CALCULUS PROBLEMS

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Authors: Pablo Ares Gastesi (pablo.aresgastesi@ceu.es)
Eduardo López Ramírez (elopez@ceu.es)
Anselmo Romero Limón (arlimon@ceu.es)
Alfredo Sánchez Alberca (asalber@ceu.es)

CEU
Universidad
San Pablo

Contents

1 One-variable Differentiable Calculus

1. Compute the derivative function of $f(x) = x^3 - 2x^2 + 1$ at the points $x = -1$, $x = 0$ and $x = 1$. Explain your result. Find an equation of the tangent line to the graph of f at each of the three given points.

   Solution

   $f'(-1) = 7$, $f'(0) = 0$ y $f'(1) = -1$.
   Tangent line at $x = -1$: $y = -2 + 7(x + 1)$.
   Tangent line at $x = 0$: $y = 1$.
   Tangent line at $x = 1$: $y = -(x - 1)$.

2. The pH measures the concentration of hydrogen ions H^+ in an aqueous solution. It is defined by

   $$\mathrm{pH} = -\log_{10}(\mathrm{H}^+).$$

   Compute the derivative of the pH as a function of the concentration of H^+. Study the growth of the pH function.

   Solution

   The pH decreases as the concentration of hydrogen ions H^+ increase.

3. The speed $v(n)$ at which a plant grows depends on the amount of nitrogen available n by the following relation:

   $$v(n) = \frac{an}{k+n}, \quad n \geq 0,$$

   where a and k are positive constants. Study the growth of this function, and explain your results.

   Solution

   The speed increases as n increases but each time with less force, so that for $n \to \infty$ the speed becomes stable.

4. Find an equation of the tangent and normal lines to the curves given below at the given point x_0.

   (a) $y = x^{\sin x}$, $x_0 = \pi/2$.

   (b) $y = \log\sqrt{\frac{1+x}{1-x}}$, $x_0 = 0$.

   Solution

   (a) Tangent: $y - \frac{\pi}{2} = x - \frac{\pi}{2}$. Normal: $y - \frac{\pi}{2} = -x + \frac{\pi}{2}$.
   (b) Tangent: $y = x$. Normal: $y = -x$.

5. Air is being pumped into a spherical balloon of radius 10cm so that the radius increases at a rate of 2 cm/s. How fast will the volume of the balloon increase?
   Remark: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

   Solution

   800π cm^3/s.

6. A liquid solution is kept in a cylindrical pipette of radius 5 cm. Suppose the liquid is taken out of the pipette at a rate of 0.5 ml per second; compute the rate of change of the level of liquid in the pipette.

Solution

-6.37 mm/s.

7. Radioactive decay is given by the following function:

$$m(t) = m_0 e^{-kt},$$

where $m(t)$ denotes the amount of matter at time t, m_0 is the initial amount of radioactive matter, and k is a constant called the *decay constant*. The variable t represents time. Compute the speed of decay at any given time t.
Recall that the *half life* of a radioactive material is the time it takes for a quantity to reduce to half its initial value. Suppose for certain radioactive material we have $k = 0.002$, compute the half life of the material.

Solution

Speed of decay: $-km_0e^{-kt}$.
Half life: 346.57 years.

8. A car is moving on a straight line direction, with position given by the following function:

$$e(t) = 4t^3 - 2t + 1.$$

Find the speed and acceleration of the car.
Remark: The acceleration is the variation rate of the instant velocity.

Solution

speed $v(t) = 12t^2 - 2$ and acceleration $a(t) = 24t$.

9. An object is thrown verticall upwards. Assuming there is no air friction, the object will travel a distance given by the following equation:

$$e(t) = v_0 t - \frac{1}{2} g t^2$$

where v_0 is the initial velocity (at which the object is thrown), $g = 9.81$ m/s^2 is the gravitational Earth constant, and t is the time lapsed since the object was thrown.

(a) Compute the speed and acceleration of the object at any time.
(b) Suppose the initial speed is 50 km/h, how high will the object get? Compute the speed at the moment of maximum height.
(c) At what time will the object fall to the ground? With what speed?

