Elementary Calculus Manual

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1.1 Vectors

Scalars

Some phenomena of Nature can be described by a number and a unit of measurement.

Definition 1 (Scalar)**.** A *scalar* is a number that expresses a magnitude without direction.

Examples The height or weight of a person, the temperature of a gas or the time it takes a vehicle to travel a distance.

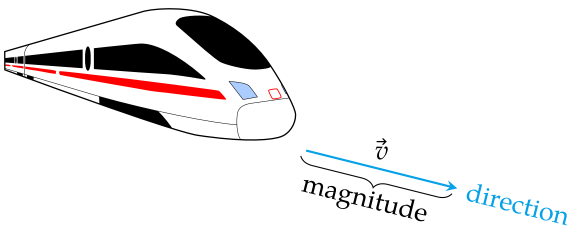
However, there are other phenomena that cannot be described adequately by an scalar. If, for instance, a sailor wants to head for seaport and only knows the intensity of wind, he won't know what direction to take. The description of wind requires two elements: intensity and direction.

Vectors

Definition 2 (Vector)**.** A *vector* is a number that expresses a magnitude and has associated an orientation and a sense.

Examples The velocity of a vehicle or the force applied to an object.

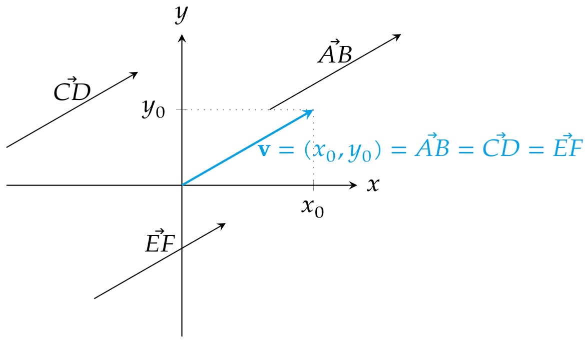
Geometrically, a vector is represented by an directed line segment, that is, an arrow.



Vector representation

An oriented segment can be located in different places in a Cartesian space. However, regardless of where it is located, if the length and the direction of the segment doesn't change, the segment represents always the same vector.

This allows to represent all vector with the same origin, the origin of the Cartesian coordinate system. Thus, a vector can be represented by the Cartesian *coordinates* of its final end in any Euclidean space.

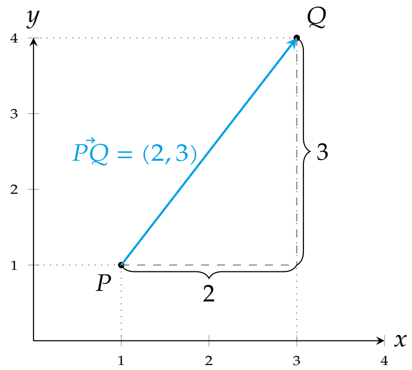


Vector from two points

Given two points P and Q of a Cartesian space, the vector that starts at P and ends at Q has coordinates $\vec{PQ} = Q - P$.

Example Given the points $P = (1, 1)$ and $Q = (3, 4)$ in the real plane $\mathbb{R}^2$, the coordinates of the vector that start at P and ends at Q are

$$\vec{PQ} = Q - P = (3, 4) - (1, 1) = (3 - 2, 4 - 1) = (2, 3).$$



Module of a vector

Definition 3 (Module of a vector)**.** Given a vector $\mathbf{v} = (v_1, \cdots, v_n)$ in $\mathbb{R}^n$, the *module* of $\mathbf{v}$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The module of a vector coincides with the length of the segment that represents the vector.

Examples Let $\mathbf{u} = (3, 4)$ be a vector in $\mathbb{R}^2$, then its module is

$$|\mathbf{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Let $\mathbf{v} = (4, 7, 4)$ be a vector in $\mathbb{R}^3$, then its module is

$$|\mathbf{v}| = \sqrt{4^2 + 7^2 + 4^2} = \sqrt{81} = 9$$

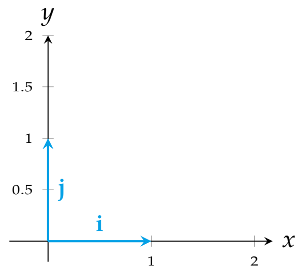
Unit vectors

Definition 4 (Unit vector)**.** A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a *unit vector* if its module is one, that is, $|\mathbf{v}| = 1$.

The unit vectors with the direction of the coordinate axes are of special importance and they form *standard basis.*

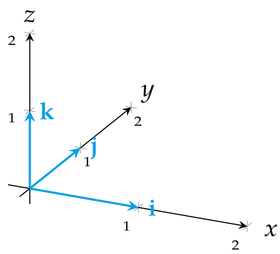
In $\mathbb{R}^2$ the standard basis is formed by two vectors

$$\mathbf{i} = (1, 0) \text{ and } \mathbf{j} = (0, 1)$$



In $\mathbb{R}^3$ the standard basis is formed by three vectors

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0) \text{ and } \mathbf{k} = (0, 0, 1)$$



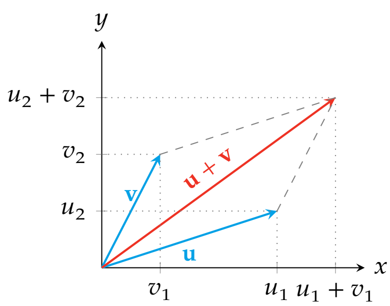
Sum of two vectors

Definition 5 (Sum of two vectors). Given two vectors $\mathbf{u} = (u_1, \cdots, u_n)$ y $\mathbf{v} = (v_1, \cdots, v_n)$ de $\mathbb{R}^n$, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n).$$

Example Let $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (2, 3)$ two vectors in $\mathbb{R}^2$, then the sum of them is

$$\mathbf{u} + \mathbf{v} = (3 + 2, 1 + 3) = (5, 4).$$



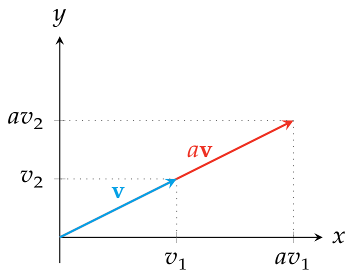
Product of a vector by an scalar

Definition 6 (Product of a vector by an scalar). Given a vector $\mathbf{v} = (v_1, \cdots, v_n)$ in $\mathbb{R}^n$, and a scalar $a \in \mathbb{R}$, the *product* of $\mathbf{v}$ by a is

$$a\mathbf{v} = (av_1, \ldots, av_n).$$

Example Let $\mathbf{v} = (2, 1)$ a vector in $\mathbb{R}^2$ and $a = 2$ a scalar, then the product of a by $\mathbf{v}$ is

$$a\mathbf{v} = 2(2, 1) = (4, 2).$$

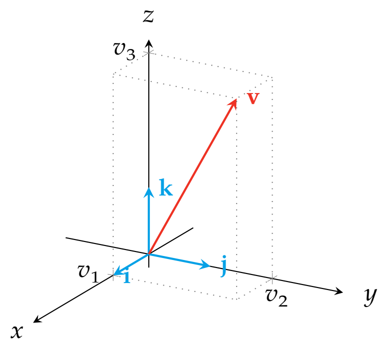


Expressing a vector as a linear combination of the standard basis

The sum of vectors and the product of vector by a scalar allow as to express any vector as a linear combination of the standard basis.

In $\mathbb{R}^3$, for instance, a vector with coordinates $\mathbf{v} = (v_1, v_2, v_3)$ can be expressed as the linear combination

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$



Dot product of two vectors

Definition 7 (Dot product of two vectors)**.** Given the vectors $\mathbf{u} = (u_1, \cdots, u_n)$ and $\mathbf{v} = (v_1, \cdots, v_n)$ in $\mathbb{R}^n$, the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \cdots + u_nv_n.$$

Example Let $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (2, 3)$ two vectors in $\mathbb{R}^2$, the the dot product of them is

$$\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + 1 \cdot 3 = 9.$$

It holds that

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \alpha$$

where α is the angle between the vectors.

