Calculus with Derive

Juan Carlos Garro Garro (garro.eps@ceu.es)
Euardo López Ramírez (elopez@ceu.es)
José Rojo Montijano (jrojo.eps@ceu.es)
Anselmo Romero Limón (arlimon@ceu.es)
Alfredo Sánchez Alberca (asalber@ceu.es)
Susana Victoria Rodríguez (victoria.eps@ceu.es)

Department of Applied Math and Statistics
CEU San Pablo

September 2016

CEU
Universidad
San Pablo

Calculus with Derive
Alfredo Sánchez Alberca (asalber@ceu.es)

Contents

Practice 1

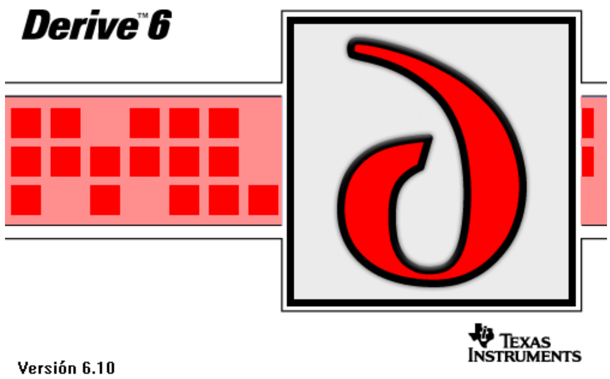
Introducction to Derive

1 Introduction

In the last decades, the computational power of computers have converted them in powerful tools for disciplines that, as Mathematics, require a large amount of complex computations.

Derive® [*] is one of the most used programs for doing numerical and symbolic computations.

Beyond their capabilities for the numerical, vectorial and matrix calculus, it also makes graphical representations. This allows to solve a lot of problems of Algebra, Analysis, Calculus, Geometry and even Statistics. The advantage of Derive versus other software as Mathematica, Mapple or MATLAB, is its simplicity, what makes it suitable for teaching Maths.



The goal of this practice is to introduce the basic usage of this program to the student.

2 Basic functionalities

Starting the program

As any other Windows applications, to start the program you have to click the Windows start button and then select All the programs » Derive 6 or simply double click the desktop shortcut if there is one.

When the program starts, the main window, that is known as *Algebra window* is shown (figure 1.1).

[*]These practices are based on version 6.1 of Derive for Windows.

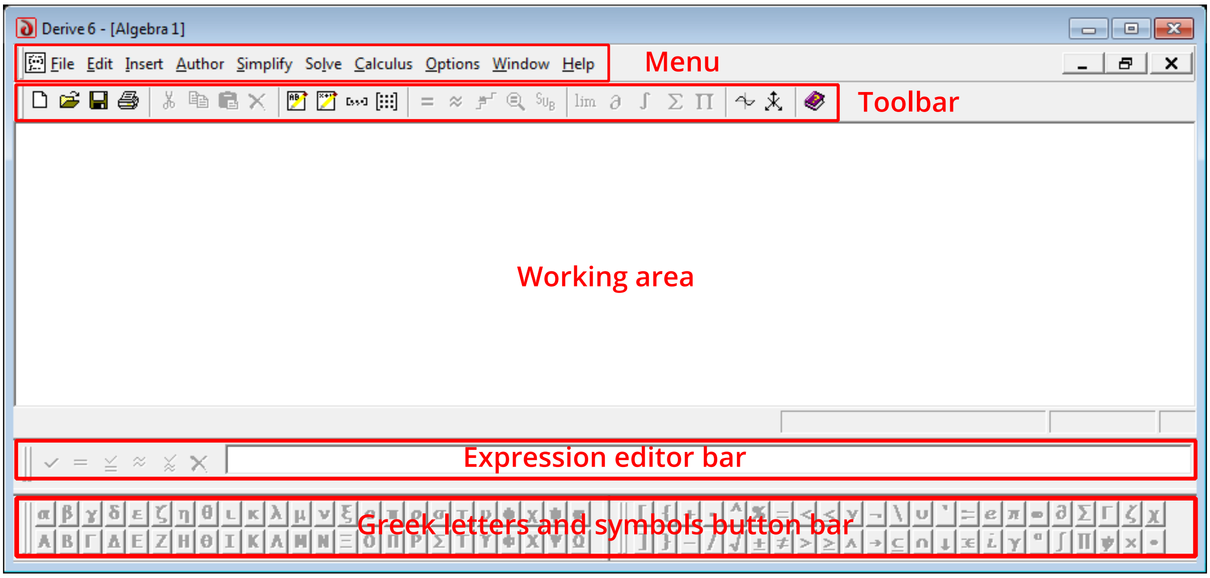


Figure 1.1 – Algebra window of Derive

The Algebra window has a title bar, a menu bar with menus for all the computations that Derive can performs (limits, derivatives, integrals, graphical representations, etc.), a tool bar with buttons for the main computations, the working area that contains the mathematical expressions that we are working with, the expression editor bar to enter mathematical expressions, the Greek letters and symbols button bar with Greek letters and symbols to enter in expressions and the status bar that shows what is the program doing at any moment.

Expression edition

Before doing any computation with a mathematical expression we have to enter it.

Entering a mathematical expression

To enter a mathematical expression we use the expression editor bar (figure 1.2), that usually appears at the bottom of the Algebra widows, over the Greeks letters and symbols bar.

Figure 1.2 – Expression editor bar.

In the expression editor bar we can write numbers, letters (that are variables) and symbols and arithmetic and logic operators. The most common operators are shown in the table below. You can also enter any Greek letter or symbol of the Greek letters and symbols button bar, just clicking on it.

Symbol	Operator
+	addition
−	subtraction
*	product
/	division
^	power

Operators have different priorities when Derive evaluates a expression. First it evaluates functions and constants, second powers, third products and quotients (from left to right) and finally sums and subtractions (from left to right). You have to take into account this priority or use parenthesis to force a subexpression to be evaluated before the rest. In the following example you have different expressions and what Derive interprets for each of them

Entered expression	**Evaluated expression**
`4x-1/x-5`	$4x - \frac{1}{x} - 5$
`(4x-1)/x-5`	$\frac{4x-1}{x} - 5$
`4x-1/(x-5)`	$4x - \frac{1}{x-5}$
`(4x-1)/(x-5)`	$\frac{4x-1}{x-5}$

After entering a expression and pressing Enter, the expression is shown in the working area of the Algebra window, labelled with a tag `#` and a number, such as is shown in the figure 1.3. After that we can reference that expression using its label instead of typing again the expression.

We can select any expression of the Algebra window click on it. If you click several times on the same expression you can select different subexpressions. It is also possible to select several consecutive expressions in a block clicking on the first expression and dragging the cursor to the last.

A useful key is `F3` that allow entering the selected expression or subexpression in the expression editor bar.

Modifying a expression

Once a expression has been entered, we can modify it clicking on the expression and selecting the menu Edit 〉Expression. The expression will be entered in the expression editor bar where you can change whatever you want. After making the changes, don't forget to press Enter.

Removing expressions

To remove a expression form the working area of the Algebra window, it is enough to select the expression and then press the Supr key or select the menu Edit 〉Delete. After removing a expression the labels of the other expressions are renumbered automatically. It is also possible to remove blocks of consecutive expressions.

Important!: If we remove a expression by mistake, it is possible to recover it with the menu Edit 〉Undelete.

Rearranging expressions

It is possible to change the position of any expression in the working area of the Algebra window just clicking on it and, when the expression or block is selected, clicking again on it an dragging it to the new position. After arranging a expression the labels of the other expressions are renumbered automatically.

Entering Comments

The are two ways of entering comments in the working area of the Algebra window. The first one is in the expression editor bar, entering the text of the comment in double quotes. If we proceed this way, the comment will be shown in the working area as any other expression, with its label. The second one is with the menu Insert 〉Text object. If we proceed this way, the comment will be shown in the working area as an object without label.

Naming variables

By default Derive uses a single letter to represent variables. Thus, the expression `xy`, is not interpreted as a variable but as the product of variables x and y. Also by default, Derive does not distinguish between lowercase and uppercase letters. For instance, Derive will interpret the same both if we write `cos(x)` or if we write `cos(X)`. Nevertheless, it is possible to configure Derive to use more than one letter for variable names and to be case sensitive with the Options › Mode settings › Input.

Defining constants and functions

It is possible to define constants and functions with the operator `:=`. To define a constant it is enough to type the name of the constant followed by `:=` and the value of the constant. For example to define the gravitational constant we can write `g:=9.81`.

To define a function, on the other hand, we have to type the name of the function followed by the list of variables separated by comma in brackets, then `:=` and the expression that defines the function. Thus, for instance, to define the function that measures the area of a triangle `a(b,h):=(b*h)/2` where `b` and `h` are the variables for the base and the height of the triangle respectively (see figure 1.3).

With respect to the definition of constants and functions we must be aware of two important facts:

- Any time we define a constant or function, the definition is active during all the working session, even if we remove the definition expression. To remove a definition we have to redefine the constant or function but letting blank the expression after `:=`. For example, to remove the definition of the gravitational constant we have to write `g:=`.
- Derive is case sensitive to function names, so that `a(b,h)` and `A(b,h)` will be different functions.

