How to Solve the Stochastic Six Vertex Model

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Introduction

These are lecture notes for PANEM-2023 at Texas A&M on the integrability and asymptotics of the stochastic six vertex model.

1 Six vertex model through different lenses

In the first lecture, we describe the stochastic six vertex model from two diverse perspectives — as a model of statistical mechanics, and as a stochastic particle system.

1.1 Gibbs measures and the six vertex model

1.1.1 Finite-volume Gibbs measures

We begin with describing the useful framework of *Gibbs measures.* For simplicity, we work on the two-dimensional lattice $\mathbb{Z}^2$. Let $\Lambda \subset \mathbb{Z}^2$ be a finite subset (for example, a rectangle). We are interested in *spin configurations* inside Λ which are encoded as $\omega = \{\sigma_e \colon e \text{ is an edge in } \Lambda\}$, where $\sigma_e \in \{0, 1\}$. By an "edge in Λ" we mean that both endpoints of this edge must be inside Λ. Each spin configuration is equipped with boundary conditions, which are fixed spins on all the boundary edges of Λ (an edge is called boundary if it connects Λ to $\mathbb{Z}^2 \setminus \Lambda$).

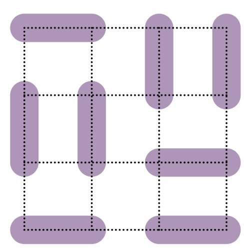
With each spin configuration ω, we associate its energy $H(\omega) \in \mathbb{R}$. This energy may depend on global parameters (e.g., inverse temperature) and local parameters (e.g., edge capacities or vertex rapidities). If a particular spin configuration ω is forbidden, we have $H(\omega) = +\infty$.

Definition 1.1. A (finite-volume) *Gibbs measure* in Λ with fixed boundary conditions and the energy function $H(\cdot)$ is the probability distribution on spin configurations whose probability weights have the form

$$\mathrm{Prob}(\omega) = \frac{1}{Z} \exp\{-H(\omega)\}.$$

Here Z is the *partition function*, which is simply the probability normalizing constant.

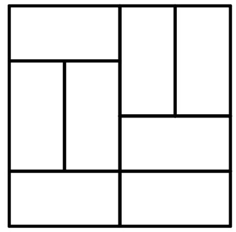
Example 1.2 (Domino tilings on the square grid)**.** A *perfect matching* on Λ is any subset M of its edges such that every vertex is covered by exactly one edge from M. For example, here is a perfect matching on the four by four rectangle:



If the set of allowed spin configurations is the set of perfect matchings (with $\sigma_e = 1$ if the edge is included in the matching, and $\sigma_e = 0$ otherwise), and

$$H(\omega) = \begin{cases} 0, & \omega \text{ is a perfect matching;} \\ +\infty, & \omega \text{ is not a perfect matching,} \end{cases}$$

then the corresponding Gibbs measure is the uniform distribution on the space of *domino tilings.* That is, we identify each covered edge with a 2×1 domino. The domino tiling corresponding to the above perfect matching is



Exercise 1.1. Compute the number of domino tilings of the thin rectangles

(a) $2 \times n$;

(b) $3 \times (2n)$.

Computing partition functions of various Gibbs measures may be challenging. For example, the number of domino tilings of the 8×8 chessboard is 12,988,816, but its theoretical computation (not via a computer program) requires several nontrivial steps [Kas61], [TF61].

Parameter-dependent partition functions represent many important quantities across all of mathematics, including various families of symmetric functions (such as Schur or Hall-Littlewood functions), and related objects.

1.1.2 Infinite-volume Gibbs measures

Besides Gibbs measures on configurations on a finite space as in Definition 1.1 with fixed boundary conditions ("*boxed distributions*"), we are interested in *infinite-volume* Gibbs measures.

Definition 1.3. A probability measure on spin configurations on an infinite subset $\Lambda_\infty \subseteq \mathbb{Z}^2$ (we will mainly consider the whole plane and the quarter plane $\mathbb{Z}^2_{\geq 0}$) is called (infinite-volume) Gibbs if for any finite $\Lambda \subset \Lambda_\infty$, the configuration inside Λ conditioned on the configuration in $\Lambda_\infty \setminus \Lambda$ is a finite-volume Gibbs measure in the sense of Definition 1.1 (with boundary conditions imposed by the outside configuration in $\Lambda_\infty \setminus \Lambda$).

Out of all possible infinite-volume Gibbs measures, we are interested in measures with special properties, such as translation invariant and/or ergodic. A Gibbs measure on $\mathbb{Z}^2$ is called *translation invariant* if its distribution does not change under arbitrary space translations. A Gibbs measure is called *ergodic* (equivalently, *extremal*) if it cannot be represented as a convex combination of two other such measures. Gibbs measures which are translation invariant and ergodic (within the class of translation invariant measures) are called *pure states*.

Classifying pure states for a given energy function $H(\cdot)$ is a very nontrivial problem, and an explicit answer is rarely available. For the general six vertex model (defined in Section 1.1.3 below), the answer is only conjectural [Res10].

