Elementary Calculus Manual

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Contents

Differential calculus with one real variable

Definition (Increment of a variable)

An *increment* of a variable x is a change in the value of the variable and is denoted Δx. The increment of a variable x along an interval $[a, b]$ is

$$\Delta x = b - a.$$

Definition (Increment of a function)

The *increment* of a function $y = f(x)$ along an interval $[a, b] \subseteq Dom(f)$ is

$$\Delta y = f(b) - f(a).$$

Example The increment of x along the interval $[2, 5]$ is $\Delta x = 5 - 2 = 3$ and the increment of the function $y = x^2$ along the same interval is $\Delta y = 5^2 - 2^2 = 21$.

The study of a function $y = f(x)$ requires to understand how the function changes, that is, how changes the dependent variable y when we change the independent variable x.

Definition (Average rate of change)

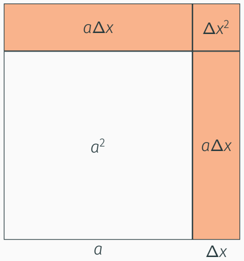
The *average rate of change* of a function $y = f(x)$ in an interval $[a, a+\Delta x] \subseteq Dom(f)$, is the quotient between the increment of y and the increment of x in that interval, and is denoted

$$\text{ARC}\ f[a, a+\Delta x] = \frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

Let $y = x^2$ be the function that measures the area of a metallic square of side x.

If at any given time the side of the square is a, and we heat the square uniformly increasing the side by dilatation a quantity Δx, how much will increase the area of the square?

$$\begin{aligned}\Delta y &= f(a+\Delta x) - f(a) = (a+\Delta x)^2 - a^2 = \\ &= a^2 + 2a\Delta x + \Delta x^2 - a^2 = 2a\Delta x + \Delta x^2.\end{aligned}$$

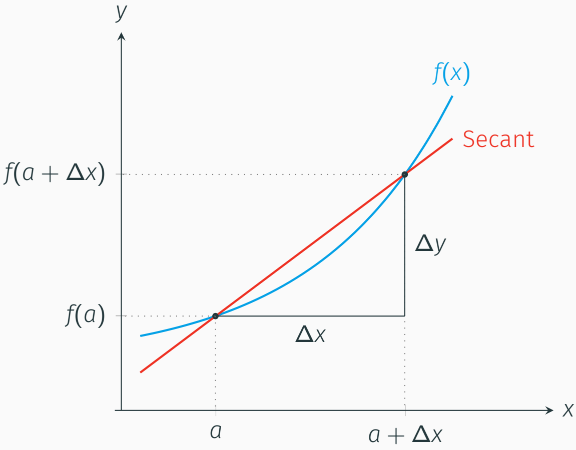


What is the average rate of change in the interval $[a, a+\Delta x]$?

$$\text{ARC } f[a, a+\Delta x] = \frac{\Delta y}{\Delta x} = \frac{2a\Delta x + \Delta x^2}{\Delta x} = 2a + \Delta x.$$

Geometric interpretation of the average rate of change

The average rate of change of a function $y = f(x)$ in an interval $[a, a + \Delta x]$ is the slope of the *secant* line of f through the points $(a, f(a))$ and $(a + \Delta x, f(a + \Delta x))$.



Instantaneous rate of change

Often is interesting to study the rate of change of a function, not in an interval, but in a point.

Knowing the tendency of change of a function in an instant can be used to predict the value of the function in next instants.

Definition (Instantaneous rate of change and derivative)

The *instantaneous rate of change* of a function f in a point a, is the limit of the average rate of change of f in the interval $[a, a+\Delta x]$, when Δx tends to 0, and is denoted

$$\mathrm{IRC}\, f(a) = \lim_{\Delta x \to 0} \mathrm{ARC}\, f[a, a+\Delta x] = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

When this limit exists, the function f is said to be *derivable* or *differentiable* at the point a, and its value is called *derivative* of f at a, and denoted $f'(a)$ (Lagrange's notation) or $\frac{df}{dx}(a)$ (Leibniz's notation).

Let's take again the function $y = x^2$ that measures the area of a metallic square of side x.

If at any given time the side of the square is a, and we heat the square uniformly increasing the side, what is the tendency of change of the area in that moment?

$$\begin{aligned}\mathrm{IRC}\, f(a) &= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \\ &= \lim_{\Delta x\to 0} \frac{2a\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x\to 0} 2a + \Delta x = 2a.\end{aligned}$$

Thus,

$$f'(a) = \frac{df}{dx}(a) = 2a,$$

indicating that the area of the square tend to increase the double of the side.

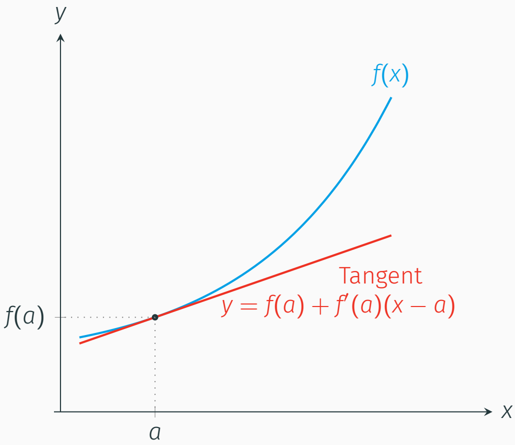
The derivative of a function $f'(a)$ shows the growth rate of f at point a:

- $f'(a) > 0$ indicates an increasing tendency (y increases as x increases).
- $f'(a) < 0$ indicates a decreasing tendency (y decreases as x increases).

Example A derivative $f'(a) = 3$ indicates that y tends to increase triple of x at point a. A derivative $f'(a) = -0.5$ indicates that y tends to decrease half of x at point a.

Geometric interpretation fo the derivative

The instantaneous rate of change or derivative of a function $y = f(x)$ at a point a is the slope of the *tangent line* f at point $(a, f(a))$.



If $y = c$, is a constant function, then $y' = 0$ at any point.

If $y = x$, is the identity function, then $y' = 1$ at any point.

If $u = f(x)$ and $v = g(x)$ are two differentiable functions, then

- $(u + v)' = u' + v'$
- $(u - v)' = u' - v'$
- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$

Theorem (Chain rule)

If the function $y = f \circ g$ is the composition of two functions $y = f(z)$ and $z = g(x)$, then

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

It's easy to proof this using the Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = f'(z)g'(x) = f'(g(x))g'(x).$$

Example If $f(z) = \sin z$ and $g(x) = x^2$, then $f \circ g(x) = \sin(x^2)$. Applying the chain rule the derivative of the composite function is

$$(f \circ g)'(x) = f'(g(x))g'(x) = \cos(g(x))2x = \cos(x^2)2x.$$

On the other hand, $g \circ f(z) = (\sin z)^2$, and applying the chain rule again, its derivative is

$$(g \circ f)'(z) = g'(f(z))f'(z) = 2f(z)\cos z = 2\sin z \cos z.$$

Theorem (Derivative of the inverse function)

Given a function $y = f(x)$ with inverse $x = f^{-1}(y)$, then

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))},$$

provided that f is differentiable at $f^{-1}(y)$ and $f'(f^{-1}(y)) \neq 0$.

It's easy to proof this using the Leibniz notation

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

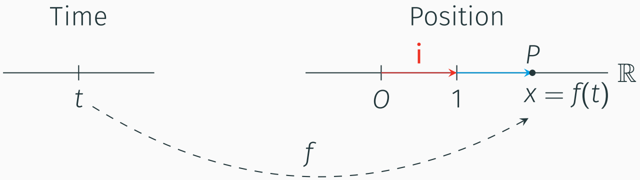
The inverse of the exponential function $y = f(x) = e^x$ is the natural logarithm $x = f^{-1}(y) = \ln y$, so that we can compute the derivative of the natural logarithm using the previous theorem and we get

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y}.$$

Example Sometimes is easier to apply the chain rule to compute the derivative of the inverse of a function. In this example, as $\ln x$ is the inverse of e^x, we know that $e^{\ln x} = x$, so that differentiating both sides and applying the chain rule to the left side we get

$$(e^{\ln x})' = x' \Leftrightarrow e^{\ln x}(\ln(x))' = 1 \Leftrightarrow (\ln(x))' = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Assume that the function $y = f(t)$ describes the position of an object moving in the real line at every time t. Taking as reference the coordinates origin O and the unitary vector $\mathbf{i} = (1)$, we can represent the position of the moving object P at every moment t with a vector $\vec{OP} = x\mathbf{i}$ where $x = f(t)$.



Observation It also makes sense when f measures other magnitudes as the temperature of a body, the concentration of a gas, or the quantity of substance in a chemical reaction at every moment t.

Kinematic interpretation of the average rate of change

In this context, if we take the instants $t = t_0$ and $t = t_0 + \Delta t$, both in Dom(f), the vector

$$\mathrm{v}_m = \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

is known as the *average velocity* of the trajectory f in the interval $[t_0, t_0 + \Delta t]$.

