$$\frac{dh}{dt} = \alpha_1 (1 - h - d) - \beta_1 h \qquad (1)$$

$$\frac{dd}{dt} = \beta_2 (1 - h - d) - \alpha_2 d \qquad (2)$$

<u>steady state</u>

$$\alpha_1 (1 - h_\infty - d_\infty) = \beta_1 h_\infty \;) \Rightarrow \alpha_1 (1 - d_\infty) = h_\infty (\alpha_1 + \beta_1) \Rightarrow h_\infty = \frac{\alpha_1 (1 - d_\infty)}{\alpha_1 + \beta_1}$$

$$\beta_2 (1 - h_\infty - d_\infty) = \alpha_2 d_\infty \;) \Rightarrow \beta_2 \left(1 - \frac{\alpha_1 (1 - d_\infty)}{\alpha_1 + \beta_1} - d_\infty\right) = \alpha_2 d_\infty$$

$$1 - \frac{\alpha_1 - \alpha_1 d_\infty + \alpha_1 d_\infty + \beta_1 d_\infty}{\alpha_1 + \beta_1} = \frac{\alpha_2}{\beta_2} d_\infty$$

$$\frac{\alpha_1 + \beta_1 d_\infty}{\alpha_1 + \beta_1} = 1 - \frac{\alpha_2}{\beta_2} d_\infty \;) \Rightarrow \cancel{\alpha_1} + \beta_1 d_\infty = \cancel{\alpha_1} + \beta_1 - \frac{\alpha_2 (\alpha_1 + \beta_1)}{\beta_2} d_\infty$$

$$d_\infty \left(\beta_1 + (\alpha_1 + \beta_1) \frac{\alpha_2}{\beta_2} \right) = \beta_1$$

$$d_\infty \cdot \frac{\beta_1 \beta_2 + \alpha_1 \alpha_2 + \alpha_2 \beta_1}{\beta_2} = \beta_1$$

$$d_\infty = \frac{\beta_1 \beta_2}{\beta_1 \beta_2 + \alpha_2 (\alpha_1 + \beta_1)}$$

$$\beta_1 = \alpha_1 K_1$$

$$\beta_2 = \alpha_2 K_2$$

$$\Rightarrow d_\infty = \frac{\alpha_1 K_1 \cancel{\alpha_2} K_2}{\alpha_1 K_1 \cancel{\alpha_2} K_2 + \cancel{\alpha_2} (\alpha_1 + \alpha_1 K_1)} =$$

$$= \frac{\cancel{\alpha_1} K_1 K_2}{\cancel{\alpha_1} K_1 K_2 + \cancel{\alpha_1} (1 + K_1)} = \frac{K_1 K_2}{K_1 K_2 + 1 + K_1} = \frac{K_1 K_2}{1 + K_1 (1 + K_2)}$$

$$h_\infty = \frac{\alpha_1 (1 - d_\infty)}{\alpha_1 + \beta_1} = \frac{\alpha_1}{\alpha_1 + K_1 \alpha_1} \left(1 - \frac{K_1 K_2}{1 + K_1 + K_1 K_2} \right) =$$

$$= \frac{1}{1 + K_1} \cdot \frac{1 + K_1 + \cancel{K_1 K_2} - \cancel{K_1 K_2}}{1 + K_1 + K_1 K_2} = \frac{1}{\cancel{1 + K_1}} \cdot \frac{\cancel{1 + K_1}}{1 + K_1 + K_1 K_2} =$$

$$= \frac{1}{1 + K_1 (1 + K_2)}$$

$$\boxed{\begin{aligned} h_\infty &= \frac{1}{1 + K_1 (1 + K_2)} \\ d_\infty &= K_1 K_2 h_\infty \end{aligned}} \qquad \begin{aligned} &(A2a) \\ &(A2b) \end{aligned}$$

$$K_1 = K_2 \Rightarrow \boxed{\begin{aligned} h_\infty &= \frac{1}{1 + K + K^2} \\ d_\infty &= K^2 h_\infty \end{aligned}} \quad (A3)$$

$$\begin{matrix}(1) \\ (2)\end{matrix} \rightarrow \frac{d}{dt} \begin{pmatrix} h \\ d \end{pmatrix} = \begin{pmatrix} -\alpha_1 - \beta_1 & -\alpha_1 \\ -\beta_2 & -\alpha_2 - \beta_2 \end{pmatrix} \begin{pmatrix} h \\ d \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \beta_2 \end{pmatrix}$$

