Lecture 3.

The main motivation for seeking the classification results below was to understand the exact solvability of a number of reaction-diffusion models on $\mathbb{Z}$:

ARW + Pairwize immigration

$\eta(I)=\sum_{i\in I}\eta_i$

$\left(f_I(\eta)=(-1)^{\eta(I)}\right)$; Branching + Coalescing RW's $\left(f_I(\eta)=\mathbb{1}(\eta(I)=0)\right)$;

Annihilation-Coalescing RW with annihilating prob. θ $\left(f_I(\eta)=(-\theta)^{\eta(I)}\right)$.

For all these systems,

(1) $$L=\sum_{i\in\mathbb{Z}}(\sigma_i-I),\quad\text{where}$$

$$\sigma_i^2=\sigma_i.$$

So it is natural to try and list all stochastic idempotent 4×4 matrices. So far, this has been done, subject to the following symmetry constraint:

(2) Let $P\in\mathrm{End}(V\otimes V)$: $P(a\otimes b)=b\otimes a$

In the standard basis,

(2') $$P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

If a Markov process is of the form (1) and satisfies

(3) $$\rho\sigma = \sigma\rho,$$

then the reflected process $\tilde{\eta}_t(x) := \eta_t(-x)$, $x \in \mathbb{Z}$ has the same law at η_t.

It makes sense to classify our models up to the particle-hole interchange: let $\tau : \eta(x) \longmapsto 1 - \eta(x)$, $x \in \mathbb{Z}$

(4) $$\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$ in the standard basis.

The new Markov process is generated by

(5) $$\tilde{L} = \sum_{i \in \mathbb{Z}} (\tau\sigma_i\tau - I)$$

and follows the evolution of 'holes' in the original system.

(6) Remark. $\rho^2 = I$, $\tau^2 = I$, $\rho\tau = \tau\rho$

Theorem1 Suppose $\sigma \in End(V \otimes V)$ is idempotent, $\sigma^2 = \sigma$, and stochastic, that is its matrix in the standard basis satisfies

$$\sigma_{ij} \geq 0 \text{ for } 1 \leq i \neq j \leq 4, \text{ and}$$

$$\sum_{j=1}^{4} \sigma_{ij} = 1 \text{ for } 1 \leq i \leq 4$$

Assume also the reflective

symmetry condition $\rho\sigma = \sigma\rho$.
Then any such non-trivial $\sigma \neq I$ must, after possibly a particle-hole conjugation $\sigma \mapsto \tau\sigma\tau$, lie in one of the following nine families (written an the standard basis):

Symmetric Exclusion Model (SEP)
rank = 3

$$\sigma = \begin{array}{c|cccc} & 11 & 10 & 01 & 00 \\ \hline 11 & 1 & 0 & 0 & 0 \\ 10 & 0 & 1/2 & 1/2 & 0 \\ 01 & 0 & 1/2 & 1/2 & 0 \\ 00 & 0 & 0 & 0 & 1 \end{array}$$

$\bullet \xrightarrow{1/2} \circ$, $\circ \xleftarrow{1/2} \bullet$, $\bullet \not\to \bullet$, $\circ \not\to \circ$

"Ferromagnetic kinetic Ising model"

Biased voter model (BVM)
rank = 3 $\quad \theta \in [0,1]$

$$\sigma = \begin{array}{c|cccc} & 11 & 10 & 01 & 00 \\ \hline 11 & 1 & 0 & 0 & 0 \\ 10 & \theta & 0 & 0 & 1-\theta \\ 01 & \theta & 0 & 0 & 1-\theta \\ 00 & 0 & 0 & 0 & 1 \end{array}$$

$10 \xrightarrow{1-\theta} 00$, $10 \xrightarrow{\theta} 11$
$01 \xrightarrow{\theta} 11$, $01 \xrightarrow{1-\theta} 00$

"Anti-ferr. kinetic Ising model"

