mathfrak{h}(\boldsymbol{t},\boldsymbol{x};\mathfrak{h}_0)$$

The constants α, β, γ, δ, σ may depend on the model, but the limit is not.
The limit $\mathfrak{h}$ is called the KPZ fixed point

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M., Quastel, Remenik

TASEP converges to the KPZ fixed point

Motivation — Variants of TASEP

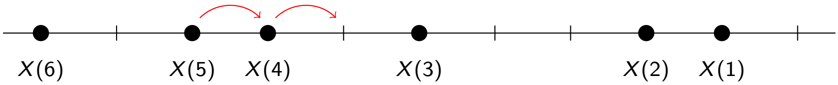
1. PushASEP
2. Discrete-time TASEP with Bernoulli/Geometric jumps and sequential/parallel updates
3. TASEP with generalized update
4. $\dots$

Motivation — Discrete-time TASEP

[M., Remenik, 2022. TASEP and generalizations: method for exact solution]

Introduced a general determinantal measure and computed its correlation kernel. In particular, it gives exact solutions for different variants of TASEP

◇ Discrete-time TASEP with sequential update (from right to left)



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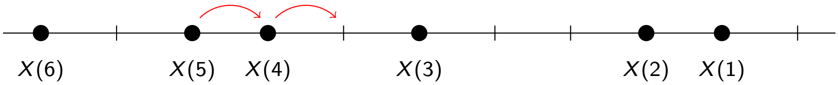
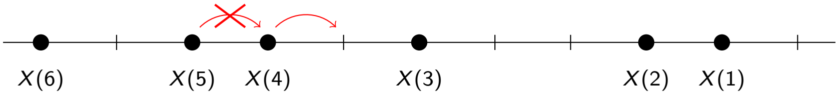


◇ Discrete-time TASEP with parallel update (from left to right)



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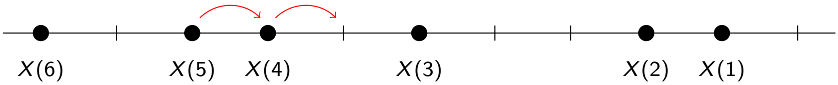

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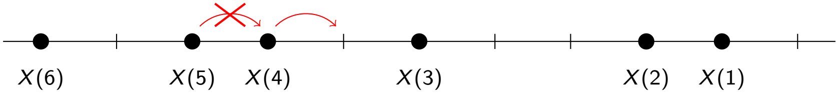
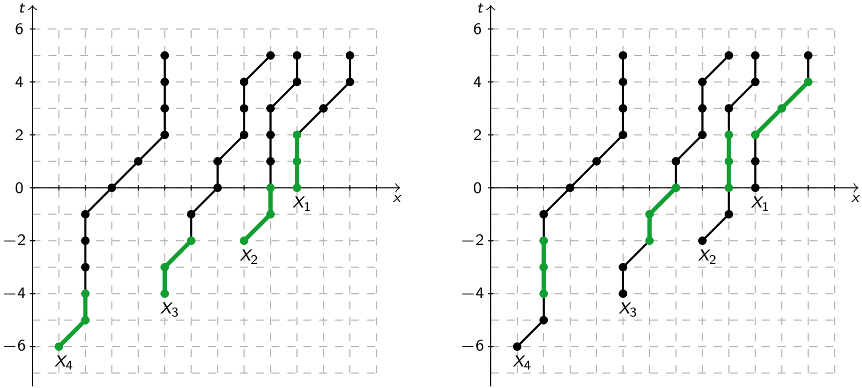


◇ Introduced particle systems with long memory length

Motivation — A general framework



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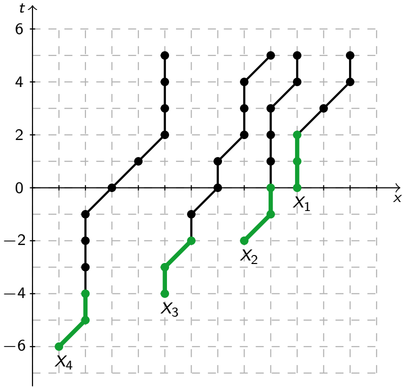

