Calculus with Derive

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Practice 1

Limits and continuity

1 Solved exercises

1. Given the function

$$f(x) = \left(1 + \frac{2}{x}\right)^{x/2},$$

plot its graph and compute the following limits:

(a) $\lim_{x\to-\infty} f(x)$

(b) $\lim_{x\to+\infty} f(x)$

(c) $\lim_{x\to-2^-} f(x)$

(d) $\lim_{x\to-2^+} f(x)$

(e) $\lim_{x\to 2} f(x)$

(f) $\lim_{x\to 0} f(x)$

(a) Enter the expression of the function in the Algebra window and select it.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

(d) For computing every limit repeat the following steps

1. Select the function in the Algebra window.
2. Select the menu Calculus » Limit or click the button Limit.
3. In the dialog shown enter the point of the limit in the field Limit Point, select the corresponding option from the list Approach From (Left for a one-sided limit from the left, Right for a one-sided limit from the right, and Both for a global or two-sided limit) and click the button Simplify.
4. Look at the graph and check if the result of the limit makes sense.

2. Given the function

$$f(x) = \begin{cases} \dfrac{x}{x-2} & \text{if } x \leq 0; \\ \dfrac{x^2}{2x-6} & \text{if } x > 0; \end{cases}$$

(a) Plot the graph and determine graphically if there are asymptotes.

1. Enter the expression `f(x) := x/(x-2) CHI(-inf,x,0) + x^2/(2x-6) CHI(0,x,inf)` to define the function in the Algebra window and select the function.
2. Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

3. Click the button Plot in the graphic window.

(b) Compute the vertical asymptotes and plot them if any.

The only point where the function is not defined is $x = 3$. Thus to check if there is a vertical asymptote at this point we have to compute the limit at this point.

1. Select the name of function in the Algebra window.
2. Select the menu Calculus » Limit or click the button Limit.
3. In the dialog shown enter 3 in the field Limit Point, select `Left` from the list Approach From and click the button Simplify.
4. Repeat the three previous steps but selecting `Right` from the list Approach From.
5. Look at the graph and check if the results of the limits make sense.

There is a vertical asymptote $x = 3$ if some of the limits is infinite. In that case enter the expression of the asymptote in the Algebra window and click the button Plot in the graphic window.

(c) Compute the horizontal asymptotes and plot them if any.

To check if there is an horizontal asymptote we have to compute the limits at infinity.

1. Select the name of function in the Algebra window.
2. Select the menu Calculus » Limit or click the button Limit.
3. In the dialog shown enter `-inf` in the field Limit Point and click the button Simplify.
4. Repeat the two previous steps but selecting entering `inf` in the field Limit Point.
5. Look at the graph and check if the results of the limits make sense.

There is an horizontal asymptote $y = a$ if some of the limits is a. In that case enter the expression of the asymptote in the Algebra window and click the button Plot in the graphic window.

(d) Compute the oblique asymptotes and plot them if any.

To check if there is an oblique asymptote we have to compute the limits at infinity of the function divided by x. To check if there is an oblique asymptote at $-\infty$, do the following steps:

1. Enter the expression `f(x)/x` in the Algebra window and select it.
2. Select the menu Calculus » Limit or click the button Limit.
3. In the dialog shown enter `-inf` in the field Limit Point and click the button Simplify.

There is an oblique asymptote $y = ax + b$ if some of the limits is a. In that case, a is the slope of the asymptote. To compute the independent term we have to compute the limits at infinity of the function minus ax.

1. Enter the expression `f(x)-ax`, where `a` is the value of the previous limit, in the Algebra window and select it.
2. Select the menu Calculus » Limit or click the button Limit.
3. In the dialog shown enter `-inf` in the field Limit Point and click the button Simplify.

The independent term of the oblique asymptote is the result of this limit.

To check if there is an oblique asymptote at ∞, repeat all the steps but entering `inf` in the field Limit Point.

If there is some oblique asymptote enter the expression of the asymptote in the Algebra window and click the button Plot in the graphic window.

3. For the following functions determine the type of discontinuity at the points given.

(a) $f(x) = \dfrac{\sin x}{x}$ at $x = 0$.

(b) $g(x) = \dfrac{1}{2^{1/x}}$ at $x = 0$.