Symmetric anti-voter model (SAVM)
rank = 3

$$\sigma = \begin{array}{c|cccc} & 11 & 10 & 01 & 00 \\ \hline 11 & 0 & 1/2 & 1/2 & 0 \\ 10 & 0 & 1 & 0 & 0 \\ 01 & 0 & 0 & 1 & 0 \\ 00 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$11 \xrightarrow{1/2} 10$, $11 \xrightarrow{1/2} 01$
$00 \xrightarrow{1/2} 10$, $00 \xrightarrow{1/2} 01$

Annihilating coalescing sym. random walk (ACSRW)
rank = 3 $\quad \theta \in [0, 1/2]$
Coalescence prob is $1-\theta$

$$\sigma = \begin{pmatrix} 0 & \frac{1-\theta}{2} & \frac{1-\theta}{2} & \theta \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$11 \xrightarrow{\frac{1-\theta}{2}} 10$, $11 \xrightarrow{\frac{1-\theta}{2}} 01$, $11 \xrightarrow{\theta} 00$
$10 \underset{1/2}{\overset{1/2}{\rightleftarrows}} 01$

Annihilating symmetric RW with pair immigration (ASRWPI)
rank = 2 $\quad$ Imm rate $= \frac{1}{2} - \theta$
Hopping rate $= \theta$

$$\sigma = \begin{pmatrix} \frac{1}{2}-\theta & 0 & 0 & \frac{1}{2}+\theta \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1/2 & 1/2 & 0 \\ \frac{1}{2}-\theta & 0 & 0 & \frac{1}{2}+\theta \end{pmatrix}$$

$11 \xrightarrow{\frac{1}{2}+\theta} 00$
$01 \underset{1/2}{\overset{1/2}{\rightleftarrows}} 10$
$00 \xrightarrow{1/2-\theta} 11$

Coalescing sym RW with Branching (CSRWB)
rank = 2 $\quad \theta$ - hopping rate
$1-2\theta$ - branching rate

$$\sigma = \begin{pmatrix} 1-2\theta & \theta & \theta & 0 \\ 1-2\theta & \theta & \theta & 0 \\ 1-2\theta & \theta & \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$11 \xrightarrow{\theta} 10$, $11 \xrightarrow{\theta} 01$
$01 \xrightarrow{\theta} 10$, $01 \xrightarrow{1-2\theta} 11$

Not a zero range model

Stationary coalescence annihilation model [SCAM] θ-coalescence rat

rank = 3

$$\sigma = \begin{pmatrix} 0 & \theta & \theta & 1-2\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 11 \xrightarrow{1-2\theta} 00, \quad 11 \xrightarrow{\theta} 10, \quad 11 \xrightarrow{\theta} 01$$

Dimer model [DM]

rank = 3

$$\sigma = \begin{pmatrix} 1-\theta & 0 & 0 & \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\theta & 0 & 0 & \theta \end{pmatrix} \qquad 11 \xrightarrow{\theta} 00, \quad 00 \xrightarrow{1-\theta} 11$$

Reshuffle model [RM]

rank = 1

$$\sigma = \begin{pmatrix} \theta_1 & \theta_2 & \theta_2 & \theta_3 \\ \theta_1 & \theta_2 & \theta_2 & \theta_3 \\ \theta_1 & \theta_2 & \theta_2 & \theta_3 \\ \theta_1 & \theta_2 & \theta_2 & \theta_3 \end{pmatrix} \qquad \theta_i \geq 0, \quad \theta_1 + 2\theta_2 + \theta_3 = 1$$

Remarks.

(i) The list is not disjoint, e.g. under PH exchange [BVM], [DM] and [RM] change their values. Moreover,

$$\sigma_{[BVM]}\Big|_{\theta=1} = \sigma_{[CSRWB]}\Big|_{\theta=0}$$

(ii) SEP → SEP under PH exchange. Some models become hard to recognize in terms of the hole dynamics. E.g.

$$\tau \sigma_{ACSRW} \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \theta & \frac{1-\theta}{2} & \frac{1-\theta}{2} & 0 \end{pmatrix}$$

$$10 \underset{1/2}{\overset{1/2}{\rightleftarrows}} 01, \qquad 00 \xrightarrow{\theta} 11, \quad 00 \xrightarrow{\frac{1-\theta}{2}} 10, \quad 00 \xrightarrow{\frac{1-\theta}{2}} 01,$$

but nothing happens to 11.