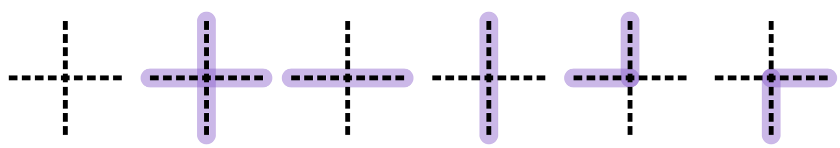
On the other hand, pure states of the six vertex model under a special *free fermion condition* (which includes the domino model from Example 1.2) admit a very explicit description through determinantal point processes (i.e., all correlation functions of these measures are diagonal minors of an explicit function in two variables), which follows from [She05], [KOS06]. One of the goals of these lecture notes is to discuss the tools and results one would require to extend this classification beyond the free fermion case.

Remark 1.4. Certain families of non translation invariant infinite-volume Gibbs measures (under the free fermion condition) power the classification of irreducible representations of infinite-dimensional unitary group and other classical groups [Voi76], [VK82], [BO12], [Pet14]. This subject is closely related to symmetric functions arising as partition functions of Gibbs measures with varying parameters (rapidities) along one of the lattice coordinate direction which we discuss in the second lecture (Section 2). There is also a direct link between these Gibbs measures and totally nonnegative triangular or full Toeplitz matrices for characters of the infinite symmetric group or, respectively, the infinite-dimensional unitary group, see [AESW51], [Edr53], [Boy83].

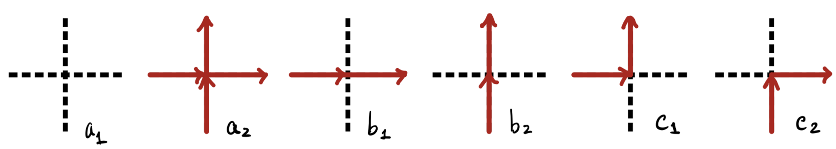
1.1.3 Six vertex model

The most general Gibbs property we consider in these notes is that of the *asymmetric six vertex model*. The six vertex model was introduced by Pauling to model the residual entropy of ice [Pau35] (see also [ASLW+15] for recent experiments with square ice between two graphene sheets, and Remark 1.5 below for an exact connection). The model has received a lot of attention since the seminal Bethe Ansatz solutions obtained in the 1960's in [Lie67], [YY66]. We refer to the book [Bax07] for an introduction, and also to [Res10] for a more recent survey of the model.

Under the asymmetric six vertex model, the allowed spin configurations on the two-dimensional lattice are such that locally around each vertex there can be one of the following six configurations:



Viewing the edges with spin $\sigma_e = 1$ as parts of up-right paths, we can think of six vertex model configurations as up-right path configurations on the lattice, where paths are allowed to touch at a vertex. We denote the six vertex types by $a_1, a_2, b_1, b_2, c_1, c_2$:



See Figure 1, right, for an example of a global configuration of up-right paths in a rectangle.

Abusing the notation, we also think of $a_1, a_2, b_1, b_2, c_1, c_2 \geq 0$ as the Gibbs weights e^{-H} assigned to each local vertex. That is, the six vertex model Gibbs measure in a rectangle Λ with given boundary conditions (that is, with prescribed spin configurations at all edges connecting Λ to $\mathbb{Z}^2 \setminus \Lambda$) has probability weights

$$\mathrm{Prob}(\omega) = \frac{a_1^{\#\{a_1 \text{ vertices}\}} a_2^{\#\{a_2 \text{ vertices}\}} b_1^{\#\{b_1 \text{ vertices}\}} b_2^{\#\{b_2 \text{ vertices}\}} c_1^{\#\{c_1 \text{ vertices}\}} c_2^{\#\{c_2 \text{ vertices}\}}}{Z(a_1, a_2, b_1, b_2, c_1, c_2)}.$$

Here $\#\{a_1 \text{ vertices}\}$ is the number of vertices of type a_1 in the configuration ω, and so on.

Remark 1.5 (Connection to square ice)**.** In the square ice, the oxygen atoms should form a perfect square grid, and each edge contains a hydrogen atom. The hydrogen atom on an edge is connected to one of the adjacent oxygens by the chemical bond, and to another oxygen by a weaker hydrogen bond. This allows to distinguish two types of edges, and assign "spins" 0 and 1 to them. Since each oxygen must have exactly two hydrogen atoms attached to it by chemical bonds, we get six possible local configurations around a vertex. See Figure 1 for an illustration.

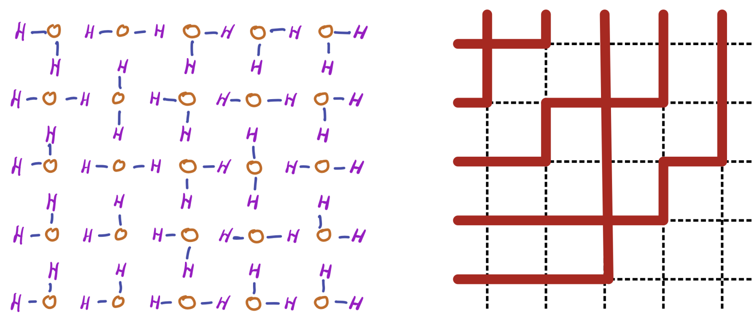


Figure 1: Left: a configuration of the square ice. Right: the corresponding configuration of the six vertex model in the square grid.

