Koopman Operator and its Approximations for Systems with Symmetries

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Abstract

Many systems of current interest are high-dimensional and highly nonlinear, making them inaccessible to traditional dynamical systems approaches. Thus, operator based approaches to dynamical systems, which consider the evolution of functions that operate on the states of the system, have gained popularity in recent years. Moreover, the approximations of these operators provide ways to apply these methods directly to data. We employ an operator based approach to systems with point symmetries. In particular, we focus on the Koopman operator, an infinite dimensional linear operator which is the adjoint of the Perron-Frobenius operator. Using tools from representation theory we study the structure of the eigendecomposition of this operator, which we show is especially revealing of the symmetries through an appropriate isotypic component basis. The Koopman operator can be well approximated via the recently proposed Extended Dynamic Mode Decomposition method that requires a dictionary of observables. We apply the knowledge of the structure of the eigenspace gained via our symmetry considerations to produce dictionaries of observables that ensure that the matrix corresponding to the Koopman operator approximation is block diagonal. That can offer computational advantages, can illuminate the attractor structure of the underlying system, and can potentially lead to new methods of detecting symmetries in high-dimensional nonlinear dynamical systems.

Preliminaries: Koopman Operator

We start from a finite dimensional nonlinear dynamical system. For discrete systems:

$$x_{t+1} = g(x_t)$$

Here, $x \in \mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^n$. The Koopman operator $\mathcal{K}$ is an infinite dimensional linear operator evolving observables $\psi(x) : \mathbb{R} \to \mathbb{C}$:

$$\mathcal{K}\psi = \psi \circ g$$

The Koopman operator eigenfunctions ϕ, eigenvalues λ, and eigenmodes v are defined by:

$$\mathcal{K}\phi = \lambda\phi$$
$$x = \sum_i v_i \phi_i(x)$$

This decomposition allows us to capture nonlinear dynamics in a linear setting.

[1] Budišić, M., Mohr, R., & Mezić, I. (2012). Applied Koopmanism. Chaos: An Interdisciplinary Journal of Nonlinear Science, 22(4), 047510.

Preliminaries: Symmetries

Let Γ be a symmetry group with elements $\gamma \in \Gamma$. Let γ_ρ be the actions of the elements of the group γ.

The system is called Γ-equivariant with respect to the action γ_ρ if for all $x \in X$ and $\gamma_\rho \in \Gamma_\rho$:

$$\boxed{g(\gamma_\rho x(t)) = \gamma_\rho g(x(t))}$$

The action on function space is defined by $\gamma_\rho f(x) = f(\gamma_\rho^{-1} x)$.

Let R_p be the p^{th} irreducible representation of Γ of dimension d_p. We can decompose the function space as $\mathcal{F} = \bigoplus \mathcal{F}_p$, where $\mathcal{F}_p$ transforms like the p^{th} irreducible representation of Γ. The subspaces $\mathcal{F}_p$ are called isotypic components of $\mathcal{F}$.

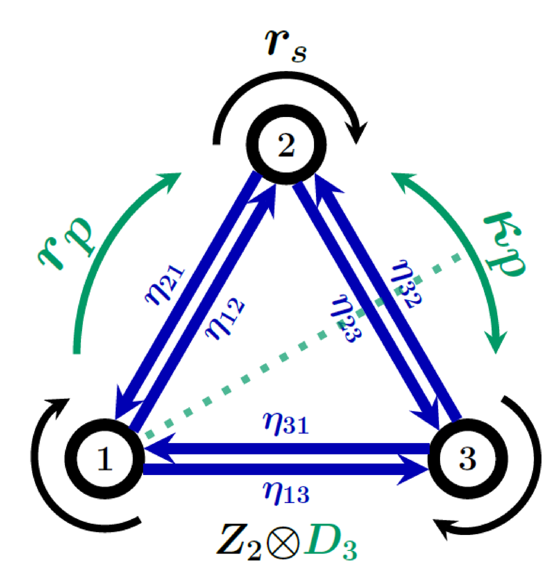
[2] Golubitsky, M., & Stewart, I. (2003). The symmetry perspective: from equilibrium to chaos in phase space and physical space (Vol. 200). Springer Science & Business Media.