Example A vehicle makes a trip from Madrid to Barcelona. Let $f(t)$ be the function that determine the position of the vehicle at every moment t. If the vehicle departs from Madrid (km 0) at 8:00 and arrive to Barcelona (km 600) at 14:00, then the average velocity of the vehicle in the path is

$$\mathrm{v}_m = \frac{f(14) - f(8)}{14 - 8} = \frac{600 - 0}{6} = 100km/h.$$

In the same context of the linear motion, the derivative of the function $f(t)$ at the moment t_0 is the vector

$$\mathrm{v} = f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t},$$

that is known, as long as the limit exists, as the *instantaneous velocity* or simply *velocity* of the trajectory f at moment t_0.

That is, the derivative of the object position with respect to time is a vector field that is called *velocity along the trajectory f*.

Example Following with the previous example, what indicates the speedometer at any instant is the modulus of the instantaneous velocity vector at that moment.

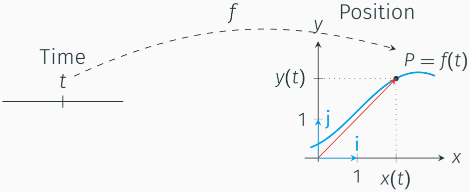
Generalization to curvilinear motion

The notion of derivative as a velocity along a trajectory in the real line can be generalized to a trajectory in any euclidean space $\mathbb{R}^n$.

In case of a two dimensional space $\mathbb{R}^2$, if $f(t)$ describes the position of a moving object in the real plane at any time t, taking as reference the coordinates origin O and the unitary vectors $\{\mathbf{i} = (1, 0), \mathbf{j} = (0, 1)\}$, we can represent the position of the moving object P at every moment t with a vector $\vec{OP} = x(t)\mathbf{i} + y(t)\mathbf{j}$, where the coordinates

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad t \in \mathrm{Dom}(f)$$

are known as *coordinate functions* of f and denoted $f(t) = (x(t), y(t))$.



In the context of a trajectory $f(t) = (x(t), y(t))$ in the real plane $\mathbb{R}^2$, the derivative of the function $f(t)$ at the moment t_0 is the vector

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t},$$

that is known, as long as the limit exists, as the *velocity* of the trajectory f at moment t_0.

As $f(t) = (x(t), y(t))$,

$$f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(x(t_0 + \Delta t), y(t_0 + \Delta t)) - (x(t_0), y(t_0))}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \left(\frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}, \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t} \right) =$$

$$= \left(\lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}, \lim_{\Delta t \to 0} \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t} \right) = (x'(t_0), y'(t_0)).$$

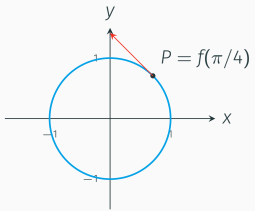
Thus,

$$\mathbf{v} = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j}.$$

Given the trajectory $f(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$, whose image is the unit circumference centered in the coordinate origin, its coordinate functions are $x(t) = \cos t$, $y(t) = \sin t$, $t \in \mathbb{R}$, and its velocity is

$$\mathbf{v} = f'(t) = (x'(t), y'(t)) = (-\sin t, \cos t).$$

In the moment $t = \pi/4$, the object is in position $f(\pi/4) = (\cos(\pi/4), \sin(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$ and it is moving with a velocity $\mathbf{v} = f'(\pi/4) = (-\sin(\pi/4), \cos(\pi/4)) = (-\sqrt{2}/2, \sqrt{2}/2)$.



Observe that the module of the velocity vector is always 1 as $|\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.

Given a trajectory $f(t)$ in the real plane, the vectors that are parallel to the velocity $\mathbf{v}$ at a moment t_0 are called *tangent vectors* to the trajectory f at the moment t_0, and the line passing through $P = f(t_0)$ directed by $\mathbf{v}$ is the tangent line to f at the moment t_0.

Definition (Tangent line to a trajectory)

Given a trajectory $f(t)$ in the real plane $\mathbb{R}^2$, the *tangent line* to f at t_0 is the line with equation

$$\begin{aligned} l : (x, y) &= f(t_0) + tf'(t_0) = (x(t_0), y(t_0)) + t(x'(t_0), y'(t_0)) \\ &= (x(t_0) + tx'(t_0), y(t_0) + ty'(t_0)). \end{aligned}$$

We have seen that for the trajectory $f(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$, whose image is unit circumference at the coordinate origin, the object position at the moment $t = \pi/4$ is $f(\pi/4) = (\sqrt{2}/2, \sqrt{2}/2)$ and its velocity $\mathbf{v} = (-\sqrt{2}/2, \sqrt{2}/2)$. Thus the equation of the tangent line to f at that moment is

$$l : (x, y) = f(\pi/4) + t\mathbf{v} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + t\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2} - t\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} + t\frac{\sqrt{2}}{2}\right)$$

From the vectorial equation of the tangent to a trajectory $f(t)$ at the moment t_0 we can get the coordinate functions

$$\begin{cases} x = x(t_0) + tx'(t_0) \\ y = y(t_0) + ty'(t_0) \end{cases} \qquad t \in \mathbb{R},$$

and solving for t and equalling both equations we get the *Cartesian equation* of the tangent

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)},$$

if $x'(t_0) \neq 0$ and $y'(t_0) \neq 0$.

From this equation is easy to get the *point-slope equation* of the tangent

$$y - y(t_0) = \frac{y'(t_0)}{x'(t_0)}(x - x(t_0)).$$

Using the vectorial equation of the tangent of the previous example

$$l : (x, y) = \left(\frac{\sqrt{2}}{2} - t\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} + t\frac{\sqrt{2}}{2}\right),$$

its Cartesian equation is

$$\frac{x - \sqrt{2}/2}{-\sqrt{2}/2} = \frac{y - \sqrt{2}/2}{\sqrt{2}/2}$$

and the point-slope equation is

$$y - \sqrt{2}/2 = \frac{-\sqrt{2}/2}{\sqrt{2}/2}(x - \sqrt{2}/2) \Rightarrow y = -x + \sqrt{2}.$$

We have seen that the tangent line to a trajectory $f(t)$ at t_0 is the line passing through the point $P = f(t_0)$ directed by the velocity vector $\mathbf{v} = f'(t_0) = (x'(t_0), y'(t_0))$. If we take as direction vector a vector orthogonal to $\mathbf{v}$, we get another line that is known as *normal line* to f at moment t_0.

Definition (Normal line to a trajectory)

Given a trajectory $f(t)$ in the real plane $\mathbb{R}^2$, the *normal line* to f at moment t_0 is the line with equation

$$l : (x, y) = (x(t_0), y(t_0)) + t(y'(t_0), -x'(t_0)) = (x(t_0) + ty'(t_0), y(t_0) - tx'(t_0)).$$

The Cartesian equation is

$$\frac{x - x(t_0)}{y'(t_0)} = \frac{y - y(t_0)}{-x'(t_0)},$$

and the point-slope equation is

$$y - y(t_0) = \frac{-x'(t_0)}{y'(t_0)}(x - x(t_0)).$$

Considering again the trajectory of the unit circumference $f(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$, the normal line to f at moment $t = \pi/4$ is

$$l : (x, y) = (\cos(\pi/2), \sin(\pi/2)) + t(\cos(\pi/2), \sin(\pi/2)) =$$
$$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + t\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2} + t\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} + t\frac{\sqrt{2}}{2}\right),$$

the Cartesian equation is

$$\frac{x - \sqrt{2}/2}{\sqrt{2}/2} = \frac{y - \sqrt{2}/2}{\sqrt{2}/2},$$

and the point-slope equation is

$$y - \sqrt{2}/2 = \frac{\sqrt{2}/2}{\sqrt{2}/2}(x - \sqrt{2}/2) \Rightarrow y = x.$$

Tangent and normal lines to a function

A particular case of tangent and normal lines to a trajectory are the tangent and normal lines to a function of one real variable. For every function $y = f(x)$, the trajectory that trace its graph is

$$g(x) = (x, f(x)) \quad x \in \mathbb{R},$$

and its velocity is

$$g'(x) = (1, f'(x)),$$

so that the tangent line to g at the moment x_0 is

$$\frac{x - x_0}{1} = \frac{y - f(x_0)}{f'(x_0)} \Rightarrow y - f(x_0) = f'(x_0)(x - x_0),$$

and the normal line is

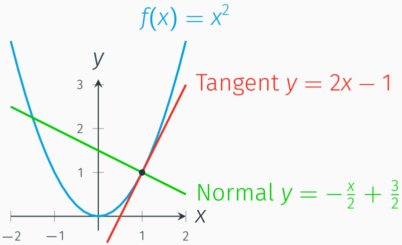
$$\frac{x - x_0}{f'(x_0)} = \frac{y - f(x_0)}{-1} \Rightarrow y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0),$$

Given the function $y = x^2$, the trajectory that traces the its graph is $g(x) = (x, x^2)$ and its velocity is $g'(x) = (1, 2x)$. At the moment $x = 1$ the trajectory passes through the point $(1, 1)$ with a velocity $(1, 2)$. Thus, the tangent line at that moment is

$$\frac{x-1}{1} = \frac{y-1}{2} \Rightarrow y - 1 = 2(x-1) \Rightarrow y = 2x - 1,$$

and the normal line is

$$\frac{x-1}{2} = \frac{y-1}{-1} \Rightarrow y - 1 = \frac{-1}{2}(x-1) \Rightarrow y = \frac{-x}{2} + \frac{3}{2}.$$



Tangent line to a trajectory in the space

The concept of tangent line to a trajectory in can be easily extended from the real plane to the three-dimensional space $\mathbb{R}^3$.

