CALCULUS PROBLEMS

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Contents

1 One-variable Differentiable Calculus

1. Compute the derivative function of $f(x) = x^3 - 2x^2 + 1$ at the points $x = -1$, $x = 0$ and $x = 1$. Explain your result. Find an equation of the tangent line to the graph of f at each of the three given points.

   Solution

   $f'(-1) = 7$, $f'(0) = 0$ y $f'(1) = -1$.
   Tangent line at $x = -1$: $y = -2 + 7(x + 1)$.
   Tangent line at $x = 0$: $y = 1$.
   Tangent line at $x = 1$: $y = -(x - 1)$.

2. The pH measures the concentration of hydrogen ions H^+ in an aqueous solution. It is defined by

   $$\mathrm{pH} = -\log_{10}(\mathrm{H}^+).$$

   Compute the derivative of the pH as a function of the concentration of H^+. Study the growth of the pH function.

   Solution

   The pH decreases as the concentration of hydrogen ions H^+ increase.

3. The speed $v(n)$ at which a plant grows depends on the amount of nitrogen available n by the following relation:

   $$v(n) = \frac{an}{k+n}, \quad n \geq 0,$$

   where a and k are positive constants. Study the growth of this function, and explain your results.

   Solution

   The speed increases as n increases but each time with less force, so that for $n \to \infty$ the speed becomes stable.

4. Find an equation of the tangent and normal lines to the curves given below at the given point x_0.

   (a) $y = x^{\sin x}$, $\quad x_0 = \pi/2$.

   (b) $y = \log\sqrt{\dfrac{1+x}{1-x}}$, $\quad x_0 = 0$.

   Solution

   (a) Tangent: $y - \frac{\pi}{2} = x - \frac{\pi}{2}$. Normal: $y - \frac{\pi}{2} = -x + \frac{\pi}{2}$.
   (b) Tangent: $y = x$. Normal: $y = -x$.

5. Air is being pumped into a spherical balloon of radius 10cm so that the radius increases at a rate of 2 cm/s. How fast will the volume of the balloon increase?
   Remark: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

   Solution

   800π cm^3/s.

6. A liquid solution is kept in a cylindrical pipette of radius 5 mm. Suppose the liquid is taken out of the pipette at a rate of 0.5 ml per second; compute the rate of change of the level of liquid in the pipette.

Solution

-0.00637 cm/s.

7. Radioactive decay is given by the following function:

$$m(t) = m_0 e^{-kt},$$

where $m(t)$ denotes the amount of matter at time t, m_0 is the initial amount of radioactive matter, and k is a constant called the *decay constant*. The variable t represents time. Compute the speed of decay at any given time t.
Recall that the *half life* of a radioactive material is the time it takes for a quantity to reduce to half its initial value. Suppose for certain radioactive material we have $k = 0.002$, compute the half life of the material.

Solution

Speed of decay: $-km_0e^{-kt}$.
Half life: 346.57 years.

8. A car is moving on a straight line direction, with position given by the following function:

$$e(t) = 4t^3 - 2t + 1.$$

Find the speed and acceleration of the car.
Remark: The acceleration is the variation rate of the instant velocity.

Solution

speed $v(t) = 12t^2 - 2$ and acceleration $a(t) = 24t$.

9. An object is thrown verticall upwards. Assuming there is no air friction, the object will travel a distance given by the following equation:

$$e(t) = v_0 t - \frac{1}{2} g t^2$$

where v_0 is the initial velocity (at which the object is thrown), $g = 9.81$ m/s^2 is the gravitational Earth constant, and t is the time lapsed since the object was thrown.

(a) Compute the speed and acceleration of the object at any time.
(b) Suppose the initial speed is 50 km/h, how high will the object get? Compute the speed at the moment of maximum height.
(c) At what time will the object fall to the ground? With what speed?

Solution

(a) Speed $v(t) = v_0 - gt$ and acceleration $a(t) = -g$.
(b) Maximum height 9.83 m at 1.42 s. The speed at that moment vanishes.
(c) The object fall to the ground at 2.83 s with speed -13.89 m/s.

10. A cylinder of radius $r = 4$ cm and height $h = 3$ is heated, and so its dimensions change with speed given by $\frac{dr}{dt} = \frac{dh}{dt} = 1$ cm/s. Find the approximate change in the volume of the cylinder at 5 and 10 seconds after the heating process starts.

Solution

$dV = 2\pi rhdt + \pi r^2 dt$ and at the initial moment $dV = 40\pi dt$. 5 seconds after the approximate rate of change is $dV(5) = 40\pi 5 = 200\pi$ cm^3/s, and 10 seconds after $dV(10) = 40\pi 10 = 400\pi$ cm^3/s.

11. The radius of a sphericall cell is equal to 5 μm, with a possible error of 0.2 μm; compute the error in the measurement of the area of the cell. More generally, if the error in the measurment of the radius is 2%, what is the error in the value of the surface of the cell?
Remark: The surface of a sphere of radius r is given by $S = 4\pi r^2$. Solve the problem by means of the linear approximation (tangent line) of a function.

Solution

For an radius error of 0.2 μm the approximate error in the area is 8π μm^2, and for a relative error of 2% the approximate relative error in the area is 4%.

