Calculus with Geogebra

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September 2018

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Practice 1

Introduction to Geogebra

1 Introduction

In the last decades, the computational power of computers have converted them in powerful tools for disciplines that, as Mathematics, require a large amount of complex computations.

Geogebra[1] is one of the most used programs for doing numerical and symbolic computations. Beyond their capabilities for the numerical, vectorial and matrix calculus, it also makes graphical representations. This allows to solve a lot of problems of Algebra, Analysis, Calculus, Geometry and even Statistics. The advantage of Geogebra versus other software as Mathematica, Mapple or MATLAB, is its simplicity, what makes it suitable for teaching Maths, and that is open source software, so that it can be modified and installed for free.

This software can be downloaded from the web `https://www.geogebra.org`. There is also in this web an on-line version of the program that can be used as a web application without installing it in the computer. This web also contains a lot of tutorials an educational resources available to the users. In fact, any user can register an upload to this site activities developed with Geogebra.

The goal of this practice is to introduce to the student the basic usage of this program for Calculus.

2 Starting the program

As any other Windows applications, to start the program you have to click the Windows start button and then select All the programs > Geogera or simply double click the desktop shortcut icon if there is one.

When the program starts, the initial windows is shown (figure 1.1), allowing the user to choose among different working environments or *Perspectives*.

[1]These practices are based on version 6.0 of Classic Geogebra

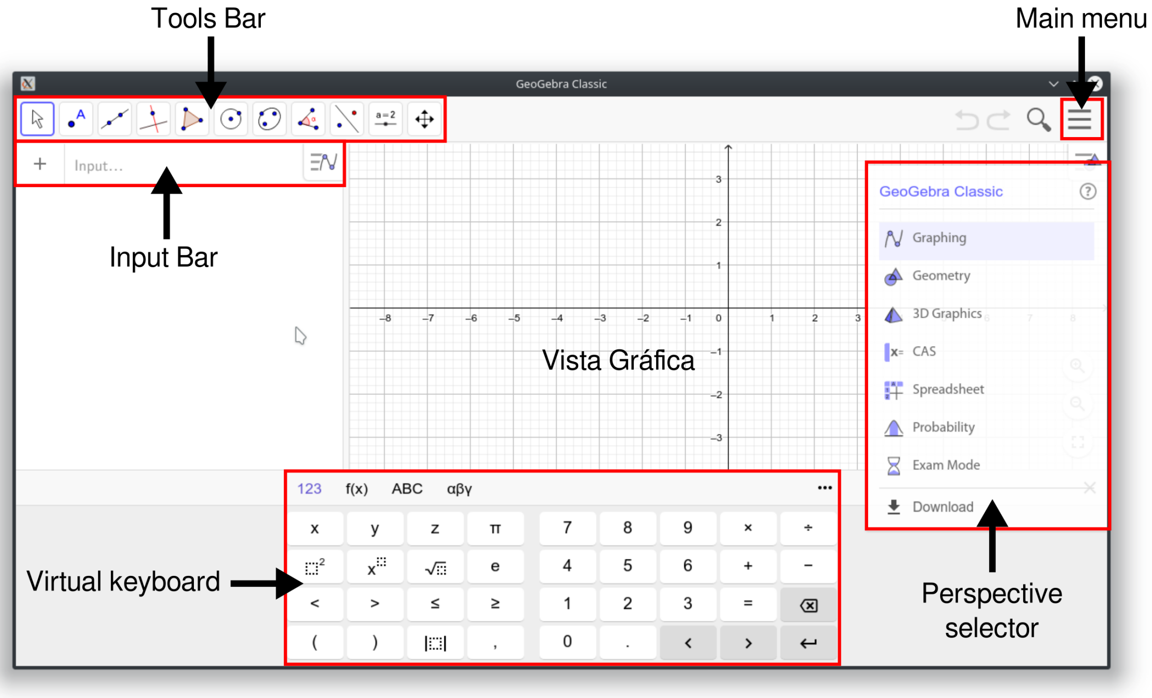

Figure 1.1 – Starting perspective of Geogegra.

3 Views

Geogebra provides several windows that are called *Views* and different working environments called *Perspectives* that combine some views. Both views and perspectives can be activated in the main menu of Geogebra that appears in the top right corner. The most important views that we are going to use during these practices are:

Algebraic View This view allows to make algebraic and geometric constructions. It provides an Input Bar where the user can enter command and algebraic expressions. This is view is active by default when the program starts.

Graphic View This view allows to represent graphically geometric objects in the real plane. Beside the algebraic view, this view is also active by default when the program starts.

3D Graphics View This view allows to represent graphically geometric objects in the real space. This view is not activated by default when the program starts, so that it must be activated by the user when it is required.

CAS View (Computer Algebra System) This view allows to do symbolic calculations. It provides an Input Bar similar to the one of the algebraic view where you can enter commands and mathematical expressions, and evaluate them. This view is not activated by default when the program starts, but *it will be the most used view during these practices*.

4 Expression edition in the CAS View

Before doing any computation with a mathematica expression, you need to know how to enter that expression and learn to manage it.

Entering expressions

Any mathematical expression must be entered in the Input Bar of the CAS View (figure 1.2).

Figure 1.2 – Input Bar.

The Input Bar allows to enter mathematical expressions, commands and text annotations. In the mathematical expressions we can enter numbers, roman letters, greek letters, mathematical operators and any symbols that appears in the virtual keyboard. It also allows to enter LaTeX[2] code to format expressions. For instance, it is possible to write superscripts with the command ^ and subscripts with the command _.

When the key `Enter` is pressed after entering a mathematical expression, Geogebra tries to evaluate it and it shows the result of the evaluation just below the expression, or a warning when there is some mistake in the expression.

Them most common operators for the construction of mathematical expressions are shown in the table below.

Symbol	Operator
+	Addition
-	Subtraction
*	Product
/	Division
^	Power

At the time of writing a mathematical expression, you must take into account that Geogebra has an order of priority to evaluate the operators. First it evaluates predefined functions and constants, after powers, after products and quotients (both with the same priority and from left to right), and finally additions and subtractions (both with the same priority and from left to right). To force the evaluation of a subexpression, skipping the order of priority, you must use parenthesis. Thus, as it can be appreciated in the table below, depending on how a expression is entered, you can get different results.

Entered expression	Evaluated expression
`4x-1/x-5`	$4x - \frac{1}{x} - 5$
`(4x-1)/x-5`	$\frac{4x-1}{x} - 5$
`4x-1/(x-5)`	$4x - \frac{1}{x-5}$
`(4x-1)/(x-5)`	$\frac{4x-1}{x-5}$

Every expression that is entered in the CAS View is labelled with a number that allows to identify it. Later, every time that we want to reference that expression we can use that identifier instead of writing again the whole expression.

[2] https://www.latex-project.org/

There are two ways of referring to an expression, that are the static and the dynamic references. To do a static reference we must write the symbol # followed by the identifier number of the expression. On the other hand, to do a dynamic reference we must write the symbol $ followed by the identifier number of the expression. A static reference will not change the expression where the reference is done even when the original expression changes, while for a dynamic reference, when the original expression changes, that change will be reflected in the expression where the reference is done.

It is possible to select any expression or subexpression of the CAS View and then copy and paste it in the Input Bar.

Entering text notes

Geogebra also allows to enter text notes or comments int he Input Bar. For that you have to right-click the Input Bar and select the option Text in the contextual menu that appears. Text annotations are very helpful to explain the steps in a mathematical construction or to interpret the results.

Removing expressions

Of course, it is possible to remove a expression from the CAS View. For that you have to go to the line with the expression to remove and click the button or right-click that line and select the option Delete row in the contextual menu that appears.