If $f(t) = (x(t), y(t), z(t))$, $t \in \mathbb{R}$, is a trajectory in the real space $\mathbb{R}^3$, then at the moment t_0, the moving object that follows this trajectory will be at the position $P = (x(t_0), y(t_0), z(t_0))$ with a velocity $\mathbf{v} = f'(t) = (x'(t), y'(t), z'(t))$. Thus, the tangent line to f at this moment have the following vectorial equation

$$\begin{aligned} l : (x, y, z) &= (x(t_0), y(t_0), z(t_0)) + t(x'(t_0), y'(t_0), z'(t_0)) = \\ &= (x(t_0) + tx'(t_0), y(t_0) + ty'(t_0), z(t_0) + tz'(t_0)), \end{aligned}$$

and the Cartesian equations are

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)},$$

provided that $x'(t_0) \neq 0$, $y'(t_0) \neq 0$ y $z'(t_0) \neq 0$.

Given the trajectory $f(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$ in the real space, at the moment $t = \pi/2$ the trajectory passes through the point

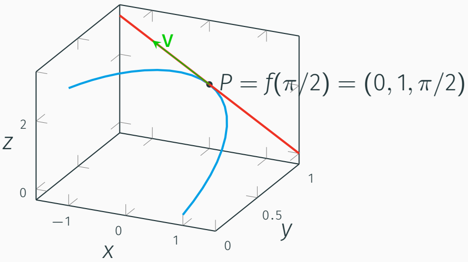
$$f(\pi/2) = (\cos(\pi/2), \sin(\pi/2), \pi/2) = (0, 1, \pi/2),$$

with a velocity

$$\mathbf{v} = f'(\pi/2) = (-\sin(\pi/2), \cos(\pi/2), 1) = (-1, 0, 1),$$

and the tangent line to f at that moment is

$$l : (x, y, z) = (0, 1, \pi/2) + t(-1, 0, 1) = (-t, 1, t + \pi/2).$$



The main application of derivatives is to determine the variation (increase or decrease) of functions. For that we use the sign of the first derivative.

Theorem

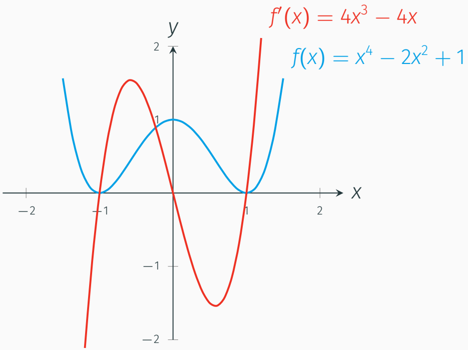
Let $f(x)$ be a function with first derivative in an interval $I \subseteq \mathbb{R}$.

- *If $\forall x \in I\ f'(x) \geq 0$ then f is increasing on I.*
- *If $\forall x \in I\ f'(x) \leq 0$ then f is decreasing on I.*

If $f'(x_0) = 0$ then x_0 is known as a *stationary point* and the function in non-increasing and non-decreasing at that point. **Example** The function $f(x) = x^3$ is increasing on $\mathbb{R}$ as $\forall x \in \mathbb{R}\ f'(x) \geq 0$.

Observation A function can be increasing or decreasing on an interval and not have first derivative.

Let's analyze the increase and decrease of the function $f(x) = x^4 - 2x^2 + 1$. Its first derivative is $f'(x) = 4x^3 - 4x$.



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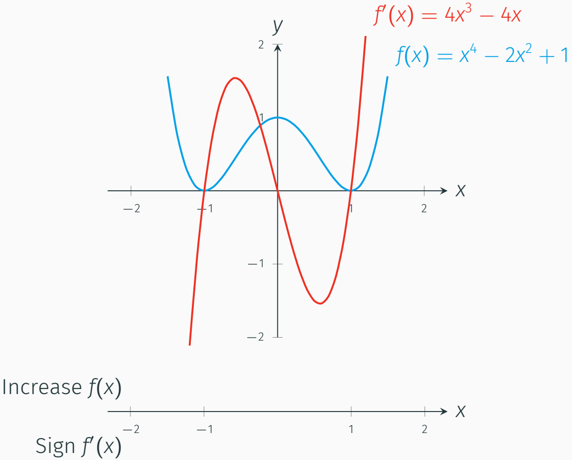

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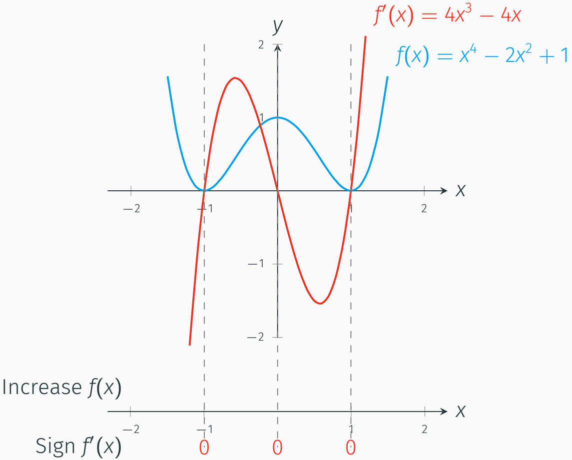

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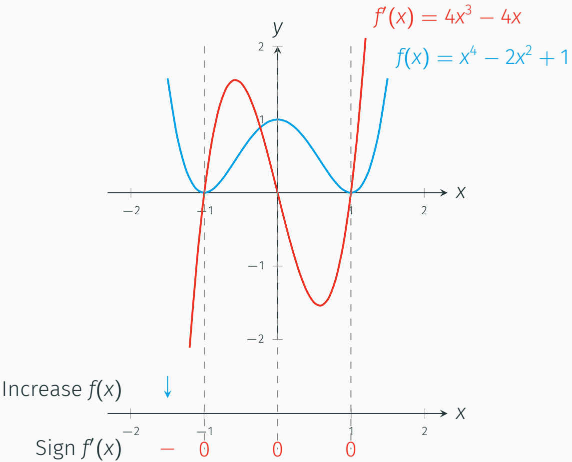

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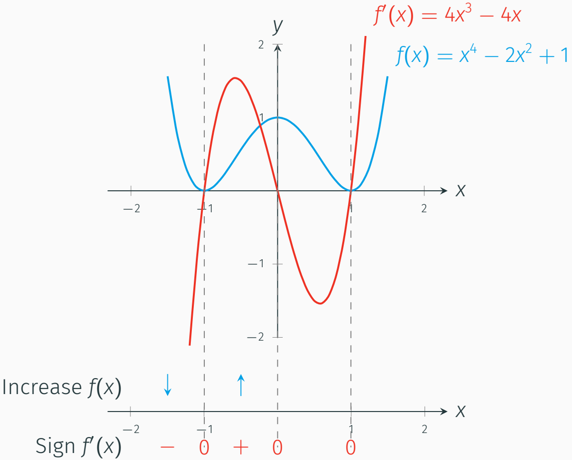

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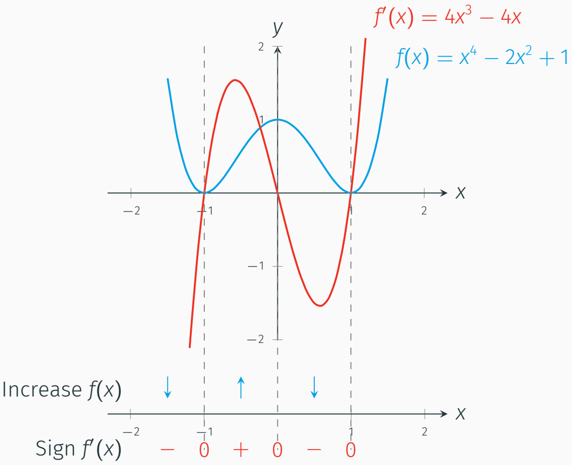
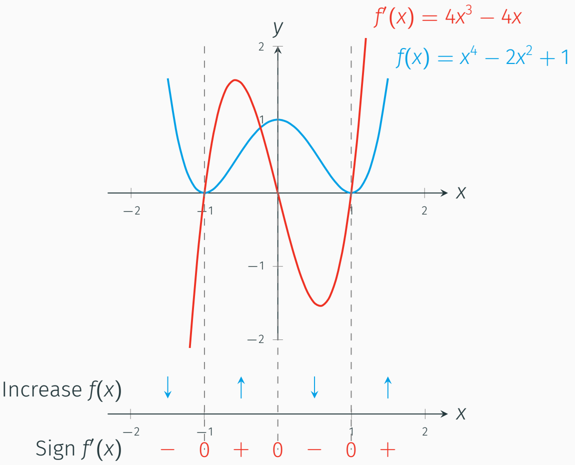


Let's analyze the increase and decrease of the function $f(x) = x^4 - 2x^2 + 1$. Its first derivative is $f'(x) = 4x^3 - 4x$.



