# How to Solve the Stochastic Six Vertex Model

Leonid Petrov

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## Introduction

These are lecture notes for PANEM-2023 at Texas A&M on the integrability and asymptotics of the stochastic six vertex model.

# 1 Six vertex model through different lenses

In the first lecture, we describe the stochastic six vertex model from two diverse perspectives — as a model of statistical mechanics, and as a stochastic particle system.

## 1.1 Gibbs measures and the six vertex model

### 1.1.1 Finite-volume Gibbs measures

We begin with describing the useful framework of *Gibbs measures*. For simplicity, we work on the two-dimensional lattice $\mathbb{Z}^2$. Let $\Lambda \subset \mathbb{Z}^2$ be a finite subset (for example, a rectangle). We are interested in *spin configurations* inside $\Lambda$ which are encoded as $\omega = \{\sigma_e \colon e \text{ is an edge in } \Lambda\}$, where $\sigma_e \in \{0, 1\}$. By an "edge in $\Lambda$" we mean that both endpoints of this edge must be inside $\Lambda$. Each spin configuration is equipped with boundary conditions, which are fixed spins on all the boundary edges of $\Lambda$ (an edge is called boundary if it connects $\Lambda$ to $\mathbb{Z}^2 \setminus \Lambda$).

With each spin configuration $\omega$, we associate its energy $H(\omega) \in \mathbb{R}$. This energy may depend on global parameters (e.g., inverse temperature) and local parameters (e.g., edge capacities or vertex rapidities). If a particular spin configuration $\omega$ is forbidden, we have $H(\omega) = +\infty$.

**Definition 1.1.** A (finite-volume) *Gibbs measure* in $\Lambda$ with fixed boundary conditions and the energy function $H(\cdot)$ is the probability distribution on spin configurations whose probability weights have the form

$$\mathrm{Prob}(\omega) = \frac{1}{Z} \exp\left\{-H(\omega)\right\}.$$

Here $Z$ is the *partition function*, which is simply the probability normalizing constant.

**Example 1.2** (Domino tilings on the square grid). A *perfect matching* on $\Lambda$ is any subset $M$ of its edges such that every vertex is covered by exactly one edge from $M$. For example, here is a perfect matching on the four by four rectangle:
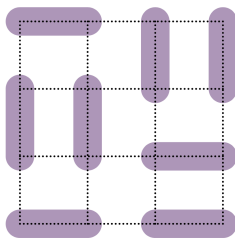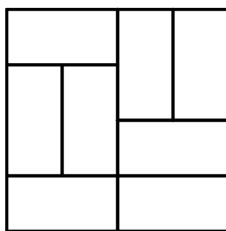


If the set of allowed spin configurations is the set of perfect matchings, and

$$H(\omega) = \begin{cases} 0, & \omega \text{ is a perfect matching;} \\ +\infty, & \omega \text{ is not a perfect matching,} \end{cases}$$

then the corresponding Gibbs measure is the uniform distribution on the space of *domino tilings*. That is, we identify each covered edge with a $2 \times 1$ domino. The domino tiling corresponding to

the above perfect matching is



Computing partition functions of various Gibbs measures may be challenging. For example, the number of domino tilings of the $8 \times 8$ chessboard is 12,988,816, but its theoretical computation (not via a computer program) requires several nontrivial steps.

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Parameter-dependent partition functions represent many important quantities across all of mathematics, including various families of symmetric functions (such as Schur or Hall-Littlewood functions), and related objects.

### 1.1.2   Infinite-volume Gibbs measures

### 1.1.3   Six vertex model

## 1.2   Stochastic six vertex model and its particle system limits

## 1.3   Gibbs properties of the stochastic six vertex model

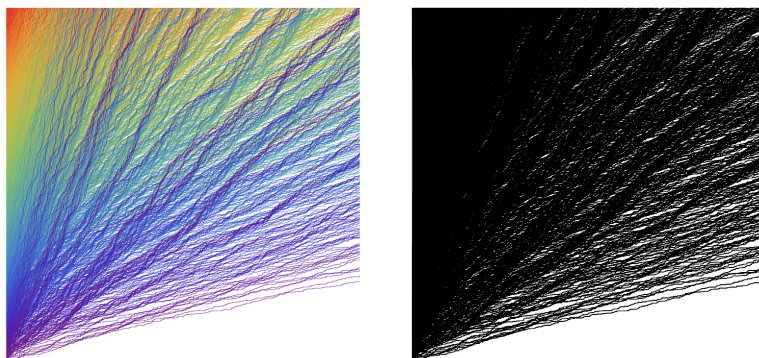## 1.4   Basic coupling and colored (multispecies) models



Figure 1: Colored stochastic six vertex model and its monochrome version.

## 1.5   Stationary distributions and hydrodynamics

Bernoulli is Stationary; also for all the limits we had.

## 1.6   Limit shape and fluctuation problem

# 2 Integrability

# 3 Asymptotics

# References

University of Virginia, Charlottesville, VA
E-mail: lenia.petrov@gmail.com