★ 12. In certain chemical process, the concentration of certain substance c depends on the concentration of two other substances a and b, by the following equation $c = \sqrt[3]{ab^2}$. Suppose that at certain moment, when $a = b = 2$ mg/mm^3, the concentrations of a and b increase at rates of 0.2 mg$\cdot$ mm^{-3}/s, and 0.4 mg$\cdot$ mm^{-3}/s, respectively. Approximate the concentration of c after 2 seconds.

Solution

$c'(t_0) = 1/3$ mg$\cdot$mm^{-3}/s.
$c(t_0 + 2) \approx 8/3$ mg$\cdot$mm^{-3}.

13. Blood flows through an artery at a speed v, which is related to the radius r of the artery by the following expression, known as Poiseuille's law,

$$v(r) = cr^2.$$

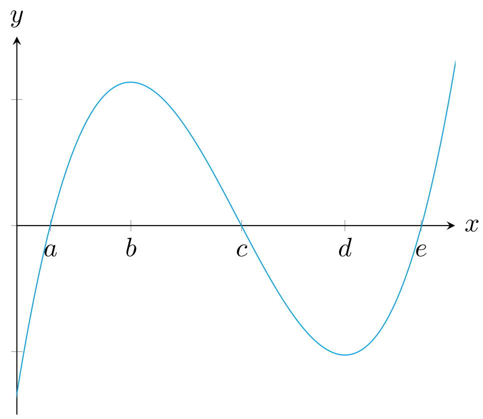
As mentioned above, v is the speed of the flow, r the radius of the artery, which we will assume to be cylindrical, and c is a constant. Assume the radius can be measured with a precision of 5%; calculate the precision in the in the computation of the speed.

Solution

10%.

★ 14. Below you can find the graph of the derivative $f'(x)$ of a function $f(x)$. Determine the behaviour of

f (increasing, decreasing, convex, concave, extrema) from that graph.



Solution

Growth: Decreasing at $(-\infty, a)$ and (c, e), and increasing at (a, c) and (e, ∞).
Extrema: Minimum at $x = a$ and $x = e$, and maximum at $x = c$.
Concavity: Concave up at $(-\infty, b)$ and (d, ∞), and concave down at (b, d).

15. Find the values of a, b and c so that the function $f(x) = x^3 + bx^2 + cx + d$ has an inflection point at $x = 3$, its graph goes through the point $(1, 0)$, and it has a maximum at $x = 1$.

Solution

$b = -9$, $c = 15$ y $d = -7$.

16. The response S of an organism to a drug depends on the dose x by the relation

$$S(x) = x(C - x),$$

where C is the maximum amount of the drug that can be given to a person. Find the dose x for which the response is maximum.

Solution

$x = C/2$.

17. The speed v at which certain chemical reaction $A + B \to AB$ takes place is a function of the concentraion x of the substance AB. This speed it is given by the following equation:

$$v(x) = 4(3 - x)(5 - x).$$

Determine the value of x that maximizes the speed of the process.

Solution

None.

18. The wheat yield C of a field depends on the level of nitrogen on the ground n, and it is given by the following relation:

$$C(n) = \frac{n}{1+n^2}, \quad n \geq 0.$$

Find the level of nitrogen that will produce the biggest yield.

Solution

$n = 1$.

19. A drug has to be given to patients in cylindrical pills. The content of the drug in each pill is 0.15 ml; determine the dimensions of the cylinder so that the amount of material used to make it (the pill) is minimal.

Solution

Radius 0.2879 cm and height 0.5760 cm.

★ 20. A mathematical model for the amount of water in certain lake, $m(t)$, in millions of cubic meters, is given as a function of the time t, measured in years lapsed since the study took place. The formula is the following:

$$m(t) = 10 + \frac{\sqrt{t}}{e^t}$$

This formula makes sense only for positive values of the variable t.

(a) How much water will there be in the lake when t goes to infinity?

(b) Use derivatives to find the time at which the amount of water in the lake is maximum, and compute the amount of water at such time.

Solution

(a) $\lim_{t\to\infty} m(t) = 10$.

(b) $\frac{dm}{dt} = e^{-t}(\frac{1}{2}t^{-1/2} - t^{1/2})$. The moment at which the amount of water in the lake will be maximum is $t = 0.5$ years and at this moment there will be 10.429 millions of m^3.

★ 21. Consider a function $f(x)$ with derivative given by

$$f'(x) = \frac{(2-x)e^{-\frac{x^2}{2}+2x-2}}{\sqrt{2\pi}}$$

(a) Determine the regions on which f is increasing, and those where f is decreasing.

(b) Find the extrema points of f.

(c) Determine the points on which f is concave up, and those on which it is concave down.

(d) Find the values of x corresponding to the inflection points of the graph of f.

Solution

(a) Increasing at $x < 2$ and decreasing at $x > 2$.

(b) Relative maximum at $x = 2$.

(c) Concave down at$(-\infty, 1)$ and $(3, \infty)$, and concave down at $(1, 3)$.

(d) Inflection points at $x = 1$ and $x = 3$.

2 Ordinary Differential Equations

22. Solve the following ODE (separation of variables):

    (a) $x\sqrt{1-y^2} + y\sqrt{1-x^2}y' = 0$ with initial condition $y(0) = 1$.

    (b) $(1+e^x)yy' = e^y$ with initial condition $y(0) = 0$.

    (c) $\mathrm{e}^y(1+x^2)y' - 2x(1+\mathrm{e}^y) = 0$.

    (d) $y - xy' = a(1+x^2y')$.