Built-in constants and functions

Derive has most of the constants and elementary functions used in Mathematics built-in. The syntax of some of them is shown in table 1.1.

Syntax	**Constant or function**
`#e`	Euler's number $e = 2.71828\ldots$
`pi`	The number $\pi = 3.14159\ldots$
`#i`	The imaginary number $i = \sqrt{-1}$
`inf`	Infinite ∞
`exp(x)`	Exponential function e^x
`log(x,a)`	Logarithmic function with base a, $\log_a x$
`ln(x)`	Natural logarithmic function $\ln x$
`sqrt(x)`	Square root function $\sqrt{x}$
`sin(x)`	Sine function $\sin x$
`cos(x)`	Cosine function $\cos x$
`tan(x)`	Tangent function $\tan x$
`asin(x)`	Arcsine function $\arcsin x$
`acos(x)`	Arccosine function $\arccos x$
`atan(x)`	Arctangent function $\arctan x$

Table 1.1 – Syntax of some predefined constants and functions.

In some cases we can also use the symbols of the symbols button bar to refer to these constants.

To know all the built-in constants and functions of Derive we can use the menu Help › Online and visit the section Built-in Functions and Constants.

Important!: In built-in functions Derive is not case sensitive. For instance, the cosine function can be written `cos(x)`, `Cos(x)` or `COS(x)`.

Vectors and matrices

Derive can also deal with vectors and matrices. To enter a vector you can use the menu Author 》Vector. Then enter the number of elements of the vector in the dialog shown and click the OK button. Finally enter the elements of the vector in the dialog shown and click again the OK button. Another way to enter vectors in the expression editor bar is to type their components separated by commas in square brackets. For instance, to enter the vector (x, y, z) we write `[x,y,z]` (see the figure 1.3).

To enter matrices we can use the menu Author 》Matrix. Then enter the number of rows and columns of the matrix in the dialog shown and click the OK button. Finally enter the elements of the matrix in the dialog shown and click again the OK button.

Another way to enter matrices in the expression editor bar is to type their row vectors separated by commas in square brackets. For instance, to enter the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

we write `[[1,2,3],[a,b,c]]` (see the figure 1.3).

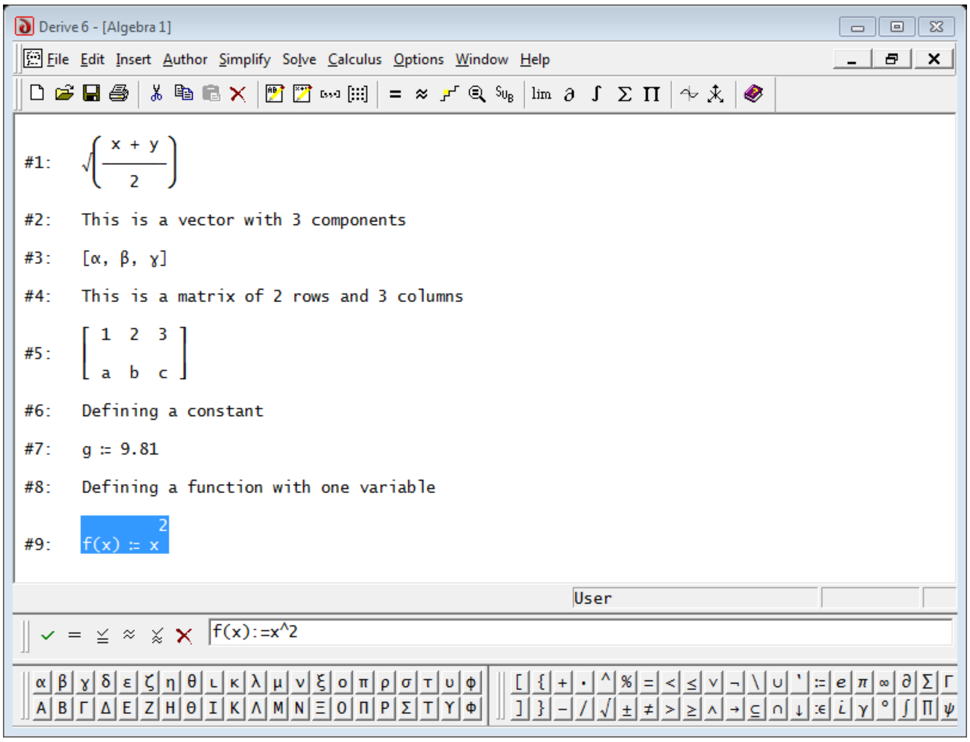


Figure 1.3 – Different types of expressions in the Algebra window.

Simplifying expressions

Derive has several ways to simplify expressions. The simplest one is the basic simplification, that can be done with the menu Simplify 》Basic. This menu performs basic simplifications as, for instance, convert the expression $x + x$ in the expression $2x$. However, it doesn't allow to convert the binomial $(x + 1)^2$ in $x^2 + 2x + 1$, as it is not clear what of these expressions is simpler.

To get the expansion of this binomial we can use the menu Simplify 》Expand that allow to expand a expression with respect to its variables.

On the contrary, if we want to get the binomial from the expanded form, we can use the menu Simplify 》Factor that allows to factor a expression with respect to its variables.

In any of these simplification types Derive woks in exact mode, what means that decimal numbers are expressed as fractions. To get the approximate value of an expression in a decimal form you can use the menu Simplify › Approximate. This menu shows a dialog where we have to enter the number of decimal places that we want.

Finally, in any expression it is possible to substitute any variable by a value with the menu Simplify › Variable substitution. In the dialog shown you have to select the variable to substitute, enter the value for that variable in the field New value and click the button OK.

Graphical representations

Derive can plot graphical representations in 2 and 3 dimensions.

2-dimensional graphical representations

To represent a function or expression with one variable, we have to select the expression and then the menu Window › New 2D-plot window. This will open a new graphic window with two Cartesian axes (x and y). Finally, to show the graph of the expression in the Cartesian plane you have to select the menu Insert › Plot or just click the corresponding button in the toolbar. Figure 1.4 shows an example of a 2-dimensional plot.

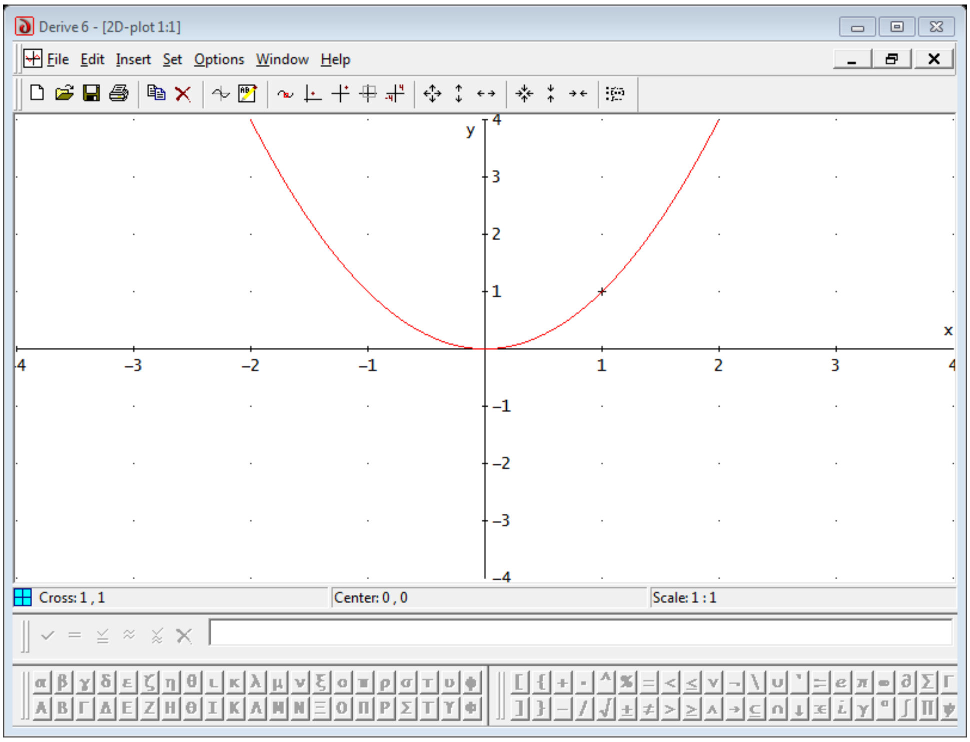


Figure 1.4 – 2-dimensional graphic window with the graph of a function.

If we want to show the plot in the Algebra window we can use the menu File › Embed.

It is possible to represent more than one function in the same 2-dimensional graphic window.

You can change from the Algebra window to the graphic window and vice versa selecting the corresponding window in the menu Window. However, when you are plotting several expressions is better to see the Algebra and graphic windows at the same time using the Window › Tile Vertically (see figure 1.5.