Koopman Operator and Symmetries

Let $\mathcal{K}$ be the Koopman operator for a Γ-equivariant system. We show that:

- $\mathcal{K}$ commutes with γ_ρ for any $\gamma \in \Gamma$
- The space of eigenfunctions of the Koopman operator with eigenvalue λ for a Γ-equivariant system is invariant under the action γ_ρ for any $\gamma \in \Gamma$
- We decompose the function space into its isotypic components $\mathcal{F} = \bigoplus \mathcal{F}_p$. $\mathcal{K}$ commutes with $\Gamma_\rho \implies \boxed{\mathcal{K}\mathcal{F}_p \subset \mathcal{F}_p}$
- Furthermore, it is possible to refine this decomposition as $\mathcal{F}_p = \bigoplus_{d_p} \mathcal{F}_{qp}$, such that $\boxed{\mathcal{K}\mathcal{F}_{qp} \subset \mathcal{F}_{qp}}$

Point symmetries: example



Duffing oscillator dynamics:

$$\ddot{x} = -\sigma\dot{x} - x(\beta + \alpha^2 x)$$

Coupled Duffing oscillators:

$$\dot{x}_i = y_i$$
$$\dot{y}_i = -\sigma y_i - x_i(\beta + \alpha^2 x_i) + \sum_{ij} \gamma(x_i - x_j)$$

The system of 3 coupled Duffing oscillators above is equivariant with respect to the following groups:

$Z_2 = \langle r | r^2 = e \rangle$

$r_s = -I_{6\times 6}$

2 irreducible representations of degree 1

$D_3 = \langle r, \kappa | r^3 = \kappa^2 = e, \kappa r \kappa = r^{-1} \rangle$

$r_p = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes I_{2\times 2}$ and $\kappa_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes I_{2\times 2}$

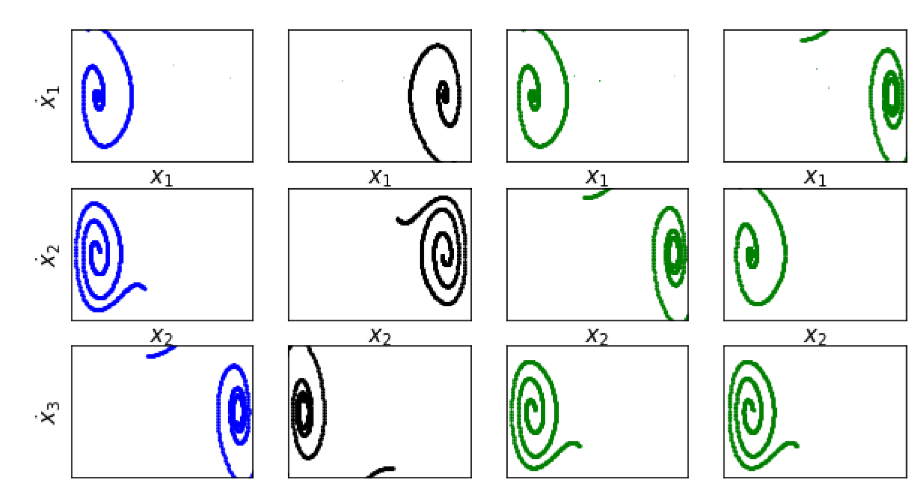
2 irreducible representations of degree 1

1 irreducible representation of degree 2

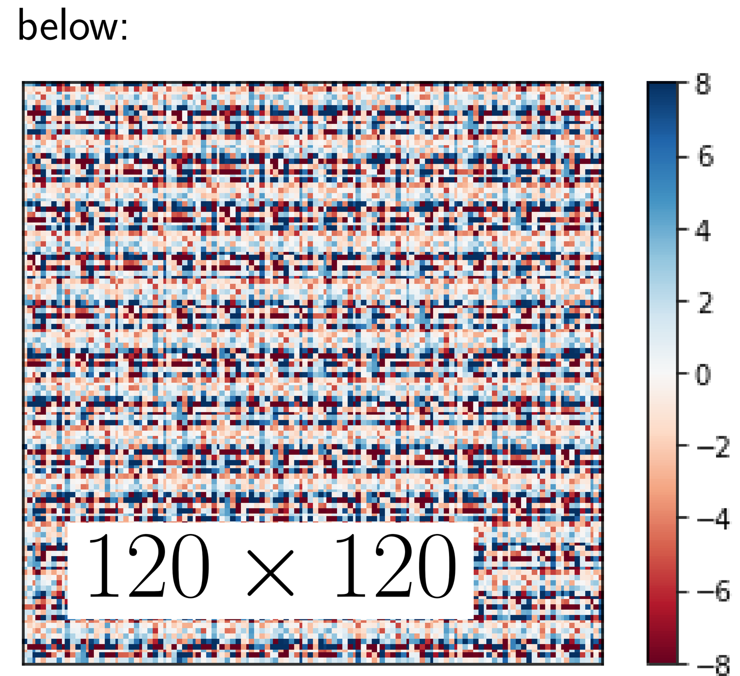
Since the system is $Z_2 \otimes D_3$-equivariant, we can obtain a decomposition of the function space $\mathcal{F}$ into 8 subspaces $\mathcal{F}_i$, s.t. $K\mathcal{F}_i \subset \mathcal{F}_i$.