The quantity

$$\Delta := \frac{a_1a_2 + b_1b_2 - c_1c_2}{2\sqrt{a_1a_2b_1b_2}} \tag{1.1}$$

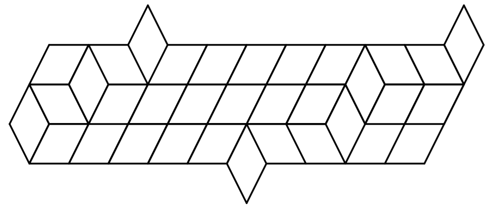
plays an important role in the (mostly conjectural) description of pure phases of the asymmetric six vertex model. Depending on Δ, there are several regimes of the model:

- Ferroelectric: $\Delta > 1$;
- Anti-ferroelectric: $\Delta < 1$;
- Disordered: $-1 < \Delta < 1$;
- Free fermion point: $\Delta = 0$.

Example 1.6 (Free fermion specializations)**.** Specializing the weights $a_1, a_2, b_1, b_2, c_1, c_2$ in two different free fermion ways, we obtain the following particular cases:

- If $a_1 = a_2 = b_1 = b_2 = 1$ and $c_1 = c_2 = \sqrt{2}$, one can map six vertex configurations into domino tilings from Example 1.2. This map is not a bijection between configurations, but instead it splits a c-type vertex into two equivalent local configurations of the domino tilings, while preserving the configuration weights. This idea goes back to [EKLP92], see also [ZJ00], [FS06]. A multiparameter generalization of the domino tiling model coming from the free fermion six vertex model was considered in [ABPW21], see also [Nap23].
- When $a_2 = 0$ and $a_1 = b_1 = b_2 = c_1 = c_2 = 1$, we forbid the intersection of paths. This model can be bijectively mapped into a model of *lozenge tilings* (e.g., see [Gor21] for the definition),

with configurations like



Exercise 1.2. (a) Work out the details of the mapping from the free fermion six vertex model with $a_2 = 0$ and $a_1 = b_1 = b_2 = c_1 = c_2 = 1$ to lozenge tilings. What happens to the boundary conditions?

(b) If we set $b_1 = c_1 = u_x v_y$, where (x, y) are the lattice coordinates of a vertex, then six vertex configurations start depending on the parameters u_i, v_j (which we assume to be generic complex numbers). How do these weights translate into the lozenge tiling picture?

In the next Section 1.2 we consider another important particular case — the *stochastic six vertex model*, which is no longer free fermion.

Remark 1.7 (Alternating sign matrices)**.** There are other very interesting specializations of the six vertex model which are not stochastic nor free fermion. Let us only mention that when $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$ (so $\Delta = 1/2$, which is in the disordered regime), the six vertex Gibbs property becomes uniform (on configurations of up-right paths which are allowed to touch at a vertex). There is a bijection from six vertex configurations to *alternating sign matrices*. This bijection allowed to compute the number of alternating sign matrices from a partition functions of the six vertex model. We refer to [Kup96] and [Pro01] for details.

1.2 Stochastic six vertex model and its particle system limits

1.3 Gibbs properties of the stochastic six vertex model

1.4 Basic coupling and colored (multispecies) models

1.5 Stationary distributions and hydrodynamics

Bernoulli is Stationary; also for all the limits we had.

1.6 Limit shape and fluctuation problem

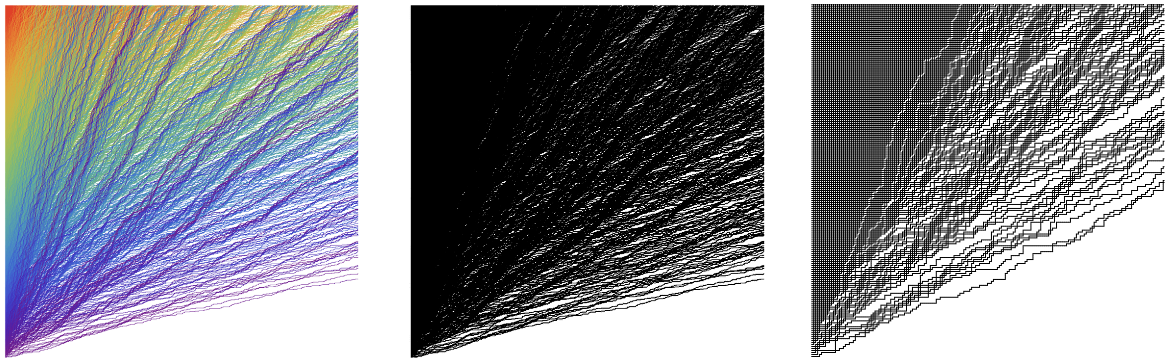


Figure 2: Colored stochastic six vertex model, its monochrome version, and a smaller simulation of the monochrome six vertex model.

2 Integrability

3 Asymptotics

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