How to Solve the Stochastic Six Vertex Model

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At 00:47, on Tuesday 11$^{\text{th}}$ July, 2023

Introduction

These are lecture notes for PANEM-2023 at Texas A&M on the integrability and asymptotics of the stochastic six vertex model.

1 Six vertex model through different lenses

In the first lecture, we describe the stochastic six vertex model from two diverse perspectives — as a model of statistical mechanics, and as a stochastic particle system.

1.1 Gibbs measures and the six vertex model

1.1.1 Finite-volume Gibbs measures

We begin with describing the useful framework of *Gibbs measures*. For simplicity, we work on the two-dimensional lattice $\mathbb{Z}^2$. Let $\Lambda \subset \mathbb{Z}^2$ be a finite subset (for example, a rectangle). We are interested in *spin configurations* inside Λ which are encoded as $\omega = \{\sigma_e \colon e \text{ is an edge in } \Lambda\}$, where $\sigma_e \in \{0, 1\}$. By an "edge in Λ" we mean that both endpoints of this edge must be inside Λ. Each spin configuration is equipped with boundary conditions, which are fixed spins on all the boundary edges of Λ (an edge is called boundary if it connects Λ to $\mathbb{Z}^2 \setminus \Lambda$).

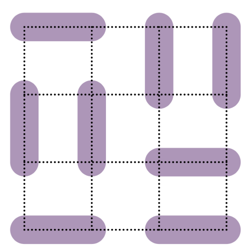
With each spin configuration ω, we associate its energy $H(\omega) \in \mathbb{R}$. This energy may depend on global parameters (e.g., inverse temperature) and local parameters (e.g., edge capacities or vertex rapidities). If a particular spin configuration ω is forbidden, we have $H(\omega) = +\infty$.

Definition 1.1. A (finite-volume) *Gibbs measure* in Λ with fixed boundary conditions and the energy function $H(\cdot)$ is the probability distribution on spin configurations whose probability weights have the form

$$\mathrm{Prob}(\omega) = \frac{1}{Z} \exp\left\{-H(\omega)\right\}.$$

Here Z is the *partition function*, which is simply the probability normalizing constant.

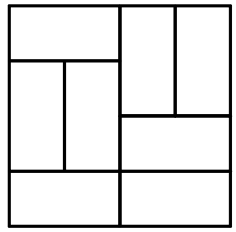
Example 1.2 (Domino tilings on the square grid)**.** A *perfect matching* on Λ is any subset M of its edges such that every vertex is covered by exactly one edge from M. For example, here is a perfect matching on the four by four rectangle:



If the set of allowed spin configurations is the set of perfect matchings, and

$$H(\omega) = \begin{cases} 0, & \omega \text{ is a perfect matching;} \\ +\infty, & \omega \text{ is not a perfect matching,} \end{cases}$$

then the corresponding Gibbs measure is the uniform distribution on the space of *domino tilings*. That is, we identify each covered edge with a 2×1 domino. The domino tiling corresponding to the above perfect matching is



Computing partition functions of various Gibbs measures may be challenging. For example, the number of domino tilings of the 8×8 chessboard is 12,988,816, but its theoretical computation (not via a computer program) requires several nontrivial steps [Kas61], [TF61].

Parameter-dependent partition functions represent many important quantities across all of mathematics, including various families of symmetric functions (such as Schur or Hall-Littlewood functions), and related objects.

1.1.2 Infinite-volume Gibbs measures

Besides Gibbs measures on configurations on a finite space as in Definition 1.1 with fixed boundary conditions ("*boxed distributions*"), we are interested in *infinite-volume* Gibbs measures.

Definition 1.3. A probability measure on spin configurations on an infinite subset $\Lambda_\infty \subseteq \mathbb{Z}^2$ (we will mainly consider the whole plane and the quarter plane $\mathbb{Z}^2_{\geq 0}$) is called (infinite-volume) Gibbs if for any finite $\Lambda \subset \Lambda_\infty$, the configuration inside Λ conditioned on the configuration in $\Lambda_\infty \setminus \Lambda$ is a finite-volume Gibbs measure in the sense of Definition 1.1 (with boundary conditions imposed by the outside configuration in $\Lambda_\infty \setminus \Lambda$).

Out of all possible infinite-volume Gibbs measures, we are interested in measures with special properties, such as translation invariant and/or ergodic. A Gibbs measure on $\mathbb{Z}^2$ is called *translation invariant* if its distribution does not change under arbitrary space translations. A Gibbs measure is called *ergodic* (equivalently, *extremal*) if it cannot be represented as a convex combination of two other such measures. Gibbs measures which are translation invariant and ergodic (within the class of translation invariant measures) are called *pure states.*

Classifying pure states for a given energy function $H(\cdot)$ is a very nontrivial problem, and an explicit answer is rarely available. For the general six vertex model (defined in Section 1.1.3 below), the answer is only conjectural [Res10].

On the other hand, pure states of the six vertex model under a special *free fermion condition* (which includes the domino model from Example 1.2) admit a very explicit description through determinantal point processes (i.e., all correlation functions of these measures are diagonal minors of an explicit function in two variables), which follows from [She05], [KOS06]. One of the goals of these lecture notes is to discuss the tools and results one would require to extend this classification beyond the free fermion case.

Remark 1.4. Certain families of non translation invariant infinite-volume Gibbs measures (under the free fermion condition) power the classification of irreducible representations of infinite-dimensional unitary group and other classical groups [Voi76], [VK82], [BO12], [Pet14]. This subject is closely related to symmetric functions arising as partition functions of Gibbs measures with varying parameters (rapidities) along one of the lattice coordinate direction which we discuss in the second lecture (Section 2). There is also a direct link between these Gibbs measures and totally nonnegative triangular or full Toeplitz matrices for characters of the infinite symmetric group or, respectively, the infinite-dimensional unitary group, see [AESW51], [Edr53], [Boy83].

1.1.3 Six vertex model

1.2 Stochastic six vertex model and its particle system limits

1.3 Gibbs properties of the stochastic six vertex model

1.4 Basic coupling and colored (multispecies) models

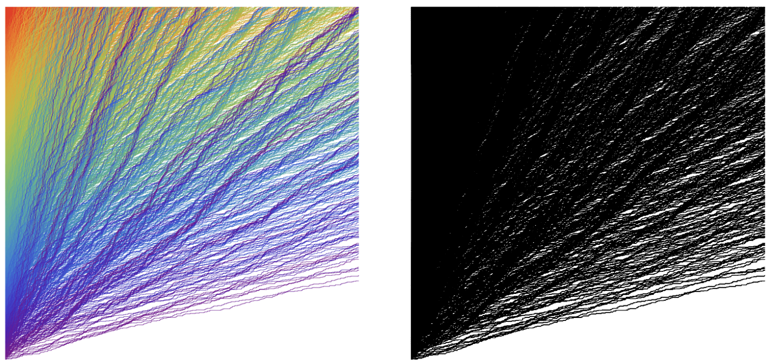


Figure 1: Colored stochastic six vertex model and its monochrome version.

1.5 Stationary distributions and hydrodynamics

Bernoulli is Stationary; also for all the limits we had.

1.6 Limit shape and fluctuation problem

2 Integrability

3 Asymptotics

References

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