Solution

(a) Speed $v(t) = v_0 - gt$ and acceleration $a(t) = -g$.
(b) Maximum height 9.83 m at 1.42 s. The speed at that moment vanishes.
(c) The object fall to the ground at 2.83 s with speed -13.89 m/s.

10. A cylinder of radius $r = 4$ cm and height $h = 3$ is heated, and so its dimensions change with speed given by $\dfrac{dr}{dt} = \dfrac{dh}{dt} = 1$ cm/s. Find the approximate rate of change in the volume of the cylinder at 5 and 10 seconds after the heating process starts.

Solution

$dV = 2\pi rhdt + \pi r^2 dt$ and at the initial moment $dV = 40\pi dt$. 5 seconds after the approximate rate of change is $dV(5) = 40\pi 5 = 200\pi$ cm^3/s, and 10 seconds after $dV(10) = 40\pi 10 = 400\pi$ cm^3/s.

11. The radius of a sphericall cell is equal to 5 μ, with a possible error of 0.2 μ; compute the error in the measurement of the area of the cell. More generally, if the error in the measurment of the radius is 2%, what is the error in the value of the surface of the cell?
Remark: The surface of a sphere of radius r is given by $S = 4\pi r^2$. Solve the problem by means of the linear approximation (tangent line) of a function.

Solution

For an radius error of 0.2 μ the approximate error in the area is 8π μ^2, and for a relative error of 2% the approximate relative error in the area is 4%.

★ 12. In certain chemical process, the concentration of certain substance c depends on the concentration of two other substances a and b, by the following equation $c = \sqrt[3]{ab^2}$. Suppose that at certain moment, when $a = b = 2$ mg/mm^3, the concentrations of a and b increase at rates of 0.2 mg$\cdot$ mm^{-3}/s, and 0.4 mg$\cdot$ mm^{-3}/s, respectively. Approximate the concentration of c after 2 seconds.

Solution

$c'(t_0) = 1/3$ mg$\cdot$mm^{-3}/s.
$c(t_0 + 2) \approx 5/3$ mg$\cdot$mm^{-3}.

13. Blood flows through an artery at a speed v, which is related to the radius r of the artery by the following expression, known as Poiseuille's law,

$$v(r) = cr^2.$$

As mentioned above, v is the speed of the flow, r the radius of the artery, which we will assume to be cylindrical, and c is a constant. Assume the radius can be measured with a precision of 5%; calculate the precision in the in the computation of the speed.

Solution

10%.

14. Consider the sine function $f(x) = \sin x$.

    (a) Compute the third degree Taylor polynomial centered at the point $x = \pi/6$. Use this polynomial to approximate $\sin(1/2)$.

    (b) Give an approximate value of $\sin(1/2)$ using a fifth degree Taylor polynomial centered at the point $x = 0$.

Solution

(a) $P^3_{f,\pi/6}(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\pi/6) - \frac{1}{4}(x-\pi/6)^2 - \frac{\sqrt{3}}{12}(x-\pi/6)^3$.
sen $1/2 \approx P^3_{f,\pi/6}(1/2) = 0.4794255322$.

(b) $P^5_{f,0}(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$.
sen $1/2 \approx P^5_{f,0}(1/2) = 0.4794270833$.

15. Compute the second degree Taylor polynomial of the funtion $f(x) = \sqrt[3]{x}$ in a neighborhood of the point $x = 1$.

Solution

$P^2_{f,1}(x) = 1 + \frac{1}{3}(x-1) - \frac{2}{18}(x-1)^2$.

16. Compute the third degree Maclaurin polynomial for the function $f(x) = \arcsin x$.

Solution

$P^3_{f,0}(x) = x + \frac{1}{6}x^3$.

★ 17. The function $C(t)$ measures the concentration (in mg/dl) of a drug in the bloodstream as function of time (in hours):

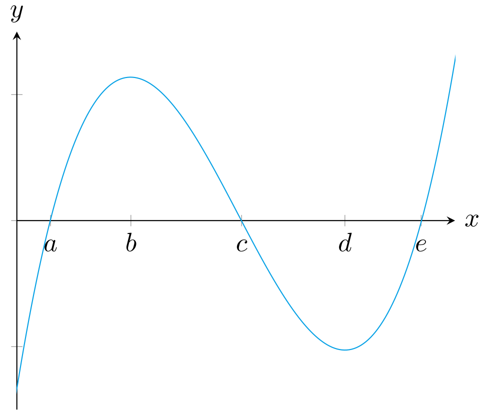
$$C(t) = \frac{1}{1+e^{-2t}}$$

(a) Compute the third degree Maclaurin polynomial for the function.
(b) Use the previous polynomial to compute approximately the concentration of drug in the bloodstream after 15 minutes.