If sometime we commit a mistake entering or deleting a wrong expression, it is possible to undo the last operations or redo them clicking the buttons or respectively.

Defining variables

To define a variable we can use roman letters or greek letters. The name of a variable can have more than one letter and, in this case, it is also possible to use numbers but it must start always by a letter. Thus, for Geogebra, the expression `xy`, is not interpreted as the product of the variables x and y, but the variable xy. In addition, it distinguishes between upper and lower case, so that xy and xY are different variables.

Defining constants and functions

To define a constant or a function the definition operator `:=` must be used. To define a constant you have to write the name of the constant followed by `:=` and the value of the constant. For example, to define the gravity constant we have to write `g:=9.81`.

On the other hand, to define a function you have to write the name of the function, followed buy the list of variables separated by commas and between parenthesis, then `:=` and finally the expression that defines the function. For example, to define the function that calculates the area of triangle with base b and high h, we have to write `a(b,h):=(b*h)/2` (ver figure 1.3).

If we have defined a constant or function, and we change the definition after, the changes will be reflected in any other expression that contains the constant or function, except if the reference is static.

To remove a definition and free the name of the constant or function, for example `c`, we can use the command `Delete(c)` or the command `c:=`.

Predefined constants and functions

Geogebra provides several predefined constants and functions that can be used in the mathematical expressions. The sintax of some of these constants and functions is shown in the table 1.1, although, instead of using those commands, we can use the operators and constants of the virtual keyboard.

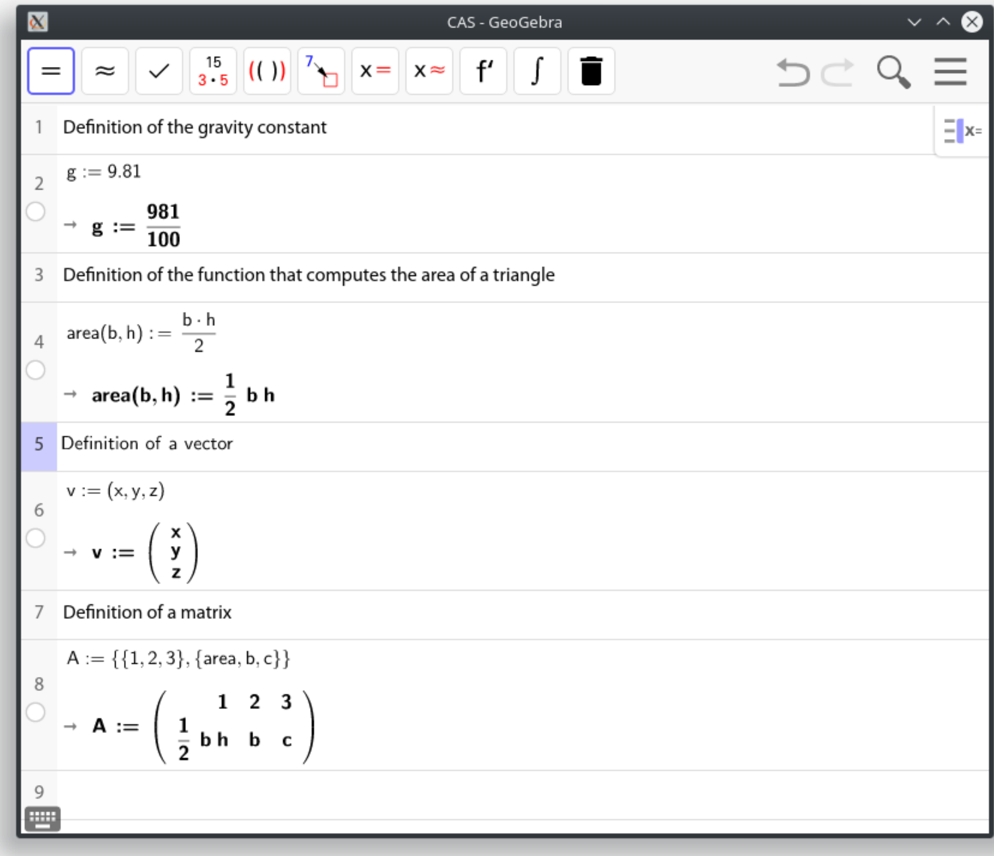

Figure 1.3 – Entering mathematical expressions in the Input Bar.

Sintaxis	**Constante o función**
`pi`	The number $\pi = 3.14159\ldots$
`Alt+e`	Euler's constant $e = 2.71828\ldots$
`Alt+i`	Imaginary number $i = \sqrt{-1}$
`inf`	Infinity ∞
`exp(x)`	Exponential function e^x
`log(a,x)`	Logarithmic function of base a, $\log_a x$
`ln(x)`	Neperian logarithmic function $\ln x$
`sqrt(x)`	Square root function $\sqrt{x}$
`sin(x)`	Sine function $\sin x$
`cos(x)`	Cosine function $\cos x$
`tan(x)`	Tangent function $\tan x$
`arcsin(x)`	Arcsine function $\arcsin x$
`arccos(x)`	Arccosine function $\arccos x$
`arctan(x)`	Arctangent function $\arctan x$

Table 1.1 – Sintax of some predefined constants and functions in Geogebra.

Entering vectors and matrices

Geogebra allows also to handle vectors and matrices. To define a vector you must write its coordinates separated by commas between parenthesis. For example, to enter the vector (x, y, z) we have to write (x,y,z) (see figure 1.3).

To define a matrix you must enter its elements by rows, separated by commas and between curly brackets. For example, to enter the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

we have to write {{1,2,3},{a,b,c}} (see figure 1.3).

Simplifying expressions

By default Geogebra always tries to simplify the mathematical expressions when it evaluates them. For example, if you enter $x + x$ the result will be $2x$. To avoid simplification you can change to the Keep Input mode clicking the button ✓ .

However, when Geogebra evaluates a mathematical expression it does not perform more complex simplifications, like, for instance, the simplification $\sin(x)^2 + \cos(x)^2 = 1$. To do this there are three commands:

Simplify This is the most simple and tries to simplify a mathematical expression the most. For example, the command Simplify(sin(x)^2+cos(x)^2) returns 1.

Expand This command tries to expand a mathematical expression computing all the possible powers, products, quotients, additions and subtractions. For example, the command Expand((x+1)^2) returns x^2+2x+1.

Factor This command tries to factorize a mathematical expression. For example, the command Factoriza(x^2+2x+1) returns (x+1)^2.

In any of these simplifications Geogebra uses by default the exact mode and returns fractional expressions. To get the approximate value of a mathematical expression, with decimals, we must change to the Numeric Evaluation mode clicking the button ≈ . The number of decimal places showed can be set in the settings menu of Geogebra.

Lastly, it is possible to replace any variable by a value with the command Substitute(<Expression>, <Substitution list>). For example, the command Substitute(2x+y, x=2, y=1) returns 5.

Entering equations and inequations

To define equations in Geogebra the equality symbol = must be used. Por example, the command 2x-y=1 defines the equation of a line.

And to define inequations we can use the symbols less than <, greater than >, less than or equal to <= or greater than or equal to >=. For example, the command x^2+y^2<=1 defines the circle with radius 1 centered at the origin.

To solve equations and inequation you can use the command Solve(<equations>). For example, the command Solve(x^2-5x+4=0) returns {x=1, x=4}. It is also possible to impose restrictions for the variables. For example, the command Solve(x^2-5x+4=0, x>3) returns only the solution {x=4}.

To solve systems of equations you must enter the list of equations separated by commans and between curly brackets. For example, the command Solve(2x+3=7, x-y=-1) returns {x=3, y=2}.

This command also solves inequations. For example, the command Solve(3x-2<1) returns {x<1}.

