

Calculus Formulas

Analytic Geometry

Vectors

Vector joining two points $P = (p_1, \dots, p_n)$,
 $Q = (q_1, \dots, q_n) \in \mathbb{R}^n$

$$\vec{PQ} = Q - P = (q_1 - p_1, \dots, q_n - p_n)$$

Dot product $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$$u \cdot v = u_1 v_1 + \dots + u_n v_n$$

Orthogonal vectors (perpendicular)

$$u \cdot v = 0$$

Lines

Vectorial equation of a line that passes through P
with direction v

$$P + tv$$

Point-slope equation of a line in \mathbb{R}^2 that passes
through (x_0, y_0) with slope m

$$y = y_0 + m(x - x_0)$$

Planes

General equation of a plane in \mathbb{R}^3 that passes
through a point (x_0, y_0, z_0) perpendicular to the vec-
tor (a, b, c)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Algebra of derivatives

Sum $(u + v)' = u' + v'$

Subtraction $(u - v)' = u' - v'$

Product $(u \cdot v)' = u' \cdot v + u \cdot v'$

Quotient $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

Chain rule $(f \circ g)'(x) = f'(g(x))g'(x)$

Secant and tangent lines

Secant line to the graph of $f(x)$ at points $(a, f(a))$
and $(a + \Delta x, f(a + \Delta x))$

$$y = f(a) + \text{ARCF}[a, a + \Delta x](x - a)$$

Tangent line to the graph of $f(x)$ at point $(a, f(a))$

$$y = f(a) + f'(a)(x - a)$$

Growth, concavity and extrema

Growth

- $\forall x \in I \ f'(x) \geq 0 \Rightarrow f$ is increasing in I .
- $\forall x \in I \ f'(x) \leq 0 \Rightarrow f$ is decreasing in I .

Concavity

- $\forall x \in I \ f''(x) \geq 0 \Rightarrow f$ is concave up in I .
- $\forall x \in I \ f''(x) \leq 0 \Rightarrow f$ is concave down in I .

Extrema If $f'(a) = 0$ (critical point)

- $f''(a) < 0 \Rightarrow f$ has a local maximum at $x = a$.
- $f''(a) > 0 \Rightarrow f$ has a local minimum at $x = a$.

Derivatives of functions of one variable

Concept of derivative

Average rate of change of a function $f(x)$ in an
interval $[a, a + \Delta x]$

$$\text{ARCF}[a, a + \Delta x] = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Instantaneous rate of change (Derivative) of a
function $f(x)$ at point $x = a$

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Function approximation

Variation of a function

$$\Delta y \approx f'(a)\Delta x$$

Taylor polynomial of order n of $f(x)$ at point $x = a$

$$P_{f,a}^n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

Maclaurin polynomial of order n of $f(x)$

$$P_{f,0}^n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^n(0)}{n!}x^n$$

Differential equations

First order differential equation

First order ordinary differential equation

$$F(x, y, y') = 0$$

Initial value problem

$$\begin{cases} F(x, y, y') = 0, & \text{First order ODE;} \\ y(x_0) = y_0, & \text{Initial condition.} \end{cases}$$

Solving first order ODE

Separable differential equation

$$y'g(y) = f(x)$$

Solution:

$$\int g(y) dy = \int f(x) dx + C.$$

Linear differential equation

$$y' + g(x)y = h(x)$$

Solution:

$$y = e^{-\int g(x) dx} \left(\int h(x) e^{\int g(x) dx} dx + C \right).$$

Tangent line and normal plane in the space

Tangent line to a trajectory in the space

$f(t) = (x(t), y(t), z(t))$ at time $t = a$

$$(x(a), y(a), z(a)) + t(x'(a), y'(a), z'(a))$$

Normal plane to a trajectory in the space

$f(t) = (x(t), y(t), z(t))$ at time $t = a$

$$x'(a)(x - x(a)) + y'(a)(y - y(a)) + z'(a)(z - z(a)) = 0$$

Derivatives of functions of several variables

Partial derivatives

Gradient vector

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

Hessian Matrix

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Hessian

$$Hf(P) = |\nabla^2 f(P)|$$

Directional derivative of f at a point P along the direction of v

$$f'_v(P) = \nabla f(P) \frac{v}{|v|}$$

Chain rule

$$f(g(t))' = \nabla f(g(t))g'(t)$$

Derivatives of vectorial functions

Derivative of a vectorial function

If $f(t) = (x_1(t), \dots, x_n(t))$ then

$$f'(t) = (x'_1(t), \dots, x'_n(t))$$

Tangent and normal lines in the plane

Tangent line to a trajectory in the plane

$f(t) = (x(t), y(t))$ at time $t = a$

$$(x(a), y(a)) + t(x'(a), y'(a)) \text{ or } (x - x(a))y'(a) - (y - y(a))x'(a) = 0$$

Normal line to a trajectory in the plane

$f(t) = (x(t), y(t))$ at time $t = a$

$$(x(a), y(a)) + t(y'(a), -x'(a)) \text{ or } (x - x(a))x'(a) + (y - y(a))y'(a) = 0$$

Tangent and normal lines in the plane

Normal line to a trajectory in the plane $f(x, y) = 0$ at point $P = (a, b)$

$$P + t\nabla f(P) = (a, b) + t\nabla f(a, b) \text{ or } (x - a)\frac{\partial f}{\partial y}(a, b) - (y - b)\frac{\partial f}{\partial x}(a, b) = 0$$

Tangent line to a trajectory in the plane $f(x, y) = 0$ at point $P = (a, b)$

$$(x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b) = 0$$

Normal line and tangent plane in the space

Normal line to a surface in the space $f(x, y, z) = 0$ at point $P = (a, b, c)$

$$P + t\nabla f(P) = (a, b, c) + t\nabla f(a, b, c)$$

Tangent plane to a surface in the space $f(x, y, z) = 0$ at point $P = (a, b, c)$

$$(x-a)\frac{\partial f}{\partial x}(a, b, c) + (y-b)\frac{\partial f}{\partial y}(a, b, c) + (z-c)\frac{\partial f}{\partial z}(a, b, c) = 0$$

Implicit derivatives

Implicit derivative of a function $f(x, y) = 0$

$$y' = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{and} \quad x' = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$$

Implicit partial derivatives of a function $f(x, y, z) = 0$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Extrema and saddle points

1. Compute the critical points $\nabla f(P) = 0$.
2. At any critical point P compute the Hessian:
 - $Hf(P) > 0$ and $\frac{\partial^2 f}{\partial x^2}(P) > 0 \Rightarrow f$ has a local minimum at P .
 - $Hf(P) > 0$ and $\frac{\partial^2 f}{\partial x^2}(P) < 0 \Rightarrow f$ has a local maximum at P .
 - $Hf(P) < 0 \Rightarrow f$ has a saddle point at P .

Function approximation

Taylor polynomial of second order of $f(x, y)$ at point $P = (a, b)$

$$P_{f,P}^2(x, y) = f(a, b) + \nabla f(a, b)(x - a, y - b) + \frac{1}{2}(x - a, y - b)\nabla^2 f(a, b)(x - a, y - b)$$

Maclaurin polynomial of second order of $f(x, y)$

$$P_{f,(0,0)}^2(x, y) = f(0, 0) + \nabla f(0, 0)(x, y) + \frac{1}{2}(x, y)\nabla^2 f(0, 0)(x, y)$$