Parallel vectors

Definition 8 (Parallel vectors)**.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are *parallel* if there is a scalar $a \in \mathbb{R}$ such that

$$\mathbf{u} = a\mathbf{v}.$$

Example The vectors $\mathbf{u} = (-4, 2)$ and $\mathbf{v} = (2, -1)$ in $\mathbb{R}^2$ are parallel, as there is a scalar -2 such that

$$\mathbf{u} = (-4, 2) = -2(2, -1) = -2\mathbf{v}.$$

Orthogonal and orthonormal vectors

Definition 9 (Orthogonal and orthonormal vectors)**.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if their dot product is zero,

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

If in addition both vectors are unit vectors, $|\mathbf{u}| = |\mathbf{v}| = 1$, then the vectors are *orthonormal*.

Orthogonal vectors are perpendicular, that is the angle between them is right.

Example The vectors $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-2, 4)$ in $\mathbb{R}^2$ are orthogonal, as

$$\mathbf{uv} = 2 \cdot -2 + 1 \cdot 4 = 0,$$

but they are not orthonormal as $|\mathbf{u}| = \sqrt{2^2 + 1^2} \neq 1$ and $|\mathbf{v}| = \sqrt{-2^2 + 4^2} \neq 1$.

The vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ in $\mathbb{R}^2$ are orthonormal, as

$$\mathbf{ij} = 1 \cdot 0 + 0 \cdot 1 = 0, \quad |\mathbf{i}| = \sqrt{1^2 + 0^2} = 1, \quad |\mathbf{j}| = \sqrt{0^2 + 1^2} = 1.$$

1.2 Lines

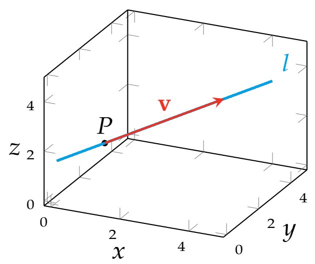
Vectorial equation of a straight line

Definition 10 (Vectorial equation of a straight line)**.** Given a point $P = (p_1, \dots, p_n)$ and a vector $\mathbf{v} = (v_1, \dots, v_n)$ of $\mathbb{R}^n$, the *vectorial equation of the line* l that passes through the point P with the direction of $\mathbf{v}$ is

$$l : X = P + t\mathbf{v} = (p_1, \dots, p_n) + t(v_1, \dots, v_n) = (p_1 + tv_1, \dots, p_n + tv_n), \quad t \in \mathbb{R}.$$

Example Let l the line of $\mathbb{R}^3$ that goes through $P = (1, 1, 2)$ with the direction of $\mathbf{v} = (3, 1, 2)$, then the vectorial equation of l is

$$\begin{aligned} l : X = P + t\mathbf{v} = (1, 1, 2) + t(3, 1, 2) = \\ = (1 + 3t, 1 + t, 2 + 2t) \quad t \in \mathbb{R}. \end{aligned}$$



Parametric and Cartesian equations of a line

From the vectorial equation of a line $l : X = P + t\mathbf{v} = (p_1 + tv_1, \dots, p_n + tv_n)$ is easy to obtain the coordinates of the the points of the line with n *parametric equations*

$$x_1(t) = p_1 + tv_1, \dots, x_n(t) = p_n + tv_n$$

from where, if $\mathbf{v}$ is a vector with non-null coordinates ($v_i \neq 0\ \forall i$), we can solve for t and equal the equations getting the *Cartesian equations*

$$\frac{x_1 - p_1}{v_1} = \cdots = \frac{x_n - p_n}{v_n}$$

Example Given a line with vectorial equation $l : X = (1,1,2) + t(3,1,2) = (1+3t, 1+t, 2+2t)$ in $\mathbb{R}^3$, its parametric equations are

$$x(t) = 1 + 3t, \quad y(t) = 1 + t, \quad z(t) = 2 + 2t,$$

and the Cartesian equations are

$$\frac{x-1}{3} = \frac{y-1}{1} = \frac{z-2}{2}$$

Point-slope equation of a line in the plane

In the particular case of the real plane $\mathbb{R}^2$, if we have a line with vectorial equation $l : X = P + t\mathbf{v} = (x_0, y_0) + t(a,b) = (x_0 + ta, y_0 + tb)$, its parametric equations are

$$x(t) = x_0 + ta, \quad y(t) = y_0 + tb$$

and its Cartesian equation is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}.$$

From this, passing b to the other side of the equation, we get

$$y - y_0 = \frac{b}{a}(x - x_0),$$

or renaming $m = b/a$,

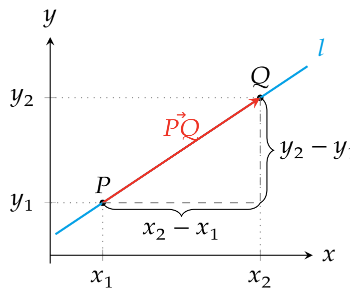
$$y - y_0 = m(x - x_0).$$

This equation is known as the *point-slope equation* of the line.

Slope of a line in the plane

Definition 11 (Slope of a line in the plane). Given a line $l : X = P + t\mathbf{v}$ in the real plane $\mathbb{R}^2$, with direction vector $\mathbf{v} = (a, b)$, the *slope* of l is b/a.

Recall that given two points $Q = (x_1, y_1)$ y $Q = (x_2, y_2)$ of the line l, we can take as a direction vector the vector from P to Q, with coordinates $\vec{PQ} = Q - P = (x_2 - x_1, y_2 - y_1)$. Thus, the slope of l is $\dfrac{y_2 - y_1}{x_2 - x_1}$, that is, the ratio between the changes in the vertical and horizontal axes respectively.



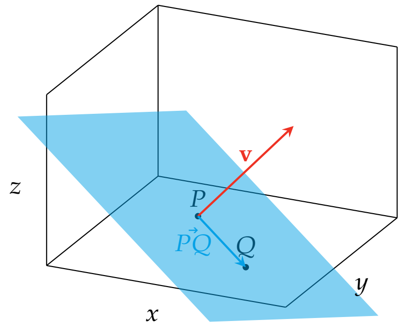
1.3 Planes

Vector equation of a plane in space

To get the equation of a plane in the real space $\mathbb{R}^3$ we can take a point of the plane $P = (x_0, y_0, z_0)$ and an orthogonal vector to the plane $\mathbf{v} = (a, b, c)$. Then, any point $Q = (x, y, z)$ of the plane meets that the vector $\vec{PQ} = (x - x_0, y - y_0, z - z_0)$ is orthogonal to $\mathbf{v}$, and therefore their dot product is zero.

$$\vec{PQ} \cdot \mathbf{v} = (x - x_0, y - y_0, z - z_0)(a, b, c) = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation is known as the *vector equation of the plane*.



Scalar equation of a plane in space

From the vector equation of the plane we can get

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \Leftrightarrow ax+by+cz=ax_0+by_0+cz_0,$$

that, renaming $d=ax_0+by_0+cz_0$, can be written

$$ax+by+cz=d,$$

and is known as the *scalar equation of the plane*.

Example Given the point $P=(2,1,1)$ and the vector $\mathbf{v}=(2,1,2)$, the vector equation of the plane that passes through P and is orthogonal to $\mathbf{v}$ is

$$(x-2,y-1,z-1)(2,1,2)=2(x-2)+(y-1)+2(z-1)=0,$$

and its scalar equation is

$$2x+y+2z=7.$$

2 Differential calculus with one real variable

2.1 Concept of derivative

Increment

Definition 12 (Increment of a variable)**.** An *increment* of a variable x is a change in the value of the variable; is it is denoted Δx. The increment of a variable x along an interval $[a,b]$ is given by

$$\Delta x = b - a.$$

Definition 13 (Increment of a function)**.** The *increment* of a function $y = f(x)$ along an interval $[a,b] \subseteq Dom(f)$ is given by

$$\Delta y = f(b) - f(a).$$

Example The increment of x along the interval $[2,5]$ is $\Delta x = 5 - 2 = 3$ and the increment of the function $y = x^2$ along the same interval is $\Delta y = 5^2 - 2^2 = 21$.

Average rate of change

The study of a function $y = f(x)$ requires to understand how the function changes, that is, how the dependent variable y changes when we change the independent variable x.

