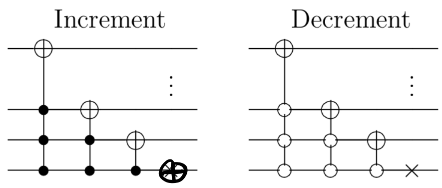
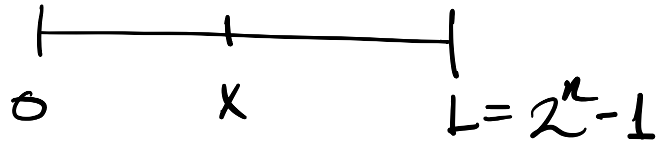
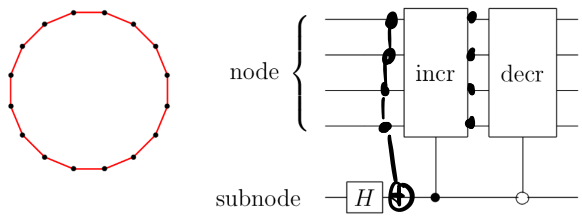


FIG. 1: Increment and decrement gates on n qubits, producing cyclic permutations in the 2^n bit-string states.



$$X = q_0 2^0 + q_1 2^1 + \cdots + q_{n-1} 2^{n-1}$$

$$|X\rangle \longleftrightarrow |q_0 q_1 \cdots q_{n-1}\rangle$$



$$(0,1,0) = 2 \to 3 = (1,1,0)$$

$$(1,0,1) = 5 \to 4 = (0,1,0)$$

$$X \longmapsto X \pm 1$$

$$q \longmapsto q^{\pm} = (q_0^{\pm}, \dots, q_{n-1}^{\pm})$$

w/ $$q_i^{+} = \begin{cases} q_i \oplus 1 & \text{if } q_0 = \cdots = q_{i-1} = 1 \\ q_i & \text{otherwise} \end{cases}$$

Notation

$\oplus$ is mod 2 addition

$$q_i^{-} = \begin{cases} q_i \oplus 1 & \text{if } q_0 = \cdots = q_{i-1} = 0 \\ q_i & \text{otherwise} \end{cases}$$

Rem

addition by 1 and subtraction by 1 is the same mod 2.

Eg $(n=4)$

$$X = 11 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3$$

$$0 \cdot q_0 + \cdots + 0 \cdot q_{n-2} + 0 q_{n-1}$$

$$0$$

$$\to q = (1,1,0,1)$$

$$\to q^{+} = (0,0,1,1) \quad \& \quad q^{-} = (0,1,0,1)$$

Eg: $(n=4)$

$X = 8 = 0\cdot 2^0 + 0\cdot 2^1 + 0\cdot 2^2 + 1\cdot 2^3$

$\to q = (0,0,0,1)$

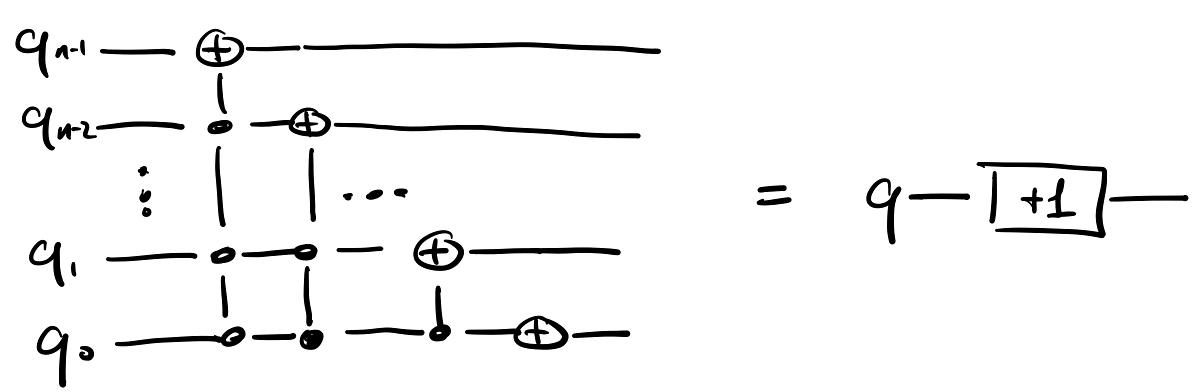
$\to q^+ = (1,0,0,1)$ & $q^- = (1,1,1,0)$

Q: How do we map $(x,y) \mapsto q = (q_0, \ldots, q_{n+m-1})$?

