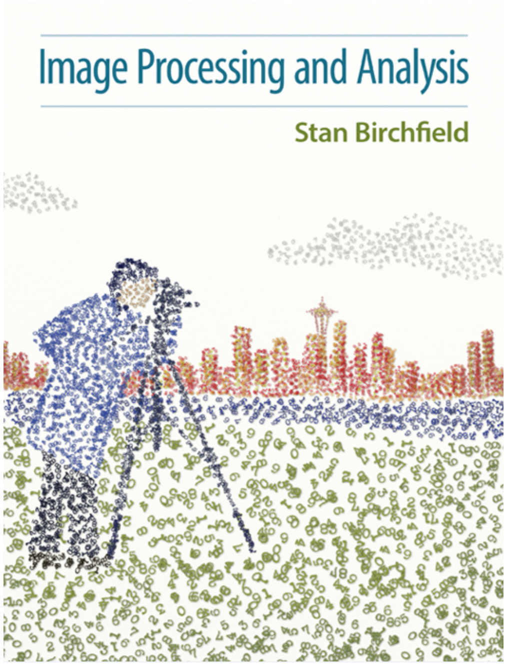
University of Stavanger

Prof. Kjersti Engan

ELE510 Image processing and computer vision

Camera and calibration, (chap 13.4, 13.5 Birchfield) 2020



Parts in course presentations is material from Cengage learning. It can be used for teaching, and it can be share it with students on access controlled web-sites (Canvas) for use in THIS course. **But it should not be copied or distributed further in any way** (by you or anybody).

(13.4) Projective Transformation

- 3D objects in a scene is *deformed by projective transformation*. Circles appears as ellipses and squares as trapezoids.

Three point perspective.

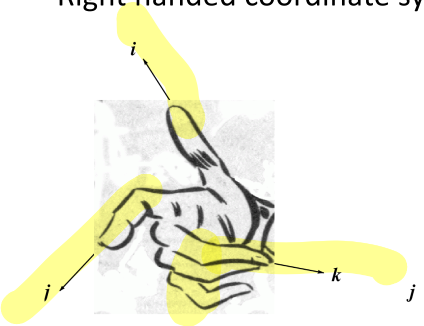
- This leads to the use of:

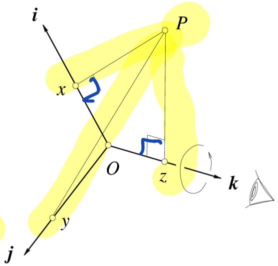
Projective space and **Homogenous coordinates**

Parallel lines,
do they meet in a point?

Euclidean Geometry

Right handed coordinate system:





$$x = \overrightarrow{OP} \cdot \mathbf{i}$$

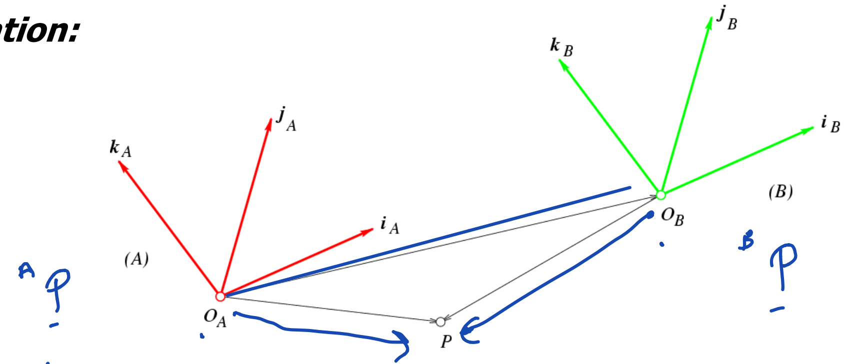
$$y = \overrightarrow{OP} \cdot \mathbf{j} \quad \Leftrightarrow \quad \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$z = \overrightarrow{OP} \cdot \mathbf{k}$$

$$\mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Coordinate changes

Translation:

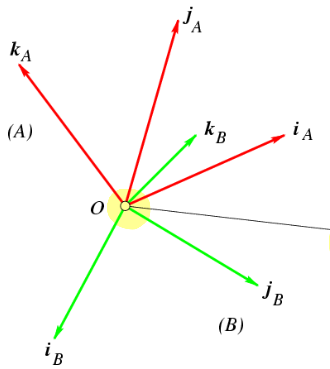


$$\overrightarrow{O_BP} = \overrightarrow{O_BO_A} + \overrightarrow{O_AP}$$

$$ {}^B\mathbf{P} = {}^A\mathbf{P} + {}^B\mathbf{O}_A = \begin{bmatrix} {}^Ax \\ {}^Ay \\ {}^Az \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} $$