Definition 14 (Average rate of change)**.** The *average rate of change* of a function $y = f(x)$ in an interval $[a, a+\Delta x] \subseteq Dom(f)$, is the quotient between the increment of y and the increment of x in that interval, and is denoted by

$$\text{ARC}\, f[a, a+\Delta x] = \frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

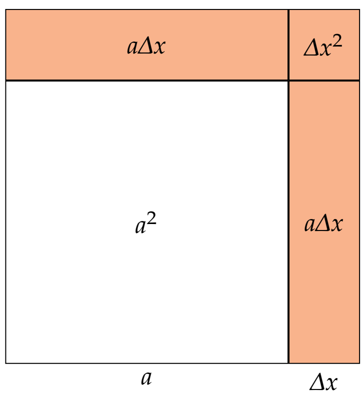
Average rate of change

Example of the area of a square

Let $y = x^2$ be the function that measures the area of a metallic square of side length x.

If at any given time the side of the square is a, and we heat the square uniformly increasing the side by dilatation a quantity Δx, how much will the area of the square increase?

$$\begin{aligned}\Delta y &= f(a+\Delta x) - f(a) = (a+\Delta x)^2 - a^2 = \\ &= a^2 + 2a\Delta x + \Delta x^2 - a^2 = 2a\Delta x + \Delta x^2.\end{aligned}$$

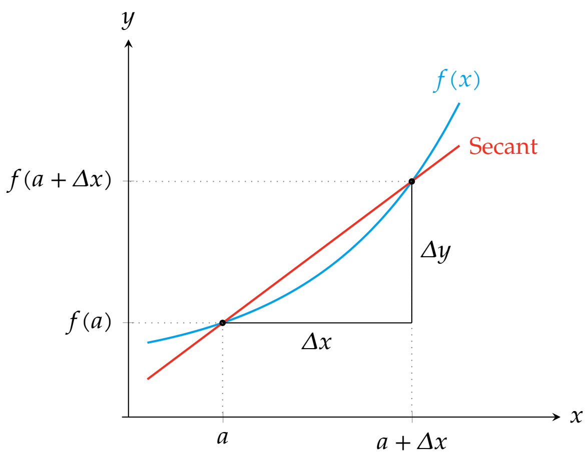


What is the average rate of change in the interval $[a, a+\Delta x]$?

$$\text{ARC}\, f[a, a+\Delta x] = \frac{\Delta y}{\Delta x} = \frac{2a\Delta x + \Delta x^2}{\Delta x} = 2a + \Delta x.$$

Geometric interpretation of the average rate of change

The average rate of change of a function $y = f(x)$ in an interval $[a, a+\Delta x]$ is the slope of the *secant* line to the graph of f through the points $(a, f(a))$ and $(a+\Delta x, f(a+\Delta x))$.



Instantaneous rate of change

Often it is interesting to study the rate of change of a function, not in an interval, but in a point.

Knowing the tendency of change of a function in an instant can be used to predict the value of the function in nearby instants.

Definition 15 (Instantaneous rate of change and derivative)**.** The *instantaneous rate of change* of a function $f(x)$ at a point $x = a$, is the limit of the average rate of change of f in the interval $[a, a+\Delta x]$, when Δx tends to 0, and is denoted by

$$\text{IRC}\, f(a) = \lim_{\Delta x \to 0} \text{ARC}\, f[a, a+\Delta x] = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

When this limit exists, the function f is said to be *differentiable* at the point a, and its value is called the *derivative* of f at a, and it is denoted $f'(a)$ (Lagrange's notation) or $\frac{df}{dx}(a)$ (Leibniz's notation).

Instantaneous rate of change

Example of the area of a square

Let's take again the function $y = x^2$ that measures the area of a metallic square of side length x.

If at any given time the side of the square is a, and we heat the square uniformly increasing the side, what is the tendency of change of the area in that moment?

$$\begin{aligned}\text{IRC}\, f(a) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \\ &= \lim_{\Delta x \to 0} \frac{2a\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 2a + \Delta x = 2a.\end{aligned}$$

Thus,

$$f'(a) = \frac{df}{dx}(a) = 2a,$$

indicating that the area of the square tends to increase the double of the side.

Interpretation of the derivative

The derivative of a function $f'(a)$ shows the growth rate of f at point a:

- $f'(a) > 0$ indicates an increasing tendency (y increases as x increases).
- $f'(a) < 0$ indicates a decreasing tendency (y decreases as x increases).

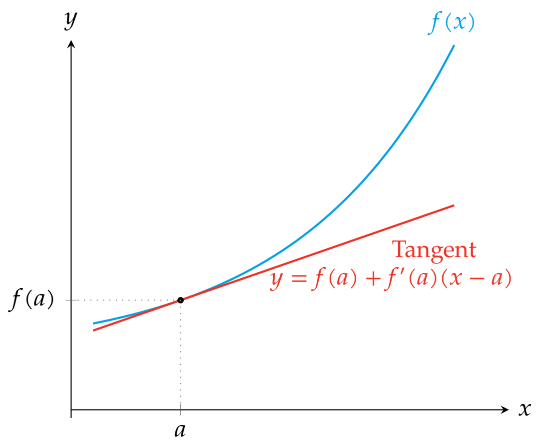
Example A derivative $f'(a) = 3$ indicates that y tends to increase triple of x at point a. A derivative $f'(a) = -0.5$ indicates that y tends to decrease half of x at point a.

Geometric interpretation of the derivative

We have seen that the average rate of change of a function $y = f(x)$ in an interval $[a, a + \Delta x]$ is the slope of the *secant* line, but when Δx tends to 0, the secant line becomes the tangent line.

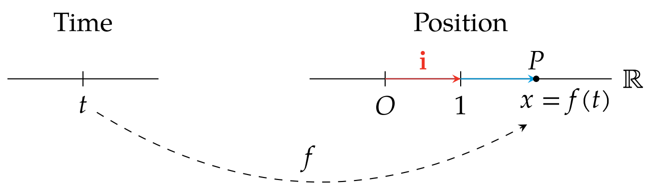
The instantaneous rate of change or derivative of a function $y = f(x)$ at $x = a$ is the slope of the *tangent line* to the graph of f at point $(a, f(a))$. Thus, the equation of the tangent line to the graph of f at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a) \Leftrightarrow y = f(a) + f'(a)(x - a)$$



Kinematic applications: Linear motion

Assume that the function $y = f(t)$ describes the position of an object moving in the real line at time t. Taking as reference the coordinates origin O and the unitary vector $\mathbf{i} = (1)$, we can represent the position of the moving object P at every moment t with a vector $\vec{OP} = x\mathbf{i}$ where $x = f(t)$.



Remark It also makes sense when f measures other magnitudes as the temperature of a body, the concentration of a gas, or the quantity of substance in a chemical reaction at every moment t.

Kinematic interpretation of the average rate of change

In this context, if we take the instants $t = a$ and $t = a + \Delta t$, both in $\mathrm{Dom}(f)$, the vector

$$\mathbf{v}_m = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

is known as the *average velocity* of the trajectory f in the interval $[a, a+\Delta t]$.

Example A vehicle makes a trip from Madrid to Barcelona. Let $f(t)$ be the function that determine the position of the vehicle at every moment t. If the vehicle departs from Madrid (km 0) at 8:00 and arrives at Barcelona (km 600) at 14:00, then the average velocity of the vehicle in the path is

$$\mathbf{v}_m = \frac{f(14) - f(8)}{14-8} = \frac{600-0}{6} = 100 km/h.$$

Kinematic interpretation of the derivative

In the same context of the linear motion, the derivative of the function $f(t)$ at the moment t_0 is the vector

$$\mathbf{v} = f'(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t},$$

that is known, as long as the limit exists, as the *instantaneous velocity* or simply *velocity* of the trajectory f at moment a.

That is, the derivative of the object position with respect to time is a vector field that is called *velocity along the trajectory f*.

Example Following with the previous example, what indicates the speedometer at any instant is the modulus of the instantaneous velocity vector at that moment.

2.2 Algebra of derivatives

Properties of the derivative

If $y = c$, is a constant function, then $y' = 0$ at any point.

If $y = x$, is the identity function, then $y' = 1$ at any point.