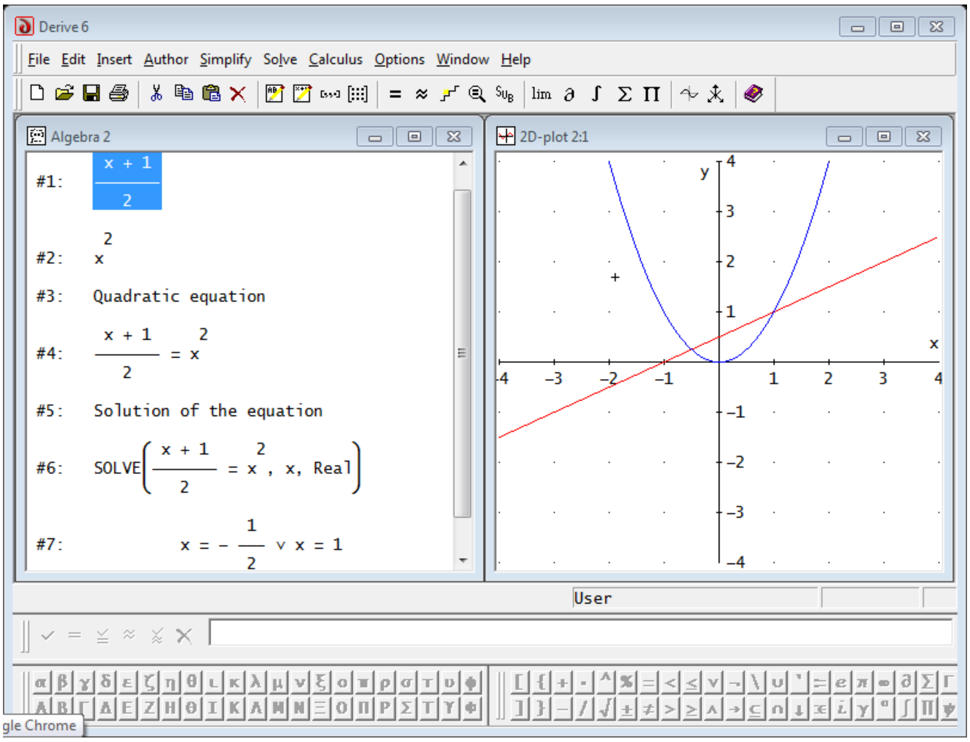


Figure 1.5 – An Algebra and a 2-dimensional graphic window showed at the same time.

Removing a plot

To remove the last plot from a graphic window you can use the menu `Edit` » `Delete plot` » `Last`. It is also possible to remove the first and all the plots but the last with the corresponding menus.

Scaling a plot

In the graphic window there are several menus and buttons to change the aspect of the plots.

One of the most interesting actions is changing the scale of axes with the menu `Set` » `Aspect Ratio`. It is also possible to change the visible area of the plot with the menu `Set` » `Plot Range` » `Minimum/Maximum`. In the dialog shown you have to enter the minimum and maximum values for every axis and click the button OK.

Tracing a plot

In the 2-dimensional graphic windows there is a small cross that we change of position just clicking on a new position of the graphic window. The coordinates of the cross are always shown in the bottom-left corner of the status bar. If you press the `F3` key, the cross changes to a small square and passes to the *trace mode*. In this mode the square follows the trajectory of a graph using the arrow keys of the keyboard. Use the left/right arrows to move the square to the left/right respectively and the up/down arrow to change the graph to follow when there are more than one plot. This can be helpful to see the value that takes a function in the graphic window or the points where two graphs intersect (see figure 1.6.

Centering the plot

It is also possible to center the plot at the position of the cross with the button Center on Cross or at the origin of coordinates with the button Center on Origin.

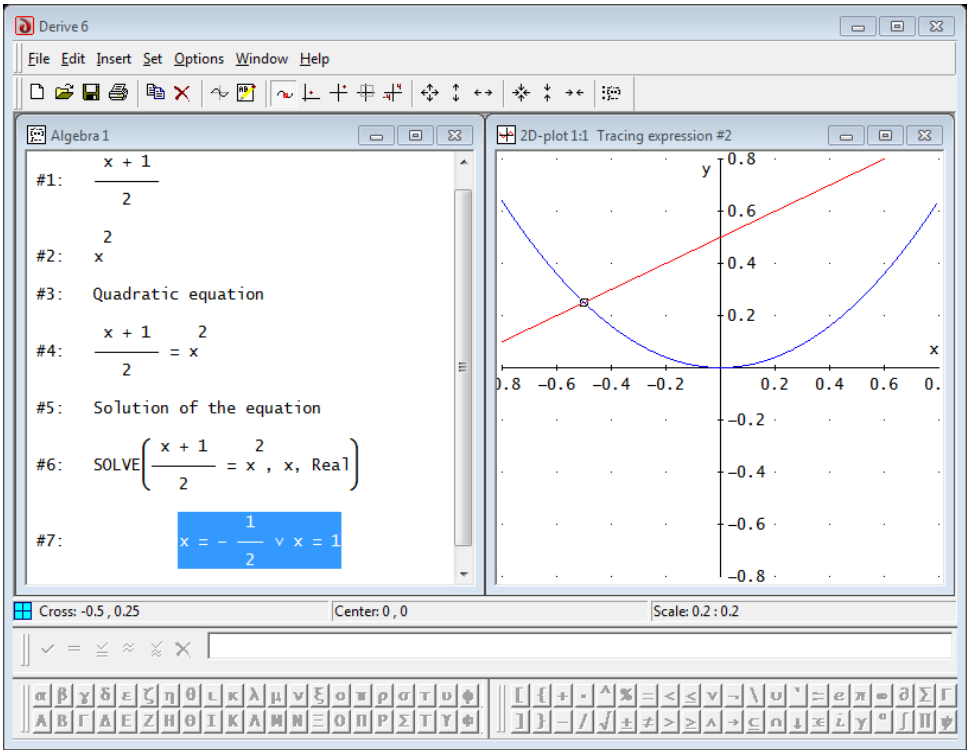


Figure 1.6 – Graphic window in trace mode showing the point where two graphs intersect (that is one solution of the equation).

2-dimensional graphical representations

To represent a function or expression with 2 variables we have to select the expression and then the menu Window » New 3D-plot window. This will open a new graphic window with three Cartesian axes (x, y and z). Finally, to show the graph of the expression in the Cartesian space you have to select the menu Insert » Plot or just click the corresponding button in the toolbar. Figure 1.7 shows an example of a 3-dimensional plot.

Again it is possible to represent more than one function in the same 3-dimensional graphic window.

Like for 2-dimensional graphic windows, there are several menus an buttons to change the aspect of the plot.

Changing the perspective of the plot

One of the most interesting actions is changing the perspective of the plot with the menu Set » Eye Position. In the dialog shown you have to enter the coordinates of the observer eye and click the button OK. It is also possible to change the perspective of the plot rotating the plot horizontally with the left/right arrow keys or vertically with the up/down arrow keys.

Changing the resolution of the grid

Derive plots the surfaces with a grid of small tiles. To change the resolution of the grid you can use the menu Edit » Plot. In the dialog shown you can enter the number of vertical and horizontal panels. The higher the number of panels the smoother the surface.

File management

The expressions and computations of an Algebra windows can be saved in a file.

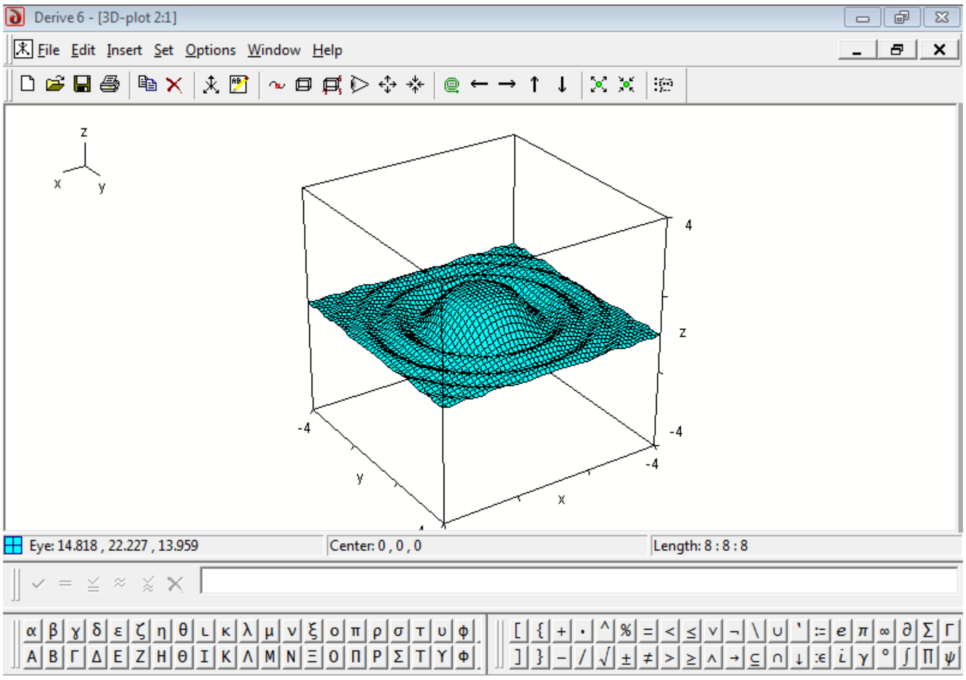


Figure 1.7 – 3-dimensional graphic window with the graph of a function.

