University of Stavanger

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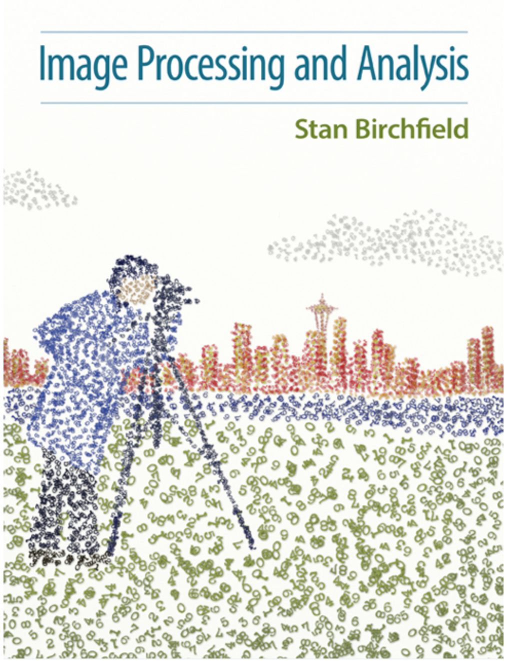
ELE510 Image processing and computer vision

Frequency Domain processing, Fourier transform (Chap 6.1-6.2 Birchfield) 2020



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(6.2) Discrete Fourier Transform

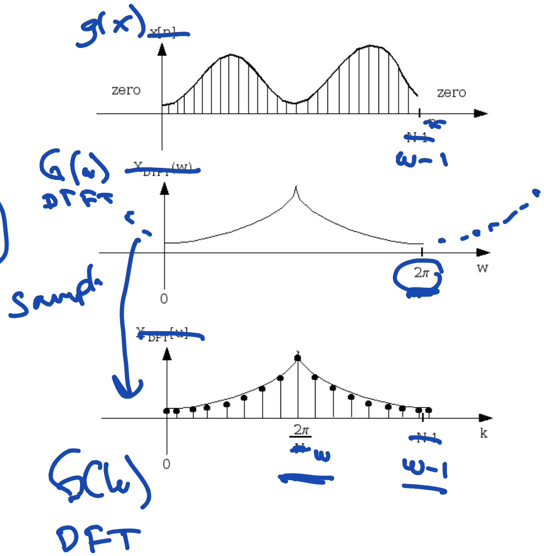
- Let $g(x)$ be a 1D discrete signal with w samples. The DFT and inverse DFT of g is defined as follows, x and k are integers :

$$G(k) = \mathcal{F}\{g(x)\} = \sum_{x=0}^{w-1} g(x) e^{-j2\pi kx/w}$$

DFT

$$g(x) = \mathcal{F}^{-1}\{G(k)\} = \frac{1}{w}\sum_{k=0}^{w-1} G(k) e^{j2\pi kx/w}$$

inverse DFT
IDFT



- All modern implementations of the DFT use some variation of the FFT algorithm

FFT – Fast Fourier Transform.

Display of DFT values

2D values of $G(p,q)$ span larger range than $g(x,y)$

for display purposes

$$\underline{d}(p,q) = \log_{10}\left(\underline{1 + |G(p,q)|}\right)$$

$$G(p,q) = 0 \Rightarrow d(p,q) = 0$$

now scale $d(p,q) \in [0, 255]$

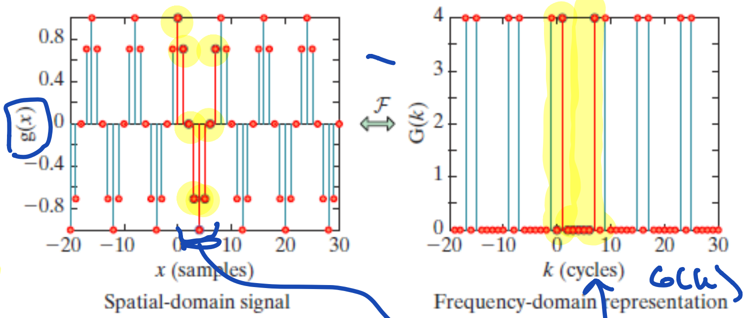
Some properties of the DFT

- The DFT of a real-valued, even-symmetric signal is also real-valued and even-symmetric.

- The DFT is **linear**. $\mathcal{F}\{ag(x) + bh(x)\} = a\mathcal{F}\{g(x)\} + b\mathcal{F}\{h(x)\}$

- The DFT is **periodic**. $g(x + nw) = g(x) \overset{DFT}{\Longleftrightarrow} G(k) = G(k + nw), \quad x, k, n, w \in \mathbb{Z}$

Figure 6.3 Periodicity of the DFT. The discrete signal consisting of eight samples $x = 0, \ldots, 7$ (red, left) gives rise to the DFT consisting of eight samples $k = 0, \ldots, 7$ (red, right). If the DFT is evaluated for other values of k, or if the inverse DFT of the DFT is evaluated for other values of x, the signal repeats with period $w = 8$.



G(k) . . . 0-7

g(x) 8 samples

- **Shift theorem**: computing the DFT of a shifted signal is the same as multiplying the DFT of the original, unshifted signal by an appropriate complex exponential.

$$g(x) \overset{DFT}{\Longleftrightarrow} G(k)$$
$$g(x - x_0) \overset{DFT}{\Longleftrightarrow} G(k)e^{-j2\pi k x_0/w}$$

$$g(x)e^{j2\pi k_0 x/w} \overset{DFT}{\Longleftrightarrow} G(k - k_0)$$
$$g(x)(-1)^x \overset{DFT}{\Longleftrightarrow} G\left(k - \frac{w}{2}\right)$$

- **Modulation**: states that multiplying a signal by a complex exponential causes a shift in the frequency domain:

- The **scaling property** says that if the signal is stretched in the spatial domain, then the Fourier transform is compressed in the frequency domain, and vice versa:

$$g(x) \overset{\mathcal{F}}{\Longleftrightarrow} G(k)$$
$$g(ax) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{a}G\left(\frac{k}{a}\right)$$

- **Parseval's theorem**: the energy is preserved in the frequency domain, where the energy is defined as the sum of the squares of the magnitudes of the elements:

$$\sum_{x=0}^{w-1} |g(x)|^2 = \sum_{k=0}^{w-1} |G(k)|^2$$

More DFT properties

- The **DC component** of the signal is captured by *G*(0), which is the sum of the values in *g(x)*.

$$G(k) = \sum_{x=0}^{w-1} g(x)\, e^{-j2\pi k \cdot x/w} \qquad k=0 \rightarrow G(0) = \sum g(x)$$

- Circular convolution in the time (or spatial) domain is equivalent to multiplication in the frequency domain, and vice versa. If standard convolution is desired, the signals must be zero padded:

$$g_1(x) \circledast g_2(x) \quad \overset{DFT}{\Longleftrightarrow} \quad G_1(k)G_2(k)$$

$$g_1(x)g_2(x) \quad \overset{DFT}{\Longleftrightarrow} \quad \frac{1}{w} G_1(k) \circledast G_2(k)$$

- It is often convenient to convert the real and imaginary components of the Fourier transform into **polar coordinates**:

$$G(k) = G_{even}(k) + jG_{odd}(k) = |G(k)|e^{j\angle G(k)}$$

$$|G(k)| = \sqrt{G^2_{even}(k) + G^2_{odd}(k)}$$

$$\angle G(k) = \tan^{-1}\left(\frac{G_{odd}(k)}{G_{even}(k)}\right)$$