If $u = f(x)$ and $v = g(x)$ are two differentiable functions, then

- $(u+v)' = u' + v'$
- $(u-v)' = u' - v'$
- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

Derivative of a composite function

The chain rule

Theorem 16 (Chain rule). *If the function $y = f \circ g$ is the composition of two functions $y = f(z)$ and $z = g(x)$, then*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

It is easy to prove this fact using the Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = f'(z)g'(x) = f'(g(x))g'(x).$$

Example If $f(z) = \sin z$ and $g(x) = x^2$, then $f \circ g(x) = \sin(x^2)$. Applying the chain rule the derivative of the composite function is

$$(f \circ g)'(x) = f'(g(x))g'(x) = \cos(g(x))2x = \cos(x^2)2x.$$

On the other hand, $g \circ f(z) = (\sin z)^2$, and applying the chain rule again, its derivative is

$$(g \circ f)'(z) = g'(f(z))f'(z) = 2f(z)\cos z = 2\sin z \cos z.$$

Derivative of the inverse of a function

Theorem 17 (Derivative of the inverse function)**.** *Given a function $y = f(x)$ with inverse $x = f^{-1}(y)$, then*

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))},$$

provided that f is differentiable at $f^{-1}(y)$ and $f'(f^{-1}(y)) \neq 0$.

Again, it is easy to prove this equality using the Leibniz notation

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

Derivative of the inverse of a function

Example

The inverse of the exponential function $y = f(x) = e^x$ is the natural logarithm $x = f^{-1}(y) = \ln y$, so we can compute the derivative of the natural logarithm using the previous theorem and we get

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y}.$$

Example Sometimes it is easier to apply the chain rule to compute the derivative of the inverse of a function. In this example, as $\ln x$ is the inverse of e^x, we know that $e^{\ln x} = x$, so differentiating both sides and applying the chain rule to the left side we get

$$(e^{\ln x})' = x' \Leftrightarrow e^{\ln x}(\ln(x))' = 1 \Leftrightarrow (\ln(x))' = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

2.3 Analysis of functions

Analysis of functions: increase and decrease

The main application of derivatives is to determine the variation (increase or decrease) of functions. For that we use the sign of the first derivative.

Theorem 18. *Let $f(x)$ be a function with first derivative in an interval $I \subseteq \mathbb{R}$.*

- *If $\forall x \in I$ $f'(x) > 0$ then f is increasing on I.*
- *If $\forall x \in I$ $f'(x) < 0$ then f is decreasing on I.*

If $f'(a) = 0$ then a is known as a *critical point* or *stationary point*. At this point the function can be increasing, decreasing or neither increasing nor decreasing.

Example The function $f(x) = x^2$ has derivative $f'(x) = 2x$; it is decreasing on $\mathbb{R}^-$ as $f'(x) < 0$ $\forall x \in \mathbb{R}^-$ and increasing on $\mathbb{R}^+$ as $f'(x) > 0$ $\forall x \in \mathbb{R}^+$.

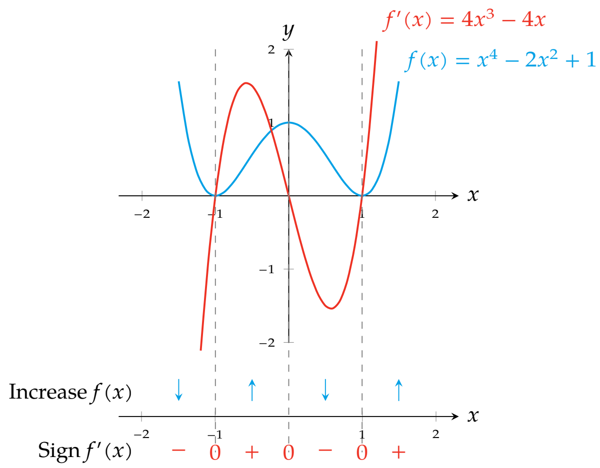
It has a critical point at $x = 0$, as $f'(0) = 0$; at this point the function is neither increasing nor decreasing.

Remark A function can be increasing or decreasing on an interval and not have first derivative.

Analysis of functions: increase and decrease

Example

Let us analyze the increase and decrease of the function $f(x) = x^4 - 2x^2 + 1$. Its first derivative is $f'(x) = 4x^3 - 4x$.



Analysis of functions: relative extrema

As a consequence of the previous result we can also use the first derivative to determine the relative extrema of a function.

Theorem 19 (First derivative test)**.** *Let $f(x)$ be a function with first derivative in an interval $I \subseteq \mathbb{R}$ and let $a \in I$ be a critical point of f ($f'(a) = 0$).*

- *If $f'(x) > 0$ on an open interval extending left from a and $f'(x) < 0$ on an open interval extending right from a, then f has a relative maximum at a.*
- *If $f'(x) < 0$ on an open interval extending left from a and $f'(x) > 0$ on an open interval extending right from a, then f has a relative minimum at a.*
- *If $f'(x)$ has the same sign on both an open interval extending left from a and an open interval extending right from a, then f has an inflection point at a.*

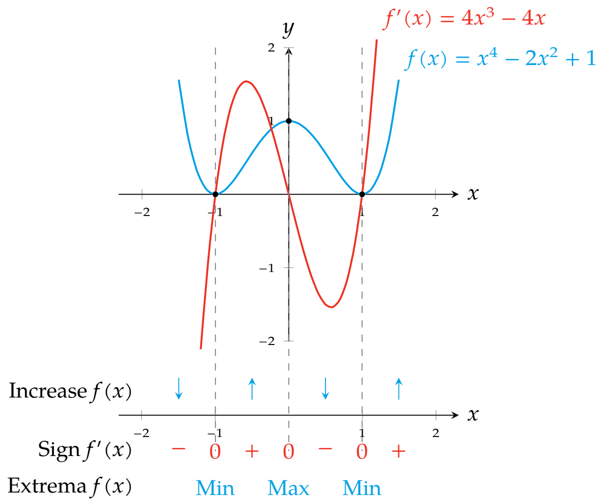
Remark *A vanishing derivative is a necessary but not sufficient condition for the function to have a relative extrema at a point.*

Example The function $f(x) = x^3$ has derivative $f'(x) = 3x^2$; it has a critical point at $x = 0$. However it does not have a relative extrema at that point, but an inflection point.

Analysis of functions: relative extrema

Example

Consider again the function $f(x) = x^4 - 2x^2 + 1$ and let's analyze its relative extrema now. Its first derivative is $f'(x) = 4x^3 - 4x$.



Analysis of functions: concavity

The concavity of a function can be determined by de second derivative.

Theorem 20. *Let $f(x)$ be a function with second derivative in an interval $I \subseteq \mathbb{R}$.*

- *If $\forall x \in I\ f''(x) > 0$ then f is concave up (convex) on I.*
- *If $\forall x \in I\ f''(x) < 0$ then f is concave down (concave) on I.*

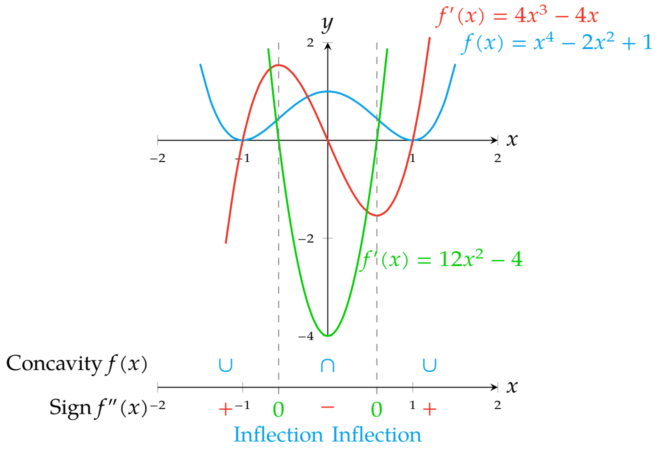
Example The function $f(x) = x^2$ has second derivative $f''(x) = 2 > 0\ \forall x \in \mathbb{R}$, so it is concave up in all $\mathbb{R}$.

