CALCULUS PROBLEMS

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Contents

1 One-variable Differentiable Calculus

1. Compute the derivative function of $f(x) = x^3 - 2x^2 + 1$ at the points $x = -1$, $x = 0$ and $x = 1$. Explain your result. Find an equation of the tangent line to the graph of f at each of the three given points.

   Solution

   $f'(-1) = 7$, $f'(0) = 0$ y $f'(1) = -1$.
   Tangent line at $x = -1$: $y = -2 + 7(x + 1)$.
   Tangent line at $x = 0$: $y = 1$.
   Tangent line at $x = 1$: $y = -(x - 1)$.

2. The pH measures the concentration of hydrogen ions H^+ in an aqueous solution. It is defined by

   $$\mathrm{pH} = -\log(\mathrm{H}^+).$$

   Compute the derivative of the pH as a function of the concentration of H^+. Study the growth of the pH function.

   Solution

   The pH increases as the concentration of hydrogen ions H^+ increase.

3. The speed $v(n)$ at which a plant grows depends on the amount of nitrogen available n by the following relation:

   $$v(n) = \frac{an}{k+n}, \quad n \geq 0,$$

   where a and k are positive constants. Study the growth of this function, and explain your results.

   Solution

   The speed increases as n increases but each time with less force, so that for $n \to \infty$ the speed becomes stable.

4. Find an equation of the tangent and normal lines to the curves given below at the given point x_0.

   (a) $y = x^{\sin x}$, $\quad x_0 = \pi/2$.

   (b) $y = \log\sqrt{\dfrac{1+x}{1-x}}$, $\quad x_0 = 0$.

   Solution

   (a) Tangent: $y - \frac{\pi}{2} = x - \frac{\pi}{2}$. Normal: $y - \frac{\pi}{2} = -x + \frac{\pi}{2}$.
   (b) Tangent: $y = x$. Normal: $y = -x$.

5. Air is being pumped into a spherical balloon of radius 10cm so that the radius increases at a rate of 2 cm/s. How fast will the volume of the balloon increase?
   Remark: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

   Solution

   800π cm^3/s.

6. A liquid solution is kept in a cylindrical pipette of radius 5 cm. Suppose the liquid is taken out of the pipette at a rate of 0.5 ml per second; compute the rate of change of the level of liquid in the pipette.

Solution

-6.37 mm/s.

7. Radioactive decay is given by the following function:

$$m(t) = m_0 e^{-kt},$$

where $m(t)$ denotes the amount of matter at time t, m_0 is the initial amount of radioactive matter, and k is a constant called the *decay constant*. The variable t represents time. Compute the speed of decay at any given time t.
Recall that the *half life* of a radioactive material is the time it takes for a quantity to reduce to half its initial value. Suppose for certain radioactive material we have $k = 0.002$, compute the half life of the material.

Solution

Speed of decay: $-km_0e^{-kt}$.
Half life: 346.57 years.

8. A car is moving on a straight line direction, with position given by the following function:

$$e(t) = 4t^3 - 2t + 1.$$

Find the speed and acceleration of the car.
Remark: The acceleration is the variation rate of the instant velocity.

Solution

speed $v(t) = 12t^2 - 2$ and acceleration $a(t) = 24t$.

9. An object is thrown verticall upwards. Assuming there is no air friction, the object will travel a distance given by the following equation:

$$e(t) = v_0 t - \frac{1}{2} g t^2$$

where v_0 is the initial velocity (at which the object is thrown), $g = 9.81$ m/s^2 is the gravitational Earth constant, and t is the time lapsed since the object was thrown.

(a) Compute the speed and acceleration of the object at any time.
(b) Suppose the initial speed is 50 km/h, how high will the object get? Compute the speed at the moment of maximum height.
(c) At what time will the object fall to the ground? With what speed?

Solution

(a) Speed $v(t) = v_0 - gt$ and acceleration $a(t) = -g$.
(b) Maximum height 9.83 m at 1.42 s. The speed at that moment vanishes.
(c) The object fall to the ground at 2.83 s with speed -13.89 m/s.

10. A cylinder of radius $r = 4$ cm and height $h = 3$ is heated, and so its dimensions change with speed given by $\dfrac{dr}{dt} = \dfrac{dh}{dt} = 1$ cm/s. Find the approximate rate of change in the volume of the cylinder at 5 and 10 seconds after the heating process starts.

Solution

$dV = 2\pi rhdt + \pi r^2 dt$ and at the initial moment $dV = 40\pi dt$. 5 seconds after the approximate rate of change is $dV(5) = 40\pi 5 = 200\pi$ cm^3/s, and 10 seconds after $dV(10) = 40\pi 10 = 400\pi$ cm^3/s.

11. The radius of a sphericall cell is equal to 5 μ, with a possible error of 0.2 μ; compute the error in the measurement of the area of the cell. More generally, if the error in the measurment of the radius is 2%, what is the error in the value of the surface of the cell?
Remark: The surface of a sphere of radius r is given by $S = 4\pi r^2$. Solve the problem by means of the linear approximation (tangent line) of a function.

Solution

For an radius error of 0.2 μ the approximate error in the area is $8\pi\ \mu^2$, and for a relative error of 2% the approximate relative error in the area is 4%.

★ 12. In certain chemical process, the concentration of certain substance c depends on the concentration of two other substances a and b, by the following equation $c = \sqrt[3]{ab^2}$. Suppose that at certain moment, when $a = b = 2$ mg/mm^3, the concentrations of a and b increase at rates of 0.2 mg$\cdot$ mm^{-3}/s, and 0.4 mg$\cdot$ mm^{-3}/s, respectively. Find the rate of change in the concentration of c after 2 seconds.

Solution

$c'(t_0) = 48$ mg$\cdot$mm^{-3}/s.
$c(t_0 + 2) \approx 192$ mg$\cdot$mm^{-3}.

13. Blood flows through an artery at a speed v, which is related to the radius r of the artery by the following expression, known as Poiseuille's law,

$$v(r) = cr^2.$$

As mentioned above, v is the speed of the flow, r the radius of the artery, which we will assume to be cylindrical, and c is a constant. Assume the radius can be measured with a precision of 5%; calculate the precision in the in the computation of the speed.

Solution

10%.

Remark: The exercises with a (★) are exam exercises of previous years.