Saving a file

To save the content of an Algebra window in a file you can use the menu File › Save. In the dialog shown give a name to the file, select the folder where to save it and click the button Save. Derive will put automatically extension `*.dfw` to the saved file. Once the file has been saved, its name will be shown in the title bar of the window.

Opening a file

To open a Derive file in an Algebra windows you can use the menu File › Open. In the dialog shown you only have to select the file that you want to open an click the button Open. The selected file will be opened in a new Algebra window.

Opening and closing new Algebra windows

Derive can manage more than one Algebra windows simultaneously. To open a new Algebra window you can use the menu File › New. Derive works with each Algebra window independently. That means that we can use the same names to refer to different constants or functions in different Algebra windows without interference.

On the other side, to close an Algebra window you only have to use the menu File › Close.

Printing

To print the content of an Algebra window you can use the menu File › Print. However, before printing it is a good idea to preview the document with the menu File › Print Preview. If everything is OK it is enough to click on the button Print to send the document to the printer. To change the margins and the orientation of the page you can use the menu File › Page Setup.

Other options like the font, the header or the footer of the page can be set with the menu Options › Printing › Header and Footer.

Getting help

Like most Windows applications you can get help about the use of the program with the menu Help.

Practice 2

Elementary functions

1 Solved exercises

1. Consider the function
$$f(t) = \frac{t^4 + 19t^2 - 5}{t^4 + 9t^2 - 10}.$$

   (a) Plot the graph of the function.

   > 1. Enter the expression of the function and select it.
   > 2. Select the menu Window > New 2D-plot Window.
   > 3. In the graphic window click the button Plot.

   (b) Looking at the graph of the function determine:

   1. Domain.

      > Look at the values of x where the function does exists, that is, where there is graph.

   2. Image.

      > Look at the values of y that are output of the function, that is, where there is graph.

   3. Asymptotes.

      > Look at the lines (horizontal, vertical or oblique) where the graph approaches (the distance between the graph and the line tends to zero as they tend to infinity).

   4. Zeros.

      > Look at the values of x where the graph cuts the horizontal axis.

   5. Sign.

      > Look at the values of x where the graph is over the horizontal axis (positive) and where it is under the horizontal axis (negative).

   6. Continuity

      > Look at the values of x where you can trace the graph without lifting your hand.

   7. Increasing and decreasing.

Look at the values of x where y increases when x increases (increasing) and the values where y decreases when x increases (decreasing).

8. Concavity

Look at the values of x where the curvature of the graph is up $\cup$ (concave up or convex) and where the curvature is down $\cap$ (concave down or simply concave).

9. Relative extrema.

Look at the values of x where the graph has a peak (relative maximum) and where the graph has a valley (relative minimum).

10. Inflection points.

Look at the values of x where the curvature changes continuously.

2. Plot in the same graphic window the graphs of the functions 2^x, e^x, 0.7^x, 0.5^x. Looking at the graphs of the functions, determine which ones are increasing and which ones are decreasing.

Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time.
For every function repeat the following steps:

(a) Enter the expression of the function in the Algebra window and select it.
(b) Click the button Plot in the graphic window.
(c) Click at some point near the graph and select the menu Insert > Annotation. In the dialog shown enter the name of the function and click the button OK.

Can you deduce for what values of a the function a^x is increasing and for what values it is decreasing?

3. Plot in the same graphic window the graphs of the following functions and determine their periods and amplitudes.

(a) $\sin(x)$, $\sin(x)+2$, $\sin(x+2)$.
(b) $\sin(2x)$, $2\sin(x)$, $\sin(x/2)$.

Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time. For every function repeat the following steps:

1. Enter the expression of the function in the Algebra window and select it.
2. Click the button Plot in the graphic window.
3. Click at some point near the graph and select the menu Insert > Annotation. In the dialog shown enter the name of the function and click the button OK.

For the period look at the length of the smaller interval in the horizontal axis where the graph is repeated.
For the amplitude look at the distance between the maximum and the minimum and divide it by two.

4. Plot the piecewise function

$$f(x) = \begin{cases} -2x & \text{if } x \leq 0; \\ x^2 & \text{if } x > 0. \end{cases}$$

To represent a piecewise function Derive uses the function `CHI`. The syntax for this function is `CHI(a,x,b)`, where `a` and `b` are the lower and upper limits where a subfunction is defined and `x` is the variable of the function. This defines the function

$$\mathrm{CHI}(a,x,b) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ 0 & \text{if } x > b \end{cases}$$

According to this, to enter the piecewise function we have to write

```
-2x CHI(-inf,x,0) + x^2 CHI(0,x,inf)
```

2 Proposed exercises

1. Plot the graphs of the following functions and determine their domains looking at their graphs.
   (a) $f(x) = \dfrac{x^2+x+1}{x^3-x}$
   (b) $g(x) = \sqrt{x^4-1}$.
   (c) $h(x) = \cos\left(\frac{x+3}{x^2+1}\right)$.
   (d) $l(x) = \arcsin\left(\frac{x}{1+x}\right)$.

2. Consider the function
$$f(x) = \frac{x^3+x+2}{5x^3-9x^2-4x+4}.$$
   Plot the function and determine looking at its graph:
   (a) Domain
   (b) Image
   (c) Asymptotes
   (d) Zeros
   (e) Sign
   (f) Continuity
   (g) Increasing and decreasing
   (h) Concavity
   (i) Relative extrema
   (j) Inflection points

3. Plot in the same graphic window the graphs of the functions $\log_{10} x$, $\log_2 x$, $\log x$, $\log_{0.5} x$.
   (a) Looking at their graphs determine which funtions are increasing and which ones are decreasing.
   (b) Deduce for what values of a the function $\log_a x$ is increasing and for what values it is decreasing?

4. Plot the following functions and complete the following sentences with the word equal or the number of times that is lower or greater in any case.
   (a) The function $\cos(2x)$ has a period .... than the function $\cos x$.
   (b) The function $\cos(2x)$ has an amplitude .... than the function $\cos x$.

(c) The function $\cos(x/2)$ has period .... than the function $\cos(3x)$.
(d) The function $\cos(x/2)$ has an amplitude .... than the function $\cos(3x)$.
(e) The function $3\cos(2x)$ has a period .... than the function $\cos(x/2)$.
(f) The function $3\cos(2x)$ has an amplitude .... than the function $\cos(x/2)$.

5. Find the solutions of the equation $e^{-1/x} = \dfrac{1}{x}$ graphically.

6. Plot the graph of the function
$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$$

Practice 3

Limits and continuity

1 Solved exercises

1. Given the function

$$f(x) = \left(1 + \frac{2}{x}\right)^{x/2},$$

plot its graph and compute the following limits:

(a) $\lim_{x \to -\infty} f(x)$

(b) $\lim_{x \to +\infty} f(x)$

(c) $\lim_{x \to -2^-} f(x)$

(d) $\lim_{x \to -2^+} f(x)$

(e) $\lim_{x \to 2} f(x)$

(f) $\lim_{x \to 0} f(x)$

(a) Enter the expression of the function in the Algebra window and select it.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

(d) For computing every limit repeat the following steps

1. Select the function in the Algebra window.
2. Select the menu Calculus » Limit or click the button Limit.
3. In the dialog shown enter the point of the limit in the field Limit Point, select the corresponding option from the list Approach From (Left for a one-sided limit from the left, Right for a one-sided limit from the right, and Both for a global or two-sided limit) and click the button Simplify.
4. Look at the graph and check if the result of the limit makes sense.

2. Given the function

$$f(x) = \begin{cases} \dfrac{x}{x-2} & \text{if } x \leq 0; \\ \dfrac{x^2}{2x-6} & \text{if } x > 0; \end{cases}$$

(a) Plot the graph and determine graphically if there are asymptotes.

1. Enter the expression `f(x) := x/(x-2) CHI(-inf,x,0) + x^2/(2x-6) CHI(0,x,inf)` to define the function in the Algebra window and select the function.
2. Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

3. Click the button Plot in the graphic window.

(b) Compute the vertical asymptotes and plot them if any.

The only point where the function is not defined is $x = 3$. Thus to check if there is a vertical asymptote at this point we have to compute the limit at this point.

1. Select the name of function in the Algebra window.
2. Select the menu Calculus › Limit or click the button Limit.
3. In the dialog shown enter 3 in the field Limit Point, select Left from the list Approach From and click the button Simplify.
4. Repeat the three previous steps but selecting Right from the list Approach From.
5. Look at the graph and check if the results of the limits make sense.

There is a vertical asymptote $x = 3$ if some of the limits is infinite. In that case enter the expression of the asymptote in the Algebra window and click the button Plot in the graphic window.

(c) Compute the horizontal asymptotes and plot them if any.

To check if there is an horizontal asymptote we have to compute the limits at infinity.

