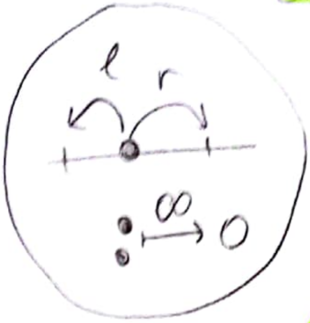
Lecture 2 Using Markov duality to solve ARW on $\mathbb{Z}$

"$A+A \to \emptyset$", a highly non-trivial model



Time rescaling $\Rightarrow$ $r+\ell=1$

$$L=\sum_{i\in\mathbb{Z}} q_i \qquad (1)$$

$q_i \in \mathrm{End}(V_i \otimes V_{i+1})$

Using the basis introduced in lecture 1 ($q_{ij,kl}$ = Rate($ij \to kl$)) we have

$$(2) \qquad q = \begin{array}{c|c} & \begin{array}{cccc} 11 & 10 & 01 & 00 \end{array} \\ \hline \begin{array}{c} 11 \\ 10 \\ 01 \\ 00 \end{array} & \begin{pmatrix} -1 & 0 & 0 & r+\ell \\ 0 & -r & r & 0 \\ 0 & \ell & -\ell & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \in \mathrm{End}(V\otimes V)$$

It will be convenient to use

$$(3) \qquad \sigma := q + I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \ell & r & 0 \\ 0 & \ell & r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(Then $L=\sum_{i\in\mathbb{Z}}(\sigma_i - I)$) $\qquad (1')$

Task: determine the law of η_t for any deterministic IC

Difficulty: $\frac{\partial}{\partial t}\mathbb{E}[\eta_t(0)]$ = a function of 1-, 2-point functions

$\frac{\partial}{\partial t}\mathbb{E}[\eta_t(0)\eta_t(1)]$ = a function of 2-, 3-point functions,...

BBGKY hierarchy

Explicitly, $\frac{\partial}{\partial t}\mathbb{E}[\eta_t(0)] =$

$$\mathbb{E}[L\,\eta_t(0)] = r\,\mathbb{E}[\eta_t(0)\oplus\eta_t(-1)] + \ell\,\mathbb{E}[\eta_t(0)\oplus\eta_t(1)] - \mathbb{E}[\eta_t(0)]$$

Notice: $\alpha\oplus\beta = \alpha+\beta-2\alpha\beta \Rightarrow$

$$\frac{\partial}{\partial t}\mathbb{E}[\eta_t(0)] = \Delta^{r,\ell}\,\mathbb{E}[\eta_t(0)] - 2\,\mathbb{E}[\eta_t(0)\eta_t(1)]$$

Assume T-invariance

Non-closure: for ARW's on $\mathbb{Z}$, $\mathbb{E}[\eta_t(0)\eta_t(1)]$ is not well approximated by $\mathbb{E}[\eta_t(0)]^2$ (This would result in $\mathbb{E}[\eta_t(0)]\sim\frac{1}{t}$ at large times, whereas the true answer is $\mathbb{E}[\eta_t(0)]\sim\frac{1}{\sqrt{t}}$ #

Idea: study representations of the generator algebra $A := \langle I, \sigma_i, i\in\mathbb{R}\rangle_{\mathbb{R}}$

Spectrum of σ:

$$\left.\begin{aligned} &\mathrm{rank}(\sigma) = 2 \\ &\sigma^2 = \sigma \quad \text{(Exercize)} \end{aligned}\right\} \quad (4.0)$$

$\Rightarrow$ Eigenvalues can only be 0 or 1 and expect two eigenvectors with e.v. 1

as $\sum_j \sigma_{ij} = 1 \quad \forall i,$

$$\sigma \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}, \quad \text{where} \quad \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = v\otimes v.$$

So, $\sigma\, v\otimes v = v\otimes v$ (4.1)

Similarly, can check that

(4.2) $\sigma\, w\otimes w = w\otimes w$, where

$$w = \begin{pmatrix}-1\\ \vdots\\ 1\end{pmatrix} \longleftrightarrow (-1)^i \quad \text{(parity function)}.$$

Consider σ in the basis $(V\otimes V,\ W\otimes W,\ V\otimes W,\ W\otimes V)$:

(4.3) $\sigma\, V\otimes W = \ell\, V\otimes V + r\, W\otimes W$

(4.4) $\sigma\, W\otimes V = \ell\, W\otimes W + r\, V\otimes V$

(Exercise)

These relations allow us to extend a representation of $\langle I, \sigma_n \rangle_{\mathbb{R}}$ on $V_n \otimes V_{n+1}$ to a representation of A on T:

Warm-up example: define

$$f_x := \left(\bigotimes_{i\le x} w_i\right) \otimes \left(\bigotimes_{j > x} v_j\right)$$

($\notin T$, but it doesn't matter for the point we're making)

