

CALCULUS PROBLEMS

Subject: Mathematics

Course: 1st

Degree: Pharmacy

Year: 2016-2017

Authors: Pablo Ares Gastesi (pablo.aresgastesi@ceu.es)
Eduardo López Ramírez (elopez@ceu.es)
Anselmo Romero Limón (arlimon@ceu.es)
Alfredo Sánchez Alberca (asalber@ceu.es)



CEU

*Universidad
San Pablo*

Contents

1	One-variable Differentiable Calculus	2
----------	---	----------

1 One-variable Differentiable Calculus

1. Compute the derivative function of $f(x) = x^3 - 2x^2 + 1$ at the points $x = -1$, $x = 0$ and $x = 1$. Explain your result. Find an equation of the tangent line to the graph of f at each of the three given points.

SOLUTION

$$f'(-1) = 7, f'(0) = 0 \text{ y } f'(1) = -1.$$

Tangent line at $x = -1$: $y = -2 + 7(x + 1)$.

Tangent line at $x = 0$: $y = 1$.

Tangent line at $x = 1$: $y = -(x - 1)$.

2. The pH measures the concentration of hydrogen ions H^+ in an aqueous solution. It is defined by

$$\text{pH} = -\log(H^+).$$

Compute the derivative of the pH as a function of the concentration of H^+ . Study the growth of the pH function.

SOLUTION

The pH increases as the concentration of hydrogen ions H^+ increase.

3. The speed $v(n)$ at which a plant grows depends on the amount of nitrogen available n by the following relation:

$$v(n) = \frac{an}{k + n}, \quad n \geq 0,$$

where a and k are positive constants. Study the growth of this function, and explain your results.

SOLUTION

The speed increases as n increases but each time with less force, so that for $n \rightarrow \infty$ the speed becomes stable.

4. Find an equation of the tangent and normal lines to the curves given below at the given point x_0 .

(a) $y = x^{\sin x}, \quad x_0 = \pi/2.$

(b) $y = \log \sqrt{\frac{1+x}{1-x}}, \quad x_0 = 0.$

SOLUTION

(a) Tangent: $y - \frac{\pi}{2} = x - \frac{\pi}{2}$. Normal: $y - \frac{\pi}{2} = -x + \frac{\pi}{2}$.

(b) Tangent: $y = x$. Normal: $y = -x$.

5. Air is being pumped into a spherical balloon of radius 10cm so that the radius increases at a rate of 2 cm/s. How fast will the volume of the balloon increase?

Remark: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

SOLUTION

$$800\pi \text{ cm}^3/\text{s}.$$

6. A liquid solution is kept in a cylindrical pipette of radius 5 cm. Suppose the liquid is taken out of the pipette at a rate of 0.5 ml per second; compute the rate of change of the level of liquid in the pipette.

SOLUTION

−6.37 mm/s.

7. Radioactive decay is given by the following function:

$$m(t) = m_0 e^{-kt},$$

where $m(t)$ denotes the amount of matter at time t , m_0 is the initial amount of radioactive matter, and k is a constant called the *decay constant*. The variable t represents time. Compute the speed of decay at any given time t .

Recall that the *half life* of a radioactive material is the time it takes for a quantity to reduce to half its initial value. Suppose for certain radioactive material we have $k = 0.002$, compute the half life of the material.

SOLUTION

Speed of decay: $-km_0 e^{-kt}$.

Half life: 346.57 years.

8. A car is moving on a straight line direction, with position given by the following function:

$$e(t) = 4t^3 - 2t + 1.$$

Find the speed and acceleration of the car.

Remark: The acceleration is the variation rate of the instant velocity.

SOLUTION

speed $v(t) = 12t^2 - 2$ and acceleration $a(t) = 24t$.

9. An object is thrown vertically upwards. Assuming there is no air friction, the object will travel a distance given by the following equation:

$$e(t) = v_0 t - \frac{1}{2} g t^2$$

where v_0 is the initial velocity (at which the object is thrown), $g = 9.81 \text{ m/s}^2$ is the gravitational Earth constant, and t is the time lapsed since the object was thrown.

- Compute the speed and acceleration of the object at any time.
- Suppose the initial speed is 50 km/h, how high will the object get? Compute the speed at the moment of maximum height.
- At what time will the object fall to the ground? With what speed?