As a consequence of the previous result we can also use the first derivative to determine the relative extrema of a function.

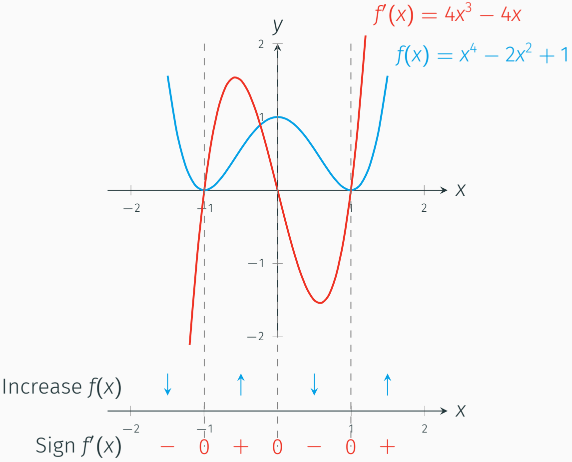
Theorem (First derivative test)

Let $f(x)$ be a function with first derivative in an interval $I \subseteq \mathbb{R}$ and let $x_0 \in I$ be a stationary point of f ($f'(x_0) = 0$).

- *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*
- *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*
- *If $f'(x)$ has the same sign on both an open interval extending left from x_0 and an open interval extending right from x_0, then f has an inflection point at $_0$.*

Observation *A vanishing derivative is a necessary but not sufficient*

Consider again the function $f(x) = x^4 - 2x^2 + 1$ and let's analyze its relative extrema now. Its first derivative is $f'(x) = 4x^3 - 4x$.



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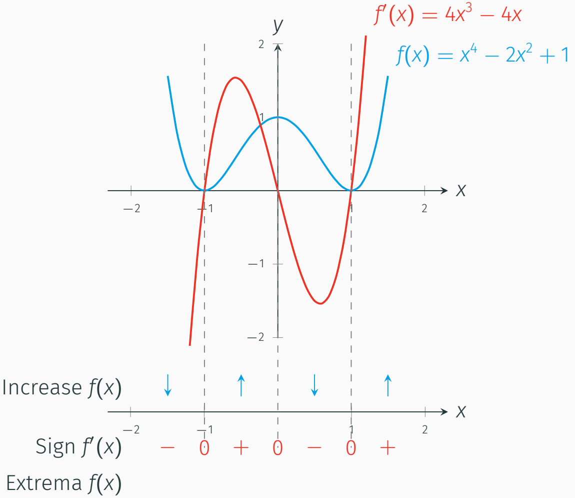

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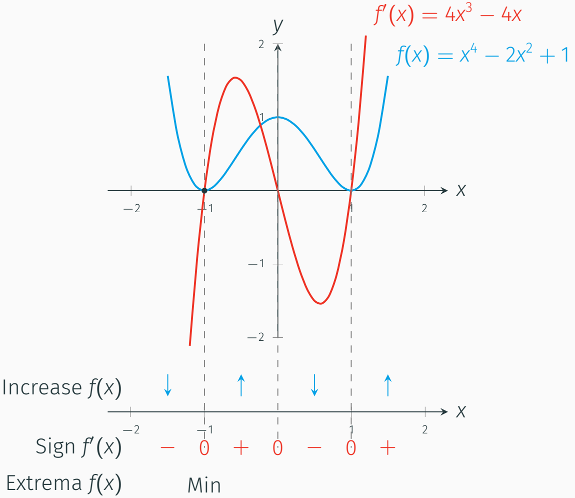
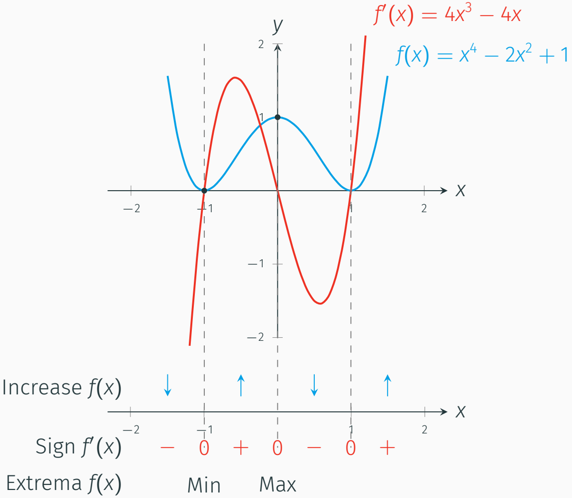


Analysis of functions: relative extrema
Example

Consider again the function $f(x) = x^4 - 2x^2 + 1$ and let's analyze its relative extrema now. Its first derivative is $f'(x) = 4x^3 - 4x$.



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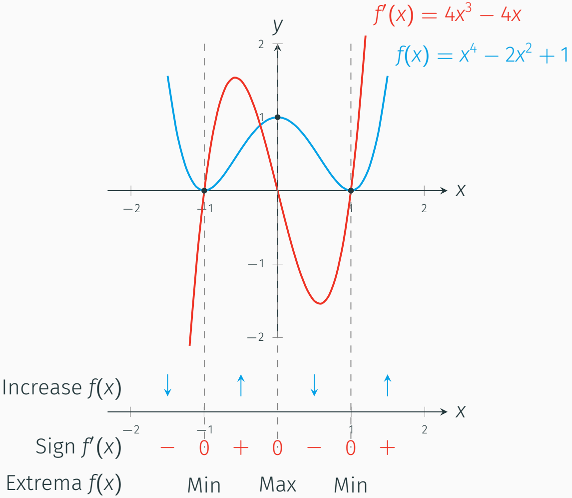


The concavity of a function can be determined by de second derivative.

Theorem

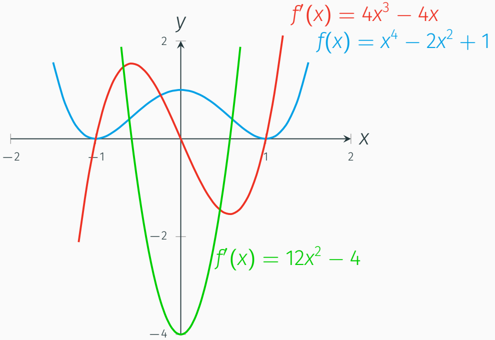
Let $f(x)$ be a function with second derivative in an interval $I \subseteq \mathbb{R}$.

- *If $\forall x \in I\ f''(x) \geq 0$ then f is concave up (convex) on I.*
- *If $\forall x \in I\ f''(x) \leq 0$ then f is concave down (concave) on I.*

Example The function $f(x) = x^2$ has second derivative $f''(x) = 2 > 0\ \forall x \in \mathbb{R}$, so it is concave up in all $\mathbb{R}$.

Observation *A function can be concave up or down and not have second derivative.*

let's analyze its concavity of the same function of previous examples $f(x) = x^4 - 2x^2 + 1$. Its second derivative is $f''(x) = 12x^2 - 4$.