Associated homogeneous equation:

$$\frac{d}{dt} \begin{pmatrix} h \\ d \end{pmatrix} = \begin{pmatrix} -\alpha_1 - \beta_1 & -\alpha_1 \\ -\beta_2 & -\alpha_2 - \beta_2 \end{pmatrix} \begin{pmatrix} h \\ d \end{pmatrix}$$

$$\begin{pmatrix} h \\ d \end{pmatrix} \equiv \vec{x}$$

$$\begin{pmatrix} -\alpha_1 - \beta_1 & -\alpha_1 \\ -\beta_2 & -\alpha_2 - \beta_2 \end{pmatrix} \equiv A$$

$\lambda' = -\lambda$ ← eigenvalues

$$\vec{x}' = A\vec{x}$$

$$\det(A + \lambda I) = \det\begin{pmatrix} -\alpha_1 - \beta_1 + \lambda & -\alpha_1 \\ -\beta_2 & -\alpha_2 - \beta_2 + \lambda \end{pmatrix} =$$

$$= (-\alpha_1 - \beta_1 + \lambda)(-\alpha_2 - \beta_2 + \lambda) - \alpha_1\beta_2 =$$

$$= \alpha_1\alpha_2 + \cancel{\alpha_1\beta_2} \underline{- \lambda\alpha_1} + \alpha_2\beta_1 + \beta_1\beta_2 \underline{- \lambda\beta_1} \underline{- \lambda\alpha_2} \underline{- \lambda\beta_2} + \lambda^2 - \cancel{\alpha_1\beta_2} =$$

$$= \lambda^2 + \lambda\underbrace{(-\alpha_1 - \beta_1 - \alpha_2 - \beta_2)}_{Tr} + \underbrace{\alpha_1\alpha_2 + \beta_1(\alpha_2 + \beta_2)}_{D} =$$

$$= \lambda^2 + Tr \cdot \lambda + D$$

$$Tr = -(\alpha_1 + \beta_1 + \alpha_2 + \beta_2) = -\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)$$

$$D = \alpha_1\alpha_2 + \beta_1(\alpha_2 + \beta_2) = \alpha_1\alpha_2 + K\alpha_1(\alpha_2 + K\alpha_2) =$$

$$= \alpha_1\alpha_2(1 + K(1+K))$$

$$K_1 = \frac{\beta_1}{\alpha_1} \Rightarrow \beta_1 = K_1\alpha_1$$

$$\downarrow$$

$$\tau_1 = \frac{1}{\alpha_1 + \beta_1} \Rightarrow \tau_1 = \frac{1}{\alpha_1(1+K_1)} \Rightarrow \alpha_1 = \frac{1}{\tau_1(1+K_1)}$$

similarly: $\alpha_2 = \dfrac{1}{\tau_2(1+K_2)}$

$$\Rightarrow$$

$$\Rightarrow \quad D = \frac{1 + K(1+K)}{\tau_1\tau_2(1+K)^2}$$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[-T_r \pm \sqrt{T_r^{\,2} - 4D} \right]$$

$$T_r^{\,2} - 4D = \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)^2 - \frac{4\left(1+K(1+K)\right)}{\tau_1 \tau_2 (1+K)^2} =$$

$$= \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)^2 + \frac{4}{\tau_1 \tau_2} - \frac{4\left(1+K(1+K)\right)}{\tau_1 \tau_2 (1+K)^2} = \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)^2 + \frac{4(1+K)^2 - 4 - 4K - 4K^2}{\tau_1 \tau_2 (1+K)^2} =$$

$$= \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)^2 + \frac{\cancel{4} + 8K + \cancel{4K^2} - \cancel{4} - 4K - \cancel{4K^2}}{\tau_1 \tau_2 (1+K)^2} = \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)^2 + \frac{4K}{\tau_1 \tau_2 (1+K)^2}$$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) \pm \sqrt{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)^2 + \frac{4K}{\tau_1 \tau_2 (1+K)^2}} \right]$$