5 Graphical representations

One of the strengths of Geogebra is its graphics capabilities, sice it allows to represent graphically a lot of geometric objects both in the plane and in the real space.

Graphical representations in the real plane

To represent geometric objects in the real plane $\mathbb{R}^2$, Geogebra uses the Graphics View. By default any function defined in the CAS View will be plotted in this view. To graphically represent other objects like constants, equations or inequations, it is required to click on the circle that appears to the left of the expression (see figure 1.4). To hide back the object in the Graphics View you have to click again on this circle.

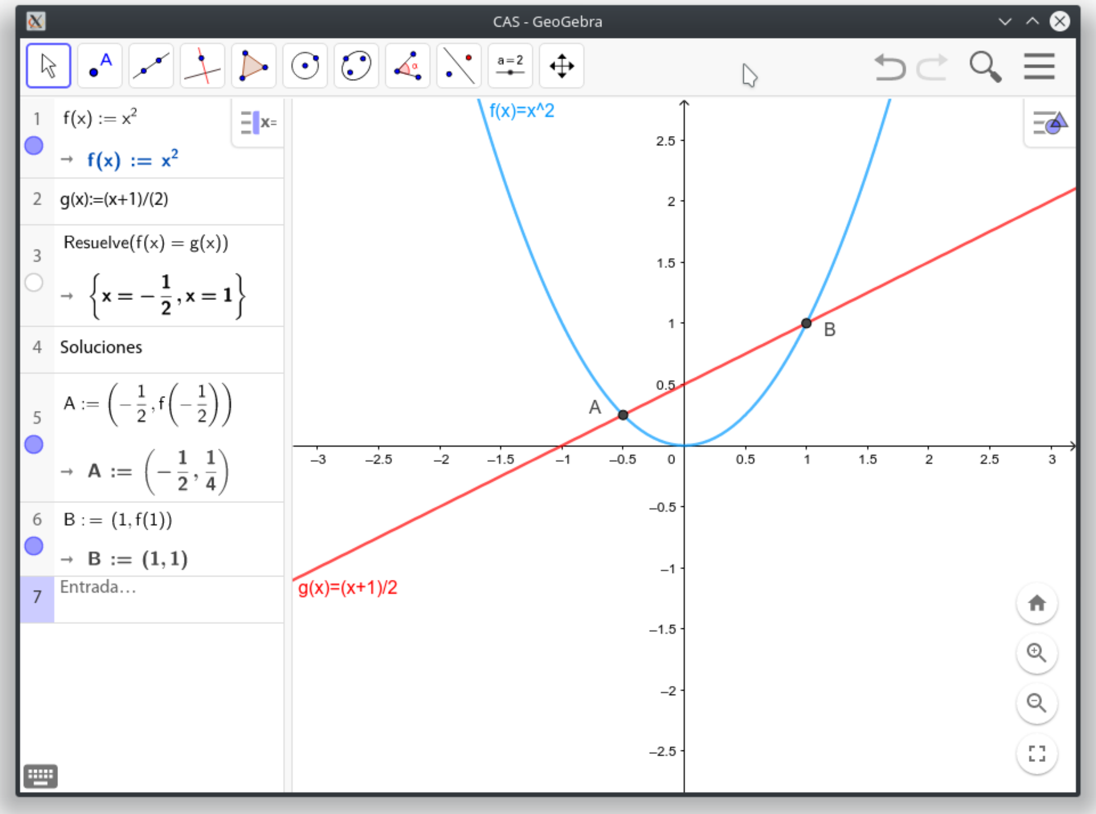

Figure 1.4 – Graphical representations in the Graphics View.

Geogebra allows also the graphical representation of parametric functions defining the vector with the coordinate functions depending on one parameter. For example, the command `g(t):=(cos(t), 2sin(t)cos(t))` plots the curve of the figure 1.5.

It is possible to change the aspect of any geometric object right-clicking on it and selecting the option Settings in the contextual menu that appears. This opens a panel that allows to change the name of the object, the colour, the thickness or the opacity of the line, or even to enter a label that will appear next to the object in the Graphics View.

The Graphics View is centered at the origin of coordinates by default, but it is possible to make a zoom in or out clicking on the buttons and respectively. It is also possible to move the view clicking at

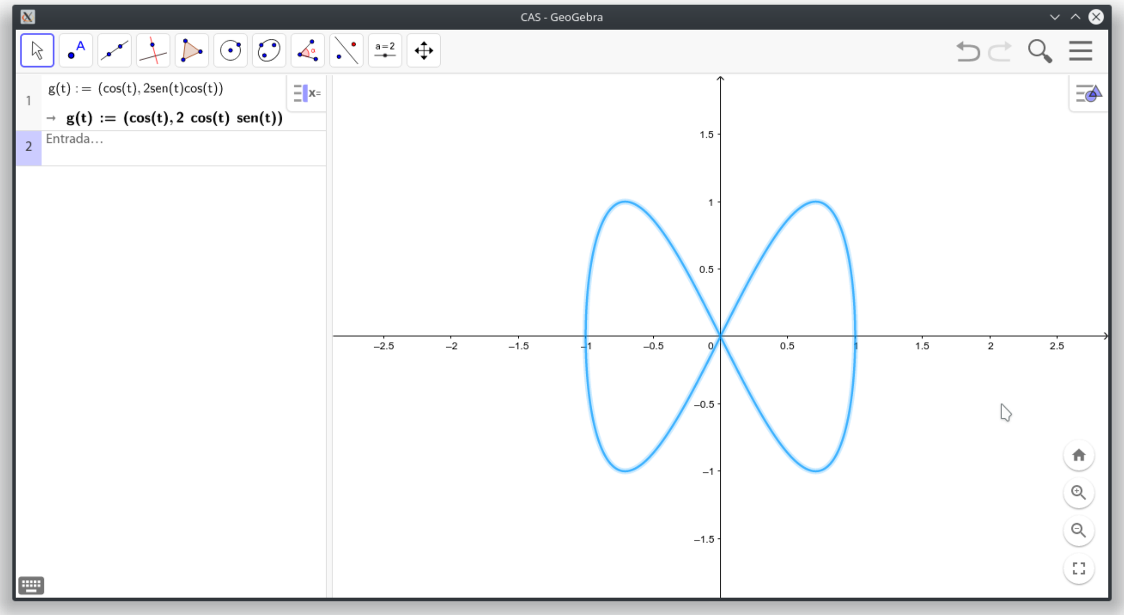

Figure 1.5 – Graphical representation of a parametric curve in the real plane.

any position in the view and dragging the mouse. To come back to the original view you can click on the button .

Graphical representations in the real space

To represent geometric objects in the real space $\mathbb{R}^3$, Geogebra uses the 3D Graphics View.

By default, any function of two variables defined in the CAS View will be plotted in this view. To graphically represent other objects like equations it is required to click on the circle that appears to the left of the expression (see figure 1.6). Para ocultar de nuevo el objeto en la Graphics View basta con volver a hacer clic sobre ese círculo.

The same than in the Graphics View it is also possible to represent parametric functions defining the vector with the coordinate functions depending on one parameter. For example, the command `h(t):=(cos(t), sin(t), t/2)` plots the curve of figure 1.7.

The same than in the Graphics View, it is possible to change the aspect of any geometric object right-clicking on it and selecting the option Settings in the contextual menu that appears. This opens a panel that allows to change the name of the object, the colour, the thickness or the opacity of the line, or even to enter a label that will appear next to the object in the 3D Graphics View.

In the same way, it is possible to make a zoom in or out clicking on the buttons and respectively. It is also possible to move the view with the button or to rotate it with the button .

6 File management

The mathematical expressions and calculus of the CAS View and the graphics of the Graphics View and 3D Graphics View can be saved into a file.