    Solution

    (a) $-\sqrt{1-y^2} = \sqrt{1-x^2} - 1$.

    (b) $e^{-y}(y+1) = \log(1+e^x) - x - \log 2 + 1$.

    (c) $y = \log(C(1+x^2) - 1)$.

    (d) $y = C\frac{x}{ax+1} + a$.

23. Radioactive decay behaves according to the following differential equation:

$$\frac{\partial x}{\partial t} + ax = 0,$$

where x stands for mass, t time and a is a positive constant. The half-life T is the time that takes for the matter to become half of its initial value. Write T as function of a, and compute a for the uranium isotope U^{238}, if it is known that $T = 4.5 \cdot 10^9$ years.

Solution

$T = \frac{\log 2}{a}$ and $a = 1.54 \cdot 10^{-10}$ years^{-1}.

24. The speed at which sugar dissolves into water is proportional to the amount of sugar left without dissolving. Suppose we have 13.6 kg of sugar that we want to mix with water, and after 4 hours there are 4.5 kg without dissolving. How long will it take, from the beginning of the process, for 95% of the sugar to be dissolved?

Solution

$C(t) = 13.6e^{-0.276t}$ and the instant at which 95% of the sugar is dissolved is $t_0 = 10.854$ hours.

★ 25. A chemical process follows the differential equation:

$$y' - 2y = 4,$$

where $y = f(t)$ is the concentration of oxygen at moment t (in seconds). Suppose there is no oxygen at the beginning of the experiment; what will the concentration (mg/lt) be equal to after 3 seconds? At what moment will the concentration be equal to 200 mg/lt?

Solution

$y(t) = 2e^{2t} - 2$. The oxygen concentration after 3 secons is $y(3) = 804$ mg/lt and the moment at which the oxygen concentration is 200 mg/lt is $t_0 = 2.3076$ s.

26. A water tank filled with 500 lts of water contains 5 kgs of salt dissolved into the water. Suppose we start pouring into the tank a solution of water with 0.4 kg of salt per liter, at a rate of 10 lts per minute. We also stir the water tank, to keep a uniform distribution of salt, and, at the same time, we release water (with salt) at the same rate of 10 lts per minute. How much salt will there be in the tank after 5 minutes? And after 1 hour?

    Remark: The variation rate of salt in the tank is equal to the difference between the amount of salt that comes into the tank and the amount of salt that is taken from the tank.

    Solution

    $C(t) = -195e^{-t/50} + 200$. The amount of salt after 5 minutes is $C(5) = 23.557$ kg and after 1 hour $C(60) = 141.267$ kg.

★ 27. Temperature and time, during certain process, are related by the following differential equation:

$$x't^2 - x't + x' - 2xt + x = 0,$$

where x denotes the temperature (in Kelvin degrees) and t the time (in seconds). Suppose the initial temperature is 100 K; compute the general expression of the temperature as a function of time. What will be the temperature after 3 seconds?

Solution

$x(t) = 100(t^2 - t + 1)$ and the system temperature after 3 seconds is $x(3) = 700$ K.

★ 28. A drug, kept in a refrigerator at 2°C, should be administered to a patient when the drug's temperature is equal to 15°C. At 9 o'clock the drug is taken out of the fridge and placed at a room, where the temperature is equal to 22°C. At 10 o'clock the drug's temperature is equal to 10°C. Assume the speed at which the drug's temperature goes up is proportional to the difference between the temperature of the drug and that of the room. At what time will the medicine be ready to be given to the patient?

Solution

At 11.06 hours.

★ 29. The amount of certain chemical compound M (in grams) in a chemical reaction is a function of time (in seconds). The amount of the substance M behaves as per the following differential equation:

$$M' - (a + b)M = 0$$

where a and b are contants. Suppose we start with 20 g of the compund, and after 10 seconds we have 40 g. Compute:

(a) The amount of the compound at any given time t.
(b) The amount of the compound after half a minute.
(c) When will the amount M be equal to 100 g?

Solution

(a) $M(t) = 20\, e^{\frac{\ln 2}{10} t}$.
(b) $M(30) = 160$ gr.
(c) $t_0 = 23.22$ s.

★ 30. During certain chemical reaction, a compound gets changed into another substance at a rate proportional to the square of the amount (of the original compund) that has not changed. We start with 20 g of the original substance, and after 1 hour wew observed that only half of it is left. At what moment in time will 75% of the substance have converted into the new compund?

Solution

$C(t) = \frac{20}{t+1}$ and the moment at which 75% of the amount of the substante has been converted is $t_0 = 3$ hours.

★ 31. The amount of polluting matter M (given in kg) in a wastewater tank follows this differential equation:

$$\frac{dM}{dt} = -0.5M + 1000,$$

where k is a contant, and t is the time (given in days). (The factor $-0.5M$ can be explained by the fact that the tank is cleaned continuously, at a rate proportional to the amount of polluting substances left. On the other hand, the $+1000$ term accounts for new polluting substances entering the tank at a rate of 1000 kg per day.) Suppose the initial amount of polluting substances is equal to $10,000$ kg:

(a) Find and expression for the amount of polluting matter at any given time t.

(b) How much polluting substance will there be in the tank after one week?

Solution

(a) $M(t) = 8000e^{-0.5t} + 2000$.

(b) $M(7) = 2241.579$ kg.