Solution

(a) $P^3_{C,0}(t) = \frac{1}{2} + \frac{1}{2}t + 0\frac{t^2}{2!} - 1\frac{t^3}{3!} = \frac{1}{2} + \frac{1}{2}t - \frac{1}{6}t^3$.
(b) $P^3_{C,0}(0.25) = 0.6223958333$ mg/dl.

★ 18. Below you can find the graph of the derivative $f'(x)$ of a function $f(x)$. Determine the behaviour of f (increasing, decreasing, convex, concave, extrema) from that graph.



Solution

Growth: Decreasing at $(-\infty, a)$ and (c, e), and increasing at (a, c) and (e, ∞).
Extrema: Minimum at $x = a$ and $x = e$, and maximum at $x = c$.
Concavity: Concave down at $(-\infty, b)$ and (d, ∞), and concave up at (b, d).

19. Find the values of a, b and c so that the function $f(x) = x^3 + bx^2 + cx + d$ has an inflection point at $x = 3$, its graph goes through the point $(1, 0)$, and it has a maximum at $x = 1$.

Solution

$b = -9$, $c = 15$ y $d = -7$.

20. The response S of an organism to a drug depends on the dose x by the relation

$$S(x) = x(C - x),$$

where C is the maximum amount of the drug that can be given to a person. Find the dose x for which the response is maximum.

Solution

$x = C/2$.

21. The speed v at which certain chemical reaction $A + B \to AB$ takes place is a function of the concentraion x of the substance AB. This speed it is given by the following equation:

$$v(x) = 4(3 - x)(5 - x).$$

Determine the value of x that maximizes the speed of the process.

Solution

None.

22. The wheat yield C of a field depends on the level of nitrogen on the ground n, and it is given by the following relation:

$$C(n) = \frac{n}{1 + n^2}, \quad n \geq 0.$$

Find the level of nitrogen that will produce the biggest yield.

Solution

$n = 1$.

23. A drug has to be given to patients in cylindrical pills. The content of the drug in each pill is 0.15 ml; determine the dimensions of the cylinder so that the amount of material used to make it (the pill) is minimal.

Solution

Radius 0.2879 cm and height 0.5760 cm.

★ 24. A mathematical model for the amount of water in certain lake, $m(t)$, in millions of cubic meters, is given as a function of the time t, measured in years lapsed since the study took place. The formula is the following:

$$m(t) = 10 + \frac{\sqrt{t}}{e^t}$$

This formula makes sense only for positive values of the variable t.

(a) How much water will there be in the lake when t goes to infinity?

(b) Use derivatives to find the time at which the amount of water in the lake is maximum, and compute the amount of water at such time.

Solution

(a) $\lim_{t\to\infty} m(t) = 10$.

(b) $\frac{dm}{dt} = e^{-t}(\frac{1}{2}t^{-1/2} - t^{1/2})$. The moment at which the amount of water in the lake will be maximum is $t = 0.5$ years and at this moment there will be 10.429 millions of m^3.

★ 25. Consider a function $f(x)$ with derivative given by

$$f'(x) = \frac{(2-x)e^{-\frac{x^2}{2}+2x-2}}{\sqrt{2\pi}}$$

(a) Determine the regions on which f is increasing, and those where f is decreasing.

(b) Find the extrema points of f.

(c) Determine the points on which f is concave up, and those on which it is concave down.

(d) Find the values of x corresponding to the inflection points of the graph of f.

Solution

(a) Increasing at $x < 2$ and decreasing at $x > 2$.

(b) Relative maximum at $x = 2$.

(c) Concave down at$(-\infty, 1)$ and $(3, \infty)$, and concave down at $(1, 3)$.

(d) Inflection points at $x = 1$ and $x = 3$.

Remark: The exercises with a (★) are exam exercises of previous years.