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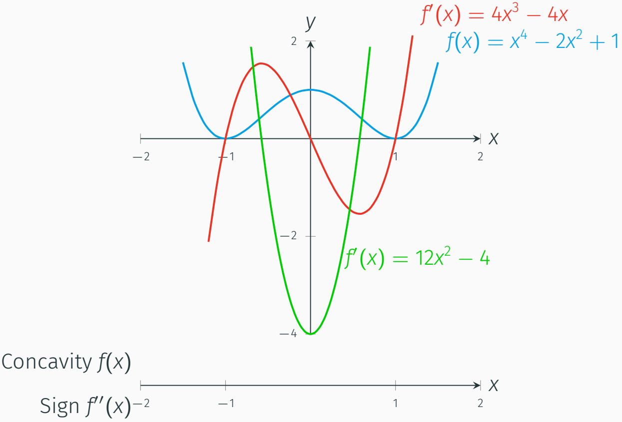

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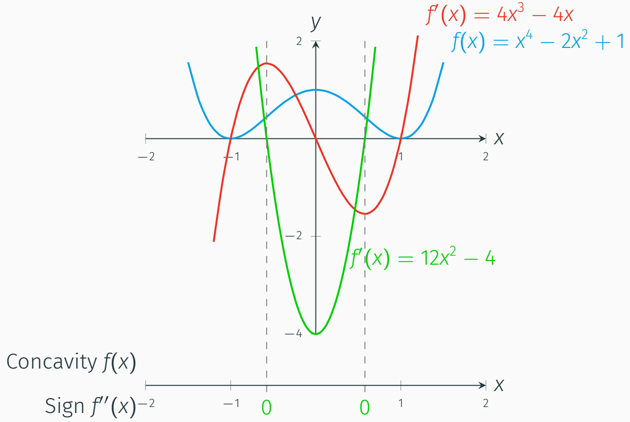

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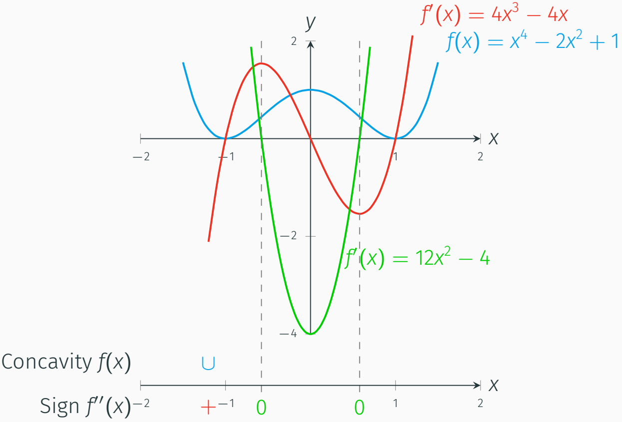

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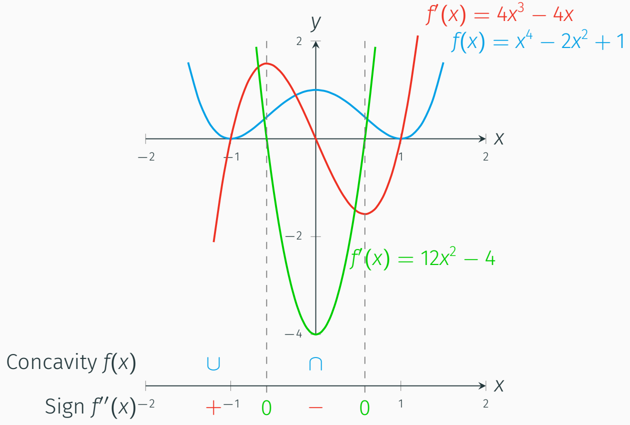

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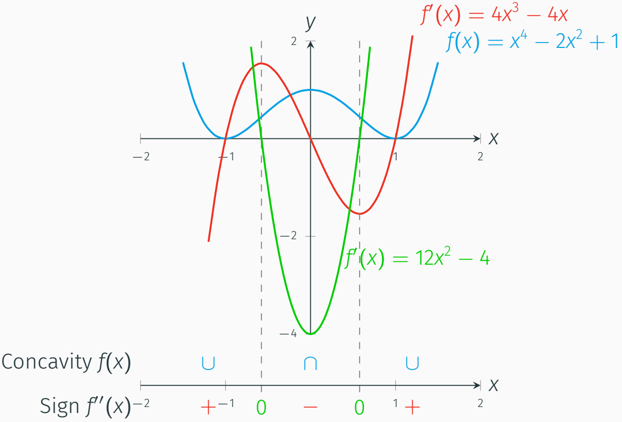

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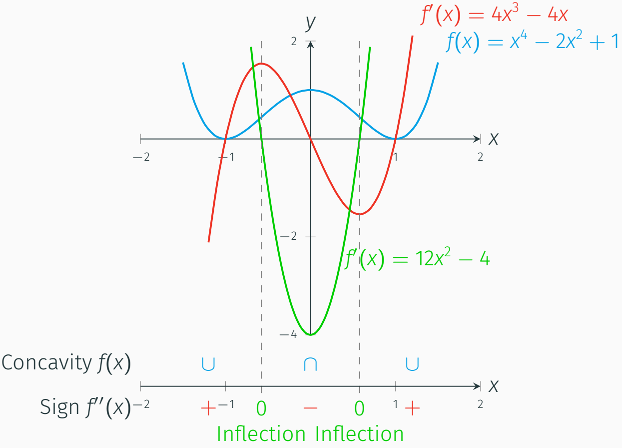


Approximating a function by a polynomial

Another useful application of the derivative is the approximation of functions by polynomials.

Polynomials are functions easy to calculate (sums and products) with very good properties:

- Defined in all the real numbers.
- Continuous.
- Differentiable of all orders with continuous derivatives.

Goal

Approximate a function $f(x)$ by a polynomial $p(x)$ near a value $x = x_0$.

A polynomial of grade 0 has equation

$$p(x) = c_0,$$

where c_0 is a constant.

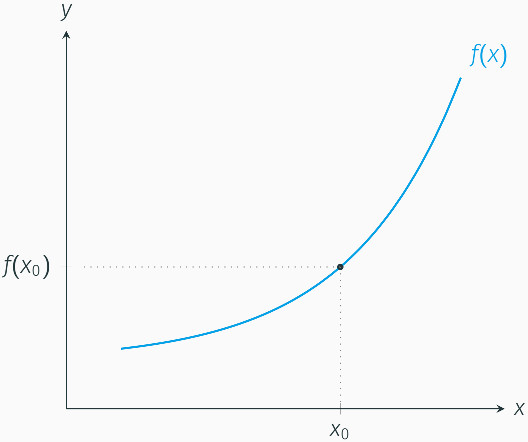
As the polynomial should coincide with the function at x_0, it must satisfy

$$p(x_0) = c_0 = f(x_0).$$

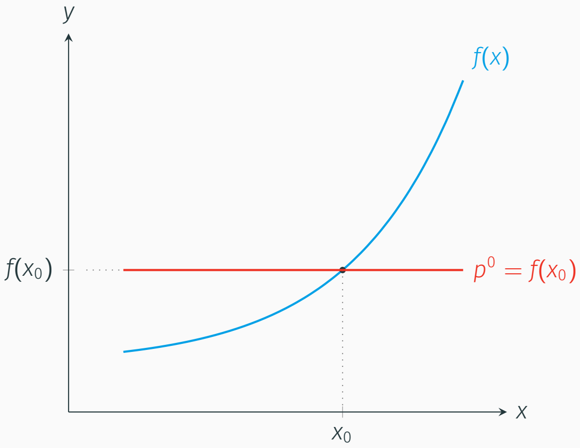
Therefore, the polynomial of grade 0 that best approximate f near x_0 is

$$p(x) = f(x_0).$$

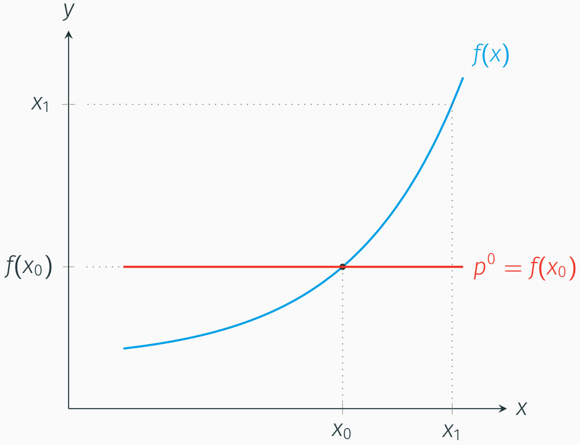
Approximating a function by a polynomial of grade 0



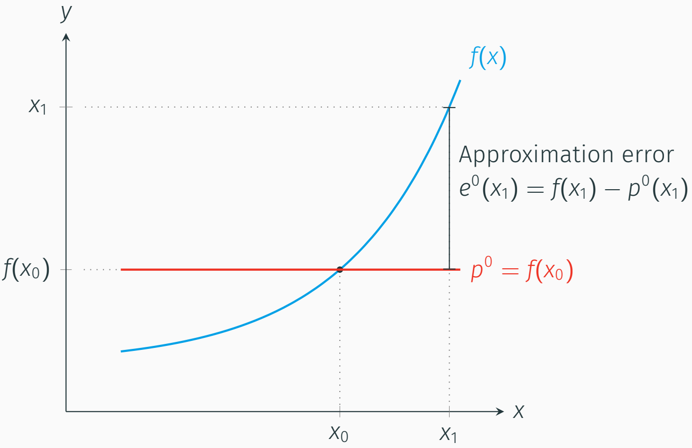
Approximating a function by a polynomial of grade 0



Approximating a function by a polynomial of grade 0



Approximating a function by a polynomial of grade 0



Approximating a function by a polynomial of grade 1

A polynomial of grade 1 has equation

$$p(x) = c_0 + c_1 x,$$

but it can also be expressed

$$p(x) = c_0 + c_1(x - x_0).$$

Among all the polynomials of grade 1, the one that best approximate $f(x)$ near x_0 is that that meets the following conditions

1. p and f coincide at x_0: $p(x_0) = f(x_0)$,
2. p and f have the same rate of change at x_0: $p'(x_0) = f'(x_0)$.