SOLUTION

- Speed $v(t) = v_0 - gt$ and acceleration $a(t) = -g$.
- Maximum height 9.83 m at 1.42 s. The speed at that moment vanishes.
- The object falls to the ground at 2.83 s with speed -13.89 m/s .

10. A cylinder of radius $r = 4$ cm and height $h = 3$ is heated, and so its dimensions change with speed given by $\frac{dr}{dt} = \frac{dh}{dt} = 1$ cm/s. Find the approximate rate of change in the volume of the cylinder at 5 and 10 seconds after the heating process starts.

SOLUTION

$dV = 2\pi r h dt + \pi r^2 dt$ and at the initial moment $dV = 40\pi dt$. 5 seconds after the approximate rate of change is $dV(5) = 40\pi 5 = 200\pi$ cm³/s, and 10 seconds after $dV(10) = 40\pi 10 = 400\pi$ cm³/s.

11. The radius of a spherical cell is equal to 5μ , with a possible error of 0.2μ ; compute the error in the measurement of the area of the cell. More generally, if the error in the measurement of the radius is 2%, what is the error in the value of the surface of the cell?

Remark: The surface of a sphere of radius r is given by $S = 4\pi r^2$. Solve the problem by means of the linear approximation (tangent line) of a function.

SOLUTION

For an radius error of 0.2μ the approximate error in the area is $8\pi \mu^2$, and for a relative error of 2% the approximate relative error in the area is 4%.

- ★ 12. In certain chemical process, the concentration of certain substance c depends on the concentration of two other substances a and b , by the following equation $c = \sqrt[3]{ab^2}$. Suppose that at certain moment, when $a = b = 2$ mg/mm³, the concentrations of a and b increase at rates of 0.2 mg·mm⁻³/s, and 0.4 mg·mm⁻³/s, respectively. Find the rate of change in the concentration of c after 2 seconds.

SOLUTION

$c'(t_0) = 48$ mg·mm⁻³/s.
 $c(t_0 + 2) \approx 192$ mg·mm⁻³.

13. Blood flows through an artery at a speed v , which is related to the radius r of the artery by the following expression, known as Poiseuille's law,

$$v(r) = cr^2.$$

As mentioned above, v is the speed of the flow, r the radius of the artery, which we will assume to be cylindrical, and c is a constant. Assume the radius can be measured with a precision of 5%; calculate the precision in the computation of the speed.

SOLUTION

10%.

14. Consider the sine function $f(x) = \sin x$.

- Compute the third degree Taylor polynomial centered at the point $x = \pi/6$. Use this polynomial to approximate $\sin(1/2)$.
- Give an approximate value of $\sin(1/2)$ using a fifth degree Taylor polynomial centered at the point $x = 0$.

SOLUTION

$$(a) \quad P_{f,\pi/6}^3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{4}(x - \pi/6)^2 - \frac{\sqrt{3}}{12}(x - \pi/6)^3.$$

$$\sin 1/2 \approx P_{f,\pi/6}^3(1/2) = 0.4794255322.$$

$$(b) \quad P_{f,0}^5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

$$\sin 1/2 \approx P_{f,0}^5(1/2) = 0.4794270833.$$

15. Compute the second degree Taylor polynomial of the function $f(x) = \sqrt[3]{x}$ in a neighborhood of the point $x = 1$.

SOLUTION

$$P_{f,1}^2(x) = 1 + \frac{1}{3}(x - 1) - \frac{2}{18}(x - 1)^2.$$

16. Compute the third degree Maclaurin polynomial for the function $f(x) = \arcsin x$.

SOLUTION

$$P_{f,0}^3(x) = x + \frac{1}{6}x^3.$$

- ★ 17. The function $C(t)$ measures the concentration (in mg/dl) of a drug in the bloodstream as function of time (in hours):

$$C(t) = \frac{1}{1 + e^{-2t}}$$

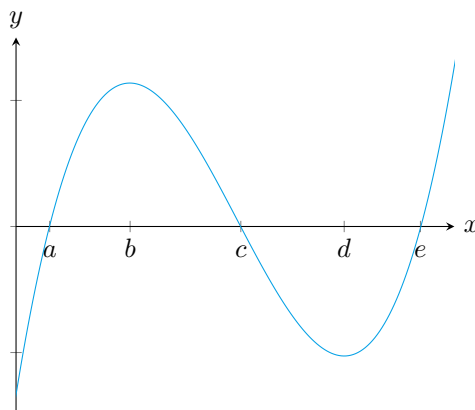
- (a) Compute the third degree Maclaurin polynomial for the function.
 (b) Use the previous polynomial to compute approximately the concentration of drug in the bloodstream after 15 minutes.