Coordinate changes

Rotation:

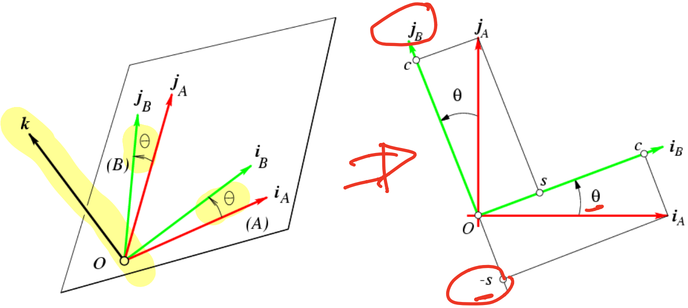


$$
{}^{B}_{A}\mathbf{R} = \begin{pmatrix} \mathbf{i}_A \cdot \mathbf{i}_B & \mathbf{j}_A \cdot \mathbf{i}_B & \mathbf{k}_A \cdot \mathbf{i}_B \\ \mathbf{i}_A \cdot \mathbf{j}_B & \mathbf{j}_A \cdot \mathbf{j}_B & \mathbf{k}_A \cdot \mathbf{j}_B \\ \mathbf{i}_A \cdot \mathbf{k}_B & \mathbf{j}_A \cdot \mathbf{k}_B & \mathbf{k}_A \cdot \mathbf{k}_B \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}
$$

$$
{}^{B}\mathbf{P} = {}^{B}_{A}\mathbf{R}\, {}^{A}\mathbf{P}
$$

Coordinate changes

Rotation about the z-axis:



$$\mathbf{R}_A^B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

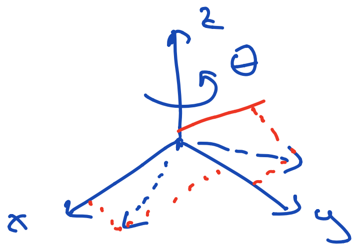
$$^{B}\mathbf{P} = {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{P}$$

Given a Rotation Matrix **R**, then: $\mathbf{R}^{-1} = \mathbf{R}^{T}$ and $\det\{\mathbf{R}\} = 1$

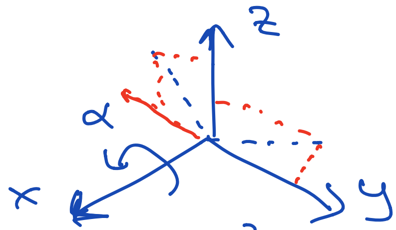
Or equivalently: Its rows (or columns) form a right-handed orthonormal coordinate system.

Fundamental rotation matrices

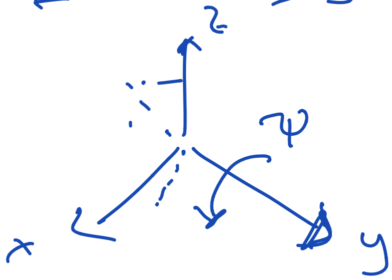
$$\mathbf{R}_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\mathbf{R}_{x,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \underset{\sin}{\cancel{\cos}\,\alpha} & \underset{\cos}{\cancel{\sin}\,\alpha} \end{pmatrix}$$



$$\mathbf{R}_{y,\psi} = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}$$



Rotation matrices

- All rows an columns are orthogonal and have unit norm
- All rotation matrices can be described as a combination (multiplication) of the three fundamental rotation matrices.
- This means there is really only 3 parameters in an arbitary rotation matrix R

Homogeneous Coordinates

- A point is represented in homogeneous coordinates by appending a 1 to the end.
- Point in 2D: $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- Point in 3D: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$
- Once in homogeneous coordinates, scaling does not matter (as long as the scaling is nonzero).
- **Projective transformation**: a matrix multiplied by the point to yield new homogeneous coordinates.

Warping transformations- Revisited

- Translation and rotation can be combined into a single **Euclidean transformation**:

$$\mathbf{x}' = \mathbf{R}(\mathbf{x} - \mathbf{c}) + \mathbf{c} + \mathbf{t} = \mathbf{R}\mathbf{x} + \tilde{\mathbf{t}}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{t}_x \\ \tilde{t}_y \end{bmatrix}$$

$$\tilde{\mathbf{t}} \equiv [\tilde{t}_x \quad \tilde{t}_y]^{\mathsf{T}} = -\mathbf{R}\mathbf{c} + \mathbf{c} + \mathbf{t}$$

Euclidean transformations preserves shape and scale of an object

- **Similarity transformations**: a superset of Euclidean transformations including translations, rotations, AND uniform scaling:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} k\cos\theta & -k\sin\theta & k\tilde{t}_x \\ k\sin\theta & k\cos\theta & k\tilde{t}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Uniform scaling: $x'=kx$, $y'=ky$