32. Human plasma is kept at a temperature of 4°C; however, in order to use it on people it should be heated to the average human body temperature 37°C. It takes 1 hour for the plasma to reach the ideal temperature, when heated in a medical heater at 50°C. How long will it take to reach the ideal temperature if the medical heater is at 60°?

Solution

For a heater temperature of 50°C $T(t) = -46e^{-0.02808t} + 50$, and for a heater temperature of 60°C $T(t) = -56e^{-0.02808t} + 60$, so it takes 31.69 min to reach the ideal temperature for the plasma.

33. Find the equation of all the functions such that, at each point (x, y), the slope of the tangent line to the graph of the function is equal to the third power of the x-component. Which one of these functions goes through the origin?

Solution

$y = x^4/4$.

★ 34. Find the equation of the function that goes through the point $P = (1, 1)$, and such that the slope of the tangent line to the graph of the function at every point of the graph is equal to the square of the y-coordinate at the point.

Solution

$y = \frac{-1}{x-2}$.

35. If a person receives glucose by an intravenous drip, the concentration of glucose $c(t)$ with respect to time follows this differential equation:

$$\frac{dc}{dt} = \frac{G}{100V} - kc.$$

Here G is the (constant) speed at which glucose is given to the patient, V is the total volume of blood in the body, and k is a positive constant that varies with each patient. Compute $c(t)$.

Solution

$c(t) = De^{kt} + \frac{G}{100Vk}$

36. The room temperature T on a winter day changes with time according to the following conditions:

$$\frac{dT}{dt} = \begin{cases} 40 - T, & \text{if the building heating is on;} \\ -T, & \text{if the building heating is off.} \end{cases}$$

The temperature in a classroom at 9 am is 5°C, so the keeper turns on the heating. Due to some unexpected malfunction, the heating does not work from 11am to noon. What will the temperature of the room be at 1pm?

Solution

From 9 to 11 the temperature function is $T(t) = -35e^{-t} + 40$ and the temperature at 11 is 35.263°C. From 11 to 12 the temperature function is $T(t) = 35.263e^{-t}$ and the temperature at 12 is 12.973°C. From 12 to 13 the temperature function is $T(t) = -27.027e^{-t} + 40$ and the temperature at 13 is 30.057°C.

★ 37. Two items made of the same ceramic material are heated in an oven at 1000°C. The first item is at 40°C, when was put in the oven, while the second was at 5°C. After one minute, the temperature of the first item has gone up to 200°C. Compute the temperature of both items five minutes after they were put into the oven.

Solution

The tempearature of the first item is $T(t) = 1000 - 960e^{-0.1823t}$ and after 5 min is 614.1559°C.
The tempearature of the second item is $T(t) = 1000 - 995e^{-0.1823t}$ and after 5 min is 600.0887°C.

★ 38. Carbon present in living organism contains an extremely small portion of the radioactive isotope C^{14}, which come from the cosmic rays present on the upper most part of the atmosphere. While the organism is alive, the proportion of the carbon C^{14} within the total amount of carbon in the body is kept constant by means of complex, natural processes. After death, these processes stop, and the radioactive carbon loses 1/8000 of its mass per year. Using this fact one can compute the age at which an organism died.

(a) Suppose that an analysis of the bones of a Neanderthal man shows that the proportion of C^{14} was 6.24% of what it would have been if he were alive; find how long ago this person died.

(b) Find the half-life of C^{14}.

Solution

(a) 22193.52 years.

(b) 5545.17 years.

39. A school of 1000 salmons has a peaceful life near the cost. The birth rate is 2% per day, while the mortality rate is 1%. Suddenly ony day a shark makes its appearance among the fish, and start eating them at a rate of 15 salmons per day. How long will it take for the shark to finish the school of salmos?

Solution

Approximately 110 days.

40. The disecting roof of a forensic is kept at a constant temperature of 5°C. While he was performing the autopsy of a murder victim, the forensic is killed and the body of the victim stolen. At 10 o'clock in the morning the forensic assistant discovered his body at a temperature of 23°C and called the police. At noon the police arrived and found the forensic body at a temperature of 18.5°C. Assuming that the forensic had a normal temperature of 37°C when he was alive, what time was he killed?

Solution

The forensic was killed at 6 o'clock in the morning approximately.

3 Several variables Differentiable Calculus

41. Compute the following partial derivatives:

(a) $\frac{\partial}{\partial x} \ln \frac{x}{y}$.

(b) $\frac{\partial}{\partial v} \frac{nRT}{v}$.

Solution

(a) $\frac{\partial}{\partial x} \log\left(\frac{x}{y}\right) = \frac{1}{x}$.

(b) $\frac{\partial}{\partial v} \left(\frac{nRT}{v}\right) = -\frac{nRT}{v^2}$.

42. The amount of CO_2 absorbed by a plant depends on the ambient temperature (t) and the intensity of light (l), according to the following function, where c is a constant:

$$f(t, l) = ctl^2,$$

Study the change on the absortion of CO_2 for different values of the intensity of light, assuming the temperature is constant. Do the reverse study; that is, keeing light constant, study the absortion depending on different values of the temperature.

Solution

$\frac{\partial f}{\partial l}(t, l) = 2ctl$ and $\frac{\partial f}{\partial t}(t, l) = cl^2$.

