Calculus Formulas

Analytic Geometry

Vectors

Vector joining two points $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n) \in \mathbb{R}^n$

$$\vec{PQ} = Q - P = (q_1 - p_1, \ldots, q_n - p_n)$$

Dot product $\mathrm{u} = (u_1, \cdots, u_n)$, $\mathrm{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$

$$\mathrm{u} \cdot \mathrm{v} = u_1 v_1 + \ldots + u_n v_n$$

Orthogonal vectors (perpendicular)

$$\mathrm{u} \cdot \mathrm{v} = 0$$

Lines

Vectorial equation of a line that passes through P with direction v

$$P + t\mathrm{v}$$

Point-slope equation of a line in $\mathbb{R}^2$ that passes through (x_0, y_0) with slope m

$$y = y_0 + m(x - x_0)$$

Planes

General equation of a plane in $\mathbb{R}^3$ that passes through a point (x_0, y_0, z_0) perpendicular to the vector (a, b, c)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Derivatives of functions of one variable

Concept of derivative

Average rate of change of a function $f(x)$ in an interval $[a, a + \Delta x]$

$$\mathrm{ARC}f[a, a + \Delta x] = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Instantaneous rate of change (Derivative) of a function $f(x)$ at point $x = a$

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Algebra of derivatives

Sum $(u + v)' = u' + v'$

Subtraction $(u - v)' = u' - v'$

Product $(u \cdot v)' = u' \cdot v + u \cdot v'$

Quotient $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

Chain rule $(f \circ g)'(x) = f'(g(x))g'(x)$

Secant and tangent lines

Secant line to the graph of $f(x)$ at points $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$

$$y = f(a) + \mathrm{ARC}f[a, a + \Delta x](x - a)$$

Tangent line to the graph of $f(x)$ at point $(a, f(a))$

$$y = f(a) + f'(a)(x - a)$$

Growth, concavity and extrema

Growth

- $\forall x \in I\ f'(x) \geq 0 \Rightarrow f$ is increasing in I.
- $\forall x \in I\ f'(x) \leq 0 \Rightarrow f$ is decreasing in I.

Concavity

- $\forall x \in I\ f''(x) \geq 0 \Rightarrow f$ is concave up in I.
- $\forall x \in I\ f''(x) \leq 0 \Rightarrow f$ is concave down in I.

Extrema If $f'(a) = 0$ (critical point)

- $f''(a) < 0 \Rightarrow f$ has a local maximum at $x = a$.
- $f''(a) > 0 \Rightarrow f$ has a local minimum at $x = a$.

Function approximation

Variation of a function

$$\Delta y \approx f'(a)\Delta x$$

Taylor polynomial of order n of $f(x)$ at point $x = a$

$$P^n_{f,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Maclaurin polynomial of order n of $f(x)$

$$P^n_{f,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^n(0)}{n!}x^n$$

Differential equations

First order differential equation

First order ordinary differential equation

$$F(x, y, y') = 0$$

Initial value problem $f(t) = (x(t), y(t))$ at time $t = a$

$$\begin{cases} F(x, y, y') = 0, & \text{First order ODE;} \\ y(x_0) = y_0, & \text{Initial condition.} \end{cases}$$

Separable differential equation

$$y'g(y) = f(x)$$

The solution is $\int g(y)\,dy = \int f(x)\,dx + C$.

Solving first order ODE

Separable differential equation

$$y'g(y) = f(x)$$

Solution:

$$\int g(y)\,dy = \int f(x)\,dx + C.$$

Linear differential equation

$$y' + g(x)y = h(x)$$

Solution:

$$y = e^{-\int g(x)\,dx}\left(\int h(x)e^{\int g(x)\,dx}\,dx + C\right).$$

Derivatives of vectorial functions

Derivative of a vectorial function

If $f(t) = (x_1(t), \ldots, x_n(t))$ then

$$f'(t) = (x_1'(t), \ldots, x_n'(t))$$

Tangent and normal lines in the plane

Tangent line to a trajectory in the plane
$f(t) = (x(t), y(t))$ at time $t = a$

$$(x(a), y(a)) + t(x'(a), y'(a)) \text{ or}$$
$$(x - x(a))y'(a) - (y - y(a))x'(a) = 0$$