The last condition guarantee that p and f have approximately the same tendency, but it requires the function f to be differentiable at x_0.

The tangent line: Best approximating polynomial of grade 1

Imposing the previous conditions we have

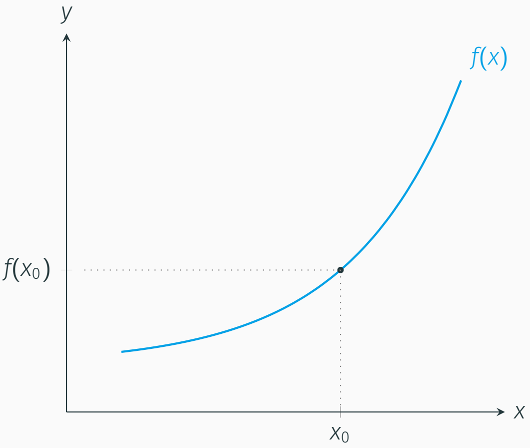
1. $p(x) = c_0 + c_1(x - x_0) \Rightarrow p(x_0) = c_0 + c_1(x_0 - x_0) = c_0 = f(x_0)$,
2. $p'(x) = c_1 \Rightarrow p'(x_0) = c_1 = f'(x_0)$.

Therefore, the polynomial of grade 1 that best approximates f near x_0 is

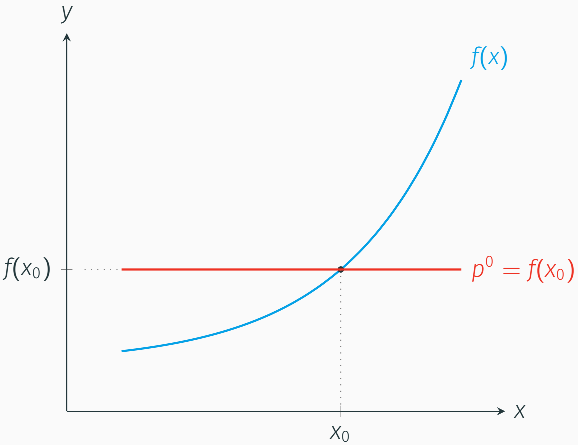
$$p(x) = f(x_0) + f'(x_0)(x - x_0),$$

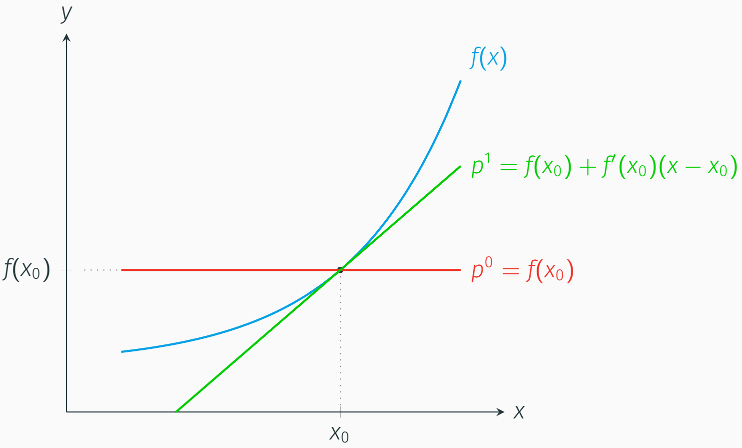
which turns out to be the tangent line to f at $(x_0, f(x_0))$.

Approximating a function by a polynomial of grade 1

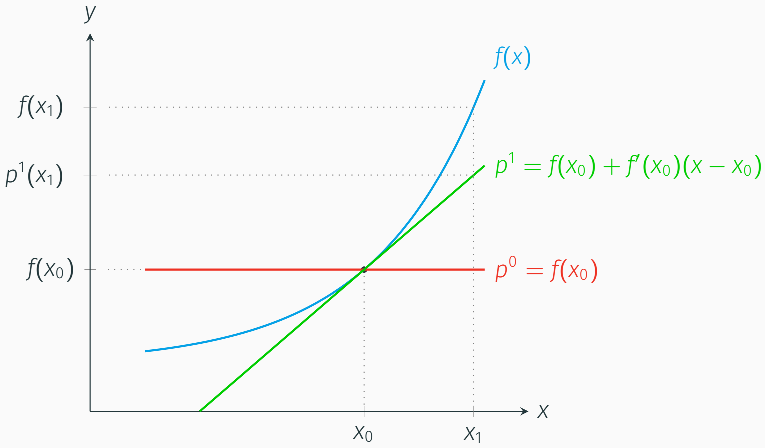


Approximating a function by a polynomial of grade 1




y
$f(x)$
$p^1 = f(x_0) + f'(x_0)(x - x_0)$
$f(x_0)$
$p^0 = f(x_0)$
x
x_0

Approximating a function by a polynomial of grade 1



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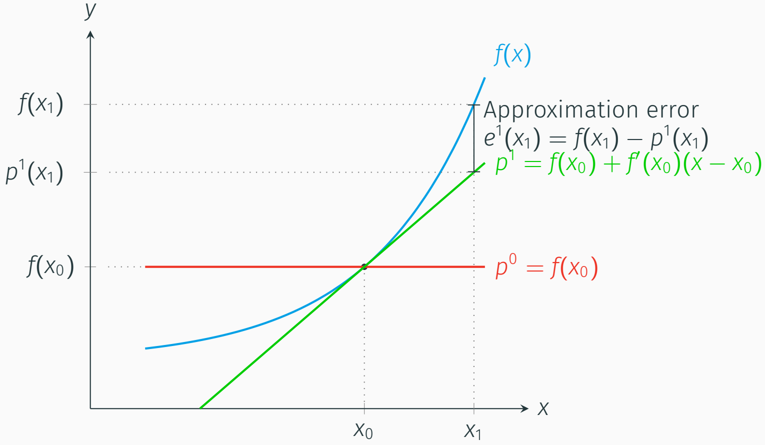


Approximating a function by a polynomial of grade 2

A polynomial of grade 2 is a parable with equation

$$p(x) = c_0 + c_1x + c_2x^2,$$

but it can also be expressed

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2.$$

Among all the polynomials of grade 2, the one that best approximate $f(x)$ near x_0 is that that meets the following conditions

1. p and f coincide at x_0: $p(x_0) = f(x_0)$,
2. p and f have the same rate of change at x_0: $p'(x_0) = f'(x_0)$.
3. p and f have the same concavity at x_0: $p''(x_0) = f''(x_0)$.

The last condition requires the function f to be differentiable twice at x_0.

Best approximating polynomial of grade 2

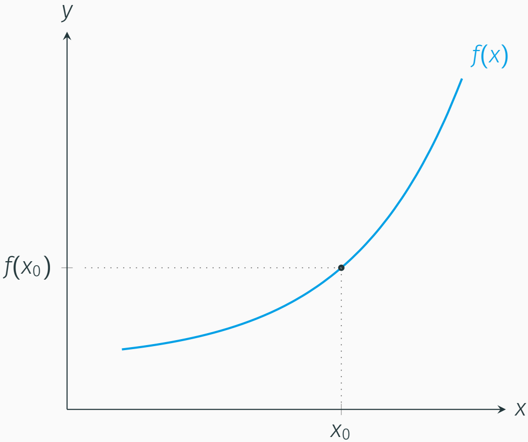
Imposing the previous conditions we have

1. $p(x) = c_0 + c_1(x - x_0) \Rightarrow p(x_0) = c_0 + c_1(x_0 - x_0) = c_0 = f(x_0)$,
2. $p'(x) = c_1 \Rightarrow p'(x_0) = c_1 = f'(x_0)$.
3. $p''(x) = 2c_2 \Rightarrow p''(x_0) = 2c_2 = f''(x_0) \Rightarrow c_2 = \frac{f''(x_0)}{2}$.