Remark *A function can be concave up or down and not have second derivative.*

Analysis of functions: concavity

Example

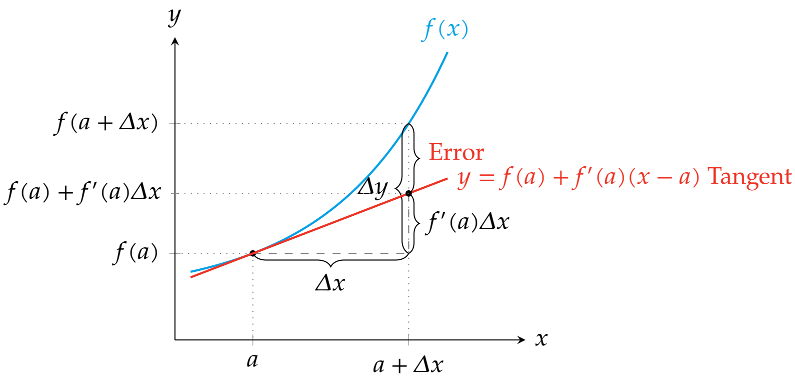
Let us analyze the concavity of the same function of previous examples $f(x) = x^4 - 2x^2 + 1$. Its second derivative is $f''(x) = 12x^2 - 4$.



2.4 Function approximation

Approximating a function with the derivative

The tangent line to the graph of a function $f(x)$ at $x = a$ can be used to approximate f in a neighbourhood of a.



Thus, the increment of a function $f(x)$ in an interval $[a, a+\Delta x]$ can be approximated multiplying the derivative of f at a by the increment of x

$$\Delta y \approx f'(a)\Delta x$$

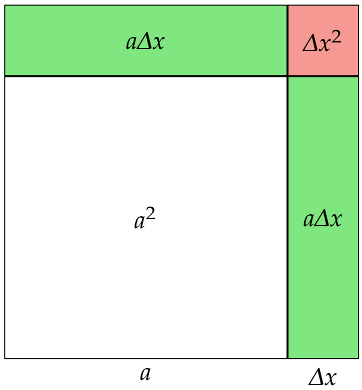
Approximating a function with the derivative

Example of the area of a square

In the previous example of the function $y = x^2$ that measures the area of a metallic square of side x, if the side of the square is a and we increment it by a quantity Δx, then the increment on the area will be approximately

$$\Delta y \approx f'(a)\Delta x = 2a\Delta x.$$

In the figure below we can see that the error of this approximation is Δx^2, which is smaller than Δx when Δx tends to 0.



Approximating a function by a polynomial

Another useful application of the derivative is the approximation of functions by polynomials. Polynomials are functions easy to calculate (sums and products) with very good properties:

- Defined in all the real numbers.
- Continuous.
- Differentiable of all orders with continuous derivatives.

Goal

Approximate a function $f(x)$ by a polynomial $p(x)$ near a point $x = a$.

Approximating a function by a polynomial of order 0

A polynomial of degree 0 has equation

$$p(x) = c_0,$$

where c_0 is a constant.

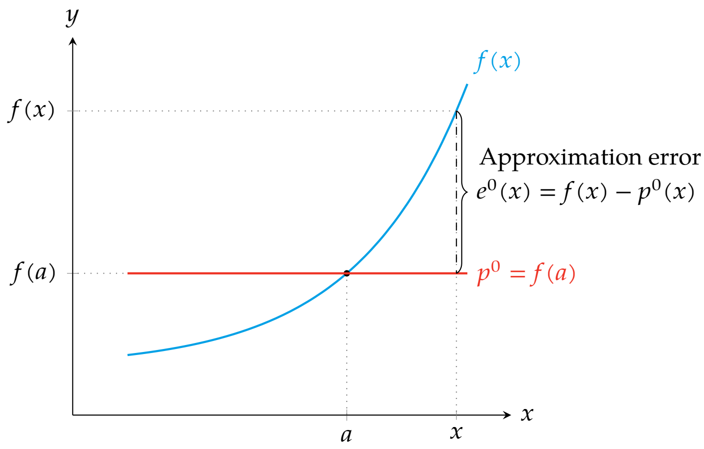
As the polynomial should coincide with the function f at a, it must satisfy

$$p(a) = c_0 = f(a).$$

Therefore, the polynomial of degree 0 that best approximates f near a is

$$p(x) = f(a).$$

Approximating a function by a polynomial of order 0



Approximating a function by a polynomial of order 1

A polynomial of degree 1 has equation

$$p(x) = c_0 + c_1 x,$$

but it can also be written as

$$p(x) = c_0 + c_1(x - a).$$

Among all the polynomials of degree 1, the one that best approximates f near a is that which meets the following conditions

1. p and f coincide at a: $p(a) = f(a)$,
2. p and f have the same rate of change at a: $p'(a) = f'(a)$.

The last condition guarantees that p and f have approximately the same tendency, but it requires the function f to be differentiable at a.

The tangent line: Best approximating polynomial of order 1

Imposing the previous conditions we have

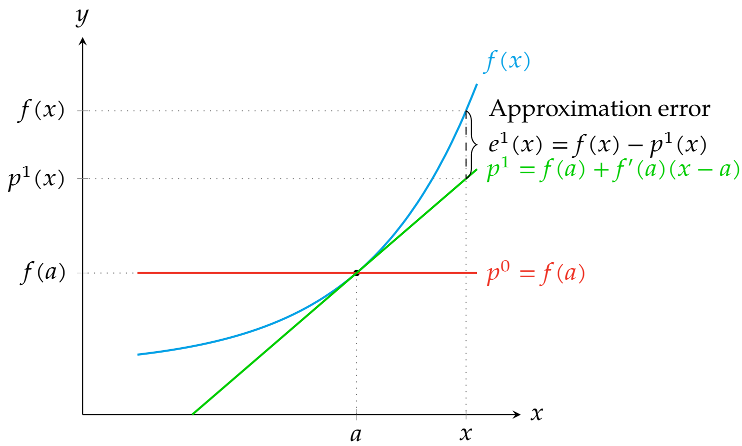
1. $p(x) = c_0 + c_1(x - a) \Rightarrow p(a) = c_0 + c_1(a - a) = c_0 = f(a)$,
2. $p'(x) = c_1 \Rightarrow p'(a) = c_1 = f'(a)$.

Therefore, the polynomial of degree 1 that best approximates f near a is

$$p(x) = f(a) + f'(a)(x - a),$$

which turns out to be the tangent line to f at $(a, f(a))$.

Approximating a function by a polynomial of order 1



Approximating a function by a polynomial of order 2

A polynomial of degree 2 is a parabola with equation

$$p(x) = c_0 + c_1x + c_2x^2,$$

but it can also be written as

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2.$$

Among all the polynomials of degree 2, the one that best approximate $f(x)$ near a is that which meets the following conditions

1. p and f coincide at a: $p(a) = f(a)$,
2. p and f have the same rate of change at a: $p'(a) = f'(a)$.
3. p and f have the same concavity at a: $p''(a) = f''(a)$.

The last condition requires the function f to be differentiable twice at a.

Best approximating polynomial of order 2

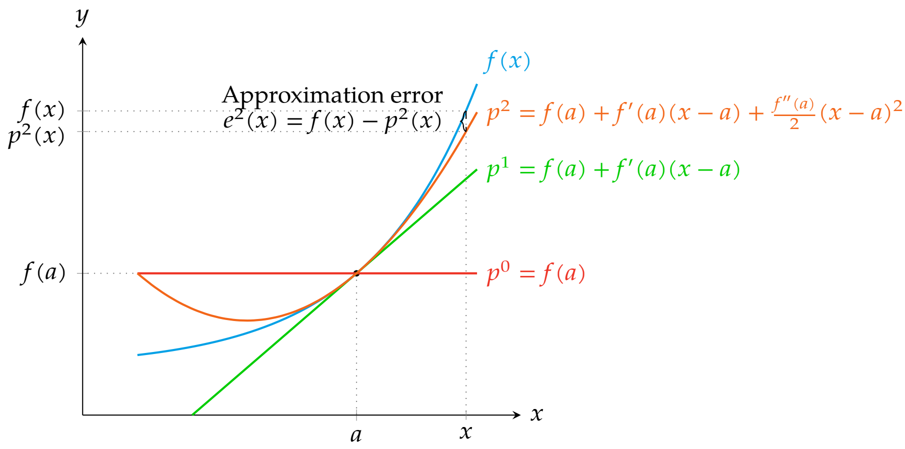
Imposing the previous conditions we have

1. $p(x) = c_0 + c_1(x-a) \Rightarrow p(a) = c_0 + c_1(a-a) = c_0 = f(a)$,
2. $p'(x) = c_1 \Rightarrow p'(a) = c_1 = f'(a)$.
3. $p''(x) = 2c_2 \Rightarrow p''(a) = 2c_2 = f''(a) \Rightarrow c_2 = \frac{f''(a)}{2}$.

