Review of counting and probability

The core of modern statistical data analysis and machine learning is probability theory. This notebook contains a couple problems that we hope will refresh your memory of some basic, but important, concepts.

Exercise 0: Odd Person Out. Suppose you roll two, fair, six-sided dice. What is the probability that their sum is even?

> Here, "fair" means unbiased or having uniformly random outcomes.

Exercise 1: Combinaisons. Consider a standard "French" deck of 52 cards: there are four (4) suits (diamonds, clubs, hearts, and spades) and, within each suit, there are 13 ranks (cards numbered 2-10 plus a Jack, a Queen, a King, and an Ace).

- **Part a)** Suppose you shuffle the deck and then select a *hand*, consisting of five (5) cards. (That is, you draw 5 cards uniformly at random from the deck of 52 without replacement.) How many distinct hands are there in total? "Distinct" means the ordering of cards within the hand does not matter.
- **Part b)** A *three-of-a-kind* hand has three (3) cards of the same rank, plus two more cards of *different* ranks. For instance, (7, 7, 7, Queen, 2) is a three-of-a-kind but (7, 7, 7, Queen, Queen) is not. How many possible three-of-a-kind hands are there in the deck?
- **Part c)** From the above, what is the probability of drawing a three-of-a-kind on the first try from a standard French deck?

> Hint: As the name of this exercise suggests, you will find the calculational tool of a combination (https://en.wikipedia.org/wiki/Combination) handy.

Exercise 2: Taxi hit-and-run. (This problem is adapted from the book, Thinking, Fast and Slow (http://www.nytimes.com/2011/11/27/books/review/thinking-fast-and-slow-by-daniel-kahneman-book-review.html?mcubz=0).)

Consider a city in which 85% of the taxis are green and 15% are blue.

One night, a taxi hits a parked car and then flees the scene. A witness claims the taxi was blue. However, it is also known that the reliability of witnesses at night is imperfect; a recent study suggests that the chance a witness makes a mistake is 20%.

What is the probability that the taxi was actually blue?

Calculus practice problems

Exercise 0: Chain (and product) of fools.... Compute the derivative $\dfrac{df}{dx}$ where $f(x) \equiv x \cdot e^{-x^2}$.

Exercise 1: Gamma, gamma, go gamma. Consider the following function.

$$\Gamma(a) \equiv \int_0^\infty t^{a-1} e^{-t} dt,$$

where a is a real number. (In fact, a can be any complex number other than the negative integers, but for this problem, let's not consider that.)

Show the following.

$$\Gamma(a) = (a - 1)\Gamma(a - 1).$$

Bonus: What is $\Gamma(n)$ when n is a positive integer?

Linear Algebra Review

The goal of this lab (homework) is to force you to refresh your linear algebra chops. Before starting it, please read the linear algebra review notes linked to below. These were originally written by Da Kuang (http://math.ucla.edu/~dakuang/), who taught CSE 6040 in Fall 2014.

- Review notes (https://cse6040.gatech.edu/fa17/kuang-linalg-notes.pdf)

Part 1: Plug and chug

While the following problems can be done using a computer, please do them *by hand*. The purpose is to reinforce the key definitions from the linear algebra notes . that we will use repeatedly.

Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 6 & 7 & 8 \end{pmatrix}.$$

Compute the following.

Exercise 0 (0 points). Compute $x + x$ of the numerical x given above.

Exercise 1 (2 points). Compute the *inner-product* (or *dot-product*), $x^T x$, of the numerical x given above.

Exercise 2 (2 points). Compute the *outer-product*, xx^T, of the numerical x given above.

Exercise 3 (2 points). Compute the *vector 1-norm*, $\|x\|_1$, of the numerical x given above.

Exercise 4 (2 points). Compute the *vector 2-norm* (or *Euclidean length* or *distance*), $\|x\|_2$, of the numerical x given above.

Exercise 5 (2 points). Compute the *infinity-norm*, $\|x\|_\infty$, of the numerical x given above.

Exercise 6 (2 points). Compute the Frobenius norm, $\|A\|_F$, of the numerical matrix A given above.

Exercise 7 (2 points). Compute the product, $A^T A$, of the numerical matrix A given above.

Exercise 8 (2 points). Compute the product, AA^T, of the numerical matrix A given above.

Exercise 9 (5 points). The *trace* of a matrix M, or trace(M), is defined to be the sum of M's diagonal entries, i.e., $\text{trace}(M) = \sum_i m_{ii}$.

Let A be any real-valued matrix of size $m \times n$. Show that $\text{trace}(A^T A) = \text{trace}(AA^T) = \|A\|_F^2$.

> In this question, show the desired claim for *any* $m \times n$ matrix A, **not** for the numerical matrix A given above.

Exercise 10 (5 points). Suppose x is now unknown and $b = (1, 1, 1)^T$. Determine a vector x such that $Ax = b$, given the numerical 3×3 matrix A above. You may use *any* method, but show your work. (Methods might include basic high school algebra starting from the definitions of a matrix-vector product, or Gaussian elimination, or LU factorization.)

Part 2: A proof [5 points]

Exercise 11 (5 points). Let $Q \in \mathcal{R}^{m \times m}$ be an orthogonal matrix, and let $A \in \mathcal{R}^{m \times n}$ be some matrix. Prove that $\|QA\|_F^2 = \|A\|_F^2$, that is, left-multiplying by an orthogonal matrix does not change the Frobenius norm.

> Hint: Consider the definitions of an orthogonal matrix, Frobenius norm, vector 2-norm, and the results of Exercise 9.