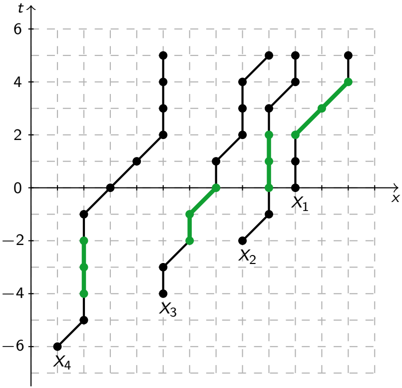

[M., Remenik, 2023. Exact solution of TASEP and variants with inhomogeneous speeds and memory lengths]

Method of solution — Transition probabilities

[Dieker, Warren, 2008]:

The transition probability of $N \geq 1$ discrete-time TASEP particles with sequential update from $y_N < \cdots < y_1$ at time s to $x_N < \cdots < x_1$ at later time t is

$$\mathbb{P}_{X_0=\vec{y}}(X_t = \vec{x}) = \left(\prod_{i=1}^{N}(1+v_i)^{t-s}\right) \det[F_{k,\ell}(y_k, x_\ell; t-s)]_{1 \leq k,\ell \leq N}$$

where p_i is the jump probability of the i^{th} particle, $v_i = \frac{p_i}{1-p_i}$ and

$$F_{k,\ell}(y_k, x_\ell; t-s) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{0,\vec{v}}} \mathrm{d}w \, \frac{(w/v_k)^{y_k}}{(w/v_\ell)^{x_\ell}} \frac{\prod_{i=1}^{\ell}(w-v_i)}{\prod_{i=1}^{k}(w-v_i)} \frac{(1+w)^{t-s}}{w^{\ell-k+1}}$$

where the contour $\Gamma_{0,\vec{v}}$ is centered at 0 and includes all v_i

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Note: To write the transition distribution of the particles with long memory, we need to convolve such determinants of different sizes

Method of solution — Particles with memory lengths

Settings:

- $N \geq 2$ particles, the i^{th} particle has speed $v_i > 0$ and length $L_i \geq 1$
- Initial configuration $\vec{y}$ satisfies $y_i - y_{i+1} \geq (L_i - 1) \vee 1$

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M., Remenik, 2023

For any $t \geq 0$, $1 \leq n_1 < \cdots < n_m \leq N$, and $a_1, \ldots, a_m \in \mathbb{Z}$,

$$\mathbb{P}\big(X_t(i) > a_i,\ i = 1, \ldots, m\big) = \det\big(I - \bar{\chi}_a K \bar{\chi}_a\big)_{\ell^2(\{n_1,\ldots,n_m\}\times\mathbb{Z})},$$

where $\bar{\chi}_a(n_i, x) = \mathbb{1}_{x \leq a_i}$ and the kernel is

$$K(n_i, x_i; n_j, x_j) = -Q_{(n_i,n_j]}(x_i, x_j)\mathbb{1}_{n_i<n_j} + \sum_{k=1}^{n_j} \Psi^{n_i}_{n_i-k}(x_i)\Phi^{n_j}_{n_j-k}(x_j)$$

$$Q_{(\ell,n]}(x, y) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, \frac{\theta^{x-y}}{w^{x-y-n+\ell+1}} \prod_{i=\ell+1}^{n} \frac{\alpha_i(1+w)^{L_{i-1}-1}}{v_i - w}$$

with $\alpha_i = \frac{v_i-\theta}{\theta}(1+\theta)^{1-L_{i-1}}$, integer $0 \leq \ell < n$, $L_0 = 1$ and any $\theta \in (0, 1)$

Method of solution — Biorthogonalization problem

The functions Ψ-s are given explicitly

$$\Psi^n_{n-k}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \, \frac{\theta^{x-y_k}(1+w)^t}{w^{x-y_k+n-k+1}} \frac{\prod_{i=1}^{n}(v_i - w)/\alpha_i(1+w)^{L_{i-1}-1}}{\prod_{i=1}^{k}(v_i - w)/\alpha_i(1+w)^{L_{i-1}-1}}$$

and the functions Φ-s are uniquely characterized by:

1. The biorthogonality relation, for $k, \ell = 0, \dots, n-1$,

$$\sum_{x\in\mathbb{Z}} \Psi^n_\ell(x)\Phi^n_k(x) = \mathbb{1}_{k=\ell}$$

2. Let $u_1 < u_2 < \dots < u_\nu$ be the distinct values among $v_1, \dots, v_n$ with multiplicities β_i. Then

$$\begin{aligned}
&\mathrm{span}\{x \in \mathbb{Z} \longmapsto \Phi^n_k(x) : 0 \le k < n\} \\
&\quad = \mathrm{span}\big\{x \in \mathbb{Z} \longmapsto x^\ell (u_k/\theta)^x : 1 \le k \le \nu,\ 0 \le \ell < \beta_k\big\} \\
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Method of solution — Solution

M., Remenik, 2023

The solution of the biorthogonalization problem is

$$\Phi_k^n(x) = (\mathcal{R}_n^{-1})^* h_k^n(0,x)$$

where

$$\mathcal{R}_n^{-1}(x,y) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \, \frac{\theta^{x-y}\psi(w)^{-1}}{w^{x-y+1}} \frac{a_n(\theta)}{a_n(w)}$$

and $\big(h_k^n(\ell,\cdot)\big)_{0\le\ell\le k}$ is the unique solution of

$$\begin{aligned} (Q_{n-\ell}^*)^{-1} h_k^n(\ell,z) &= h_k^n(\ell+1,z), && \ell<k,\ z\in\mathbb{Z} \\ h_k^n(k,z) &= (\theta/v_{n-k})^{y_{n-k}-z}, && z\in\mathbb{Z} \\ h_k^n(\ell,y_{n-\ell}) &= 0, && \ell<k \end{aligned}$$

for fixed $0\le k<n$, and

$$\mathrm{span}\{x\in\mathbb{Z}\longmapsto h_k^n(\ell,x) : \ell\le k<n\} \subseteq \mathbb{V}_{n-\ell}(\vec{v},\theta)$$

Method of solution — Solution

M., Remenik, 2023

The solution to the biorthogonalization problem is

$$\Phi_k^n(x) = \sum_{\eta > y_{n-k}} \bar{Q}^+_{[n-k,n]} \mathcal{R}_n^{-1}(\eta, x)$$
$$- \mathbb{1}_{k\geq 1} \sum_{\eta > y_{n-k}} \sum_{\eta' \in \mathbb{Z}} Q^+_{n-k}(\eta,\eta') \mathbb{E}_{B^+_{n-k}=\eta'} \left[\bar{Q}^+_{(\tau^+,n]} \mathcal{R}_n^{-1}(B^+_{\tau^+}, x) \mathbb{1}_{\tau^+<n} \right]$$

where the functions are

$$Q^+_\ell(x,y) = \frac{\alpha^+_\ell}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \, \frac{\theta^{x-y}}{w^{x-y}} \frac{a_\ell(w)}{v_\ell - w}, \qquad \alpha^+_\ell = \frac{v_\ell - \theta}{\theta a_\ell(\theta)},$$

$$\bar{Q}^+_{(\ell,n]}(x,y) = -\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{\vec{v}}} \mathrm{d}w \, \frac{\theta^{x-y}}{w^{x-y-n+\ell+1}} \prod_{i=\ell+1}^{n} \frac{\alpha^+_i a_i(w)}{v_i - w},$$

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where

- B^+_m is the time-inhomogeneous random walk which has transitions from time $m-1$ to time m with step distribution Q^+_m
- the stopping time is

$$\tau^+ = \min\left\{ m = 0, \ldots, N-1 : B^+_m > y_{m+1} \right\}$$

Method of solution — The final formula

Recall the formula of the kernel

$$K(n_i, x_i; n_j, x_j) = -Q_{(n_i,n_j]}(x_i, x_j)\mathbb{1}_{n_i<n_j} + \sum_{k=1}^{n_j} \Psi^{n_i}_{n_i-k}(x_i)\Phi^{n_j}_{n_j-k}(x_j)$$