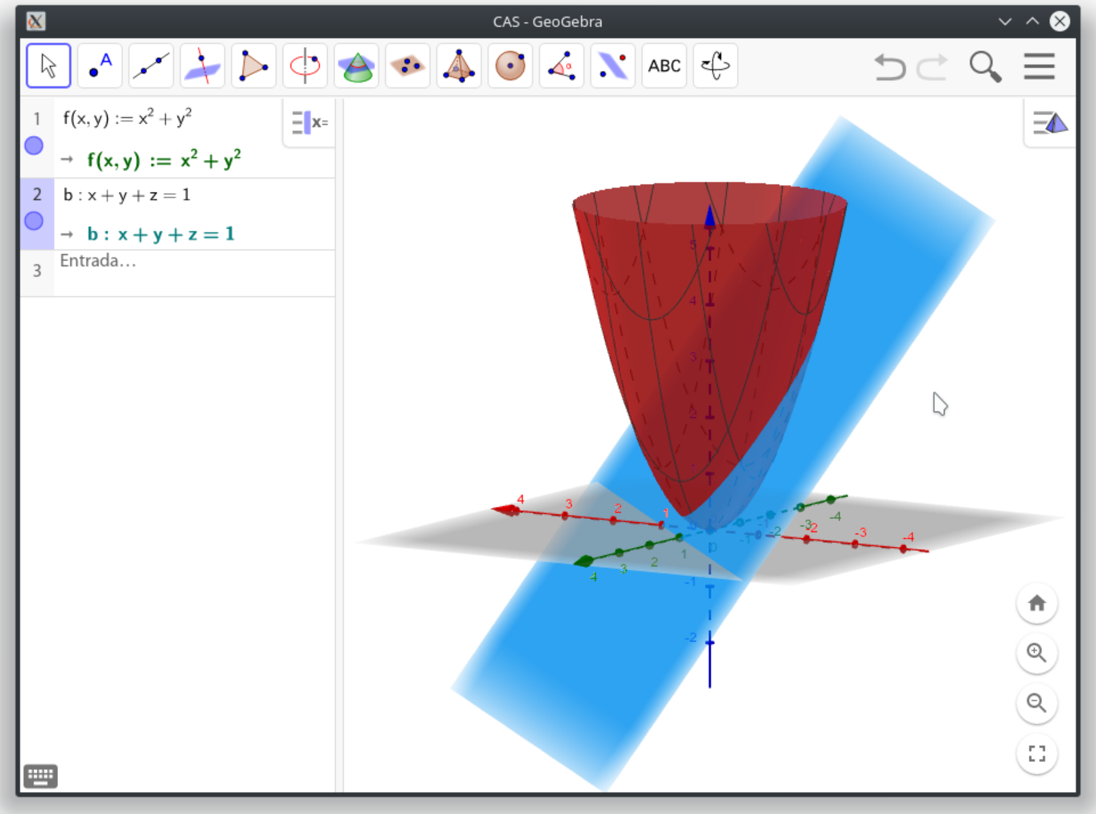

Figure 1.6 – Grahpical representations in the 3D Graphics View.

Saving a file

To save the mathematical expressions, calculus and graphics of a working session you must select the menu File » Save. If you have not logged in to the Geogebra web site a dialog appears asking for the user name an password to log in (see figure 1.8). If the you has no account in this site now is possible to register, but if you do not want to log in just click the link Continue without signing in now.

If you have logged in to the Geogebra web site, it will ask for the file name and the file will be uploaded to the Geogebra web site. This way the file will be available whenever we are connected to this site with our user account.

If you have not logged in to the Geogebra web site, then a dialog is shown where you can enter the file name and select the local folder where to save the file in your computer. Geogebras' files have extension *.ggb.

Once the file is saved, its name will appear in the title bar of the Geogebra window.

Opening a file

To open a file in Geogebra you can use the menu File » Open. In the dialog shown you can choose to open a file from the Geogebra web site or to open a local file.

If you has logged in to the Geogebra web site, the files saved in our user account will appear automatically. Eve if you are not logged in, you can open any public file in the Geogebra cloud. For that we can search for a file entering a keyword in the search bar and Geogebra will show a list with all the files that contains that word. Selecting any of them will download the file and open it.

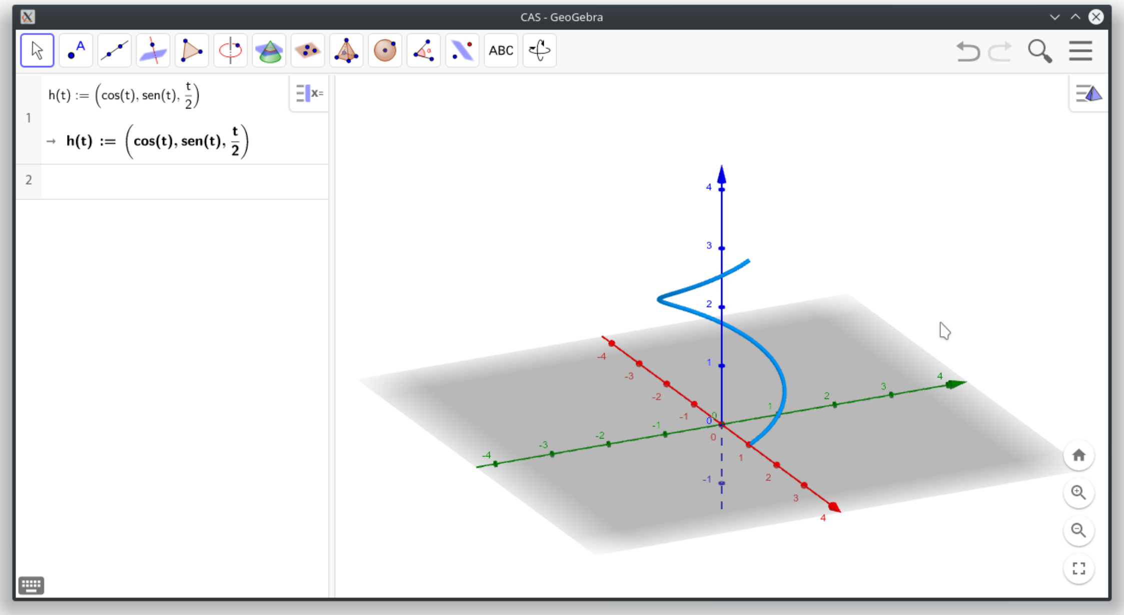

Figure 1.7 – Graphical representation of a parametric curve in the real space.

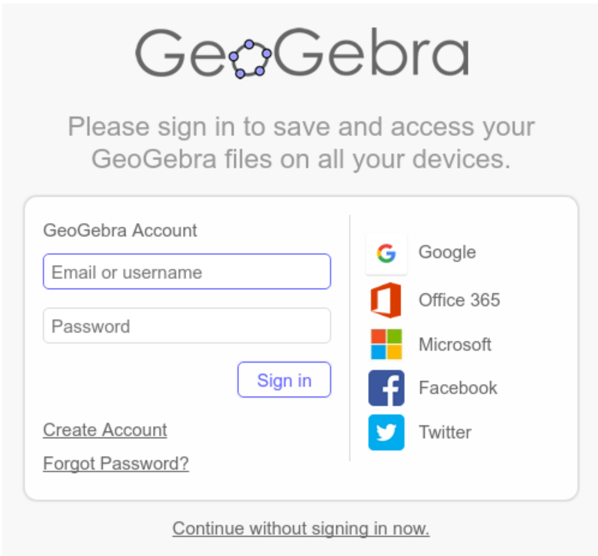

Figure 1.8 – Login dialog of Geogebra web site.