43. The number of plants of certain species on a field depends on the level of nitrogen on the ground, and the level of movement on the field. An increment on nitrogen, or on movement, results on a negative effect for the plant. Suppose that at certain point on time the level of nitrogen increases, and there is an increase on the amount of movement due to the presence of cattle; how do these two factors affect the change in the number of plants in the field?

Solution

The number of plants will decrease.

44. The speed at which certain organism grows is a function of the amount of available food and the number of other organisms fighting for food. How will this speed change when the food available increases in quantity, and competitors decrease in number?

Solution

The growth speed will increase.

★ 45. Compute the gradient of the following function

$$f(x,y,z) = \log\frac{\sqrt{x}}{yz} + \arcsin(xz).$$

Solution

$\nabla f(x,y,z) = \left(\frac{z}{\sqrt{1-x^2z^2}} + \frac{1}{2x}, \frac{-1}{y}, \frac{x}{\sqrt{1-x^2\,z^2}} - \frac{1}{z}\right).$

★ 46. Consider the following function:

$$f(x,y,z) = \log\sqrt{xy - \frac{z^2}{xy}}$$

(a) Compute its gradient.

(b) Find a point on which the gradient of $f(x,y,z)$ is parallel to the bisectriz of the plane XY; compute the gradient at that point.

Solution

(a) $\nabla f(x,y,z) = \left(-\frac{z^2+x^2y^2}{2xz^2-2x^3y^2}, -\frac{z^2+x^2y^2}{2yz^2-2x^2y^3}, \frac{z}{z^2-x^2y^2}\right).$

(b) The gradient is parallel to the bisectriz of the plane XY at any point $(a,a,0)$, $a \in \mathbb{R}$. $\nabla f(1,1,0) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

47. A spaceship, traveling near the sun, is in trouble. The temperature at position (x,y,z) is given by $T(x,y,z) = \mathrm{e}^{-x^2-2y^2-3z^2}$, where the variables are measured in thousands of kilometers, and we assume the sun is at position $(0,0,0)$. If the ship is at position $(1,1,1)$, find the direction in which it should move so that the temperature will decrease as fast as possible.

Solution

It should move in direction $-\nabla f(1,1,1) = e^{-6}(2,4,6)$.

48. A bug moving on a surface follows always the direction of steepest descent. If the equation of the surface is given by

$$f(x, y) = x^2 - y^2,$$

find the direction the bug will follow from the point $(2, 3)$.

Solution

It will follow the direcction $-\nabla f(2, 3) = (-4, 6)$.

49. The surface of a mountain peak is given by the equation displayed below, where a, b and c are constants; and x and y are the East-West and North-South coordinates, respectively.

$$S : z = a - bx^2 - cy^2,$$

Find the direction of steepest increase of the height of the mountain if we are located at the point $P = (1, 1)$.

Solution

$(-2b, -2c)$.

50. Find the directions of maximum increase and decrease of the following functions, at the given point P:

(a) $f(x, y) = x^2 + xy + y^2$, $P = (-1, 1)$.
(b) $f(x, y) = x^2y + e^{xy} \sin y$, $P = (1, 0)$.
(c) $f(x, y, z) = \log(xy) + \log(yz) + \log(xz)$, $P = (1, 1, 1)$.
(d) $f(x, y, z) = \log(x^2 + y^2 - 1) + y + 6z$, $P = (1, 1, 0)$.

Solution

(a) Maximun increase in direction $(-1, 1)$ and maximum decrease in direction $(1, -1)$.
(b) Maximun increase in direction $(0, 2)$ and maximum decrease in direction $(0, -2)$.
(c) Maximun increase in direction$(2, 2, 2)$ and maximum decrease in direction $(-2, -2, -2)$.
(d) Maximun increase in direction $(2, 3, 6)$ and maximum decrease in direction $(-2, -3, -6)$.

51. Compute $(f \circ g)'(t)$, assuming $f(x, y, z) = x^3y^2z$ and $g(t) = (e^t, \cos t, \sin t)$.

Solution

$(f \circ g)'(t) = e^{3t}(3 \sin t \cos^2 t - 2 \sin^2 t \cos t + \cos^3 t)$.

★ 52. Chemiotaxis is the movement of an organism in response to a chemical stimulus. Usually this movement takes place in the direction in which the concentration of the checmical increases the fastest. The Dictyoselium discoideum mold shows this type of behaviour. The single-celled amoeba of this mold moves following the concentration of a chemical substance denoted by AMP. Suppose the concentration of AMP at the point of coordinates (x, y, z) is given by:

$$C(x, y, z) = \frac{4}{\sqrt{x^2 + y^2 + z^4 + 1}}$$

An amoeba is placed at the point $(-1, 0, 1)$, in which direction will it move?

Solution

$(4/\sqrt{27}, 0, -8/\sqrt{27})$.

★ 53. The function below measures the air pressure at position (x, y, z).

$$f(x, y, z) = x^2 + y^2 - z^3$$

Consider an object A moving along the following trajectory:

$$\begin{cases} x = t \\ y = 1 \\ z = 1/t \end{cases} \quad t > 0.$$

(a) Give an equation of the tangent line to the trajectory of A at the point $(1, 1, 1)$.

(b) Is the trajectory of A at the point $(1, 1, 1)$ moving in the direction of maximum increase of the function f?

Solution

(a) $(1 + t, 1, 1 - t)$.

(b) No, as the direction of maximum increase of f is $\nabla f(1, 1, 1) = (2, 2, -3)$ and the direccion of the trajectory is $(1, 0, -1)$.