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Using the solution of the biorthogonalization formula we can write

$$K(n_i, x_i; n_j, x_j) = -Q_{(n_i,n_j]}(x_i, x_j)\mathbb{1}_{n_i<n_j} + (\mathcal{S}_{-n_i})^* \bar{\mathcal{S}}^{\mathrm{epi}(\vec{y})}_{n_j}(x_i, x_j)$$

where

$$\mathcal{S}_{-n}(x_1, x_2) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \, \frac{\theta^{x_2-x_1}\psi(w)}{w^{x_2-x_1+n+1}} \frac{\prod_{i=1}^{n}(v_i - w)}{\prod_{i=1}^{n} \alpha_i^+ \prod_{i=1}^{n-1} a_i(w)}$$

$$\bar{\mathcal{S}}_{(m,n]}(x_1, x_2) = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{\vec{v}}} \mathrm{d}w \, \frac{\theta^{x_1-x_2}\psi(w)^{-1}}{w^{x_1-x_2-n+m+1}} \frac{\prod_{i=m+1}^{n} \alpha_i^+ \prod_{i=m+1}^{n-1} a_i(w)}{\prod_{i=m+1}^{n}(v_i - w)}$$

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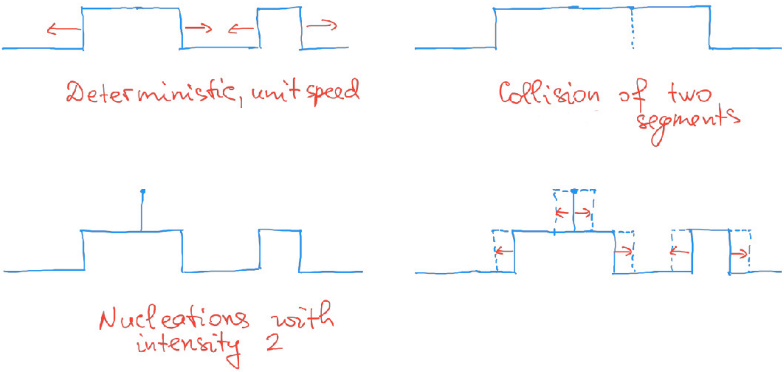
where

$$\bar{\mathcal{S}}^{\mathrm{epi}(\vec{y})}_{n}(x_1, x_2) = \mathbb{E}_{B_0^+=x_1}\left[\bar{\mathcal{S}}_{(\tau^+,n]}(B^+_{\tau^+}, x_2)\mathbb{1}_{\tau^+<n}\right]$$

Some interesting findings — Polynuclear growth (PNG)

◇ Introduced by Gates-Westcott 1995, Prähofer-Spohn 2002

The evolution of the height function $h : \mathbb{R} \to \mathbb{Z} \cup \{-\infty\}$ is



Some interesting findings — Polynuclear growth (PNG)

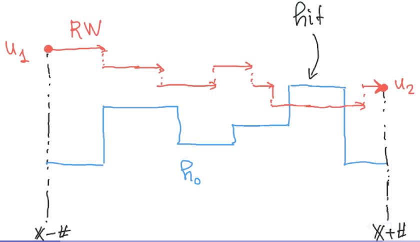
[M., Quastel, Remenik, 2022. Polynuclear growth and the Toda lattice]

$$\mathbb{P}_{h_0}(h(t,x_i) \le r_i, i = 1, \ldots, m) = \det\big(I - \chi_r K \chi_r\big)_{\ell^2(\{x_1,\ldots,x_m\}\times\mathbb{Z})}$$

where $\chi_r(x_i, z) = \mathbb{1}_{z > r_i}$ and

$$K(x_i, \cdot; x_j, \cdot) = -e^{(x_j - x_i)\Delta}\mathbb{1}_{x_i < x_j} + e^{-2t\nabla - x_i\Delta} \underbrace{\left(e^{(x_i - t)\Delta} P^{\mathrm{hit}(h_0)}_{x_i - t, x_j + t} e^{-(x_j + t)\Delta}\right)}_{\text{scattering transform}} e^{2\boldsymbol{t}\nabla + \boldsymbol{x}_j\Delta}$$

where $P^{\mathrm{hit}(h_0)}_{x_i - t, x_j + t}(u_1, u_2)$ is the probability of



Questions

1. What are the inhomogeneous analogues of PNG and the KPZ fixed point?
2. How about space inhomogeneity? Does it give the same limit?
3. Borodin, Ferrari, Sasamoto (2009) studied Two speed TASEP.
   Can we say something about shocks produced by blocks of particles with different speeds?