Similarity transformations preserves shape of an object

Transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

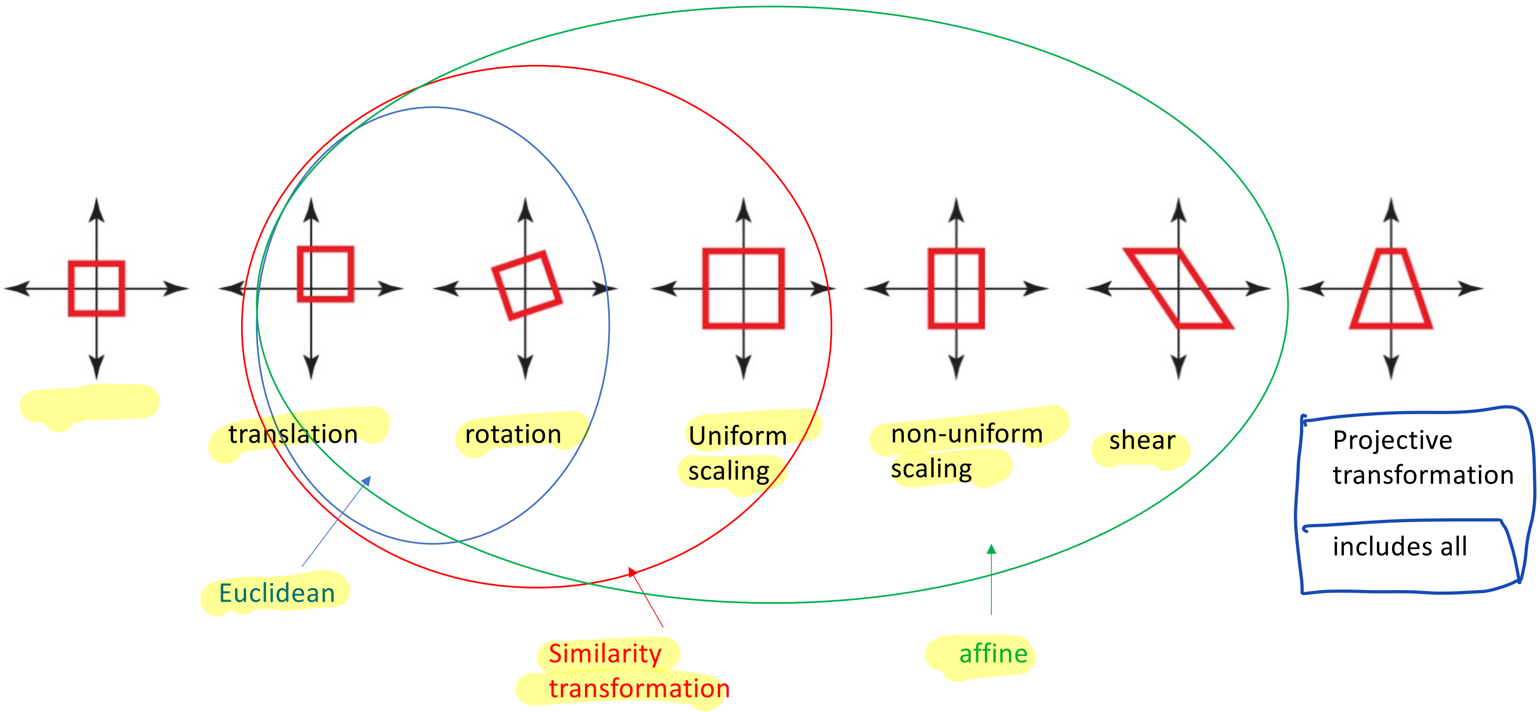
- **Affine transformations**. Any 2x2 invertable matrix. In homogeneous coordinates:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} & a_{23}a_{12} - a_{22}a_{13} \\ -a_{21} & a_{11} & -a_{23}a_{11} + a_{21}a_{13} \\ 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

- Include: rotation, translation
  - Uniform scaling
  - Non-uniform scaling $x' = ax$ $y' = by$
  - Shear $x' = x + ay$, $y' = y$