54. Find for which directions the directional derivative of the function f below at the point $P = (1, 1)$ vanishes?

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

Solution

Direction $(1/\sqrt{2}, 1/\sqrt{2})$.

55. Does there exist a direction on which the directional derivative of the function f below takes the value 14 at the point $P = (1, 2)$?

$$f(x, y) = x^2 - 3xy + 4y^2$$

Solution

No.

56. The maximum value of the directional derivative of a function f at the point P is equal to $2\sqrt{3}$, and it takes place in the direction of the vector $(1, 1, -1)$. Compute the value of the directional derivative of f at P in the direction $(1, 1, 0)$?

Solution

$2\sqrt{2}$.

57. Given the scalar field
$$f(x, y, z) = x^2 - y^2 + xyz^3 - zx$$
and the point $P = (1, 2, 3)$, compute the following:

(a) The directional derivative of f at P in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}(1, -1, 0)$.

(b) The direction for which the directional derivative of f at P takes its maximum value. Find such value.

Solution

(a) $15\sqrt{2}$.

(b) The directional derivative takes its maximum value in direction $(53, 23, 53)$ and its value is $\sqrt{6147}$.

★ 58. Compute the equation of the tangent plane and the normal line to the surface
$$S : xyz = 8$$
at point $P = (4, -2, -1)$.

Solution

Normal line $l : (4 + 2t, -2 - 4t, -1 - 8t)$. Tangent plane $\pi : 2x - 4y - 8z + 24 = 0$.

59. In the following expressions, compute the derivative of y as a function of x using implicit differentiation techniques.

(a) $x^3 - 3xy^2 + y^3 = 1$.

(b) $y = \dfrac{\sin(x + y)}{x^2 + y^2}$.

Solution

(a) $\frac{dy}{dx} = \frac{-x^2+y^2}{-2xy+y^2}$.

(b) $\frac{dy}{dx} = \frac{-2xy+\cos(x+y)}{(x^2+y^2)+2y^2-\cos(x+y)}$.

★ 60. Compute the equations of the tangent and normal lines, at the point $x = 0$, to the graph of the function y, given by $xy + e^x - \log y = 0$.

Solution

Tangent: $y = (e^2 + e)x + e$. Normal: $y = \frac{-x}{e^2+e} + e$.

★ 61. The temperature T and the volume V of a gas kept in a closed container of variable volume are related by the following formula, where T is given in Celsius degrees and V in cubic meters:
$$T^2(V^2 - \pi^2) - V\cos(TV) = 0$$

(a) Compute the derivative of the volume with respect to the temperature when the volume is equal to π m^3 and the temperature is equal to half Celsius degree.

(b) Compute an equation of the line tangent to the graph of the volume, as a function of the temperature, in the point of part (a).

(c) If we assume that temperature and volume are both functions of the pressure, find the equation that would relate the derivative of the temperature (with respect to pressure) with the derivative of the volume (with respect to pressure).

Solution

(a) $\frac{dV}{dT} = \frac{-2T(V^2-\pi^2)-V^2\sin(TV)}{2T^2V-\cos(TV)+TV\sin(TV)}$ and $\frac{dV}{dT}(V=\pi, T=0.5) = -\pi$ m^3/^{o}C.

(b) Tangent: $V = \pi(-T+1.5)$.

(c) $2T\frac{dT}{dP}(V^2-\pi^2)+T^2(2V\frac{dV}{dT}) - \frac{dV}{dT}\cos(VT) - V(-\sin(TV)(\frac{dT}{dP}V + T\frac{dV}{dP})) = 0$.

62. Compute the equations of the tangent and normal lines to the curve C at the point P given below:

(a) $C : \dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$, $P = (-3, 0)$.

(b) $C : x^3 - y^5 + xy^2 = 8$, $P = (2, 0)$.

(c) $C : x = y^2$, $P = (0, 0)$.

(d) $C : x^{2/3} + y^{2/3} = 1$, $P = (\sqrt{2}/4, \sqrt{2}/4)$.

Solution

(a) Tangent $x = -3$ and normal $y = 0$.

(b) Tangent $x = 2$ and normal $y = 0$.

(c) Tangent $x = 0$ and normal $y = 0$.

(d) Tangent $y = -x + \sqrt{2}/2$ and normal $y = x$.

63. On each of the following cases below, find the equation of the tangent plane and the normal line to the surface S at the given point P:

(a) $S : x - y + z = 1$, $P = (0, 0, 1)$.

(b) $S : x^2 + y^2 + z^2 = 1$, $P = (0, 1, 0)$.

(c) $S : z = \log(x^2 + y^2)$, $P = (1, 0, 0)$.

(d) $S : z = e^{-(x^2+y^2)}$, $P = (0, 0, 1)$.

(e) $S : z = e^{x+y}\sin x$, $P = (\pi, 0, 0)$.

Solution

(a) Tangent plane $x - y + z - 1 = 0$ and normal line $\frac{x}{-1} = \frac{y}{1} = \frac{z-1}{-1}$.

(b) Tangent plane $y = 1$ and normal line $(x = 0, y = 1 + 2t, z = 0)$.

(c) Tangent plane $2x - z - 2 = 0$ and normal line $(x = 1 + 2t, y = 0, z = -t)$.

(d) Tangent plane $z = 1$ and normal line $(x = 0, y = 0, z = 1 + t)$.