Some findings — Mixture of lengths

We consider equal jump probabilities $p \in (0,1)$, and blocks of particles with equal memory lengths:

1. a particles with length 1
2. b particles with length 2
3. a particles with length 1
4. b particles with length 2
5. ...

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2. b particles with length 2
3. a particles with length 1
4. b particles with length 2
5. ...

- if $b = 0$, then we have TASEP with sequential update
- if $a = 0$, then we have TASEP with parallel update

Some findings — Mixture of lengths

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Our method gives a solution to this model, and we can show convergence to the KPZ fixed point

Some consequences — Mixture of lengths

[M., Remenik, 2023]: Convergence to the KPZ fixed point:

$$-\gamma^{-1}\sigma^{-1}\varepsilon^{1/2}\Big(X_{\varepsilon^{-3/2}\boldsymbol{t}}(\alpha\varepsilon^{-3/2}\boldsymbol{t}-\sigma^2\varepsilon^{-1}\boldsymbol{x})-\beta\varepsilon^{-3/2}\boldsymbol{t}-2\sigma^2\varepsilon^{-1}\boldsymbol{x}\Big) \xrightarrow[\varepsilon\to 0]{} \mathfrak{h}(\boldsymbol{t},\boldsymbol{x};\mathfrak{h}_0)$$

where $q = 1 - p$ and

$$\varrho = \frac{b}{a+b}, \qquad \theta = \frac{\sqrt{p^2(1-\varrho)^2+4q}+p(1-\varrho)-2q}{2q(2-\varrho)}$$

$$\alpha = \frac{(p-q\theta)^2}{pq(1+\theta)^2+\varrho(p-q\theta)^2}, \qquad \beta = \frac{p(q(1+\theta)^2-1)}{pq(1+\theta)^2+\varrho(p-q\theta)^2}$$

$$\gamma = \left(\frac{pq\theta}{2(p-q\theta)^2}+\frac{\theta\varrho}{2(1+\theta)^2}\right)^{1/2}, \qquad \sigma = \left(\frac{2pq\theta(1+\theta)(p-q\theta)}{\gamma(pq(1+\theta)^2+\varrho(p-q\theta)^2)}\right)^{1/3}$$

Questions

1. What are the scaling limits for general lengths?

2. What is the effect of one long caterpillar?

Relation to classic integrable systems

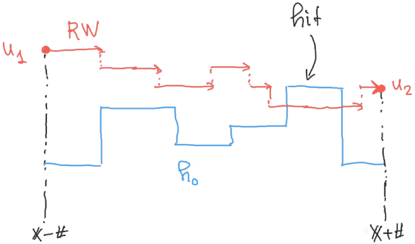
Polynuclear growth (PNG)

$$\mathbb{P}_{h_0}\big(h(t,x_i) \le r_i, i = 1,\ldots,m\big) = \det\big(I - \chi_r K \chi_r\big)_{\ell^2(\{x_1,\ldots,x_m\}\times\mathbb{Z})}$$

where $\chi_r(x_i, z) = \mathbb{1}_{z>r_i}$ and

$$K(x_i,\cdot;x_j,\cdot) = -e^{(x_j-x_i)\Delta}\mathbb{1}_{x_i<x_j}$$
$$+\, e^{-2t\nabla - x_i\Delta}\underbrace{\left(e^{(x_i-t)\Delta}P^{\mathrm{hit}(h_0)}_{x_i-t,x_j+t}e^{-(x_j+t)\Delta}\right)}_{\text{scattering transform}}e^{2\boldsymbol{t}\nabla+\boldsymbol{x}_j\Delta}$$

where $P^{\mathrm{hit}(h_0)}_{x_i-t,x_j+t}(u_1,u_2)$ is the probability of



Connection with the Toda lattice for general initial state

For PNG with an arbitrary (deterministic) initial condition h_0 we set $F_r(t,x) = \mathbb{P}_{h_0}(h(t,x) \le r)$ and $r_0(t,x) = \sup_{y \in [x-t, x+t]} h_0(y)$

Theorem [M.-Quastel-Remenik 2022]

$F_r(t,x)$ satisfies the 2D Toda equation: for $t > 0$ and $r > r_0(t,x)$

$$\tfrac{1}{4}(\partial_t^2 - \partial_x^2) \log F_r = \frac{F_{r-1} F_{r+1}}{F_r^2} - 1$$