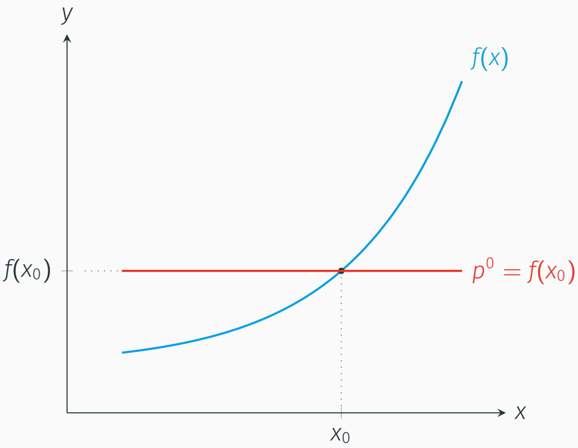
Therefore, the polynomial of grade 2 that best approximates f near x_0 is

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

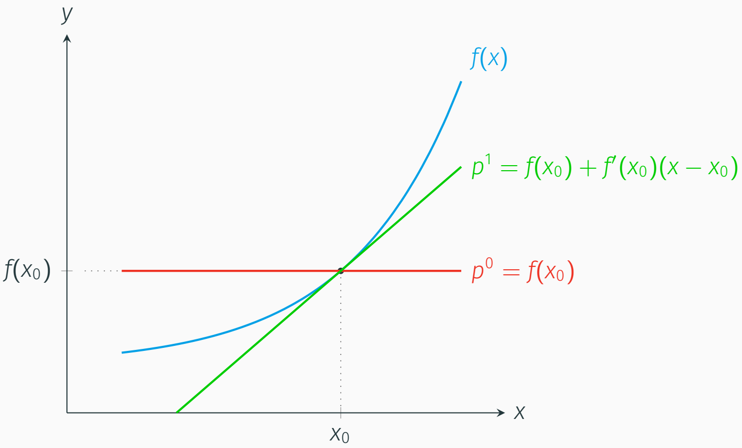
Approximating a function by a polynomial of grade 2

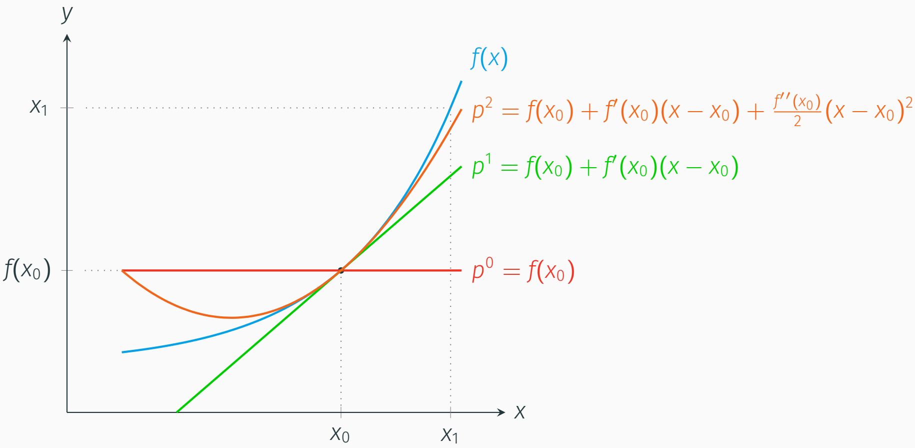


Approximating a function by a polynomial of grade 2

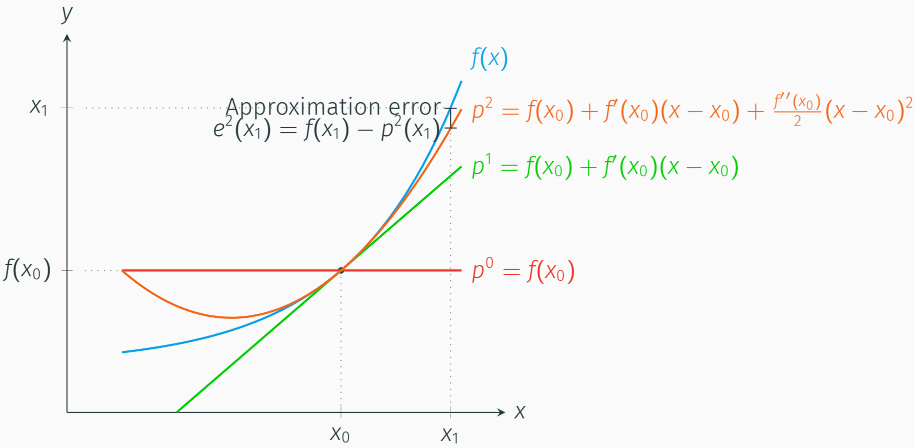


Approximating a function by a polynomial of grade 2





y
x
f(x)
x1
f(x0)
x0
x1
p2 = f(x0) + f′(x0)(x − x0) + f′′(x0)/2 (x − x0)2
p1 = f(x0) + f′(x0)(x − x0)
p0 = f(x0)



y
$f(x)$
x_1
Approximation error
$e^2(x_1) = f(x_1) - p^2(x_1)$
$p^2 = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$
$p^1 = f(x_0) + f'(x_0)(x - x_0)$
$f(x_0)$
$p^0 = f(x_0)$
x
x_0
x_1

A polynomial of grade n has equation

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n,$$

but it can also be expressed

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n.$$

Among all the polynomials of grade 2, the one that best approximate $f(x)$ near x_0 is that that meets the following $n + 1$ conditions

1. $p(x_0) = f(x_0)$,
2. $p'(x_0) = f'(x_0)$,
3. $p''(x_0) = f''(x_0)$,

   $\ldots$

n+1. $p^{(n)}(x_0) = f^{(n)}(x_0)$.

Observe that these conditions require the function f to be differentiable n times at x_0.

Coefficients calculation for the best approximating polynomial of grade n

The successive derivatives of p are

$$
\begin{aligned}
p(x) &= c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots + c_n(x-x_0)^n, \\
p'(x) &= c_1 + 2c_2(x-x_0) + \cdots + nc_n(x-x_0)^{n-1}, \\
p''(x) &= 2c_2 + \cdots + n(n-1)c_n(x-x_0)^{n-2}, \\
&\vdots \\
p^{(n)}(x) &= n(n-1)(n-2)\cdots 1 c_n = n!c_n.
\end{aligned}
$$

Imposing the previous conditions we have

1. $p(x_0) = c_0 + c_1(x_0-x_0) + c_2(x_0-x_0)^2 + \cdots + c_n(x_0-x_0)^n = c_0 = f(x_0)$,
2. $p'(x_0) = c_1 + 2c_2(x_0-x_0) + \cdots + nc_n(x_0-x_0)^{n-1} = c_1 = f'(x_0)$,
3. $p''(x_0) = 2c_2 + \cdots + n(n-1)c_n(x_0-x_0)^{n-2} = 2c_2 = f''(x_0) \Rightarrow c_2 = f''(x_0)/2$,

   $\ldots$

n+1. $p^{(n)}(x_0) = n!c_n = f^{(n)}(x_0) = c_n = \frac{f^{(n)}(x_0)}{n!}$.

Definition (Taylor polynomial)

Given a function $f(x)$ differentiable n times at x_0, the *Taylor polynomial* of order n of f at x_0 is the polynomial with equation

$$p^n_{f,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n =$$
$$= \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i,$$

The Taylor polynomial of order n of f at x_0 is the nth degree polynomial that best approximates f near x_0, as is the only one that meets the previous conditions.

Let's approximate the function $f(x) = \log x$ near the value 1 by a polynomial of order 3.

The equation of the Taylor polynomial of order 3 of f at $x_0 = 1$ is

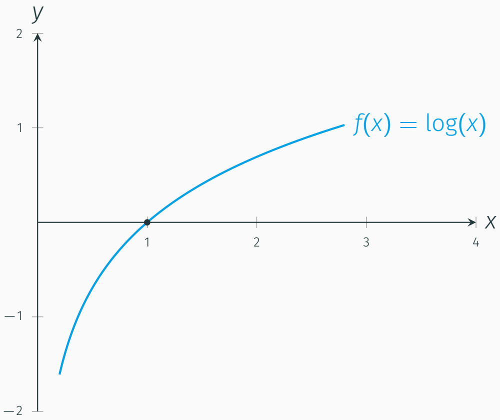
$$p^3_{f,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3.$$

The derivatives of f at 1 up to order 3 are

$$\begin{aligned} f(x) &= \log x & f(1) &= \log 1 = 0,\\ f'(x) &= 1/x & f'(1) &= 1/1 = 1,\\ f''(x) &= -1/x^2 & f''(1) &= -1/1^2 = -1,\\ f'''(x) &= 2/x^3 & f'''(1) &= 2/1^3 = 2. \end{aligned}$$