(c) $h(x) = \dfrac{1}{1 + e^{\frac{1}{1-x}}}$ at $x = 1$.

For every function repeat the following steps:

(a) Enter the expression of the function in the Algebra window and select it.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

(d) Select the function in the Algebra window.

(e) Select the menu Calculus » Limit or click the button Limit.

(f) In the dialog shown enter the given point in the field Limit Point, select `Left` from the list Approach From and click the button Simplify.

(g) Repeat the three previous steps but selecting `Right` from the list Approach From.

If both limits exist and are the same, then there is a *removable discontinuity*. If both limits exist but are different, then there is a *jump discontinuity*. If some of the limits doesn't exist or is infinite, then there is an *essential discontinuity*.

4. Determine the points where the following function has a discontinuity and classify it.

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-1}, & \text{if } x < 0; \\ \dfrac{1}{e^{1/(x^2-1)}}, & \text{if } x \geq 0. \end{cases}$$

(a) Enter the expression `f(x) := (x+1)/(x^2-1) CHI(-inf,x,0) + 1/exp(1/(x^2-1)) CHI(0,x,inf)` to define the function in the Algebra window and select the function.

(b) Open a new graphic window with the menu Window » New 2d-plot Window and select the menu Window - » Tile Vertically to see the Algebra and the graphic windows at the same time.

(c) Click the button Plot in the graphic window.

The function is not defined in $x = -1$ and $x = 1$, so there is a discontinuity at each of these points. As the functions is piecewise, also we have to study the points where the expression of the function changes, that is, at $x = 0$. To classify the type of discontinuity for each of these points, repeat the following steps:

(a) Select the function in the Algebra window.

(b) Select the menu Calculus » Limit or click the button Limit.

(c) In the dialog shown enter the given point in the field Limit Point, select `Left` from the list Approach From and click the button Simplify.

(d) Repeat the three previous steps but selecting `Right` from the list Approach From.

If both limits exist and are the same, then there is a *removable discontinuity*. If both limits exist but are different, then there is a *jump discontinuity*. If some of the limits doesn't exist or is infinite, then there is an *essential discontinuity*.

2 Proposed exercises

1. Compute the following limits:

   (a) $\displaystyle\lim_{x\to 1}\frac{x^3-3x+2}{x^4-4x+3}$.

   (b) $\displaystyle\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$.

   (c) $\displaystyle\lim_{x\to\infty}\frac{x^2-3x+2}{e^{2x}}$.

   (d) $\displaystyle\lim_{x\to\infty}\frac{\log(x^2-1)}{x+2}$.

   (e) $\displaystyle\lim_{x\to 1}\frac{\log(1/x)}{\tan(x+\frac{\pi}{2})}$.

   (f) $\displaystyle\lim_{x\to a}\frac{x^n-a^n}{x-a}\qquad n\in\mathbb{N}$.

   (g) $\displaystyle\lim_{x\to 1}\frac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}\qquad n,m\in\mathbb{Z}$.

   (h) $\displaystyle\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$.

   (i) $\displaystyle\lim_{x\to\pi/4}\frac{\sin x-\cos x}{1-\tan x}$.

   (j) $\displaystyle\lim_{x\to 0}x^2e^{1/x^2}$.

   (k) $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

   (l) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

   (m) $\displaystyle\lim_{x\to 0}(\cos x)^{1/\operatorname{sen} x}$.

   (n) $\displaystyle\lim_{x\to 0}\frac{6}{4+e^{-1/x}}$.

   (o) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-2x-1}\right)$.

2. Given the function

$$f(x)=\begin{cases}\dfrac{x^2+1}{x+3} & \text{if } x<0;\\[2mm] \dfrac{1}{e^{1/(x^2-1)}} & \text{if } x\geq 0;\end{cases}$$

   compute its asymptotes.

3. The following functions are not defined at $x=0$. Determine, when possible, the value that should take the function at that point to be continuous.

   (a) $f(x)=\dfrac{(1+x)^n-1}{x}$.

   (b) $h(x)=\dfrac{e^x-e^{-x}}{x}$.

   (c) $j(x)=\dfrac{\log(1+x)-\log(1-x)}{x}$.

   (d) $k(x)=x^2\sin\dfrac{1}{x}$.