Lemma 1

The generator algebras of [SEP], [SAVM], [ACSRW] are factor algebras of $\mathbb{H}_\infty$ with $Q = 1/4$

The gen. algebras of BVM is a factor algebra of $\mathbb{H}_\infty$ with $Q = \theta(1-\theta)$

The gen algebra of CSRWB is a factor of $\mathbb{H}_\infty$ with $Q = \theta(1-\theta)$

The gen algebra of ASRWPI is a factor of $\mathbb{H}_\infty$ with $Q = \theta^2$

The "exceptional" models do not generically satisfy the deformed braid relation

Steps of the proof of the classification theorem:

Step 1 Lemma 2 Let $\sigma \in End(V \otimes V)$ be a stochastic matrix: $\sigma^2 = \sigma$ and $p\sigma = \sigma p$.

$v = \binom{1}{1}$ Then either

(i) $\exists w \in V$ s.t. v and w are linearly indep and $\sigma\, w \otimes w = w \otimes w$ or

(ii) σ corresponds to [SAVM], $[BVM]_{\theta = 1/2}$ or the [RW]

Step 2 **Lemma 3.** Let $\sigma \in End(V \otimes V)$ be a stochastic idempotent matrix s.t. $p\sigma = \sigma p$. Assume that $\exists w \in V$ lin. ind of σ s.t. $\sigma w \otimes w = w \otimes w$. Then, after possibly a PH conjugation $\exists$ 7 non-trivial matrices $\sigma \neq I$ as listed below:

Hecke

There can be multiple solutions for w

SEP $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\sigma v \otimes w = \sigma w \otimes v = \frac{1}{2}(v \otimes w + w \otimes v)$

BVM $/_{\theta \neq 1/2}$ $w = \begin{pmatrix} 1-\theta \\ \theta \end{pmatrix}$, $\sigma v \otimes w = \sigma w \otimes v = \theta(1-\theta)\, v \otimes v + w \otimes w$

ACSRW $w = \begin{pmatrix} -\theta \\ 1 \end{pmatrix}$ $\sigma v \otimes w = \sigma w \otimes v = \frac{1}{2} v \otimes v + \frac{1}{2} w \otimes w$

CSRWB $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\sigma v \otimes w = \sigma w \otimes v = \theta\, \sigma \otimes \sigma + (1-\theta)\, w \otimes w$

ASRW PI $w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\sigma v \otimes w = \sigma w \otimes v = \theta\, \sigma \otimes \sigma + \theta\, w \otimes w$

Non Hecke

SCAM $w = \begin{pmatrix} 2\theta - 1 \\ 1 \end{pmatrix}$

DM $w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Bad action not preserving the order of w, v's

Remarks.

(i) Voter models can be mapped to ARW's using domain wall coordinates:

VM	o • • • • o • • o o	Voters
	x x x x	Annih. particles
AVM	o • • o • o • o o •	Antivoters
	x x	Annih particles

(ii) [ACRW], [CSRWB], [ASRWPI] are Pfaffian point processes. The analysis is v. similar to the case of ARW carried out in Lec 2.

(iii) [SCAM] is exactly solvable:

$\sigma e^{(1)} \otimes e^{(1)} = 0$ and

$\sigma \sigma \otimes e^{(1)} = \sigma \otimes e^{(1)} - (1-\theta) e^{(1)} \otimes e^{(1)}$

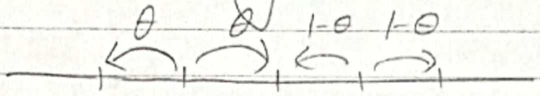
$\sigma e^{(1)} \otimes \sigma = e^{(1)} \otimes \sigma - (1-\theta) e^{(1)} \otimes e^{(1)}$