If you want to open a local file you have to click on the folder icon. This will open a dialog where you can select the file to open.

7 Solved exercises

1. Enter and evaluate the following expressions.

   (a) $4x - \frac{1}{x} - 5$.

   Enter the expression `4x-1/x-5` in the Input Bar of the CAS View.

   (b) $\frac{4x-1}{x} - 5$.

   Enter the expression `(4x-1)/x-5` in the Input Bar of the CAS View.

   (c) $4x - \frac{1}{x-5}$.

   Enter the expression `4x-1/(x-5)` in the Input Bar of the CAS View.

   (d) $\frac{4x-1}{x-5}$.

   Enter the expression `(4x-1)/(x-5)` in the Input Bar of the CAS View.

2. Define the following mathematical objects and plot them.

   (a) The constants $a = 2$ and $b = 3$.

   i. Enter the command `a:=2` in the Input Bar of the CAS View and activate the Graphics View.
   ii. To plot the slider of the constant click the circle that appears to the left of the previous expression.
   iii. Enter the command `b:=3` in the Input Bar.
   iv. To plot the slider of the constant click the circle that appears to the left of the previous expression.

   (b) The line $f(x) = a + bx$. Use the sliders of the constants to see how the line changes.

   Enter the command `f(x):=a+b*x` in the Input Bar.

   (c) The equation $ax^2 + by^2 = 8$. Use the sliders of the constants to see how the conic changes.

   Enter the command `a*x^2+b*y^2=8` in the Input Bar.

3. Define the following functions and plot them.

   (a) $f(x) := x^2$.

   Enter the command `f(x):=x^2` in the Input Bar of the CAS View and activate the Graphics View.

(b) $g(x) := \log(x)$.

Enter the command `g(x):=log(x)` in the Input Bar.

(c) $h(x) := \sin(x)$.

Enter the command `g(x):=sin(x)` in the Input Bar.

(d) $g \circ f(x)$.

Enter the command `g(f(x))` in the Input Bar and click the circle that appears to the left of the expression.

(e) $h \circ g \circ f(x)$.

Enter the command `h(g(f(x)))` in the Input Bar and click the circle that appears to the left of the expression.

(f) $f \circ g \circ h(x)$.

Enter the command `f(g(h(x)))` in the Input Bar and click the circle that appears to the left of the expression.

4. Given the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

and the vector $\mathbf{v} = (x, y, z)$, se pide:

(a) Define the matrices A and B, and the vector v.

i. Enter the command `A:={{a_11,a_12},{a_21,a_22},{a_31,a_32}}` in the Input Bar of the CAS View.
ii. Enter the command `B:={{1,2,3},{4,5,6}}` in the Input Bar.
iii. Enter the command `v:=(x,y,z)` in the Input Bar.

(b) Compute $A \cdot B$.

Enter the command `A*B` in the Input Bar.

(c) Compute $B \cdot A$.

Enter the command `B*A` in the Input Bar.

(d) Compute $\mathbf{v} \cdot A$.

Enter the command `v*A` in the Input Bar.

(e) Compute $B \cdot \mathbf{v}$.

Enter the command `B*v` in the Input Bar.

(f) Substitute $x = 1$, $y = 1$ and $z = 0$ in the previous vector and plot it.

> Enter the command `Substitute($,{x=1,y=1,z=0})` in the Input Bar and click the circle that appears to the left of the expression.

(g) Compute the modulus of the previous vector.

> Enter the command `|$|` in the Input Bar and click the circle that appears to the left of the expression.

(h) Change the previous substitution for $x = 0$, $y = 0$ and $z = 1$ and observe how changes the modulus of the previous vector.

> Edit the line with the substitution and change it for `Substitute($,{x=0,y=0,z=1})` in the Input Bar.

5. Find out the points where the graphs of the functions $f(x) = x^2$ and $g(x) = \dfrac{x+1}{2}$ intersect and plot them.

   (a) Enter the command `f(x):=x^2` in the Input Bar of the CAS View and activate the Graphics View.

   (b) Enter the command `g(x):=(x+1)/2` in the Input Bar.

   (c) To solve the equation, enter the command `Solve(f=g)` in the Input Bar.

   (d) To plot the intersection points, enter the command `Intersect(f,g)` in the Input Bar and click on the circle that appears to the left of the expression.

6. Plot the parametric function

$$g(t) = \begin{cases} \cos(t) \\ 2\sin(t)\cos(t) \end{cases} \qquad t \in \mathbb{R}$$

> Enter the command `g(t):=(cos(t), 2sin(t)cos(t))` in the Input Bar of the CAS View and activate the Graphics View.

7. Plot the following surfaces

$$f(x,y) = \frac{2\sin(x^2+y^2)}{\sqrt{x^2+y^2}}, \qquad x^2 + y2 + (z-2)^2 = 1$$

and and the parametric curve

$$h(t) = \begin{cases} \sin(t) \\ \cos(t) \\ t/2 \end{cases} \qquad t \in \mathbb{R}$$

> (a) Enter the command `f(x,y):=2sin(x^2+y^2)/sqrt(x^2+y^2)` in the Input Bar of the CAS View and activate the 3D Graphics View.
>
> (b) Enter the command `x^2+y^2+(z-2)^2=1` in the Input Bar.
>
> (c) Enter the command `h(t):=(sin(t),cos(t),t/2)` in the Input Bar.

Practice 2

Elementary functions

1 Solved exercises

1. Plot the graph of the function

$$f(t) = \frac{t^4 + 19t^2 - 5}{t^4 + 9t^2 - 10}.$$

and determine, looking at the graph, the following:

(a) Domain.

> i. Enter the expression `(t^4+19t^2-5)/(t^4+9t^2-10)` in the Input Bar of the CAS View and activate the Graphics View.
> ii. To determine the domain look at the values of x where the function does exists, that is, where there is graph.
> iii. Remember that, for this and the other parts of this exercise, we pretend to get approximate conclusions looking at the graph of the function.

(b) Image.

> Look at the values of y that are output of the function, that is, where there is graph.

(c) Asymptotes.

> Look at the lines (horizontal, vertical or oblique) where the graph approaches (the distance between the graph and the line tends to zero as they tend to infinity).

(d) Zeros.

> Look at the values of x where the graph cuts the horizontal axis.

(e) Sign.

> Look at the values of x where the graph is over the horizontal axis (positive) and where it is under the horizontal axis (negative).

(f) Growth.

> Look at the values of x where y increases when x increases (increasing) and the values where y decreases when x increases (decreasing).

(g) Concavity.

Look at the values of x where the curvature of the graph is up $\cup$ (concave up or convex) and where the curvature is $\cap$ (concave down or simply concave).

(h) Local extrema.

Look at the values of x where the graph has a peak (relative maximum) and where the graph has a valley (relative minimum).

(i) Inflection points.

Look at the values of x where the curvature changes continuously.

2. Plot the graphs of the exponential functions 2^x, e^x, 0.7^x, 0.5^x. Looking at the graphs of the functions, determine for which values of the base the exponential function increases and for which ones decreases.

   (a) Enter the expression `2^x` in the Input Bar of the CAS View.
   (b) Enter the expression `e^x` in the Input Bar.
   (c) Enter the expression `0.7^x` in the Input Bar.
   (d) Enter the expression `0.5^x` in the Input Bar.
   (e) The exponential function a^x increases for $a > 1$ and decreases for $0 < a < 1$.

3. Plot the graphs of the following functions and determine their periods and amplitudes.

   (a) $\sin x$, $\sin x + 2$, $\sin(x + 2)$.

      i. Enter the expression `sin(x)` en la in the Input Bar of the CAS View.
      ii. Enter the expression `sin(x)+2` in the Input Bar.
      iii. Enter the expression `sen(x+2)` in the Input Bar.
      iv. When a value is added to the sine or to the argument of the sine, the period and the amplitude does not change.