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In the flat case ($h_0 \equiv 0$) F_r is independent of x, and the function

$$g_r(t) = \log F_r(2t,0) - \log F_{r-1}(2t,0)$$

satisfies

$$g_r'' = e^{g_{r+1} - g_r} - e^{g_r - g_{r-1}} \qquad \text{the classic Toda lattice}$$

This equation describes the deterministic motion of a chain of particles with nearest neighbor interaction

There is also an equation for the multipoint distributions

Let $F(t; x_1, \ldots, x_n; r_1, \ldots, r_n) = \mathbb{P}_{h_0}(h(t, x_i) \leq r_i,\ i = 1, \ldots, n)$

Theorem [M.-Quastel-Remenik 2022]

For $t > 0$ and $r_i > r_0(t, x_i)$, $i = 1, \ldots, n$, there exists $Q_r \in GL(n)$ such that

$$\frac{F(t; x_1, \ldots, x_n; r_1 + 1, \ldots, r_n + 1)}{F(t; x_1, \ldots, x_n; r_1, \ldots, r_n)} = \det Q_r$$

and Q_r satisfies the non-Abelian 2D Toda equations

$$\partial_{t-x}\left(\partial_{t+x} Q_r Q_r^{-1}\right) + Q_r Q_{r-1}^{-1} - Q_{r+1} Q_r^{-1} = 0$$

where $\partial_{t\pm x} = \frac{1}{2}(\partial_t \pm (\partial_{x_1} + \ldots + \partial_{x_n}))$, $r = (r_1, \ldots, r_n)$ and $r \pm 1 = (r_1 \pm 1, \ldots, r_n \pm 1)$

Bauhardt, Pöppe, 1987: Derivation of the Toda equation from a Fredholm determinant

1:2:3 limits of Toda equations

There exists $\mathsf{Q} \in GL(n)$ such that

$$\partial_r \log \mathbb{P}\big(\mathfrak{h}(t, x_i) \leq r_i, i = 1, \ldots, n\big) = \operatorname{tr} \mathsf{Q},$$

where $\mathsf{q} = \partial_r \mathsf{Q}$ solves the matrix Kadomtsev–Petviashvili (KP) equation

$$\partial_t \mathsf{q} + \tfrac{1}{2}\partial_r \mathsf{q}^2 + \tfrac{1}{12}\partial_r^3 \mathsf{q} + \tfrac{1}{4}\partial_x^2 \mathsf{Q} + \tfrac{1}{2}\big(\mathsf{q}\,\partial_x \mathsf{Q} - \partial_x \mathsf{Q}\,\mathsf{q}\big) = 0$$

where $\partial_r = \partial_{r_1} + \ldots + \partial_{r_n}$ and $\partial_x = \partial_{x_1} + \ldots + \partial_{x_n}$

1:2:3 limits of Toda equations

There exists $Q \in GL(n)$ such that

$$\partial_r \log \mathbb{P}\big(\mathfrak{h}(t, x_i) \le r_i, i = 1, \ldots, n\big) = \operatorname{tr} Q,$$

where $q = \partial_r Q$ solves the matrix Kadomtsev–Petviashvili (KP) equation

$$\partial_t q + \tfrac{1}{2}\partial_r q^2 + \tfrac{1}{12}\partial_r^3 q + \tfrac{1}{4}\partial_x^2 Q + \tfrac{1}{2}\big(q\,\partial_x Q - \partial_x Q\, q\big) = 0$$

where $\partial_r = \partial_{r_1} + \ldots + \partial_{r_n}$ and $\partial_x = \partial_{x_1} + \ldots + \partial_{x_n}$

◇ For $n = 1$ this is the standard (scalar) KP-II equation

$$\partial_t q + \tfrac{1}{2}\partial_r q^2 + \tfrac{1}{12}\partial_r^3 q + \tfrac{1}{4}\partial_r^{-1}\partial_x^2 q = 0$$

◇ The flat case $h_0 \equiv 0$ corresponds to the KdV equation

$$\partial_t q + \tfrac{1}{2}\partial_r q^2 + \tfrac{1}{12}\partial_r^3 q = 0$$