To justify formal manips. with $(f_x)_{x\in\mathbb{Z}}$ need to consider ARW with the IC containing the leftmost particle

Using the above,

(5)
$$\begin{cases} \sigma_x f_x = \ell f_{x+1} + r f_{x-1} \\ \sigma_y f_x = f_x \quad \forall y \neq x \end{cases}$$

So, A acts in
$\mathrm{Span}_{\mathbb{R}}(f_x,\ x \in \mathbb{Z})$ - a representation of "dimension" $|\mathbb{Z}|$ rather than $2^{|\mathbb{Z}|}$.
Consequently, the generator also acts in this space:

$$(5') \qquad \mathcal{L} f_x = \Delta^{r,\ell} f_x$$

More generally, let $x_1 \leq x_2 \leq \ldots \leq x_{2n}$, $n \in \mathbb{N}_0$, and define

$$f^{(2n)}_{x_1, x_2 \ldots x_{2n}} = \ldots \left(\otimes_{x_1+1 \leq j_1 \leq x_2} W_{j_1}\right) \ldots \left(\otimes_{x_3+1 \leq j_2 \leq x_4} W_{j_2}\right) \ldots$$

$$(6) \qquad \ldots \left(\otimes_{x_{2n-1}+1 \leq j_n \leq x_{2n}} W_{j_n}\right)$$

$\ldots$ - factors of $\otimes\sigma$

Then $\sigma_x f^{(2n)}_{x_1 \ldots x_{2n}} \in \underbrace{\mathrm{Span}\{f_{y_1 \ldots y_{2n}},\ y_1 \leq y_2 \leq \ldots \leq y_{2n}\}}_{\text{Contains } f^{(2m)} \text{ for } \forall m \leq n.}$

Example $\sigma_3 f^{(2)}_{34} = \sigma_3 (\ldots \otimes V_3 \otimes W_4 \otimes V_5 \otimes \ldots)$

$$= \ell \underbrace{\ldots \otimes V_3 \otimes V_4 \otimes}_{f^{(0)}} + r \underbrace{\ldots \otimes W_3 \otimes W_4 \otimes V_5 \otimes \ldots}_{f^{(2)}_{24}}$$

Using explicit formulae for σ-action,

$$(7) \begin{cases} \mathcal{L} f^{(2n)}_{x_1 \ldots x_{2n}} = \sum_{k=1}^{2n} \underbrace{\Delta^{r,\ell}_k}_{\text{Acts on the } k\text{-th label}} f^{(2n)}_{x_1, x_2 \ldots x_{2n}}, \quad x_1 < x_2 < \ldots < x_{2n} & (i) \\ f^{(2n)}_{x_1 \ldots x_i = x_{i+1} \ldots x_{2n}} = f^{(2n-2)}_{x_1 \ldots x_{i-1} x_{i+2} \ldots x_{2n}} & (ii) \end{cases}$$

Let us interprete the above construction:

Algebra: For each $n \in \mathbb{N}$, we've constructed a lower triangular rep. of $\mathbb{A}$ in T: if $T_{2n} := \mathrm{Span}_{\mathbb{R}}\left(f^{(2n)}_{x_1 \ldots x_{2n}} : x_1 < x_2 < \ldots < x_{2n}\right)$, then $\mathbb{A}\, T_{2n} \subset T_{2n} \oplus T_{2n-2} \oplus \ldots$

In fact one can construct irreps of $\mathbb{A}$ of "dimensions" $\binom{|\mathbb{Z}|}{2n}$, see Lecture 3

Probability: Let us interprete the r.h.s. of (7). (i): we've $2n$ particles performing indep RW's (r, l) until they meet ($x_1 < x_2 < \ldots < x_{2n}$ are coordinates of the particles).

(ii): if particles $i, i+1$ meet, they immediately annihilate ($: \xrightarrow{\infty} \emptyset$)

So (7) tells us that there is an infinite sequence $f^{(2n)}$ of functions on $\Omega \times \overline{W_{2n}}$ establishing Markov duality between ARW on $\mathbb{Z}$ and ARW on $\mathbb{Z}$ with at most $2n$ particles, $n \in \mathbb{N}$.

⑥

As $W \leftrightarrow (-1)^{\eta}, \ldots$

$$f^{(2n)}_{x_1 \ldots x_{2n}} \longleftrightarrow \prod_{k=1}^{n} (-1)^{\sum_{j_k = x_{2k-1}+1}^{x_{2k}} \eta(j_k)}, \quad (8)$$

these are well-known duality functions measuring the parity of the # of annihilating particles in a subset of $\mathbb{Z}$.

As (9) $\underbrace{\eta(x)}_{\text{The indicator of particle at } x.} = \frac{1-(-1)^{\eta}}{2}$, the knowledge of expectations of all duality functions determines the correlation functions of the point process corresponding to the distribution of $\eta_t \Rightarrow$ The law of η_t is determined by the expectations of all duality functions, which therefore form a complete set.