Useful duality function:

$$\phi_t^{(2n)}(x_1, \dots, x_n) = \prod_{i=1}^{n} \mathbb{1}\left(\eta(y) = 1 \text{ for } x_{2i-1} < y \leq x_{2i}\right)$$

Conclusion: if $\eta_0 \equiv 1$, then [SCAM] is a stationary renewal process $\forall t > 0$

(iv) Dimer model can be mapped to inhomog SEP



(v) The reshuffle model with

$$\sigma = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix},$$

where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$ is a det point process, coinciding at $t = \infty$ with a thinning of Borodin-Diaconis-Fulman "descent" determinantal PP:

$$\eta_k^{(BDF)} = \mathbb{1}(U_k > U_{k+1}),\ k \in \mathbb{Z},$$

where $(U_k)_{k\in\mathbb{Z}}$ is a sequence of iid $U[0,1]$ variables

Det structure is obvious from the graphical construction (see the slides) and the BDF theorem stating that every one-dependent point process on $\mathbb{Z}$ is determinantal.

Useful duality functions are of product moment type

Algebraic remarks

(i) Consider [ARW] on $\mathbb{Z}_N$ with

$$L_N = \sum_{k=1}^{N-1} (\sigma_k - I)$$

Then $A_N = \langle \sigma_i,\ 1 \le i \le N-1 \rangle$ has irreps of dimensions

$$1^{(2)},\ (N-1)^{(2)},\ \frac{(N-1)(N-1)}{2}^{(2)}, \dots, \binom{N-1}{N-1}^{(2)}$$

These irreps can be constructed as

Span {Duality functions with k jumps}

/ Span {Duality functions with (k-1) jumps},

$0 \le k \le N-1$

(ii) Interpreting duality functions as intertwiners can construct (novel?) reps of Hecke algebras:

- $\left(\hat{q}_x = \mathbb{1}_x \cdot \Delta^{(r,\ell)}\right)_{x\in\mathbb{Z}}$ generates a rep of TL_∞ algebra in $L_2(\mathbb{Z})$

$Q = r\ell$

$$(\hat{q}_i \hat{q}_{i\pm1} \hat{q}_i - Q q_i = 0,\ i \in \mathbb{Z})$$

$n > 1$ fixed

- $\left(\hat{q}_x = \sum_{k=1}^{n} \mathbb{1}_x^{(k)} \Delta_k^{(r,\ell)}\right)_{x\in\mathbb{Z}}$ generates a rep. of H_∞ algebra with par Q in $L_2^{(0)}(W^n)$. Here $\mathbb{1}_x^{(k)}(y) = \mathbb{1}_x(y_k)$, $y \in W^n$

W^n: Weil Chamber; $L_2^{(0)}$: Vanishing at the boundary

(iii) Baxterization:

Thm (V. Jones, 1990)

Consider the set $(s_i)_{i=1}^{M}$ of canonical generators of Hecke algebra. Then

$$R_n(x) := s_n - x\, s_n^{-1}, \quad 1 \leq n \leq M, \; x \in \mathbb{C}$$

"Homog" YBE

solve the YB eq-n

$$R_n(x)\, R_{n-1}(xy)\, R_n(y) = R_{n-1}(y)\, R_n(xy)\, R_{n-1}(x)$$

$$2 \leq n \leq M, \quad x, y \in \mathbb{C}.$$

Example For ASRWPI,

$$R(x) = \begin{array}{c|cccc} & 11 & 10 & 01 & 00 \\ \hline 11 & -(1-x)+(1+x)\frac{q-q^{-1}}{2} & 0 & 0 & (1-x)\left(1-\frac{q+q^{-1}}{2}\right) \\ 10 & 0 & (1+x)\frac{q-q^{-1}}{2} & -(1-x)\frac{q+q^{-1}}{2} & 0 \\ 01 & 0 & -(1-x)\frac{q+q^{-1}}{2} & (1+x)\frac{q-q^{-1}}{2} & 0 \\ 00 & -(1-x)\left(1+\frac{q+q^{-1}}{2}\right) & 0 & 0 & (1-x)+(1+x)\frac{q-q^{-1}}{2} \end{array}$$