SOLUTION

$$(a) \quad P_{C,0}^3(t) = \frac{1}{2} + \frac{1}{2}t + 0\frac{t^2}{2!} - 1\frac{t^3}{3!} = \frac{1}{2} + \frac{1}{2}t - \frac{1}{6}t^3.$$

$$(b) \quad P_{C,0}^3(0.25) = 0.6223958333 \text{ mg/dl.}$$

- ★ 18. Below you can find the graph of the derivative $f'(x)$ of a function $f(x)$. Determine the behaviour of f (increasing, decreasing, convex, concave, extrema) from that graph.



SOLUTION

Growth: Decreasing at $(-\infty, a)$ and (c, e) , and increasing at (a, c) and (e, ∞) .

Extrema: Minimum at $x = a$ and $x = e$, and maximum at $x = c$.

Concavity: Concave down at $(-\infty, b)$ and (d, ∞) , and concave up at (b, d) .

19. Find the values of a , b and c so that the function $f(x) = x^3 + bx^2 + cx + d$ has an inflection point at $x = 3$, its graph goes through the point $(1, 0)$, and it has a maximum at $x = 1$.

SOLUTION

$b = -9$, $c = 15$ y $d = -7$.

20. The response S of an organism to a drug depends on the dose x by the relation

$$S(x) = x(C - x),$$

where C is the maximum amount of the drug that can be given to a person. Find the dose x for which the response is maximum.

SOLUTION

$x = C/2$.

21. The speed v at which certain chemical reaction $A + B \rightarrow AB$ takes place is a function of the concentration x of the substance AB . This speed it is given by the following equation:

$$v(x) = 4(3 - x)(5 - x).$$

Determine the value of x that maximizes the speed of the process.

SOLUTION

None.

22. The wheat yield C of a field depends on the level of nitrogen on the ground n , and it is given by the following relation:

$$C(n) = \frac{n}{1 + n^2}, \quad n \geq 0.$$

Find the level of nitrogen that will produce the biggest yield.

SOLUTION

$n = 1$.

23. A drug has to be given to patients in cylindrical pills. The content of the drug in each pill is 0.15 ml; determine the dimensions of the cylinder so that the amount of material used to make it (the pill) is minimal.

SOLUTION

Radius 0.2879 cm and height 0.5760 cm.

- ★ 24. A mathematical model for the amount of water in certain lake, $m(t)$, in millions of cubic meters, is given as a function of the time t , measured in years lapsed since the study took place. The formula is the following:

$$m(t) = 10 + \frac{\sqrt{t}}{e^t}$$

This formula makes sense only for positive values of the variable t .

- (a) How much water will there be in the lake when t goes to infinity?
- (b) Use derivatives to find the time at which the amount of water in the lake is maximum, and compute the amount of water at such time.

SOLUTION

- (a) $\lim_{t \rightarrow \infty} m(t) = 10$.
 - (b) $\frac{dm}{dt} = e^{-t}(\frac{1}{2}t^{-1/2} - t^{1/2})$. The moment at which the amount of water in the lake will be maximum is $t = 0.5$ years and at this moment there will be 10.429 millions of m^3 .
-

- ★ 25. Consider a function $f(x)$ with derivative given by

$$f'(x) = \frac{(2-x)e^{-\frac{x^2}{2}+2x-2}}{\sqrt{2\pi}}$$

- (a) Determine the regions on which f is increasing, and those where f is decreasing.
- (b) Find the extrema points of f .
- (c) Determine the points on which f is concave up, and those on which it is concave down.
- (d) Find the values of x corresponding to the inflection points of the graph of f .

SOLUTION

- (a) Increasing at $x < 2$ and decreasing at $x > 2$.
 - (b) Relative maximum at $x = 2$.
 - (c) Concave down at $(-\infty, 1)$ and $(3, \infty)$, and concave up at $(1, 3)$.
 - (d) Inflection points at $x = 1$ and $x = 3$.
-

REMARK: The exercises with a (★) are exam exercises of previous years.