1. Select the name of function in the Algebra window.
2. Select the menu Calculus › Limit or click the button Limit.
3. In the dialog shown enter `-inf` in the field Limit Point and click the button Simplify.
4. Repeat the two previous steps but selecting entering `inf` in the field Limit Point.
5. Look at the graph and check if the results of the limits make sense.

There is an horizontal asymptote $y = a$ if some of the limits is a. In that case enter the expression of the asymptote in the Algebra window and click the button Plot in the graphic window.

(d) Compute the oblique asymptotes and plot them if any.

To check if there is an oblique asymptote we have to compute the limits at infinity of the function divided by x. To check if there is an oblique asymptote at $-\infty$, do the following steps:

1. Enter the expression `f(x)/x` in the Algebra window and select it.
2. Select the menu Calculus › Limit or click the button Limit.
3. In the dialog shown enter `-inf` in the field Limit Point and click the button Simplify.

There is an oblique asymptote $y = ax + b$ if some of the limits is a. In that case, a is the slope of the asymptote. To compute the independent term we have to compute the limits at infinity of the function minus ax.

1. Enter the expression `f(x)-ax`, where `a` is the value of the previous limit, in the Algebra window and select it.
2. Select the menu Calculus › Limit or click the button Limit.
3. In the dialog shown enter `-inf` in the field Limit Point and click the button Simplify.

The independent term of the oblique asymptote is the result of this limit.

To check if there is an oblique asymptote at ∞, repeat all the steps but entering `inf` in the field Limit Point.

If there is some oblique asymptote enter the expression of the asymptote in the Algebra window and click the button Plot in the graphic window.

3. For the following functions determine the type of discontinuity at the points given.

(a) $f(x) = \dfrac{\sin x}{x}$ at $x = 0$.

(b) $g(x) = \dfrac{1}{2^{1/x}}$ at $x = 0$.

(c) $h(x) = \dfrac{1}{1 + e^{\frac{1}{1-x}}}$ at $x = 1$.

For each function repeat the following steps:

(a) Enter the expression of the function in the Algebra window and select it.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

(d) Select the function in the Algebra window.

(e) Select the menu Calculus » Limit or click the button Limit.

(f) In the dialog shown enter the given point in the field Limit Point, select Left from the list Approach From and click the button Simplify.

(g) Repeat the three previous steps but selecting Right from the list Approach From.

If both limits exist and are the same, then there is a *removable discontinuity*. If both limits exist but are different, then there is a *jump discontinuity*. If some of the limits doesn't exist or is infinite, then there is an *essential discontinuity*.

4. Determine the points where the following function has a discontinuity and classify it.

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-1}, & \text{if } x < 0; \\ \dfrac{1}{e^{1/(x^2-1)}}, & \text{if } x \geq 0. \end{cases}$$

(a) Enter the expression `f(x) := (x+1)/(x^2-1) CHI(-inf,x,0) + 1/exp(1/(x^2-1)) CHI(0,x,inf)` to define the function in the Algebra window and select the function.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

The function is not defined in $x = -1$ and $x = 1$, so there is a discontinuity at each of these points. As the functions is piecewise, also we have to study the points where the expression of the function changes, that is, at $x = 0$. To classify the type of discontinuity for each of these points, repeat the following steps:

(a) Select the function in the Algebra window.

(b) Select the menu Calculus » Limit or click the button Limit.

(c) In the dialog shown enter the given point in the field Limit Point, select Left from the list Approach From and click the button Simplify.

(d) Repeat the three previous steps but selecting Right from the list Approach From.

If both limits exist and are the same, then there is a *removable discontinuity*. If both limits exist but are different, then there is a *jump discontinuity*. If some of the limits doesn't exist or is infinite, then there is an *essential discontinuity*.

2 Proposed exercises

1. Compute the following limits:

(a) $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{x^4-4x+3}$.

(b) $\lim\limits_{x\to a} \dfrac{\sin x-\sin a}{x-a}$.

(c) $\lim\limits_{x\to\infty} \dfrac{x^2-3x+2}{e^{2x}}$.

(d) $\lim\limits_{x\to\infty} \dfrac{\log(x^2-1)}{x+2}$.

(e) $\lim\limits_{x\to 1} \dfrac{\log(1/x)}{\tan(x+\frac{\pi}{2})}$.

(f) $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a}$ $\quad n\in\mathbb{N}$.

(g) $\lim\limits_{x\to 1} \dfrac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ $\quad n,m\in\mathbb{Z}$.

(h) $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$.

(i) $\lim\limits_{x\to\pi/4} \dfrac{\sin x-\cos x}{1-\tan x}$.

(j) $\lim\limits_{x\to 0} x^2 e^{1/x^2}$.

(k) $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^x$.

(l) $\lim\limits_{x\to 0} \left(\dfrac{1}{x}\right)^{\tan x}$.

(m) $\lim\limits_{x\to 0} (\cos x)^{1/\operatorname{sen} x}$.

(n) $\lim\limits_{x\to 0} \dfrac{6}{4+e^{-1/x}}$.

(o) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-2x-1}\right)$.

2. Given the function

$$f(x)=\begin{cases}\dfrac{x^2+1}{x+3} & \text{if } x<0;\\[2mm] \dfrac{1}{e^{1/(x^2-1)}} & \text{if } x\geq 0;\end{cases}$$

compute its asymptotes.

3. The following functions are not defined at $x=0$. Determine, when possible, the value that should take the function at that point to be continuous.

(a) $f(x)=\dfrac{(1+x)^n-1}{x}$.

(b) $h(x)=\dfrac{e^x-e^{-x}}{x}$.

(c) $j(x)=\dfrac{\log(1+x)-\log(1-x)}{x}$.

(d) $k(x)=x^2\sin\dfrac{1}{x}$.

Practice 4

Derivatives of functions of one variable

1 Solved exercises

1. Study the differentiability of the following function using limits.

   (a) $f(x) = |x-1|$ at $x = 1$.

   (b) $g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 \end{cases}$

For the function $f(x)$ take the following steps:

(a) Define the function in the Algebra window naming it `f(x)`.

(b) Enter the expression `(f(1+h)-f(1))/h`, that correspond to the average rate of change of f at 1, in the Algebra window and select it.

(c) Select the menu Calculus » Limit or click the button Limit.

(d) In the dialog shown enter the point O in the field Limit Point, select `Left` from the list Approach From and click the button Simplify.

(e) Repeat the three previous steps but selecting `Right` from the list Approach From.

As the lateral limits are not the same, the is no derivative at that point.

For the function $g(x)$ repeat the previous steps but naming the function `g(x)` and entering the expression `g(h)/h`.

2. Compute the derivatives of the following function until order 4.

   (a) $a^x \log a$.

   (b) $\dfrac{\sin x + \cos x}{2}$.

   (c) $\dfrac{1}{\sqrt{1+x}}$.

   Can you infer the formula for the derivative of order n in each case?

For each function repeat the following steps:

(a) Define the function in the Algebra window naming it `f(x)`.

(b) For the first derivative enter the expression `f'(x)` and click the button Simplify.

(c) For the second derivative enter the expression `f''(x)` and click the button Simplify.

(d) For the third derivative enter the expression `f'''(x)` and click the button Simplify.
(e) For the fourth derivative enter the expression `f''''(x)` and click the button Simplify.

3. Compute the tangent line to the graph of the function $f(x) = \log(\sqrt{x+1})$ at $x = 1$. Plot the graph of the function and the tangent line.

(a) Define the function in the Algebra window naming it `f(x)`.
(b) Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window > Tile Vertically to see the Algebra and the graphic windows at the same time.
(c) Click the button Plot in the graphic window.
(d) Enter the expression `f(1)+f'(1)(x-1)`, corresponding to the equation of the tangent line to the graph of f at $x = 1$, in the Algebra window and click the button Simplify.
(e) Click the button Plot in the graphic window.

4. Consider the function

$$g(x) = \frac{2x^3 - 3x}{x^2 + 1}.$$

(a) Plot the graph of g.

Define the function in the Algebra window naming it `g(x)`.

Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time.

Click the button Plot in the graphic window.

(b) Compute the first derivative of g an plot its graph.

1. Enter the expression `g'(x)` in the Algebra window and click the button Simplify.
2. Click the button Plot in the graphic window.

(c) Compute the critical points of g.

1. Select the first derivative of the function in the Algebra window.
2. Select the menu Solve > Expression.
3. In the dialog shown select Real from the Solution Domain list and click the buttonSolve.

(d) Study the increase and decrease of g and determine its relative extrema.

Consider the intervals defined by the critical points of the function. For each interval study the sign of the first derivative. If the first derivative is positive the function is increasing. If the first derivative is negative the function is decreasing.

For the relative extrema consider the critical points of the function. At each critical point study the sign of the first derivative to the left an to the right. If the first derivative is positive to the left and negative to the right, the is a relative maximum. If the first derivative is negative to the left and positive to the right, the is a relative minimum.

(e) Compute the second derivative of g an plot its graph.

1. Enter the expression `g''(x)` and click the button Simplify.

2. Click the button Plot in the graphic window.

(f) Compute the zeros of the second derivative of g.