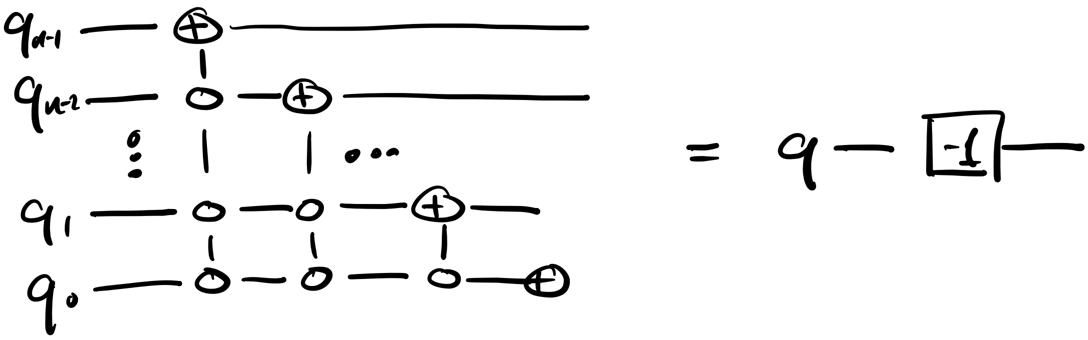
- $0 \le x \le 2^n - 1$ & $0 \le y \le 2^m - 1$
- $N := 2^n$ & $M = 2^m$
- $(x,y) = (2^0 \cdot q_0 + \cdots + 2^{n-1} q_{n-1}, \; 2 \cdot p_0 + 2^1 p_1 + \cdots + 2^{m-1} p_{m-1})$

$\mapsto (q_0, \ldots, q_{n-1}, p_0, \ldots, p_{m-1})$

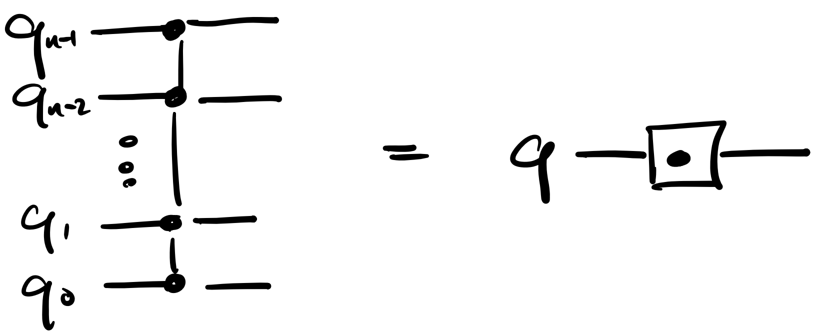
- Increment:



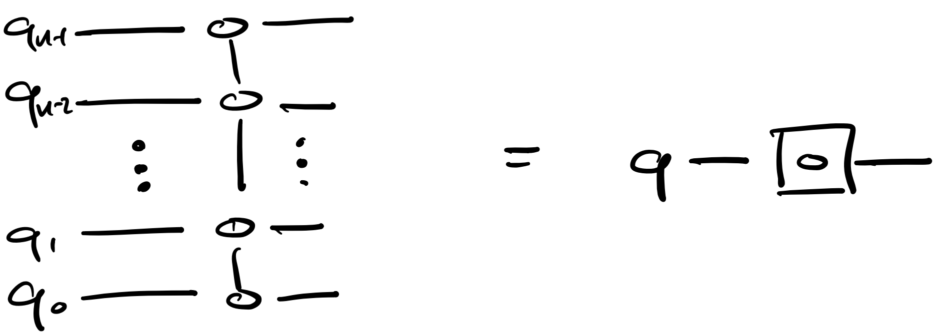
- Decrement:



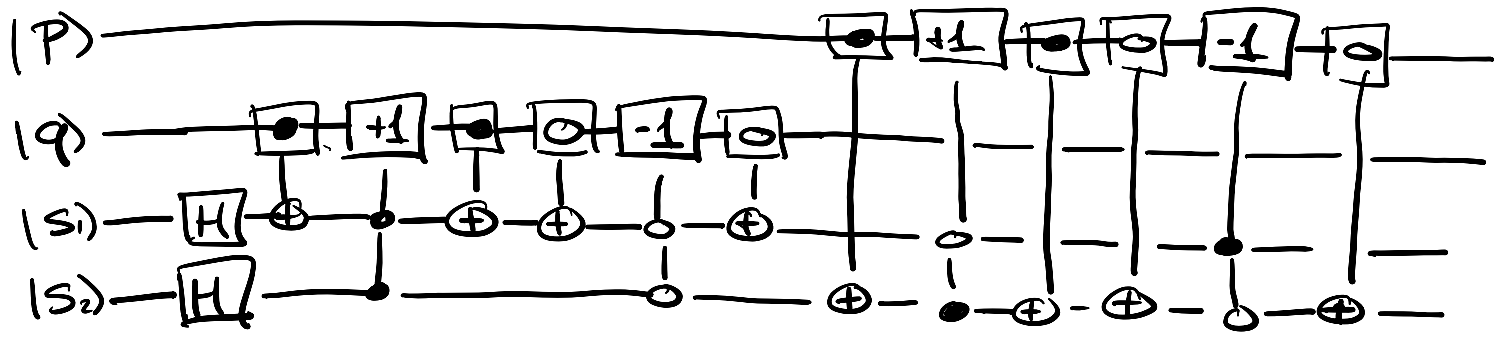
- Control on the last number:



- Control on the first element:

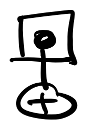
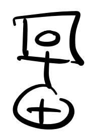


- Quantum Circuit for a 2-dimensional (discrete and homogeneous) quantum random walk on a $2^n \times 2^m$ grid



$|q\rangle = |q_0, \dots, q_{n-1}\rangle$ & $|p\rangle = |p_0, \dots, p_{m-1}\rangle$

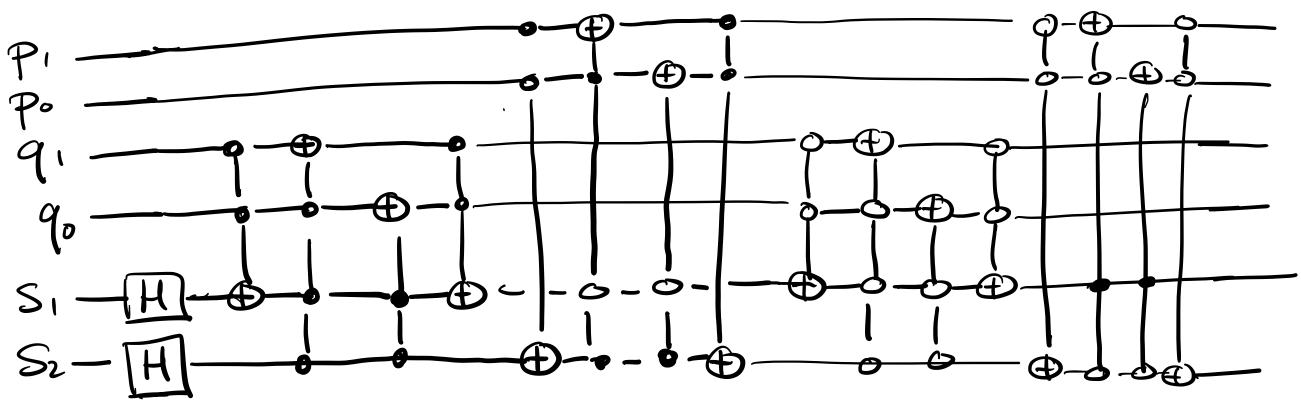
$|S_i\rangle = |S_i\rangle$ ← submodule qbits, one qbit each

Rem: The controlled gates  &  are added to avoid the increments & decrements along the edges, $N \leftrightarrow 0$ & $M \leftrightarrow 0$, respectively.

Eg ($N = M = 4$)

- $(X, Y) \mapsto (q_0 2^0 + q_1 2^1 + \dots + q_{n-1} 2^{n-1}, p_0 2^0 + \dots + p_{m-1} 2^{m-1})$

  $\mapsto |q_0, \dots, q_{n-1}, p_0, \dots, p_{m-1}\rangle$



∘ $|x,y,\overset{s_1}{0},\overset{s_2}{0}\rangle \overset{H^{\otimes 2}}{\longmapsto} \frac{1}{2}\left(|x,y,0,0\rangle + |x,y,1,0\rangle + |x,y,0,1\rangle + |x,y,1,1\rangle\right)$

(in the bulk) $\longmapsto \frac{1}{2}\left(|x-1,y,0,0\rangle + |x,y-1,1,0\rangle + |x,y+1,0,1\rangle + |x+1,y,1,1\rangle\right)$