   (b) $\sin 2x$, $2\sin x$, $\sin \frac{x}{2}$.

      i. Enter the expression `sen(2x)` in the Input Bar.
      ii. Enter the expression `2sin(x)` in the Input Bar.
      iii. Enter the expression `sen(x/2)` in the Input Bar.
      iv. When the sine is mulitiplied by a value, the amplitude is multiplied by that value, and when the argument of the sine is multiplied by a value, the period is divided by that value.

4. Plot the graph of the piecewise function

$$f(x) = \begin{cases} -2x & \text{si } x \leq 0; \\ x^2 & \text{si } x > 0. \end{cases}$$

Enter the expression `If(x<=0, -2x, x^2)` in the Input Bar.

2 Proposed exercises

1. Plot the graphs of the following functions and determine their domains looking at their graphs.

   (a) $f(x) = \dfrac{x^2 + x + 1}{x^3 - x}$

   (b) $g(x) = \sqrt{x^4 - 1}$.

   (c) $h(x) = \cos\left(\frac{x+3}{x^2+1}\right)$.

   (d) $l(x) = \arcsin\left(\frac{x}{1+x}\right)$.

2. Consider the function
$$f(x) = \frac{x^3 + x + 2}{5x^3 - 9x^2 - 4x + 4}.$$
   Plot the function and determine looking at its graph:

   (a) Domain.

   (b) Image.

   (c) Asymptotes

   (d) Zeros.

   (e) Sign.

   (f) Continuity

   (g) Growth

   (h) Concavity

   (i) Relative extrema

   (j) Inflection points

3. Plot in the same graphic window the graphs of the functions $\log_{10} x$, $\log_2 x$, $\log x$, $\log_{0.5} x$.

   (a) Looking at their graphs determine which funtions are increasing and which ones are decreasing.

   (b) Deduce for what values of a the function $\log_a x$ is increasing and for what values it is decreasing?

4. Plot the following functions and complete the following sentences with the word equal or the number of times that is lower or greater in any case.

   (a) The function $\cos(2x)$ has a period .... than the function $\cos x$.

   (b) The function $\cos(2x)$ has an amplitude .... than the function $\cos x$.

   (c) The function $\cos(x/2)$ has period .... than the function $\cos(3x)$.

   (d) The function $\cos(x/2)$ has an amplitude .... than the function $\cos(3x)$.

   (e) The function $3\cos(2x)$ has a period .... than the function $\cos(x/2)$.

   (f) The function $3\cos(2x)$ has an amplitude .... than the function $\cos(x/2)$.

5. Find the solutions of the equation $e^{-1/x} = \dfrac{1}{x}$ graphically.

6. Plot the graph of the function
$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$$

Practice 3

Limits and continuity

1 Solved exercises

1. Given the function
$$f(x) = \left(1 + \frac{2}{x}\right)^{x/2},$$

   (a) Plot its graph and try to guess the values of the following limits looking at the graph:

      i. $\lim_{x\to -2^-} f(x)$
      ii. $\lim_{x\to -2^+} f(x)$
      iii. $\lim_{x\to -\infty} f(x)$
      iv. $\lim_{x\to +\infty} f(x)$
      v. $\lim_{x\to 2} f(x)$
      vi. $\lim_{x\to 0} f(x)$

      i. To plot the graph, enter the function `f(x):=(1+2/x)^(x/2)` in the Input Bar of the CAS View and activate the Graphics View.
      ii. To guess the limits, create a slider entering the expression `a:=0` in the Input Bar.
      iii. Enter the point `A:=(a,f(a))` to plot the point on the graph of the function.
      iv. Finally, move the slider and observe the value of the y coordinate when x approaches any of the values in the limits.

   (b) Compute the previous limits. Do you get the same results that you guess looking at the graph?

      i. To compute $\lim_{x\to -2^-} f(x)$ enter the command `LimitBelow(f(x), -2)` in the Input Bar.
      ii. To compute $\lim_{x\to -2^+} f(x)$ enter the command `LimitAbove(f(x), -2)` in the Input Bar.
      iii. To compute $\lim_{x\to -\infty} f(x)$ enter the command `Limit(f(x), -inf)` in the Input Bar.
      iv. To compute $\lim_{x\to \infty} f(x)$ enter the command `Limit(f(x), inf)` in the Input Bar.
      v. To compute $\lim_{x\to 2} f(x)$ enter the command `Limit(f(x), 2)` in the Input Bar.
      vi. To compute $\lim_{x\to 0} f(x)$ enter the command `Limit(f(x), 0)` in the Input Bar.

2. Given the function
$$g(x) = \begin{cases} \dfrac{x}{x-2} & \text{if } x \leq 0; \\ \dfrac{x^2}{2x-6} & \text{if } x > 0; \end{cases}$$

   (a) Plot the graph and determine graphically if there are asymptotes.

      i. To plot the graph, enter the function `g(x):=If(x<=0, x/(x-2), x^2/(2x-6))` in the Input Bar of the CAS View and activate the Graphics View.

ii. To check if there are vertical, horizontal or oblique asymptotes look at the lines (horizontal, vertical or oblique) where the graph approaches (the distance between the graph and the line tends to zero as they tend to infinity).

(b) Compute the vertical asymptotes of g and plot them if any.

i. The function is not defined in the values where the denominators vanish. To find the zeros of the denominator of the firt piece enter the command `Root(x-2)` in the Input Bar. The only root is at $x = 2$ but it falls outside the region of this piece.
ii. To find the zeros of the denominator of the second piece enter the command `Root(2x-6)` in the Input Bar. The onliy root is at $x = 3$, so that the function is not defined a that point and it could exist a vertical asymptote at this point.
iii. To check if there is a vertical asymptote a this point you have to compute the lateral limits $\lim_{x\to 3^-} g(x)$ and $\lim_{x\to 3^+} g(x)$.
iv. To compute the lateral limit to the left enter the command `LimitBelow(g, 3)` in the Input Bar.
v. To compute the lateral limit to the right enter the command `LimitAbove(g, 3)` in the Input Bar.
vi. Since $\lim_{x\to 3^-} g(x) = -\infty$ and $\lim_{x\to 3^+} g(x) = \infty$, there exist a vertical asymptote at $x = 3$. To plot it, enter the expression `x=3` in the Input Bar.

(c) Compute the horizontal asymptotes of g and plot them if any.

i. To check if there is an horizontal asymptote we have to compute the limits at infinity $\lim_{x\to -\infty} g(x)$ y $\lim_{x\to\infty} g(x)$.
ii. To compute the limit at $-\infty$ enter the command `Limit(g, -inf)` in the Input Bar.
iii. To compute the limit at ∞ enter the command `Limit(g, inf)` in the Input Bar.
iv. Since $\lim_{x\to -\infty} g(x) = 1$, there exist an horizontal asymptote at $y = 1$ to the left. To plot it, enter the expression `y=1` in the Input Bar.
v. Since $\lim_{x\to\infty} g(x) = \infty$, there is no horizontal asymptote to the right.

(d) Compute the oblique asymptotes of g and plot them if any.

i. To the left there is no asymptote since there is an horizontal asymptote. To check if there is an oblique asymptote to the right you have to compute the limits $\lim_{x\to\infty} \frac{g(x)}{x}$. For it, enter the `Limit(g/x, inf)` in the Input Bar.
ii. Since $\lim_{x\to\infty} \frac{g(x)}{x} = 0.5$, there exist an oblique asymptote to the right and 0.5.
iii. To compute the intercept you have to compute the limit $\lim_{x\to\infty} g(x) - 0.5x$. For it, enter the command `Limit(g(x)-0.5x, inf)` in the Input Bar.
iv. Since $\lim_{x\to\infty} g(x) - 0.5x = 1.5$ the equation of the oblique asymptote is $y = 0.5x + 1.5$. To plot it, enter the expression `y=0.5x+1.5` in the Input Bar.