Therefore, the polynomial of degree 2 that best approximates f near a is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Approximating a function by a polynomial of order 2



Approximating a function by a polynomial of order n

A polynomial of degree n has equation

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n,$$

but it can also be written as

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n.$$

Among all the polynomials of degree n, the one that best approximate $f(x)$ near a is that which meets the following $n+1$ conditions

1. $p(a) = f(a)$,
2. $p'(a) = f'(a)$,
3. $p''(a) = f''(a)$,

   ...

n+1. $p^{(n)}(a) = f^{(n)}(a)$.

Observe that these conditions require the function f to be differentiable n times at a.

Coefficients calculation for the best approximating polynomial of order n

The successive derivatives of p are

$$\begin{aligned}
p(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n,\\
p'(x) &= c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1},\\
p''(x) &= 2c_2 + \cdots + n(n-1)c_n(x-a)^{n-2},\\
&\vdots\\
p^{(n)}(x) &= n(n-1)(n-2)\cdots 1 c_n = n!\,c_n.
\end{aligned}$$

Imposing the previous conditions we have

1. $p(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots + c_n(a-a)^n = c_0 = f(a)$,
2. $p'(a) = c_1 + 2c_2(a-a) + \cdots + nc_n(a-a)^{n-1} = c_1 = f'(a)$,
3. $p''(a) = 2c_2 + \cdots + n(n-1)c_n(a-a)^{n-2} = 2c_2 = f''(a) \Rightarrow c_2 = f''(a)/2$,

   ...

n+1. $p^{(n)}(a) = n!\,c_n = f^{(n)}(a) = c_n = \frac{f^{(n)}(a)}{n!}$.

Taylor polynomial of order n

Definition 21 (Taylor polynomial)**.** Given a function $f(x)$ differentiable n times at a, the *Taylor polynomial* of order n of f at a is the polynomial with equation

$$\begin{aligned} p_{f,a}^n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \\ &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i. \end{aligned}$$

The Taylor polynomial of order n of f at a is the nth degree polynomial that best approximates f near a, as is the only one that meets the previous conditions.

Taylor polynomial calculation

Example

Let us approximate the function $f(x) = \log x$ near the value 1 by a polynomial of order 3.

The equation of the Taylor polynomial of order 3 of f at $a = 1$ is

$$p_{f,1}^3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3.$$

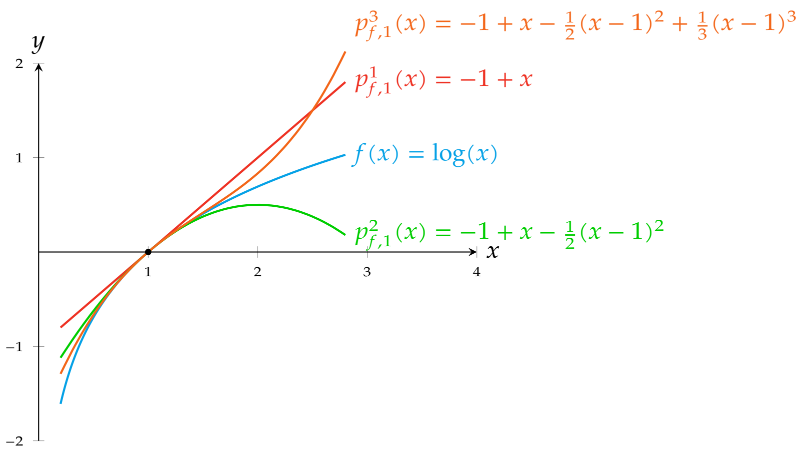
The derivatives of f at 1 up to order 3 are

$$\begin{array}{ll} f(x) = \log x & f(1) = \log 1 = 0, \\ f'(x) = 1/x & f'(1) = 1/1 = 1, \\ f''(x) = -1/x^2 & f''(1) = -1/1^2 = -1, \\ f'''(x) = 2/x^3 & f'''(1) = 2/1^3 = 2. \end{array}$$

And substituting into the polynomial equation we get

$$p_{f,1}^3(x) = 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 = \frac{2}{3}x^3 - \frac{3}{2}x^2 + 3x - \frac{11}{6}.$$

Taylor polynomials of the logarithmic function



Maclaurin polynomial of order n

The Taylor polynomial equation has a simpler form when the polynomial is calculated at 0. This special case of Taylor polynomial at 0 is known as the *Maclaurin polynomial*.

Definition 22 (Maclaurin polynomial)**.** Given a function $f(x)$ differentiable n times at 0, the *Maclaurin polynomial* of order n of f is the polynomial with equation

$$\begin{aligned} p_{f,0}^n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \\ &= \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!}x^i. \end{aligned}$$

Maclaurin polynomial calculation

Example

Let us approximate the function $f(x) = \sin x$ near the value 0 by a polynomial of order 3.

The Maclaurin polynomial equation of order 3 of f is

$$p_{f,0}^3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3.$$

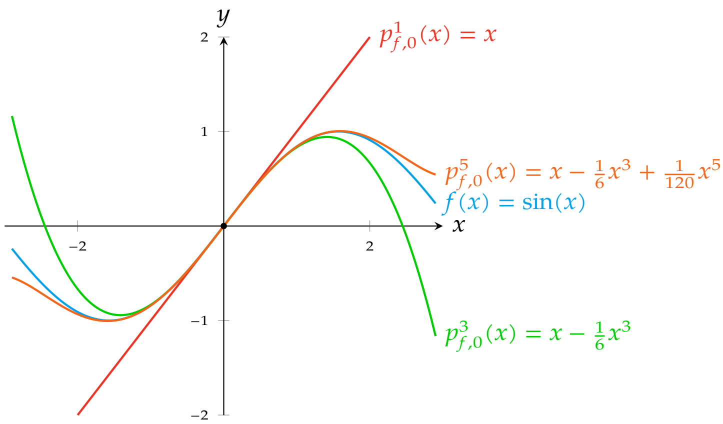
The derivatives of f at 0 up to order 3 are

$$\begin{array}{ll} f(x) = \sin x & f(0) = \sin 0 = 0, \\ f'(x) = \cos x & f'(0) = \cos 0 = 1, \\ f''(x) = -\sin x & f''(0) = -\sin 0 = 0, \\ f'''(x) = -\cos x & f'''(0) = -\cos 0 = -1. \end{array}$$

And substituting into the polynomial equation we get

$$p_{f,0}^3(x) = 0 + 1 \cdot x + \frac{0}{2}x^2 + \frac{-1}{3!}x^3 = x - \frac{x^3}{6}.$$

Maclaurin polynomial of the sine function



Maclaurin polynomials of elementary functions

$f(x)$	$p_{f,0}^n(x)$
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^k \dfrac{x^{2k-1}}{(2k-1)!}$ if $n = 2k$ or $n = 2k-1$
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^k \dfrac{x^{2k}}{(2k)!}$ if $n = 2k$ or $n = 2k+1$
$\arctan x$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots + (-1)^k \dfrac{x^{2k-1}}{(2k-1)}$ if $n = 2k$ or $n = 2k-1$
e^x	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$
$\log(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots + (-1)^{n-1}\dfrac{x^n}{n}$

Taylor remainder and Taylor formula

Taylor polynomials allow to approximate a function in a neighborhood of a value a, but most of the times there is an error in the approximation.

Definition 23 (Taylor remainder)**.** Given a function $f(x)$ and its Taylor polynomial of order n at a, $p_{f,a}^n(x)$, the *Taylor remainder* of order n of f at a is the difference between the function and the polynomial,

$$r_{f,a}^n(x) = f(x) - p_{f,a}^n(x).$$

The Taylor remainder measures the error in the approximation of $f(x)$ by the Taylor polynomial and allow us to express the function as the Taylor polynomial plus the Taylor remainder

$$f(x) = p_{f,a}^n(x) + r_{f,a}^n(x).$$

This expression is known as the *Taylor formula* of order n or f at a.

It can be proved that

$$\lim_{h\to 0} \frac{r_{f,a}^n(a+h)}{h^n} = 0,$$

which means that the remainder $r_{f,a}^n(a+h)$ is much smaller than h^n.