(from the paper)

$$\tau_1^{-1}, \tau_2^{-2} = \frac{1}{2}\left[(\lambda_1 + \lambda_2) \pm \sqrt{(\lambda_1 + \lambda_2)^2 - \frac{4(1+K)^2 \lambda_1 \lambda_2}{1+K(1+K)}} \right]$$

Measurable time constants in experiments are λ_1^{-1} and λ_2^{-1} (i.e. τ_R and τ_r).

general solution for d and R:

$$A_0^{(\ell)} + A_1^{(\ell)} \exp(-\lambda_1 t) + A_2^{(\ell)} \exp(-\lambda_2 t)$$

where $\ell = \{d, R\}$

Approximations:

$$\boxed{\begin{aligned}\tilde{\tau}_1 &\simeq \lambda_1^{-1}\\ \tilde{\tau}_2 &\simeq \lambda_2^{-1}\left[1+K(1+K)\right]/(1+K)^2\end{aligned}}$$

$$\dot{h} = \alpha_1(1-h-d) - \beta_1 h \qquad (1)$$

$$\dot{d} = \beta_2(1-h-d) - \alpha_2 d \qquad (2)$$

As (1) describes much faster process than (2), the authors used rapid equilibrium hypothesis on the former: assume steady state $\frac{dh}{dt}=0$. Then:

$$\left.\begin{aligned} h &= \frac{\alpha_1(1-d)}{\alpha_1+\beta_1}\\ &(2)\end{aligned}\right) \Rightarrow$$

$$\Rightarrow \quad \dot{d} = \beta_2\left(1 - \frac{\alpha_1(1-d)}{\alpha_1+\beta_1} - d\right) - \alpha_2 d =$$

$$= \beta_2 - \frac{\alpha_1\beta_2(1-d)}{\alpha_1+\beta_1} - \beta_2 d - \alpha_2 d =$$

$$= \frac{\cancel{\alpha_1\beta_2} + \beta_1\beta_2 - \cancel{\alpha_1\beta_2} + \cancel{\alpha_1\beta_2 d} - \cancel{\alpha_1\beta_2 d} - \beta_1\beta_2 d - \alpha_1\alpha_2 d - \alpha_2\beta_1 d}{\alpha_1+\beta_1} =$$

$$= \frac{\alpha_1\alpha_2 K^2(1-d) - \alpha_1\alpha_2 d - \alpha_1\alpha_2 K d}{\alpha_1+\beta_1}$$

$$\dot{d} = \frac{\alpha_1 \alpha_2 K^2 (1-d) - \alpha_1 \alpha_2 d - \alpha_1 \alpha_2 K d}{\alpha_1 + \alpha_1 K} =$$

$$= \frac{\alpha_2 K^2 (1-d) - \alpha_2 d - \alpha_2 K d}{1+K} = \frac{\alpha_2 \left(K^2 - K^2 d - d - K d \right)}{1+K} =$$

$$= \frac{\alpha_2 \left(K^2 - d\left(1 + K(1+K)\right)\right)}{1+K} = - \frac{\alpha_2 \left(1 + K(1+K)\right)}{1+K} d + \frac{\alpha_2 K^2}{1+K} .$$

$$\frac{1+K}{\alpha_2 \left(1+K(1+K)\right)} \dot{d} = -d + \frac{\cancel{\alpha_2} K^2}{\cancel{1+K}} \cdot \frac{\cancel{1+K}}{\cancel{\alpha_2} \left(1+K(1+K)\right)}$$

$$\frac{1+K}{\alpha_2 \left(1+K(1+K)\right)} \equiv \lambda_2^{-1}$$

$$\boxed{\lambda_2^{-1} \dot{d} = -d + \frac{K^2}{1+K(1+K)}} \quad (*)$$

$$\alpha_1 = \frac{1}{\tau_1 (1+K_1)}$$

$$\left. \begin{aligned} \lambda_2^{-1} &= \frac{1+K}{\alpha_2 \left(1+K(1+K)\right)} \\ \alpha_2 &= \frac{1}{\tau_2 (1+K)} \end{aligned} \right) \Rightarrow \lambda_2^{-1} = \frac{\tau_2 (1+K)^2}{1+K(1+K)}$$

$$\boxed{\tau_2 = \frac{\lambda_2^{-1} \left(1+K(1+K)\right)}{(1+K)^2}}$$