(e) Tangent plane $z = -e^{\pi}(x - \pi)$ and normal line $(x = \pi - e^{\pi}t, y = 0, z = -t)$.

64. Suppose the equation $F(x,y,z) = 0$ defines the variable z as a function of the variables x and y ($z = f(x,y)$). Show that
$$\frac{\partial z}{\partial x} = \frac{-\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}.$$
Use these identities to compute $\dfrac{\partial f}{\partial x}(2,1)$, when we have $x^2yz = 4$ and $f(2,1) = 1$.

Solution

$\frac{\partial f}{\partial x}(2,1) = -1$.

★ 65. The equation
$$x \log y + \frac{2e^{y^2+z}}{x} - \frac{x}{z^2} = -1$$
defines z as a function of x and y in a neighborhoud of the point $(2,1,-1)$. Compute the gradient of z at that point, and explain its meaning.

Solution

$\nabla z(2,1,-1) = (-1/2, 4/3)$.

★ 66. Consider two surfaces, S and M, where S is given by the equation $xy + 8z = 0$; and M is the ellipsoid with equation $x^2 + 2y^2 + 4z^2 = 7$. Find the equation of the plane tangent to S, and parallel to the tangent plane to M at the point $P = (1,1,1)$.

67. Compute the gradient vector and Hessian matrix of the following two functions:

(a) $e^{x^2+y^2+z^2}$

(b) $\sin((x^2-y^2)z)$

Solution

(a) $\nabla e^{x^2+y^2+z^2} = \left(2\,x\,e^{z^2+y^2+x^2}, 2\,y\,e^{z^2+y^2+x^2}, 2\,z\,e^{z^2+y^2+x^2}\right)$,
$$He^{x^2+y^2+z^2} = \begin{pmatrix} (4x^2+2)e^{x^2+y^2+z^2} & 4xye^{x^2+y^2+z^2} & 4xze^{x^2+y^2+z^2} \\ 4xye^{x^2+y^2+z^2} & (4y^2+2)e^{x^2+y^2+z^2} & 4yze^{x^2+y^2+z^2} \\ 4xze^{x^2+y^2+z^2} & 4yze^{x^2+y^2+z^2} & (4z^2+2)e^{x^2+y^2+z^2} \end{pmatrix}.$$

(b) $\nabla \sin((x^2-y^2)z) = \left(2\,x\,z\,\cos\left(\left(x^2-y^2\right)\,z\right), -2\,y\,z\,\cos\left(\left(x^2-y^2\right)\,z\right), \left(x^2-y^2\right)\,\cos\left(\left(x^2-y^2\right)\,z\right)\right)$
$H\sin((x^2-y^2)z) =$
$$\begin{pmatrix} 4x^2\sin((x^2-y^2)z) + 2\cos((x^2-y^2)z) & 4xy\sin((x^2-y^2)z) & -2x(x^2-y^2)\sin((x^2-y^2)z) \\ 4xy\sin((x^2-y^2)z) & -4y^2\sin((x^2-y^2)z) - 2\cos((x^2-y^2)z) & 2y(x^2-y^2)\sin((x^2-y^2)z) \\ -2x(x^2-y^2)\sin((x^2-y^2)z) & 2y(x^2-y^2)\sin((x^2-y^2)z) & -(x^2-y^2)^2\sin((x^2-y^2)z) \end{pmatrix}.$$

★ 68. The following functions indicates the temperature on a plane:
$$f(x,y) = e^{x+2y}\cos(x^2+y^2).$$

(a) Compute the gradient of f.

(b) Suppose we are at the origin point, in which direction will the temperature increase the maximum? What if we were at the point $(0,1)$?

(c) Compute the Hessian matrix of f, and its determinant, at the origin.

Solution

(a) $\nabla f(x,y) = e^{x+2y}\left(\cos(x^2+y^2) - 2x\sin(x^2+y^2), 2\cos(x^2+y^2) - 2y\sin(x^2+y^2)\right)$.

(b) $\nabla f(0,0) = (1,2)$ y $\nabla f(0,1) = (3.99\,, -4.45)$.

(c) $Hf(0,0) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad |Hf(0,0)| = 0$.

★ 69. The electricl potential, as a function of the distance, is given by $V = \log D$, where $D = \sqrt{x^2+y^2}$.

(a) Compute the gradient of V.

(b) Find the direction of biggest change of the electric potential at the point $(x,y) = (\sqrt{3}, \sqrt{3})$.

(c) Compute the Hessian of V and its determinant at the point of part (b).

(d) In which point of the curve $y = x+1$ will the potential be minimal?

Solution

(a) $\nabla V(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$.

(b) $\nabla V(\sqrt{3}, \sqrt{3}) = \sqrt{3}/6(1,1)$.

(c) $HV(x,y) = \begin{pmatrix} \frac{y^2-x^2}{y^4+2x^2y^2+x^4} & \frac{-2xy}{y^4+2x^2y^2+x^4} \\ \frac{-2xy}{y^4+2x^2y^2+x^4} & \frac{x^2-y^2}{y^4+2x^2y^2+x^4} \end{pmatrix}$, $\begin{pmatrix} 0 & -1/6 \\ -1/6 & 0 \end{pmatrix}$, y $|H(\sqrt{3}, \sqrt{3})| = -1/36$.