3. For the following functions determine the type of discontinuity at the points given.

(a) $f(x) = \dfrac{\sin x}{x}$ at $x = 0$.

i. To plot the graph enter the function `f(x):=sin(x)/x` in the Input Bar of the CAS View and activate the Graphics View.
ii. To compute the limit $\lim_{x\to 0^-} f(x)$ enter the command `LimitBelow(f, 0)` in the Input Bar.

iii. To compute the limit $\lim_{x\to 0^+} f(x)$ enter the command `LimitAbove(f, 0)` in the Input Bar.

iv. Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$, f has a removable discontinuity at $x = 0$.

(b) $g(x) = \dfrac{1}{2^{1/x}}$ at $x = 0$.

i. To plot the graph enter the function `g(x):=1/2^(1/x)` in the Input Bar of the CAS View.

ii. To compute the limit $\lim_{x\to 0^-} g(x)$ enter the command `LimitBelow(g, 0)` in the Input Bar.

iii. To compute the limit $\lim_{x\to 0^+} g(x)$ enter the command `LimitAbove(g, 0)` in the Input Bar.

iv. Since $\lim_{x\to 0^-} g(x) = \infty$, g has an essential discontinuity at $x = 0$.

(c) $h(x) = \dfrac{1}{1+e^{\frac{1}{1-x}}}$ at $x = 1$.

i. To plot the graph enter the function `h(x):=1/(1+e^(1/(1-x)))` in the Input Bar of the CAS View.

ii. To compute the limit $\lim_{x\to 1^-} h(x)$ enter the command `LimitBelow(h, 1)` in the Input Bar.

iii. To compute the limit $\lim_{x\to 1^+} h(x)$ enter the command `LimitAbove(h, 1)` in the Input Bar.

iv. Since $\lim_{x\to 1^-} h(x) = 0$ y $\lim_{x\to 1^+} f(x) = 1$, h has a jump discontinuity at $x = 1$.

4. Plot the graph of the function

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-1}, & \text{if } x < 0; \\ \dfrac{1}{e^{1/(x^2-1)}}, & \text{if } x \geq 0. \end{cases}$$

and determine the points where it has a discontinuity and classify them.

(a) To plot the graph enter the function `f(x):=If(x<0, (x+1)/(x^2-1), 1/e^(1/(x^2-1)))` in the Input Bar of the CAS View and activate the Graphics View.

(b) First you have to find the points that are not in the domain for every piece. For it, you have to compute the zeros of the denominators that appears in any piece. To find the zeros of $x^2 - 1$ enter the command `Root(x^2-1)` in the Input Bar.

(c) There are to roots at $x = -1$ and $x = 1$. At this values the function is not defined and, therefore, is discontinuous. In addition to this points you have to study also what happens at $x = 0$ where the definition of the function changes.

(d) To compute the limit $\lim_{x\to -1^-} f(x)$ enter the command `LimitBelow(f, -1)` in the Input Bar.

(e) To compute the limit $\lim_{x\to -1^+} f(x)$ enter the command `LimitAbove(f, -1)` in the Input Bar.

(f) Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = -0.5$, f has a removable discontinuity at $x = -1$.

(g) To compute the limit $\lim_{x\to 0^-} f(x)$ enter the command `LimitBelow(f, 0)` in the Input Bar.

(h) To compute the limit $\lim_{x\to 0^+} f(x)$ enter the command `LimitAbove(f, 0)` in the Input Bar.

(i) Since $\lim_{x\to 0^-} f(x) = -1$ y $\lim_{x\to 0^+} f(x) = e$, f has a jump discontinuity at $x = 0$.

(j) To compute the limit $\lim_{x\to 1^-} f(x)$ enter the command `LimitBelow(f, 1)` in the Input Bar.

(k) To compute the limit $\lim_{x\to 1^+} f(x)$ enter the command `LimitAbove(f, 1)` in the Input Bar.

(l) Since $\lim_{x\to 1^-} f(x) = \infty$, f has an essential discontinuity at $x = 1$.

2 Proposed exercises

1. Compute the following limits:

   (a) $\displaystyle\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$.

   (b) $\displaystyle\lim_{x\to a} \frac{\sin x - \sin a}{x-a}$.

   (c) $\displaystyle\lim_{x\to\infty} \frac{x^2-3x+2}{e^{2x}}$.

   (d) $\displaystyle\lim_{x\to\infty} \frac{\log(x^2-1)}{x+2}$.

   (e) $\displaystyle\lim_{x\to 1} \frac{\log(1/x)}{\tan(x+\frac{\pi}{2})}$.

   (f) $\displaystyle\lim_{x\to a} \frac{x^n-a^n}{x-a}$ $\quad n\in\mathbb{N}$.

   (g) $\displaystyle\lim_{x\to 1} \frac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ $\quad n,m\in\mathbb{Z}$.

   (h) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

   (i) $\displaystyle\lim_{x\to \pi/4} \frac{\sin x - \cos x}{1-\tan x}$.

   (j) $\displaystyle\lim_{x\to 0} x^2 e^{1/x^2}$.

   (k) $\displaystyle\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x$.

   (l) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x}$.

   (m) $\displaystyle\lim_{x\to 0} (\cos x)^{1/\operatorname{sen} x}$.

   (n) $\displaystyle\lim_{x\to 0} \frac{6}{4+e^{-1/x}}$.

   (o) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-2x-1}\right)$.

2. Given the function
$$f(x) = \begin{cases} \dfrac{x^2+1}{x+3} & \text{if } x<0; \\ \dfrac{1}{e^{1/(x^2-1)}} & \text{if } x\geq 0; \end{cases}$$
   compute its asymptotes.

3. The following functions are not defined at $x = 0$. Determine, when possible, the value that should take the function at that point to be continuous.

   (a) $f(x) = \dfrac{(1+x)^n-1}{x}$.

   (b) $h(x) = \dfrac{e^x-e^{-x}}{x}$.

   (c) $j(x) = \dfrac{\log(1+x)-\log(1-x)}{x}$.

   (d) $k(x) = x^2 \sin\dfrac{1}{x}$.

Practice 4

Derivatives of functions of one variable

1 Solved exercises

1. Representar gráficamente y estudiar mediante la definición de derivada la derivabilidad de las siguientes funciones en los puntos que se indica:

   (a) $f(x) = |x - 1|$ en $x = 1$.

   i. Para representarla gráficamente, enter the function `f(x):=|x-1|` in the Input Bar of the CAS View and activate the Graphics View.
   ii. To compute la derivada de f por la izquierda en $x = 1$ enter the command `LimitBelow((f(1+h)-f(1))/h, 0)` in the Input Bar.
   iii. To compute la derivada de f por la derecha en $x = 1$ enter the command `LimitAbove((f(1+h)-f(1))/h, 0)` in the Input Bar.
   iv. Como $lim_{x\to 0^-}\frac{f(1+h)-f(1)}{h} = -1$ es distinto de $lim_{x\to 0^+}\frac{f(1+h)-f(1)}{h} = 1$, la función no es derivable en $x = 1$.