3 Integrals

Integrals

Contents

3.1 Antiderivative of a function

Antiderivative of a function

Definition 24 (Antiderivative of a function)**.** Given a function $f(x)$, it is said that the function $F(X)$ is an *antiderivative* or *primitive function* of f if it satisfies that $F'(x) = f(x)\ \forall x \in \mathrm{Dom}(f)$.

Example The function $F(x) = x^2$ is an antiderivative of the function $f(x) = 2x$ as $F'(x) = 2x$ for all $\mathbb{R}$.

Roughly speaking, the calculus of antiderivatives is the reverse process of differentiation, and that is the reason for the name of antiderivative.

Indefinite integral of a function

As two functions that differs in a constant term have the same derivative, if $F(x)$ is an antiderivative of $f(x)$, so will be any function of the family $F(x) + k\ \forall k \in \mathbb{R}$. This means that, when a function has an antiderivative, it has an infinite number of antiderivatives.

Definition 25 (Indefinite integral)**.** The *indefinite integral* of a function $f(x)$ is the set of all its antiderivatives; it is denoted by

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is an antiderivative of $f(x)$ and C is a constant.

Example The indefinite integral of the function $f(x) = 2x$ is

$$\int 2x\,dx = x^2 + C.$$

Interpretation of the integral

We have seen in a previous chapter that the derivative of a function is the instantaneous rate of change of the function. Thus, if we know the instantaneous rate of change of the function at any point, we can compute the change of the function.

Example What is the space covered by an object free falling?

Assume that the only force acting upon an object drop is the gravity, with an acceleration of $9.8\ \mathrm{m/s^2}$. As acceleration is the the rate of change of the speed, that is constant at any moment, the antiderivative is the speed of the object, Th

$$v(t) = 9.8t\ \mathrm{m/s}$$

And as the speed is the rate of change of the space covered by object during the fall, the antiderivative of the speed is the space covered by the object,

$$s(t) = \int 9.8t\,dt = 9,8\frac{t^2}{2}.$$

Thus, for instance, after 2 seconds, the covered space is $s(2) = 9.8\frac{2^2}{2} = 19.6$ m.

Linearity of integration

Given two integrable functions $f(x)$ and $g(x)$ and a constant $k \in \mathbb{R}$, it is satisfied that

1. $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$,
2. $\int kf(x)\,dx = k \int f(x)\,dx$.

This means that the integral of any linear combination of functions equals the same linear combination of the integrals of the functions.

3.2 Elementary integrals

Elementary integrals

- $\int a\,dx = ax + C$, with a constant.
- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad$ if $n \neq -1$.
- $\int \dfrac{1}{x}\,dx = \ln|x| + C$.
- $\int e^x\,dx = e^x + C$.
- $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$.
- $\int \sin x\,dx = -\cos x + C$.
- $\int \cos x\,dx = \sin x + C$.
- $\int \tan x\,dx = \ln|\sec x| + C$.
- $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.
- $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.
- $\int \cot x\,dx = \ln|\sin x| + C$.
- $\int \sec^2 x\,dx = \tan x + C$.
- $\int \csc^2 x\,dx = -\cot x + C$.
- $\int \sec x \tan x\,dx = \sec x + C$.
- $\int \csc x \cot x\,dx = -\csc x + C$.
- $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin \dfrac{x}{a} + C$.
- $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a} \arctan \dfrac{x}{a} + C$.
- $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a} \sec^{-1} \dfrac{x}{a} + C$.
- $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a} \ln|\dfrac{x+a}{x-a}| + C$.

3.3 Techniques of integration

Techniques of integration

Unfortunately, unlike differential calculus, the is not an infallible procedure to compute the antiderivative of a function. However, there are some techniques that allow to integrate some types of functions. The most common are

- Integration by parts
- Integration by reduction
- Integration by substitution
- Integration of rational functions
- Integration of trigonometric functions

Integration by parts

Given two differentiable functions $u(x)$ and $v(x)$, from the rule for differentiating a product we can get

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx,$$

or, writing $u'(x)dx = du$ and $v'(x)dx = dv$,

$$\int u\,dv = uv - \int v\,du.$$

To apply this method we have to choose the functions u and dv in a way that the final integral was easier to compute than the original one.

Example To integrate $\int x\sin x\,dx$ we have to choose $u = x$ and $dv = \sin x\,dx$, so $du = dx$ and $v = -\cos x$, getting

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x.$$

If we had chosen $u = \sin x$ and $dv = x\,dx$, we would have got a more difficult integral.

Integration by reduction

The reduction technique serves when we have to apply the integration by parts several times.

If we want to compute the antiderivative I_n that depends on a natural number n, the reduction formulas allow us to write I_n as a function of I_{n-1}, that is, we have a recurrent relation

$$I_n = f(I_{n-1}, x, n)$$

such as computing the first antiderivative I_0 we can compute the others.

Example To compute $I_n = \int x^n e^x\,dx$ applying the integration by parts, we have to choose $u = x^n$ y $dv = e^x\,dx$, so $du = nx^{n-1}\,dx$ and $v = e^x$, getting

$$I_n = \int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx = x^n e^x - nI_{n-1}.$$

Thus, for instance, for $n = 3$ we have

$$\int x^3e^x\,dx = I_3 = x^3e^x - 3I_2 = x^3e^x - 3(x^2e^x - 2I_1) = x^3e^x - 3(x^2e^x - (xe^x - I_0) =$$
$$= x^3e^x - 3(x^2e^x - (xe^x - e^x) = e^x(x^3 - 3x^2 + 6x - 6).$$

Integration by substitution

From the chain rule for differentiating the composition of two functions

$$f(g(x))' = f'(g(x))g'(x),$$

we can make a variable change $u = g(x)$, so $du = g'(x)dx$, and

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C.$$

Example To compute the integral of $\int \frac{1}{x\log x}\,dx$ we can make the substitution $u = \log x$, so $du = \frac{1}{x}dx$, and we have

$$\int \frac{dx}{x\log x} = \int \frac{1}{\log x}\frac{1}{x}\,dx = \int \frac{1}{u}\,du = \log|u| + C.$$

Finally, undoing the substitution we get

$$\int \frac{1}{x\log x}\,dx = \log|\log x| + C.$$

Integration of rational functions

Partial fractions decomposition

A rational function can be written as the sum of a polynomial (with an immediate antiderivative) plus a proper rational function, that is, a rational function in which the degree of the numerator is less than the degree of the denominator.

On the other hand, depending of the factorization fo the denominator, a proper rational function can be expressed as a sum of simpler fractions of the following types

- Denominator with a single linear factor: $\dfrac{A}{(x-a)}$
- Denominator with a linear factor repeated n times : $\dfrac{A}{(x-a)^n}$
- Denominator with a single quadratic factor: $\dfrac{Ax+B}{x^2+cx+d}$
- Denominator with a quadratic factor repeated n times: $\dfrac{Ax+B}{(x^2+cx+d)^n}$

Integration of rational functions

Antiderivatives of partial fractions

Using the linearity of integration, we can compute the antiderivative of the rational function from the antiderivative of these partial fractions

$$\int \frac{A}{x-a}\,dx = A\log|x-a| + C,$$

$$\int \frac{A}{(x-a)^n}\,dx = \frac{-A}{(n-1)(x-a)^{n-1}} + C \text{ si } n \neq 1.$$

$$\int \frac{Ax+B}{x^2+cx+d} = \frac{A}{2}\log|x^2+cx+d| + \frac{2B-Ac}{\sqrt{4d-c^2}}\arctan\frac{2x+c}{\sqrt{4d-c^2}} + C.$$

Integration of rational functions

Example of denominator with linear factors

Consider the function $f(x) = \dfrac{x^2+3x-5}{x^3-3x+2}$.