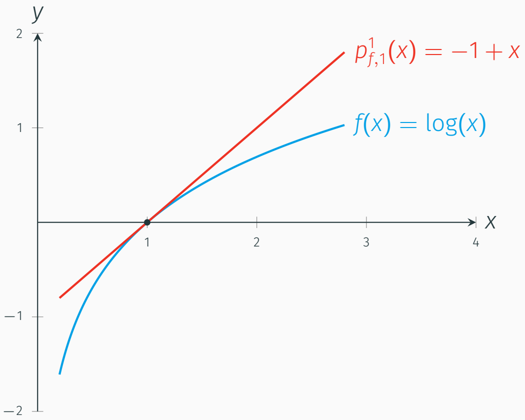
And substituting into the polynomial equation we get

$$p^3_{f,1}(x) = 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 = \frac{2}{3}x^3 - \frac{3}{2}x^2 + 3x - \frac{11}{6}.$$

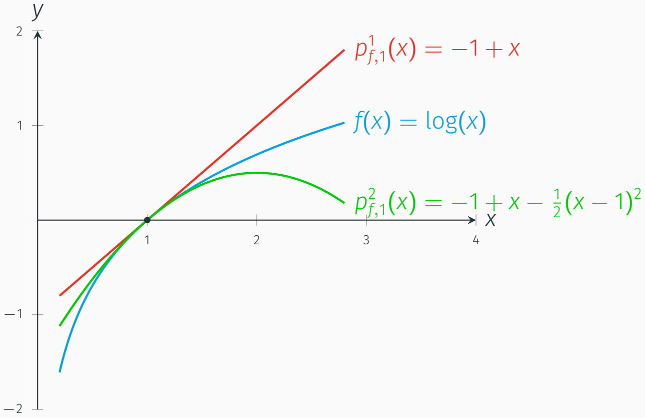


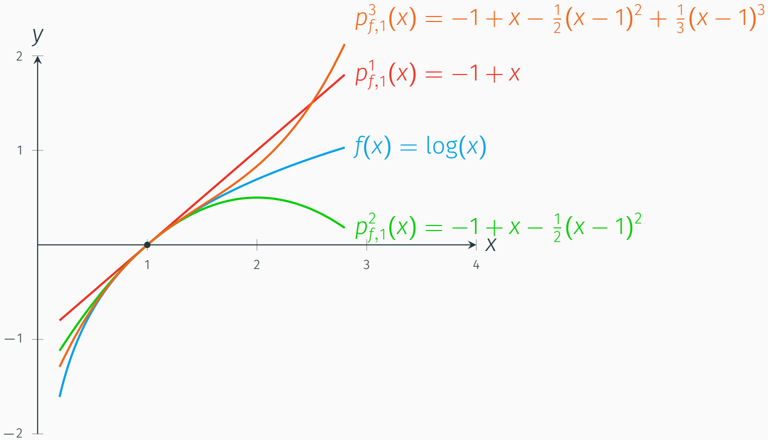
y
2
1
−1
−2
1
2
3
4
x
$f(x) = \log(x)$

Taylor polynomials of the logarithmic function



Taylor polynomials of the logarithmic function





$p^3_{f,1}(x) = -1 + x - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$
$p^1_{f,1}(x) = -1 + x$
$f(x) = \log(x)$
$p^2_{f,1}(x) = -1 + x - \frac{1}{2}(x-1)^2$
y
x
2
1
−1
−2
1
2
3
4

The Taylor polynomial equation simplifies when the polynomial is calculated at 0. This special case of Taylor polynomial at 0 is known as the *Mcalaurin polynomial*.

Definition (Maclaurin polynomial)

Given a function $f(x)$ differentiable n times at 0, the *Maclaurin polynomial* of order n of f is the polynomial with equation

$$p^n_{f,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n =$$
$$= \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!}x^i.$$

Let's approximate the function $f(x) = \sin x$ near the value 0 by a polynomial of order 3.

The Maclaurin polynomial equation of order 3 of f is

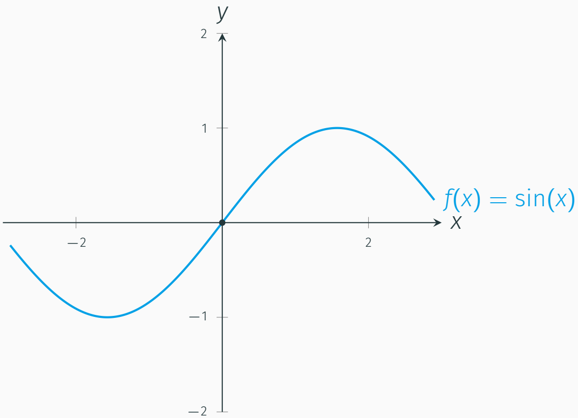
$$p_{f,0}^3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3.$$

The derivatives of f at 0 up to order 3 are

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0, \\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1, \\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0, \\ f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1. \end{aligned}$$

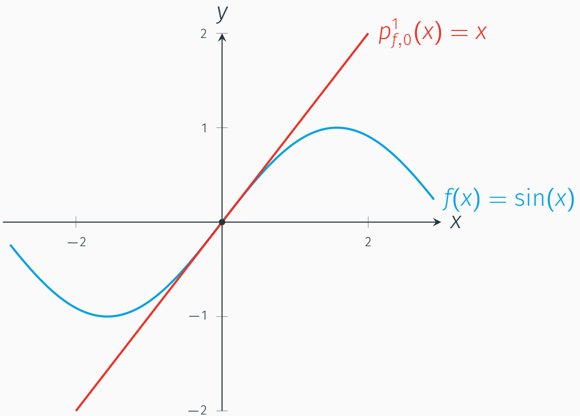
And substituting into the polynomial equation we get

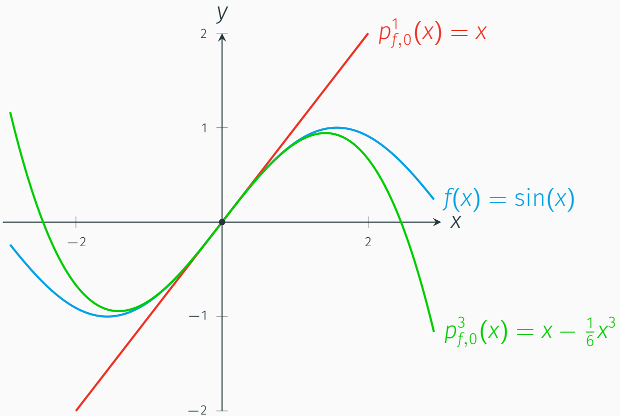
$$p_{f,0}^3(x) = 0 + 1 \cdot x + \frac{0}{2}x^2 + \frac{-1}{3!}x^3 = x - \frac{x^3}{6}.$$



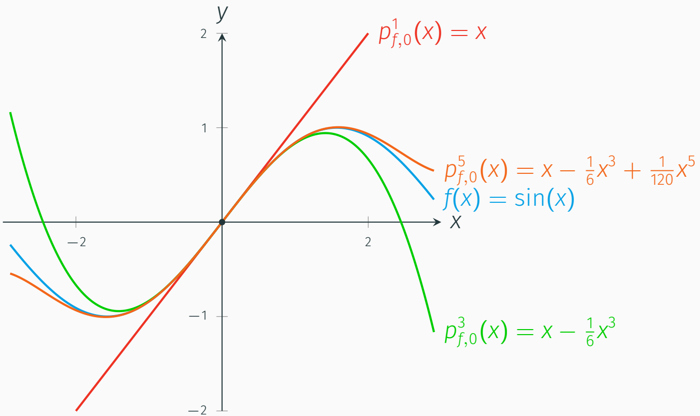
y
2
1
-1
-2
-2
2
x
$f(x) = \sin(x)$

Maclaurin polynomial of the sine function




y
2
1
−1
−2
−2
2
x
$p^1_{f,0}(x) = x$
$f(x) = \sin(x)$
$p^3_{f,0}(x) = x - \frac{1}{6}x^3$

Maclaurin polynomial of the sine function



$f(x)$	$p^n_{f,0}(x)$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k-1}}{(2k-1)!}$ if $n = 2k$ or $n = 2k-1$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$ if $n = 2k$ or $n = 2k+1$
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^k \frac{x^{2k-1}}{(2k-1)}$ if $n = 2k$ or $n = 2k-1$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\log(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n}$

Taylor remainder and Taylor formula

Taylor polynomials allow to approximate a function in a neighborhood of a value x_0, but there is always an error in the approxition.

Definition (Taylor remainder)

Given a function $f(x)$ and its Taylor polynomial of order n at x_0, $p^n_{f,x_0}(x)$, the *Taylor remainder* of order n of f at x_0 is de difference between the function and the polynomial,

$$r^n_{f,x_0}(x) = f(x) - p^n_{f,x_0}(x).$$

The Taylor remainder measures the error int the approximation of $f(x)$ by the Taylor polynomial and allow us to express the function as the Taylor polynomial plus the Taylor remainder

$$f(x) = p^n_{f,x_0}(x) + r^n_{f,x_0}(x).$$

This expression is known as *Taylor formula* of order n or f at x_0.

It can be proved that

$$\lim_{h\to 0} \frac{r^n_{f,x_0}(x_0+h)}{h^n} = 0,$$