   (b) $g(x) = \begin{cases} x \sin\dfrac{1}{x}, & \text{si } x \neq 0; \\ 0, & \text{si } x = 0. \end{cases}$ en $x = 0$.

   i. Para representarla gráficamente, enter the function `g(x):=If(x!=0, sen(1/x), 0)` in the Input Bar of the CAS View.
   ii. To compute la derivada de g por la izquierda en $x = 0$ enter the command `LimitBelow((g(h)-g(0))/h, 0)` in the Input Bar.
   iii. To compute la derivada de g por la derecha en $x = 0$ enter the command `LimitAbove((g(h)-g(0))/h, 0)` in the Input Bar.
   iv. Como ninguno de los dos límites anteriores existe g no es derivable en $x = 0$.

2. Calcular las derivadas de las siguientes funciones hasta el orden 4 y conjeturar cuál sería el valor de la derivada de orden n.

   (a) $f(x) = a^x \log(a)$.

   i. Introducir la función `f(x):=a^xlog(a)` in the Input Bar of the CAS View.
   ii. Para la primera derivada enter the expression `f'(x)` in the Input Bar.
   iii. Para la segunda derivada enter the expression `f''(x)` in the Input Bar.
   iv. Para la tercera derivada enter the expression `f'''(x)` in the Input Bar.
   v. Para la cuarta derivada enter the expression `f''''(x)` in the Input Bar.
   vi. La derivada de orden n será por tanto $f^n(x) = a^x \log(a)^{n+1}$.

(b) $g(x) = \dfrac{\sin x + \cos x}{2}$.

> i. Introducir la función `g(x):=(sin(x)+cos(x))/2` in the Input Bar of the CAS View.of the CAS View.
> ii. Para la primera deof the CAS View.g′(x) in the Input Bar.
> iii. Para la segunda deof the CAS View.g″(x) in the Input Bar.
> iv. Para la tercera deof the CAS View.g‴(x) in the Input Bar.
> v. Para la cuarta derivada enter the expression `g''''(x)` in the Input Bar.
> vi. A partir de aquí las derivadas se repiten, por lo que la derivada de orden n será
>
> $$f^n(x) = \begin{cases} \frac{\sin(x)+\cos(x)}{2} & \text{si } x = 4k \\ \frac{\cos(x)-\sin(x)}{2} & \text{si } x = 4k+1 \\ \frac{-\sin(x)-\cos(x)}{2} & \text{si } x = 4k+2 \\ \frac{-\cos(x)+\sin(x)}{2} & \text{si } x = 4k+3 \end{cases} \quad \text{con } k \in \mathbb{Z}$$

3. Calcular las rectas tangente y normal a la gráfica de la función $f(x) = \log(\sqrt{x+1})$ en $x = 1$. Dibujar la gráfica de la función y de la recta tangente.

> (a) Introducir la función `f(x):=log(sqrt(x+1))` in the Input Bar of the CAS View.
> (b) Para obtener la ecuación de la recta tangente a f en $x = 1$ introducir la ecuación $y = f(1) + f'(1)(x-1)$ in the Input Bar.
> (c) Para dibujar la recta tangente hacer clic en el círculo que aparece a la izquierda de la expresión anterior.
> (d) Para obtener la ecuación de la recta normal a f en $x = 1$ introducir la ecuación $y = f(1) - 1/f'(1)(x-1)$ in the Input Bar.
> (e) Para dibujar la recta normal hacer clic en el círculo que aparece a la izquierda de la expresión anterior.

4. Given the function

$$g(x) = \frac{2x^3 - 3x}{x^2 + 1}$$

(a) Representar la gráfica de g.

> Para representarla gráficamente, enter the function `g(x):=(2x^3-3x)/(x^2+1)` in the Input Bar of the CAS View and activate the Graphics View.

(b) Calcular la función derivada $g'(x)$ y representar su gráfica.

> i. Para obtener la derivada de g enter the expression `g'(x):=Derivada(g)` in the Input Bar.
> ii. Para dibujar la gráfica de g' hacer clic en el círculo que aparece a la izquierda de la expresión anterior.

(c) Calcular las raíces de $g'(x)$.

> i. To compute las raíces de g' enter the expression `Root(g'(x))` in the Input Bar y hacer clic sobre el botón de evaluación aproximada.
> ii. g' tiene dos raíces en $x = -0.56$ y $x = 0.56$ aproximadamente.

(d) A la vista de las raíces y de la gráfica de la función derivada, determinar los extremos relativos de la función y los intervalos de crecimiento.

i. $g'(x) > 0$ para $x \in (-\infty, -0.56)$, luego g es creciente en ese intervalo.
ii. $g'(x) < 0$ para $x \in (-0.56, 0.56)$, luego g es decreciente en ese intervalo.
iii. $g'(x) > 0$ para $x \in (-0.56, \infty)$, luego g es creciente en ese intervalo.
iv. g tiene un máximo en $x = -0.56$ ya que g' se anula en este punto y a la izquierda la función es creciente y a la derecha decreciente.
v. g tiene un mínimo en $x = 0.56$ ya que g' se anula en este punto y a la izquierda la función es decreciente y a la derecha creciente.

(e) Calcular la segunda derivada $g''(x)$ y representar su gráfica.

i. Para obtener la segunda derivada de g enter the expression `g''(x):=Derivada(g, 2)` in the Input Bar.
ii. Para dibujar la gráfica de g'' hacer clic en el círculo que aparece a la izquierda de la expresión anterior.

(f) Calcular las raíces de $g''(x)$.

i. To compute las raíces de g'' enter the expression `Root(g''(x))` in the Input Bar y hacer clic sobre el botón de evaluación aproximada.
ii. g'' tiene tres raíces en $x = -\sqrt{3}$, $x = 0$ y $x = \sqrt{3}$.

(g) A la vista de las raíces y de la gráfica de la segunda derivada, determinar los intervalos de concavidad de la función y los puntos de inflexión.

i. $g''(x) > 0$ para $x \in (-\infty, -\sqrt{3})$, luego g es cóncava en ese intervalo.
ii. $g''(x) < 0$ para $x \in (-\sqrt{3}, 0)$, luego g es convexa en ese intervalo.
iii. $g''(x) > 0$ para $x \in (0, \sqrt{3})$, luego g es cóncava en ese intervalo.
iv. $g''(x) < 0$ para $x \in (\sqrt{3}, \infty)$, luego g es convexa en ese intervalo.
v. g tiene puntos de inflexión en $x = -\sqrt{3}$, $x = 0$ y $x = \sqrt{3}$ pues g'' se anula en estos puntos y la concavidad a la izquierda y derecha de ellos cambia.

2 Ejercicios propuestos

1. Probar que no es derivable en $x = 0$ la siguiente función:

$$f(x) = \begin{cases} e^x - 1 & \text{si } x \geq 0, \\ x^3 & \text{si } x < 0. \end{cases}$$

2. Para cada una de las siguientes curvas, hallar las ecuaciones de las rectas tangente y normal en el punto x_0 indicado.

   (a) $y = x^{\sin x}$, $\quad x_0 = \pi/2$.

   (b) $y = (3 - x^2)^4 \sqrt[3]{5x - 4}$, $\quad x_0 = 1$.

   (c) $y = \log \sqrt{\dfrac{1 + x}{1 - x}} + \arctan x$, $\quad x_0 = 0$.

3. Estudiar el crecimiento, decrecimiento, extremos relativos, concavidad y puntos de inflexión de la función $f(x) = \dfrac{x}{x^2 - 2}$.

4. Se ha diseñado un envoltorio cilíndrico para cápsulas. Si el contenido de las cápsulas debe ser de 0.15 ml, hallar las dimensiones del cilindro para que el material empleado en el envoltorio sea mínimo.
5. La cantidad de trigo en una cosecha C depende de la cantidad de nitrógeno en el suelo n según la ecuación

$$C(n) = \frac{n}{1+n^2}, \quad n \geq 0$$

¿Para qué cantidad de nitrógeno se obtendrá la mayor cosecha de trigo?