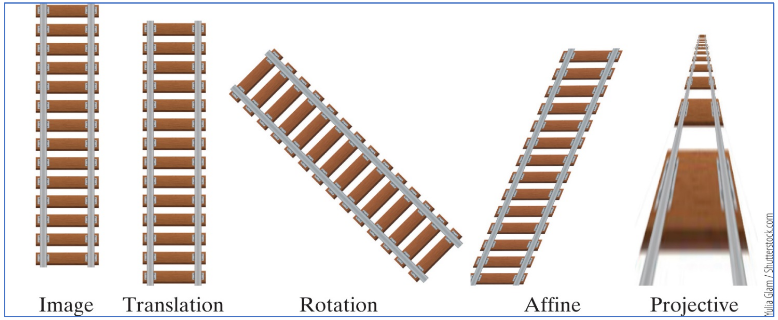
Parallel lines in input -> parallel lines in output


translation
rotation
Uniform scaling
non-uniform scaling
shear
Projective transformation
includes all
Euclidean
Similarity transformation
affine

Transformations

- **Projective Transformations**. Relax the constraint of the bottom row of the transformation matrix (3x3 invertable matrix, homography)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \propto \underbrace{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Includes all affine transformations

+ parallel lines in input -> intersecting lines in output

0 0 1

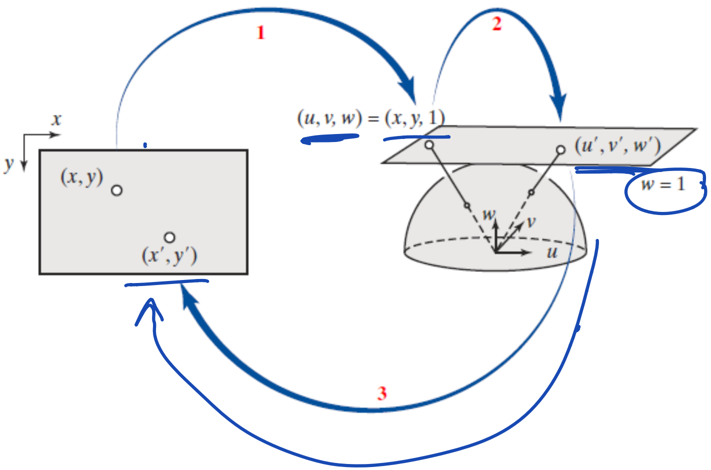
Projective transformation

- **H** is the 3 X 3 projective transformation matrix known as a **homography.**

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \mathbf{H}_{\{3\times 3\}} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

- Both (*u, v, w*) and (*u', v', w'*) are points in the **projective plane**.

Figure 13.20 Using projective geometry involves 3 steps: 1) augmenting the coordinates of a point by appending a 1 to the end; 2) applying a projective transformation, by multiplying a matrix by the augmented point; and 3) dividing by the final coordinate to yield the transformed point.



Example, projective transformation

$$\mathbf{p_2} = \mathbf{H}\mathbf{p_1}$$

Hierarchy of Transformations

- A general invertible matrix **H** is a **projective transformation**.
- If the bottom row of **H** consists of all zeros followed by a single 1, then it is also an **affine transformation**, which is a special case of projective.
- If the top-left corner consists of a rotation matrix with an overall scaling, then it is known as a **similarity transformation**, which is a special case of affine.
- Finally, if this scaling is 1, then it is a **Euclidean transformation**, which is a special case of similarity.

- These transformations, are summarized as follows for the case of 2D:

R $\qquad$ t

$$\underbrace{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}}_{\text{projective}} \quad \underbrace{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{\text{affine}} \quad \underbrace{\begin{bmatrix} kc_\theta & -ks_\theta & t_x \\ ks_\theta & kc_\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{similarity}} \quad \underbrace{\begin{bmatrix} c_\theta & -s_\theta & t_x \\ s_\theta & c_\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Euclidean}}$$

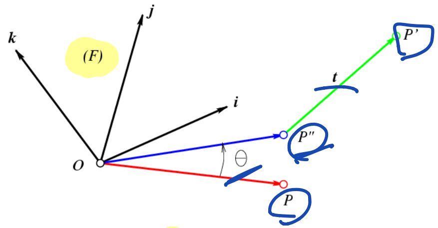
Homogeneous representation of Rigid Transformations

Rotation and translation:

$$\begin{bmatrix} {}^B\mathbf{P} \\ 1 \end{bmatrix} = \begin{pmatrix} {}^B_A\mathbf{R} & {}^B\mathbf{O}_A \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{bmatrix} {}^A\mathbf{P} \\ 1 \end{bmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} {}^A x \\ {}^A y \\ {}^A z \\ 1 \end{bmatrix}$$

${}^A P$

Rigid Transformations as mappings:



$${}^F P' = \mathcal{R}\, {}^F P + \mathbf{t} \quad \Leftrightarrow \quad \begin{pmatrix} {}^F P' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^F P \\ 1 \end{pmatrix}$$

Fundamental rotation matrices

$$\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \cos\theta & \sin\theta \end{pmatrix}$$

$$\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$