(d) The potential is minimum at $(x = -1/2, y = 1/2)$ and its value is $V(-1/2, 1/2) = -\dfrac{\log 2}{2}$.

★ 70. A long piece of metal is heated in such way that, at time t minutes, and at x meters from its left endpoint, the temperature is given by $H(x,t) = 100e^{-0.1t}\sin(\pi xt)$ with $0 \le x \le 1$.

(a) Compute $\dfrac{\partial H}{\partial x}(0.2, 1)$ and $\dfrac{\partial H}{\partial x}(0.8, 1)$. What is the meaning (in terms of the temperature) of these two partial derivatives? Explain the sign of each partial derivative.

(b) Compute the Hessian of H.

Solution

(a) $\frac{\partial H}{\partial x}(0.2,\, 1) = 100e^{-0.1}\cos(0.2\pi)\pi = 229.9736$
$\frac{\partial H}{\partial x}(0.8,\, 1) = 100e^{-0.1}\cos(0.8\pi)\pi = -229.9736$.

(b) $\begin{pmatrix} -100e^{-0.1t}\pi^2t^2\sin(\pi xt) & 100e^{-0.1t}\left((-0.1\pi t + \pi)\cos(\pi xt) - \pi^2 xt\sin(\pi xt)\right) \\ 100e^{-0.1t}\left((-0.1\pi t + \pi)\cos(\pi xt) - \pi^2 xt\sin(\pi xt)\right) & 100e^{-0.1t}\left(0.01\sin(\pi xt) - (0.2 + \pi^2x^2)\cos(\pi xt)\right) \end{pmatrix}$

71. Compute the second degree Taylor polynomial of f at point P in each of the following cases:

(a) $f(x,y) = \sin(x+2y)$, $P = (0,0)$.

(b) $f(x,y) = e^x\cos y$, $P(0,0)$.

(c) $f(x,y) = \sin(xy)$, $P(1, \pi/2)$.

Solution

(a) $P^2_{f,P}(x,y) = x + 2y$.

(b) $P^2_{f,P}(x,y) = 1 + x + \frac{x^2}{2} - \frac{y^2}{2}$.

(c) $P^2_{f,P}(x,y) = 1 + \frac{1}{2}\left(-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right)$.

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72. Given the function
$$f(x,y,z) = e^x\sqrt{yz},$$
approximate the value of $f(0.01, 24.8, 1.02)$ using the linear Taylor series at point $P = (0, 25, 1)$.

Solution

5.08.

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73. Compute the linear and quadratic approximations of
$$\frac{(3.98-1)^2}{(5.97-3)^2}$$
using Taylor series. Compare the approximation with the actual value.

Solution

The linear approximation is 1.00666666 and the quadratic is 1.006744438.

★ 74. Given the function $f(x,y) = \sqrt{xy}$:

(a) Compute the first degree Taylor polynomial centered at point $(4, 9)$.

(b) Compute the approximate value of $f(4.01,\ 8.99)$ using the previous polynomial.

Solution

(a) $P(x,y) = \frac{3}{4}x + \frac{1}{3}y$.

(b) $f(4.01,\ 8.99) \approx 6.00416667$.

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75. The yield of a field R depends on the concentration of nitrogen n and phosphorus p on the ground, by the following expression:
$$R(n,p) = n \cdot p \cdot e^{-(n+p)}$$

(a) Find the first and second order partial derivative of the function $R(n,p)$.

(b) A necessary condition for a function to have a maximum at a point is that all its partial derivatives of first order vanish at the point. Compute the values of the concentrations n and p that maximize the yield of the field.

Solution

The yield of the field is maximum for $n = p = 1$.

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76. Determine the relative extrema and saddle points of f in the following cases:

(a) $f(x,y) = x^2 + y^2$.

(b) $f(x,y) = x^2 - y^2$.

(c) $f(x,y) = x^2 - 2xy + y^2$.
(d) $f(x,y) = \log(x^2 + y^2 + 1)$.

Solution

(a) Relative minimum at $(0,0)$.
(b) Saddle point at $(0,0)$.
(c) Impossible to say anything with the Hessian.
(d) Relative minimum at $(0,0)$.

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77. The function

$$f(x,y) = \frac{x^3}{3} - x - \left(\frac{y^3}{3} - y\right)$$

has a relative maximum, a relative minimum and two saddle points. Find them.

Solution

Relative maximum at $(-1,1)$, relative minimum at $(1,-1)$ and saddle points at $(1,1)$ y $(-1,-1)$.

★ 78. Find the relative extrema and the saddle points of the function

$$f(x,y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2),$$

where we assume that $a \neq 0$.

Solution

Relative minima at $(-a,0)$ and $(a,0)$. Saddle point at $(0,0)$.

★ 79. Find the relative extrema and the saddle points of the scalar field h given by

$$h(x,y) = xy + \frac{xy^2}{2} - 2x^2.$$

Solution

Relative maximum at $(-1/8,-1)$. Saddle points at $(0,0)$ and $(0,-2)$. $(0,0)$.

★ 80. Consider the function $f(x,y) = \frac{ax^3}{3} + \frac{by^3}{3} - 4ax - 4by$, where a and b are two positive parameters. Find the extrema and saddle points of f.

Solution

Local maximum at $(-2,-2)$, local minimum at $(2,2)$ and saddle points at $(-2,2)$ and $(2,-2)$.

Remark: The exercises with a (★) are exam exercises of previous years.