Normal line to a trajectory in the plane
$f(t) = (x(t), y(t))$ at time $t = a$

$$(x(a), y(a)) + t(y'(a), -x'(a)) \text{ or}$$
$$(x - x(a))x'(a) + (y - y(a))y'(a) = 0$$

Tangent line and normal plane in the space

Tangent line to a trajectory in the space
$f(t) = (x(t), y(t), z(t))$ at time $t = a$

$$(x(a), y(a), z(a)) + t(x'(a), y'(a), z'(a))$$

Normal plane to a trajectory in the space
$f(t) = (x(t), y(t), z(t))$ at time $t = a$

$$x'(a)(x - x(a)) + y'(a)(y - y(a)) + z'(a)(z - z(a)) = 0$$

Derivatives of functions of several variables

Partial derivatives

Gradient vector

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Hessian Matrix

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Hessian

$$Hf(P) = |\nabla^2 f(P)|$$

Directional derivative of f at a point P along the direction of v

$$f_{\mathrm{v}}'(P) = \nabla f(P)\frac{\mathrm{v}}{|\mathrm{v}|}$$

Chain rule

$$f(g(t))' = \nabla f(g(t))g'(t)$$

Tangent and normal lines in the plane

Normal line to a trajectory in the plane $f(x, y) = 0$ at point $P = (a, b)$

$$P + t\nabla f(P) = (a, b) + t\nabla f(a, b) \text{ or}$$
$$(x - a)\tfrac{\partial f}{\partial y}(a, b) - (y - b)\tfrac{\partial f}{\partial x}(a, b) = 0$$

Tangent line to a trajectory in the plane $f(x, y) = 0$ at point $P = (a, b)$

$$(x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b) = 0$$

Normal line and tangent plane in the space

Normal line to a surface in the space $f(x,y,z)=0$ at point $P=(a,b,c)$

$$P+t\nabla f(P)=(a,b,c)+t\nabla f(a,b,c)$$

Tangent plane to a surface in the space $f(x,y,z)=0$ at point $P=(a,b,c)$

$$(x-a)\frac{\partial f}{\partial x}(a,b,c)+(y-b)\frac{\partial f}{\partial y}(a,b,c)+(z-c)\frac{\partial f}{\partial z}(a,b,c)=0$$

Implicit derivatives

Implicit derivative of a function $f(x,y)=0$

$$y'=\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{and} \quad x'=\frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$$

Implicit partial derivatives of a function $f(x,y,z)=0$

$$\frac{\partial z}{\partial x}=\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial x}=\frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Extrema and saddle points

1. Compute the critical points $\nabla f(P)=0$.
2. At any critical point P compute the Hessian:
   - $Hf(P)>0$ and $\frac{\partial^2 f}{\partial x^2}(P)>0 \Rightarrow f$ has a local minimum at P.
   - $Hf(P)>0$ and $\frac{\partial^2 f}{\partial x^2}(P)<0 \Rightarrow f$ has a local maximum at P.
   - $Hf(P)<0 \Rightarrow f$ has a saddle point at P.

Function approximation

Taylor polynomial of second order of $f(x,y)$ at point $P=(a,b)$

$$\begin{aligned}P^2_{f,P}(x,y)&=f(a,b)+\nabla f(a,b)(x-a,y-b)+\\&+\frac{1}{2}(x-a,y-b)\nabla^2 f(a,b)(x-a,y-b)\end{aligned}$$

Maclaurin polynomial of second order of $f(x,y)$

$$P^2_{f,(0,0)}(x,y)=f(0,0)+\nabla f(0,0)(x,y)+\frac{1}{2}(x,y)\nabla^2 f(0,0)(x,y)$$