1. Select the second derivative of the function in the Algebra window.
2. Select the menu Solve › Expression.
3. In the dialog shown select Real from the Solution Domain list and click the buttonSolve.

(g) Study the concavity of g and determine its inflection points.

Consider the intervals defined by the zeros of the second derivative. For each interval study the sign of the second derivative. If the second derivative is positive the function is concave up. If the derivative derivative is negative the function is concave down.
For the inflection points consider the zeros of the second derivative. At each zero study the sign of the second derivative to the left an to the right. If the sign of the second derivative is different to the left and to the right, there is an inflection point.

2 Proposed exercises

1. Prove that the following function is not differentiable at $x = 0$.

$$f(x) = \begin{cases} e^x - 1 & \text{if } x \geq 0; \\ x^3 & \text{if } x < 0. \end{cases}$$

2. For each of the following functions compute the equation of the tangent and normal lines at the points given.

   (a) $y = x^{\sin x}, \quad x_0 = \pi/2.$

   (b) $y = (3 - x^2)^4 \sqrt[3]{5x - 4}, \quad x_0 = 1.$

   (c) $y = \log \sqrt{\dfrac{1+x}{1-x}} + \arctan x, \quad x_0 = 0.$

3. Study the increase, decrease, relative extrema, concavity and inflection points of the function $f(x) = \dfrac{x}{x^2 - 2}$.

4. A drug has to be given to patients in cylindrical pills. The content of the drug in each pill is 0.15 ml; determine the dimensions of the cylinder so that the amount of material used to make it (the pill) is minimal.

5. The wheat yield C of a field depends on the level of nitrogen on the ground n, and it is given by the following relation:

$$C(n) = \frac{n}{1 + n^2}, \quad n \geq 0.$$

   Find the level of nitrogen that will produce the biggest yield.

Practice 5

Taylor polynomials

1 Solved exercises

1. Compute the Taylor polynomials of the function $f(x) = \log x$ at the point $x = 1$, up to order 4 and plot their graphs. Which polynomial approximates better the function in a neighbourhood of 1?

   (a) Define the function in the Algebra window entering the expression `f(x):=log(x)`.

   (b) Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time.

   (c) Click the button Plot in the graphic window.

   (d) Select the name of the function in the Algebra window and select the menu Calculus > Taylor Series.

   (e) In the dialog shown enter the 1 in the Expansion Point field, enter 0 in the Order field and click the button Simplify.

   (f) Click the button Plot in the graphic window.

2. Compute the value of $\log 1.2$ approximately using the previous polynomials and give the error of the approximation in each case filling the table below.

Point	Grade	Approximation	Error

   For each polynomial do the following steps:

   (a) Give a name to the polynomial entering the expression `p(x):=#i`, where `#i` is the label corresponding to the polynomial, or use the the expression `p(x):=TAYLOR(f(x),x,1,n)`, where `n` is the grade of the polynomial.

   (b) Enter the expression `p(1.2)` in the Algebra windows and click the button Approximate to get the approximation.

   (c) Enter the expression `ABS(p(1.2)-f(1.2))` in the Algebra windows and click the button Approximate to get the approximation error.

3. Compute the Maclaurin polynomial of order 3 of the function $\sin(x)$, and use it to approximate the value of $\sin(1/2)$. Compute the approximation error.

(a) Define the function in the Algebra window entering the expression `f(x):=sin(x)`.

(b) Open a new graphic window with the menu Window › New 2d-plot Window and select the menu Window - › Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

(d) Select the name of the function in the Algebra window and select the menu Calculus › Taylor Series.

(e) In the dialog shown enter the 1 in the Expansion Point field, enter 0 in the Order field and click the button Simplify.

(f) Click the button Plot in the graphic window.

(g) Enter the expression `p3(x):=TAYLOR(f(x),x,0,3)` in the Algebra window and click the button Simplify.

(h) Enter the expression `p3(1/2)` in the Algebra window and click the button Approximate to get the approximation.

(i) Enter the expression `ABS(p3(1/2)-f(1.2))` in the Algebra windows and click the button Approximate to get the approximation error.

4. Given the function $f(x,y) = \sqrt{xy}$:

(a) Plot its graph.

1. Define the function entering the expression `f(x,y):=sqrt(xy)` in the Algebra window.
2. Open a new graphic window with the menu Window › New 3D-plot Window and select the menu Window - › Tile Vertically to see the Algebra and the graphic windows at the same time.
3. Click the button Plot in the graphic window.

(b) Compute first degree Taylor polynomial of f at the point $(8,2)$ and plot its graph. Check that you get the tangent plane to the graph of f at the point $(8,2)$.

1. Enter the expression `[8,2,f(8,2)]` in the Algebra window.
2. Click the button Plot in the graphic window.
3. Enter the expression `p1(x,y):=f(8,2)+f'(8,2)[x-8,y-2]` in the Algebra window and click the button Simplify.
4. Click the button Plot in the graphic window.

(c) Use the previous polynomial to approximate the value of $\sqrt{8.02 \cdot 1.99}$.

Enter the expression `p1(8.02,1.99)` in the Algebra window and click the button Approximate.

(d) Compute the error of the previous approximation.

Enter the expression `abs(p1(8.02,1.99)-f(8.02,1.99))` in the Algebra window and click the button Approximate.

(e) Compute second degree Taylor polynomial of f at the point $(8,2)$ and plot its graph.

1. Enter the expression `p2(x,y):=p1(x,y)+1/2[x-8,y-2]f''(8,2)[x-8,y-2]` in the Algebra window and click the button Simplify.
2. Click the button Plot in the graphic window.

(f) Use the previous polynomial to approximate the value of $\sqrt{8.02 \cdot 1.99}$.

Enter the expression `p2(8.02,1.99)` in the Algebra window and click the button Approximate.

(g) Compute the error of the previous approximation. Check that the error of the second degree Taylor polynomial is less than the error with the first degree Taylor polynomial.

Enter the expression `abs(p2(8.02,1.99)-f(8.02,1.99))` in the Algebra window and click the button Approximate.

2 Proposed exercises

1. Given the function $f(x) = \sqrt{x+1}$:
   (a) Compute the 4th degree Taylor polynomial of f at point $x = 0$.
   (b) Approximate the value of $\sqrt{1.02}$ using the 2nd degree and 4th degree Taylor polynomials at $x = 0$ and compute the approximation error en each case.
2. Given the functions $f(x) = e^x$ and $g(x) = \cos x$:
   (a) Compute the 2nd degree Maclaurin polynomials of f and g.
   (b) Use the previous polynomials to compute
   $$\lim_{x\to 0} \frac{e^x - \cos x}{x}.$$
3. Compute the value of $\log(0.09^3 + 0.99^3)$ approximately using:
   (a) A first degree Taylor polynomial.
   (b) A second degree Taylor polynomial.

Practice 6

Integrals

1 Solved exercises

1. Compute the following integrals:

   (a) $\int x^2 \log x \, dx$

   > 1. Enter the expression `x^2 log(x)` in the Algebra window.
   > 2. Select the menu Calculus » Integrate or click the button Find Integral.
   > 3. In the dialog shown check the option Indefinite, enter the constant `C` in the field Constant and click the button Simplify.
   >
   > A faster way is entering the expression `INT(x^2 log(x), x, C)` in the Algebra window and clicking the button Simplify.

   (b) $\displaystyle\int \frac{\log(\log x)}{x} \, dx$

   > Enter the expression `INT(log(log(x)), x, C)` in the Algebra window and click the button Simplify.

   (c) $\displaystyle\int \frac{5x^2+4x+1}{x^5-2x^4+2x^3-2x^2+x} \, dx$

   > Enter the expression `INT((5x^2+4x+1)/(x^5-2x^4+2x^3-2x^2+x), x, C)` in the Algebra window and click the button Simplify.

   (d) $\displaystyle\int \frac{6x+5}{(x^2+x+1)^2} \, dx$

   > Enter the expression `INT((6x+5)/((x^2+x+1)^2), x, C)` in the Algebra window and click the button Simplify.

2. Compute the following definite integrals:

   (a) $\displaystyle\int_{-\frac{1}{2}}^{0} \frac{x^3}{x^2+x+1} \, dx$

   > 1. Enter the expression `x^3/(x^2+x+1)` in the Algebra window.
   > 2. Select the menu Calculus » Integrate or click the button Find Integral.

3. In the dialog shown check the option Definite, enter -1/2 in the field Lower Limit, enter 0 in the field Upper Limit and click the button Simplify.

A faster way is entering the expression `INT(x^3/(x^2+x+1), x, -1/2, 0)` in the Algebra window and clicking the button Simplify.

(b) $\displaystyle\int_2^4 \frac{\sqrt{16-x^2}}{x}\,dx$

Enter the expression `INT(sqrt(16-x^2)/x, x, 2, 4)` in the Algebra window and click the button Simplify.

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3+\cos(2x)}$

Enter the expression `INT(sqrt(1/(3+cos(2x), x, 0, pi/2)` in the Algebra window and click the button Simplify.