The factorization of the denominator is $x^3-3x+2=(x-1)^2(x+2)$, so it has a single linear factor $(x+2)$ and a linear factor $(x-1)$ repeated two times. In this case the decomposition in partial fractions is:

$$\begin{aligned}\frac{x^2+3x-5}{x^3-3x+2} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2} = \\ &= \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)} = \\ &= \frac{(A+C)x^2+(A+B-2C)x+(-2A+2B+C)}{(x-1)^2(x+2)}\end{aligned}$$

and equating the numerators we get $A=16/9$, $B=-1/3$ and $C=-7/9$, so

$$\frac{x^2+3x-5}{x^3-3x+2} = \frac{16/9}{x-1}+\frac{-1/3}{(x-1)^2}+\frac{-7/9}{x+2}.$$

Finally, integrating every partial fraction we have

$$\begin{aligned}\int \frac{x^2+3x-5}{x^3-3x+2}\,dx &= \int \frac{16/9}{x-1}\,dx+\int \frac{-1/3}{(x-1)^2}\,dx+\int \frac{-7/9}{x+2}\,dx = \\ &= \frac{16}{9}\int \frac{1}{x-1}\,dx-\frac{1}{3}\int (x-1)^{-2}\,dx-\frac{7}{9}\int \frac{1}{x+2}\,dx = \\ &= \frac{16}{9}\ln|x-1|+\frac{1}{3(x-1)}-\frac{7}{9}\ln|x+2|+C.\end{aligned}$$

Integration of rational functions

Example of denominator with simple quadratic factors

Consider the function $f(x) = \dfrac{x+1}{x^2-4x+8}$.

In this case the denominator can not be factorised as a product of linear factors, but we can write

$$x^2-4x+8=(x-2)^2+4,$$

so

$$\begin{aligned}\int \frac{x+1}{x^2-4x+8}\,dx &= \int \frac{x-2+3}{(x-2)^2+4}\,dx = \\ &= \int \frac{x-2}{(x-2)^2+4}\,dx+\int \frac{3}{(x-2)^2+4}\,dx = \\ &= \frac{1}{2}\ln|(x-2)^2+4|+\frac{3}{2}\arctan\left(\frac{x-2}{2}\right)+C.\end{aligned}$$

Integration of trigonometric functions

Integration of $\sin^n x\cos^m x$ *with* n *or* m *odd*

If $f(x)=\sin^n x\cos^m x$ with n or m odd, then we can make the substitution $t=\sin x$ or $t=\cos x$, to convert the function into a polynomial.

Example

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx,$$

and making the substitution $t = \sin x$, so $dt = \cos x dx$, we have

$$\int \sin^2 x (1 - \sin^2 x) \cos x \, dx = \int t^2 (1 - t^2) \, dt = \int t^2 - t^4 \, dt = \frac{t^3}{3} - \frac{t^5}{5} + C.$$

Finally, undoing the substitution we have

$$\int \sin^2 x \cos^3 x \, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

Integration of trigonometric functions

Integration of $\sin^n x \cos^m x$ *with n and m even*

If $f(x) = \sin^n x \cos^m x$ with n and m even, then we can make the following substitutions to simplify the integration

$$\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\ \cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\ \sin x \cos x &= \frac{1}{2} \sin(2x) \end{aligned}$$

Example

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx = \int (\sin x \cos x)^2 \cos^2 x \, dx = \int \left(\frac{1}{2} \sin(2x) \right)^2 \frac{1}{2}(1 + \cos(2x)) \, dx = \\ = \frac{1}{8} \int \sin^2(2x) \, dx + \frac{1}{8} \int \sin^2(2x) \cos(2x) \, dx, \end{aligned}$$

been the first integral of the same type and te second one of the previous type,

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{32} x - \frac{1}{32} \sin(2x) + \frac{1}{24} \sin^3(2x).$$

Integration of trigonometric functions

Products of sines and cosines

The equalities

$$\begin{aligned} \sin x \cos y &= \frac{1}{2}(\sin(x - y) + \sin(x + y)) \\ \sin x \sin y &= \frac{1}{2}(\cos(x - y) - \cos(x + y)) \\ \cos x \cos y &= \frac{1}{2}(\cos(x - y) + \cos(x + y)) \end{aligned}$$

transform products in sums, simplifying the integration

Example

$$\begin{aligned}\int \sin x \cos 2x\, dx &= \int \frac{1}{2}(\sin(x-2x)+\sin(x+2x))\, dx = \\ &= \frac{1}{2}\int \sin(-x)\, dx + \frac{1}{2}\int \sin 3x\, dx = \\ &= \frac{1}{2}\cos(-x) - \frac{1}{6}\cos 3x + C.\end{aligned}$$

Integration of trigonometric functions

Rational functions of sines and cosines

If $f(x,y)$ is a rational function then the function $f(\sin x, \cos x)$ can be transformed in an rational function of t with the following substitutions

$$\tan\frac{x}{2} = t \quad \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad dx = \frac{2}{1+t^2}dt.$$

Example

$$\int \frac{1}{\sin x}\, dx = \int \frac{1}{\frac{2t}{1+t^2}}\frac{2}{1+t^2}\, dt = \int \frac{1}{t}\, dt = \log|t| + C = \log|\tan\frac{x}{2}| + C.$$

3.4 Definite integral

Definite integral

Definition 26 (Definite integral)**.** Let $f(x)$ be a function which is continuous on an interval $[a,b]$. Divide this interval into n subintervals of equal width Δx and choose an arbitrary point x_i from each interval. The *definite integral* of f from a to b is defined to be the limit

$$\int_a^b f(x)\, dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x.$$

Definite integral

Theorem 27 (First fundamental theorem of Calculus)**.** *If $f(x)$ is continuous on the interval $[a,b]$ and $F(x)$ is the antiderivative of f on $[a,b]$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Example. Given the function $f(x) = x^2$, we have

$$\int_1^2 x^2\, dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}.$$

Properties of the definite integral

Given two functions $f(x)$ and $g(x)$ integrable on $[a,b]$ and $k \in \mathbb{R}$ the following properties are satisfied:

- $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$ (linearity)
- $\int_a^b kf(x)\, dx = k \int_a^b f(x)\, dx$ (linearity)
- $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$ si $f(x) \leq g(x)$ $\forall x \in [a,b]$ (monotony)
- $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ for any $c \in (a,b)$ (additivity)
- $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

3.5 Area calculation

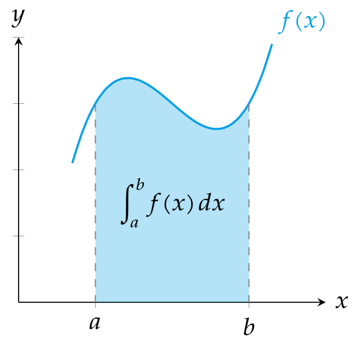
Area calculation

Area between a positive function and the x axis

If $f(x)$ is an integrable function on the interval $[a,b]$ and $f(x) \geq 0$ $\forall x \in [a,b]$, then the definite integral

$$\int_a^b f(x)\, dx$$

measures the area between the graph of f and the x axis on the interval $[a,b]$.

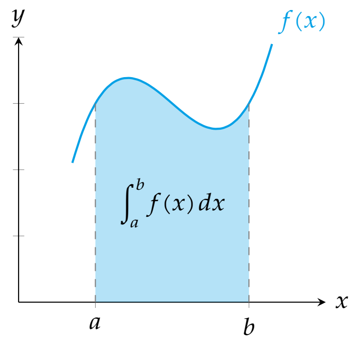


Area calculation

Area between a negative function and the x axis

If $f(x)$ is an integrable function on the interval $[a,b]$ and $f(x) \leq 0$ $\forall x \in [a,b]$, then the area between the graph of f and the x axis on the interval $[a,b]$ is

$$-\int_a^b f(x)\, dx.$$

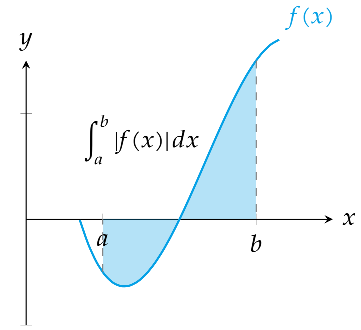


Area calculation

Area between a function and the x axis

In general, if $f(x)$ is an integrable function on the interval $[a, b]$, no matter the sign of f on $[a, b]$, the area between the graph of f and the x axis on the interval $[a, b]$ is

$$\int_a^b |f(x)|\,dx.$$



Area calculation

Area between two functions

If $f(x)$ and $g(x)$ are two integrable functions on the interval $[a, b]$, then the area between the graph of f and g on the interval $[a, b]$ is

$$\int_a^b |f(x) - g(x)|\,dx.$$