Let $\phi^{(2n)}_t(x_1 \ldots x_{2n}) = \mathbb{E}_{\underset{\text{Def. I.C}}{0}}\left[f^{(2n)}_{x_1 \ldots x_{2n}}(\eta_t) \right]$ (10)

Then (7) $\Rightarrow$

$$(11) \begin{cases} \left(\partial_t - \sum_{k=1}^{2n} \Delta^{re}_k \right) \phi^{(2n)}_t (x_1 \ldots x_{2n}) = 0 & x_1 < x_2 < \ldots < x_{2n} \\ \phi^{(2n)}_t (\ldots x_i = x_{i+1} \ldots) = \phi^{(2n-2)}_t (\ldots \hat{x}_i \hat{x}_{i+1} \ldots) & t > 0 \\ \phi^{(2n)}_0 = \text{Deterministic I.C.} \end{cases}$$

Claim. (9) has a unique solution

$$(12)\quad \phi_t^{(2n)}(x_1 \dots x_{2n}) = Pf\left[\phi_t^{(2)}(x_i, x_j),\ 1 \le i < j \le 2n\right],$$

("Fermionic Wick's theorem") $\quad n \in \mathbb{N},\ x_1 \le \dots \le x_{2n}$

Using (9) can get all CF's of ARW at a fixed time and discover that the law of η_t is a Pfaffian point process

Consequences: assume that $\eta_0(x) = 1 \ \forall x$. Then

(i) $\rho_t^{(n)}(x_1 \dots x_{2n}) \underset{t \uparrow \infty}{\sim} |\Delta(x)|\, t^{-\frac{n(n-1)}{4} - \frac{n}{2}}$

(ii) Gap probabilities for η_t are given by Fredholm Pfaffians

$\swarrow$

The distribution of the rightmost particle for a step-like initial distribution (Exponential tails.)

$\searrow$

Persistence exponents (related to the prob. that a particle hasn't crossed 0 before time t) $(P_t \sim t^{-1/4})$

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We refer to duality functions (6) as alternating intervals dualities.

Our construction worked because of the following two facts: (i) Existence of a second factorized EV $W \otimes W$ lin. indep. of $V \otimes V$; (ii) σ action on $V \otimes W$ ($W \otimes V$) doesn't generate terms proportional to $W \otimes V$ ($V \otimes W$) (Order preservation.) Can we explain this luck? Partially.

Firstly, notice that many IPS admit dualities of product moment type,

$$\ldots \otimes W_{x_1} \otimes \ldots \otimes W_{x_2} \otimes \ldots\ldots \otimes W_{x_n}$$

$\uparrow$ Insertions of V

$$\updownarrow \qquad\qquad\qquad\qquad (13)$$

$$\prod_{1 \le i \le n} \eta(x_i)$$

(SEP, voter model, ...)

Secondly, an explicit calculation shows that

(14) $\sigma_n \sigma_{n+1} \sigma_n - q\sigma_n = \sigma_{n+1} \sigma_n \sigma_{n+1} - q\sigma_{n+1}$

(15) Noticing that $[\sigma_n, \sigma_m] = 0$, $|n-m| \ge 2$

(16) and $\sigma_n^2 = \sigma_n$, we conclude that

$\Rightarrow \mathbb{A}$ is a quotient of Hecke algebra $\mathbb{H}_\infty$. We have the following

Proposition. Let $\{\sigma_n\}_{n\in\mathbb{Z}}$; $\sigma_n \in \mathrm{End}(V_n \otimes V_{n+1})$, satisfy (α), (β). S'pose $\dim \ker(\sigma - I) = 2$ and there are two lin. indep. eigenvectors of the form $V\otimes V$ and $W\otimes W$. Then the action of σ on $V\otimes W$ and $W\otimes V$ must take one of the following forms:

WOV doesn't apper

$$(*)\begin{cases} \sigma\, V\otimes W = \alpha\, V\otimes V + \beta\, W\otimes W + \gamma\, V\otimes W \\ \sigma\, W\otimes V = \tilde{\alpha}\, W\otimes W + \tilde{\beta}\, V\otimes V + \tilde{\gamma}\, W\otimes V \end{cases}$$

OR

$$(**)\begin{cases} \sigma\, V\otimes W = \varepsilon\, V\otimes W + \delta\, W\otimes V \\ \sigma\, W\otimes V = \tilde{\varepsilon}\, W\otimes V + \tilde{\delta}\, V\otimes W, \end{cases}$$

where $\alpha, \beta, \gamma, \varepsilon, \delta$
$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\varepsilon}, \tilde{\delta}$ $\in \mathbb{R}$.

$(*) \Rightarrow$ Duality functions of alternating interval type

$(**) \Rightarrow$ Duality functions of product moment type.