3. Compute the following improper integral $\int_2^\infty x^2 e^{-x}\,dx$.

Enter the expression `INT(x^2 exp(-x), x, 2, inf)` in the Algebra window and click the button Simplify.

4. Plot the graph of the parabola $y = x^2 - 7x + 6$ and compute the area between the parabola and the horizontal axis, limited by the lines $x = 2$ and $x = 7$.

To plot the graph of the parabola:

(a) Define the function in the Algebra window entering the expression `f(x):=x^2-7x+6`.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

To compute the area enter the expression `INT(ABS(f(x)), x, 2, 7)` in the Algebra window and click the button Simplify.

To plot the area enter the expression `x>2 ∧ x<7 ∧ y>MIN(0,f(x)) ∧ y<MAX(0,f(x))` in the Algebra window and click the button Plot in the graphic window.

5. Compute and plot the area between the graphs of the functions $\sin x$ and $\cos x$ in the interval $[0, 2\pi]$.

To plot the area:

(a) Define the first function in the Algebra window entering the expression `f(x):=sin x`.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

(d) Define the second function in the Algebra window entering the expression `f(x):=cos x` and click the button Plot in the graphic window.

(e) Enter the expression `x>0 ∧ x<2pi ∧ y>MIN(f(x),g(x)) ∧ y<MAX(f(x),g(x))` in the Algebra window and click the button Plot in the graphic window.

> To compute the area enter the expression `INT(ABS(f(x)-g(x)), x, 0, 2pi)` in the Algebra window and click the button Simplify.

6. Plot the region of the first quadrant limited by the parabola $y^2 = 8x$ the horizontal axis and the line $x = 2$. Compute the volume of the solid of revolution generated rotating the area around the horizontal axis.

> To plot the region:
>
> (a) Define the function in the Algebra window entering the expression `f(x):=sqrt(8x)`.
>
> (b) Open a new graphic window with the menu Window ≫ New 2d-plot Window and select the menu Window - ≫ Tile Vertically to see the Algebra and the graphic windows at the same time.
>
> (c) Click the button Plot in the graphic window.
>
> (d) Enter the expression `x>0 ∧ x<2 ∧ y>0 ∧ y<f(x)` in the Algebra window and click the button Plot in the graphic window.
>
> To compute the volume of the solid of revolution enter the expression `INT(pi f(x)^2, x, 0, 2)` in the Algebra window and click the button Simplify.

2 Proposed exercises

1. Compute the following integrals:

   (a) $\displaystyle\int \frac{2x^3+2x^2+16}{x(x^2+4)^2\,dx}$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$

2. Compute the area limited by the parabola $y = 9 - x^2$ and the line $y = -x$.

3. Compute the area limited by the graph of the function $y = e^{-|x|}$ and its asymptote.

4. Plot the region limited by the parabola $y = 2x^2$, the lines $x = 0$, $x = 5$ and the horizontal axis. Compute the volume of the solid or revolution generated rotating that region around the horizontal axis.

5. Compute the volume of the solid of revolution generated rotating around the vertical axis the region limited by the parabola $y^2 = 8x$ and the line $x = 2$.

Practice 7

Ordinary Differential Equations

1 Solved exercises

To solve ordinary differential equations with Derive, we use the following commands

`DSOLVE1_GEN(p,q,x,y,c)` gives the general solution of a differential equation with form $p(x,y) + q(x,y)y' = 0$.

`DSOLVE1(p,q,x,y,`x_0, y_0`)` gives the particular solution of a differential equation with form $p(x,y) + q(x,y)y' = 0$, with initial condition $y_0 = y(x_0)$.

1. Solve the following separable differential equations and plot their integral curves for the constants $c = -1$, $c = -2$ y $c = -3$:

   (a) $-2x(1+e^y) + e^y(1+x^2)y' = 0$.

   Observe that the equation is written as $p(x,y) + q(x,y)y' = 0$, with $p(x,y) = -2x(1+e^y)$ and $q(x,y) = e^y(1+x^2)$.

   To solve the differential equation enter the expression `DSOLVE1_GEN(-2x(1+#e^y),#e^y(1+x^2),x,y,c)` and click the button Simplify.

   To plot the integral curves:

   1. Select the menu Simplify > Variable Substitution o click the button Varible Substitution.
   2. In the dialog shown select the variable c, enter the value -1 into the field New Value and click the button Simplify.
   3. Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time.
   4. Click the button Plot in the graphic window.

   Repeat the previous steps but entering the values -2 and -3 for c.

   (b) $y - xy' = 1 + x^2y'$.

   Before solving the equation we have to write it in the form $p(x,y) + q(x,y)y' = 0$,

   $$y - xy' = 1 + x^2y' \Leftrightarrow 1 + x^2y' - y + xy' = 0 \Leftrightarrow 1 - y + (x^2 + x)y' = 0$$

   so $p(x,y) = 1 - y$ and $q(x,y) = x^2 + x$.

   To solve the differential equation enter the expression `DSOLVE1_GEN(1-y,x^2+x,x,y,c)` and click the button Simplify.

   To plot the integral curves:

1. Select the menu Simplify > Variable Substitution o click the button Varible Substitution.
2. In the dialog shown select the variable c, enter the value -1 into the field New Value and click the button Simplify.
3. Open a new graphic window with the menu Window > New 2d-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time.
4. Click the button Plot in the graphic window.

Repeat the previous steps but entering the values -2 and -3 for c.

2. Solve the following differential equations with the initial conditions given:

   (a) $x\sqrt{1-y^2} + y\sqrt{1-x^2}y' = 0$, with the initial condition $y(0) = 1$.

   Observe that the equation is written as $p(x,y) + q(x,y)y' = 0$, with $p(x,y) = x\sqrt{1-y^2}$ and $q(x,y) = y\sqrt{1-x^2}$.
   To solve the differential equation enter the expression `DSOLVE1(xsqrt(1-y^2),ysqrt(1-x^2),x,y,0,1)` and click the button Simplify.

   (b) $(1+e^x)yy' = e^y$, with the initial condition $y(0) = 0$.

   Before solving the equation we have to write it in the form $p(x,y) + q(x,y)y' = 0$,

$$(1+e^x)yy' = e^y \Leftrightarrow -e^y + (1+e^x)yy' = 0,$$

   so $p(x,y) = -e^y$ and $q(x,y) = (1+e^x)y$.
   To solve the differential equation enter the expression `DSOLVE1(-#e^y,(1+#e^x)y,x,y,0,0)` and click the button Simplify.

   (c) $y' + y\cos x = \sin x \cos x$ with the initial condition $y(0) = 1$.

   Before solving the equation we have to write it in the form $p(x,y) + q(x,y)y' = 0$,

$$y' + y\cos x = \sin x \cos x \Leftrightarrow -\sin x \cos x + y\cos x + y' = 0,$$

   so $p(x,y) = -\sin x \cos x + y\cos x$ and $q(x,y) = 1$.
   To solve the differential equation enter the expression `DSOLVE1(-sinxcosx+ycosx,1,x,y,0,1)` and click the button Simplify.

3. The speed at which sugar dissolves into water is proportional to the amount of sugar left without dissolving. Suppose we have 13.6 kg of sugar tat we want to mix with water, and after 4 hours there are 4.5 kg without dissolving. How long will it take, from the beginning of the process, for the 95% of the sugar to be dissolved?

The differential equation that explains the dissolution of sugar into water is $y' = ky$, where y is the amount of sugar, t is time and k is the dissolution constant of sugar. This equation can be written in the form $p(t,y) + q(t,y)y' = 0$,

$$y' = ky \Leftrightarrow -ky + y' = 0,$$

so $p(t,y) = -ky$ and $q(t,y) = 1$.
To solve the differential equation enter the expression `DSOLVE1(-ky,1,t,y,0,13.6)` and click the button Simplify.
To get the dissolution constant of sugar we impose the initial condition $y(4) = 4.5$:

(a) Select the menu Simplify ≫ Variable Substitution o click the button Varible Substitution.

(b) In the dialog shown select the variable t, enter the value 4 into the field New Value, select the variable y, enter the value 4.5 into the field New Value and click the button Simplify.

(c) Click the button Approximate.

Finally, to get the time required to have 5% of sugar without dissolving,

(d) Select the expression corresponding to the particular solution of the differential equation.

(e) Select the menu Simplify ≫ Variable Substitution o click the button Varible Substitution.

(f) In the dialog shown select the variable k, enter the previous value got for k in the field New Value, select the variable y, enter the value $13.6 * 0.05$ into the field New Value and click the button Simplify.

(g) Click the button Approximate.

