

# How to Solve the Stochastic Six Vertex Model

Leonid Petrov

At 07:21, on Monday 10<sup>th</sup> July, 2023

## Introduction

These are lecture notes for **PANEM-2023** at Texas A&M on the integrability and asymptotics of the stochastic six vertex model.

# 1 Six vertex model through different lenses

In the first lecture, we describe the stochastic six vertex model from two diverse perspectives — as a model of statistical mechanics, and as a stochastic particle system.

## 1.1 Gibbs measures and the six vertex model

### 1.1.1 Finite-volume Gibbs measures

We begin with describing the useful framework of *Gibbs measures*. For simplicity, we work on the two-dimensional lattice  $\mathbb{Z}^2$ . Let  $\Lambda \subset \mathbb{Z}^2$  be a finite subset (for example, a rectangle). We are interested in *spin configurations* inside  $\Lambda$  which are encoded as  $\omega = \{\sigma_e : e \text{ is an edge in } \Lambda\}$ , where  $\sigma_e \in \{0, 1\}$ . By an “edge in  $\Lambda$ ” we mean that both endpoints of this edge must be inside  $\Lambda$ . Each spin configuration is equipped with boundary conditions, which are fixed spins on all the boundary edges of  $\Lambda$  (an edge is called boundary if it connects  $\Lambda$  to  $\mathbb{Z}^2 \setminus \Lambda$ ).

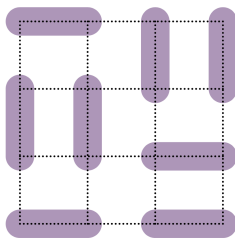
With each spin configuration  $\omega$ , we associate its energy  $H(\omega) \in \mathbb{R}$ . This energy may depend on global parameters (e.g., inverse temperature) and local parameters (e.g., edge capacities or vertex rapidities). If a particular spin configuration  $\omega$  is forbidden, we have  $H(\omega) = +\infty$ .

**Definition 1.1.** A (finite-volume) *Gibbs measure* in  $\Lambda$  with fixed boundary conditions and the energy function  $H(\cdot)$  is the probability distribution on spin configurations whose probability weights have the form

$$\text{Prob}(\omega) = \frac{1}{Z} \exp \{-H(\omega)\}.$$

Here  $Z$  is the *partition function*, which is simply the probability normalizing constant.

**Example 1.2** (Domino tilings on the square grid). A *perfect matching* on  $\Lambda$  is any subset  $M$  of its edges such that every vertex is covered by exactly one edge from  $M$ . For example, here is a perfect matching on the four by four rectangle:

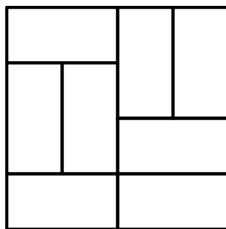


If the set of allowed spin configurations is the set of perfect matchings, and

$$H(\omega) = \begin{cases} 0, & \omega \text{ is a perfect matching;} \\ +\infty, & \omega \text{ is not a perfect matching,} \end{cases}$$

then the corresponding Gibbs measure is the uniform distribution on the space of *domino tilings*. That is, we identify each covered edge with a  $2 \times 1$  domino. The domino tiling corresponding to

the above perfect matching is



Computing partition functions of various Gibbs measures may be very nontrivial. For example, the number of domino tilings of the  $8 \times 8$  chessboard is 12,998,816, but its theoretical computation (not via a computer program) requires several steps.

ref

Parameter-dependent partition functions represent many important quantities across all of mathematics, including various families of symmetric functions (such as Schur or Hall-Littlewood functions), and related objects.

### 1.1.2 Infinite-volume Gibbs measures

### 1.1.3 Six vertex model

## 1.2 Stochastic six vertex model and its particle system limits

### 1.3 Gibbs properties of the stochastic six vertex model

### 1.4 Basic coupling and colored (multispecies) models

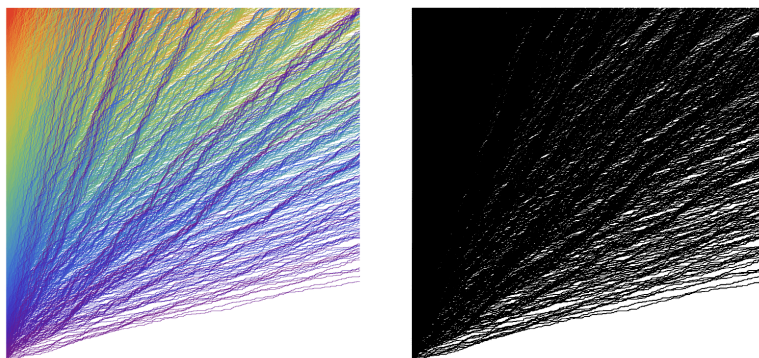


Figure 1: Colored stochastic six vertex model and its monochrome version.

## 1.5 Stationary distributions and hydrodynamics

### 1.6 Limit shape and fluctuation problem

## 2 Integrability

### 3 Asymptotics

## References

UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA  
E-mail: lenia.petrov@gmail.com