2 Proposed exercises

1. Solve the following differential equations:
   (a) $(1+y^2)+xyy'=0$.
   (b) $xy'-4y+2x^2+4=0$.
   (c) $(y^2+xy^2)y'+x^2-yx^2=0$.
   (d) $(x^3-y^3)dx+2x^2ydy=0$.
   (e) $(x^2+y^2+x)+xydy=0$.
2. Compute the curves (x,y) such that the slope of the tangent line is equal to the value of x at any point. Which of these curves passes through the origin of coordinates?
3. If a person receives glucose by an intravenous drip, the concentration of glucose $c(t)$ with respect to time follows this differential equation:

$$\frac{dc}{dt}=\frac{G}{100V}-kc.$$

   Here G is the (constant) speed at which glucose is given to the patient, V is the total volume of blood in the body, and k is a positive constant that varies with each patient. Compute $c(t)$.
4. A water tank of 50l contains 10 l of water. Suppose we start pouring into the tank a solution of water with 100 g of salt per liter, at a rate of 4 l per minute. We also stir the water tank, to keep a uniform distribution of salt, and, at the same time, we release water (with salt) at a rate of 2 l per minute. How long will it take to the tank to be full? How much salt will there be in the tank in that moment?

   Remark: The variation rate of salt in the tank is equal to the difference between the amount of salt that comes into the tank and the amount of salt that is taken from the tank.

Practice 8

Several variables differentiable calculus

1 Solved exercises

1. Compute the tangent line and the normal plane to the trajectory of

$$f(t) = \begin{cases} x = \sin(t), \\ y = \cos(t), \\ z = \sqrt(t), \end{cases} \qquad t \in \mathbb{R};$$

   at the time $t = 1$ and plot them.

> To plot the trajectory:
>
> (a) Define the function entering the expression `f(t):=[sin(t),cos(t),sqrt(t)]` in the Algebra window.
>
> (b) Open a new graphic window with the menu Window » New 3D-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.
>
> (c) Click the button Plot in the graphic window.
>
> To compute the tangent line and plot its graph:
>
> (a) Enter the expression `f(1)` in the Algebra window.
>
> (b) Click the button Plot in the graphic window.
>
> (c) Enter the expression `f(1)+tf'(1)` in the Algebra window and click the button Simplify.
>
> (d) Click the button Plot in the graphic window.
>
> To compute the normal plane and plot its graph:
>
> (a) Enter the expression `([x,y,z]-f(1))f'(1)=0` in the Algebra window and click the button Simplify.
>
> (b) Click the button Plot in the graphic window.

2. Given the function $f(x,y) = y^2 - x^2$,

   (a) Plot its graph.

> 1. Enter the expression `f(x,y):=y^2-x^2` in the Algebra window.
> 2. Open a new graphic window with the menu Window » New 3D-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.
> 3. Click the button Plot in the graphic window.

(b) Plot the plane with equation $x = 1$. What shape has the curve that results from the intersection of this plane and the graph of f?

1. Enter the expression `x=1` in the Algebra window.
2. Click the button Plot in the 3D graphic window.

(c) Compute the derivative of $f(1, y)$ at $y = 2$.

1. Enter the expression `f(1,y)` in the Algebra window.
2. Select the menu Calculus » Differentiate or click the button Find Derivative.
3. In the dialog shown click the button Simplify.
4. Select the menu Simplify » Variable Substitution or click the button Variable Substitution.
5. In the dialog shown enter the value 2 in the field New Value and click the button Simplify.

(d) Plot the plane with equation $y = 2$. What shape has the curve that results from the intersection of this plane and the graph of f?

1. Enter the expression `y=2` in the Algebra window.
2. Click the button Plot in the 3D graphic window.

(e) Compute la derivative of $f(x, 2)$ at $x = 1$.

1. Enter the expression `f(x,2)` in the Algebra window.
2. Select the menu Calculus » Differentiate or click the button Find Derivative.
3. In the dialog shown click the button Simplify.
4. Select the menu Simplify » Variable Substitution or click the button Variable Substitution.
5. In the dialog shown enter the value 1 in the field New Value and click the button Simplify.

(f) Compute the partial derivatives of f at the point $(1, 2)$. What conclusions can you draw?

For the partial derivative of f with respect to x:

1. Enter the expression `f(x,y)` or select the expression corresponding to f in the Algebra window.
2. Select the menu Calculus » Differentiate or click the button Find Derivative.
3. In the dialog shown select the variable x in the drop-down list and click the button Simplify.
4. Select the menu Simplify » Variable Substitution or click the button Variable Substitution.
5. In the dialog shown select the variable x in the list Variables and enter the value 1 in the field New Value, then select the variable y in the list Variables and enter the value 2 in the field New Value, and click the button Simplify.

A faster way of computing the partial derivative is entering the command `DIF(f(x,y),x)`

For the partial derivative of f with respect to y repeat the previous steps but selecting the variable y in the drop-down list of the Calculus Differentiate dialog.

3. Compute the following partial derivatives:

(a) $\dfrac{\partial}{\partial V}\dfrac{nRT}{V}$.

Enter the expression `DIF(nRT/V, V)` in the Algebra window and click the button Simplify.

(b) $\dfrac{\partial^2}{\partial x \partial y} e^{x+y} \sin(x/y)$.

Enter the expression `DIF(DIF(exp(x+y)sin(x/y), y), x)` in the Algebra window and click the button Simplify.

4. Given the function $f(x,y) = 20 - 4x^2 - y^2$, compute at the point $(2,-3)$:

(a) Gradient.

1. Define the function entering the expression `f(x,y):=20-4x^2-y^2` in the Algebra window.
2. Enter the expression `f'(2,-3)` in the Algebra window and click the button Simplify.

(b) Hessian matrix.

Enter the expression `f''(2,-3)` in the Algebra window and click the button Simplify.

(c) Hessian.

Enter the expression `DET(f''(2,-3))` in the Algebra window and click the button Simplify.

5. Compute the normal line and the tangent plane to the surface $S : x + 2y - \log z + 4 = 0$ at the point $(0,-2,1)$ and plot them.

To plot the surface:

(a) Define the function entering the expression `f(x,y,z):=x+2y-log(z)+4` in the Algebra window.

(b) Enter the expression `f(x,y,z)=0` in the Algebra window.

(c) Select the menu Solve > Expression.

(d) In the dialog shown select the variable z in the list Solution Variables, check the option Real in the field Solution Domain and click the button Solve.

(e) Open a new graphic window with the menu Window > New 3D-plot Window and select the menu Window - > Tile Vertically to see the Algebra and the graphic windows at the same time.

(f) Click the button Plot in the graphic window.

To compute normal line and plot its graph:

(g) Enter the expression `[0,-2,1]+tf'(0,-2,1)` and click the button Simplify.

(h) Click the button Plot in the graphic window.

To compute the tangent plane and plot its graph:

(i) Enter the expression `([x,y,z]-[0,-2,1])f'(0,-2,1)=0` in the Algebra window and click the button Simplify.

(j) Click the button Plot in the graphic window.

6. Compute directional derivative of the function $h(x,y) = 3x^2 + y$ at the point $(0,0)$, along the vector $(1,1)$.

(a) Define the function entering the expression `h(x,y):=3x^2+y` in the Algebra window.
(b) Enter the expression `h'(0,0)SIGN([1,1])` in the Algebra window and click the button Simplify.

7. Given the function $f(x,y) = x^3 + y^3 - 3xy$:

   (a) Plot its graph. Looking at the graph, can you determine the relative extrema of f?

   1. Define the function entering the expression `f(x,y):=x^3+y^3-3xy`.
   2. Open a new graphic window with the menu Window » New 3D-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.
   3. Click the button Plot in the graphic window.

   (b) Compute critical points of f.

   The critical points are the points that make zero the gradient.

   1. Enter the expression `f'(x,y)=0` in the Algebra window.
   2. Select the menu Solve » Expression.
   3. In the dialog shown select the variables x and y in the list Solution Variables, check the option Real in the field Solution Domain and click the button Solve.

   (c) Determine the relative extrema and the saddle points of f.

   For each critical point (a,b) you have to compute the Hessian matrix and the Hessian:

   1. Enter the expression `f''(a,b)` in the Algebra window and click the button Simplify.
   2. Enter the expression `DET(f''(a,b))` in the Algebra window and click the button Simplify.

   If the Hessian is positive and the second partial derivative of f with respect to x two times is positive the function has a relative minimum at the critical point. If the Hessian is positive and the second partial derivative of f with respect to x two times is negative the function has a relative maximum at the critical point. If the Hessian is negative the function has a saddle point at the critical point.

2 Proposed exercises

1. A spaceship, traveling near the sun, is in trouble. The temperature at position (x,y,z) is given by

$$T(x,y,z) = e^{-x^2-2y^2-3z^2},$$

   where the variables are measured in thousands of kilometers, and we assume the sun is at position $(0,0,0)$. If the ship is at position $(1,1,1)$, find the direction in which it should move so that the temperature will decrease as fast as possible.

2. Compute the gradient, the Hessian matrix and the Hessian of the function

$$g(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}^3}$$

   at the point $(1,1,1)$ and at the point $(0,3,4)$.

3. Determine the points of the ellipsoid $S : x^2 + 2y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $\Pi : x - y + 2z^2 = 0$.

4. Determine the relative extrema of the function

$$f(x) = -\frac{y}{9 + x^2 + y^2}.$$

5. Compute the directional derivative of the scalar field $f(x, y, z) = x^2 - y^2 + xyz^3 - zx$ at the point $(1, 2, 3)$ along the vector $(1, -1, 0)$.