Example: Z_2 has 2 irresucible representations: the trivial representation $R_{tr}(e) = 1$ and $R_{tr}(r) = 1$ and the sign representation $R_s(e) = 1$ and $R_s(r) = -1$. The isotypic components in function space corresponding to Z_2 generated by the sign flip are the subspaces of even and odd functions, and $\mathcal{F} = \mathcal{F}_{even} \bigoplus \mathcal{F}_{odd}$

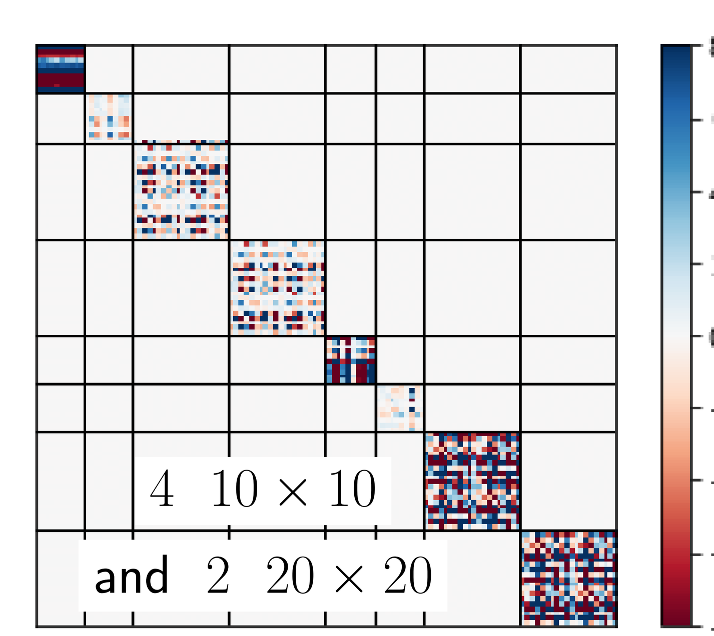
We obtain trajectories to use in EDMD. Given a trajectory we can form a set of trajectories that respects the symmetry of the system (example to the right). We also form a dictionary that respects the symmetries of the system.



Below is an example illustrating the form of K for a network of Duffing oscillators with $Z_2 \otimes D_3$ symmetry. We obtain K using the EDMD from a basis of 120 radial basis functions and data from 500 trajectories. The resulting matrices K are plotted below:



(a) K for a standard dictionary of observables



(b) K for a symmetry adapted dictionary of observables

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EDMD for systems with symmetries

Extended Dynamic Mode Decomposition (EDMD) is a method of approximating the Koopman operator defined as follows:

$$K = G^+ A, \quad G = \sum_m \Psi(x_m)^* \Psi(x_m), \quad A = \sum_m \Psi(x_m)^* \Psi(y_m)$$

* denotes the complex conjugate transpose, $^+$ denotes the Moore-Penrose pseudoinverse

The approximation method requires:

- A dictionary of observables $\mathcal{D} = \{\psi_1, ..., \psi_N\}$
- A set of snapshot pairs $\{(x_i, y_i)\}$ where $y_i = g(x_i)$

Standard EDMD

- Pick a dictionary of N observables
- Evaluate the observables at data points x_i and y_i
- Evaluate the entries of G, A: N^2 elements
- Obtain G^+, $K = G^+ A$, find the eigendecomposition of K: $N \times N$ matrices

For the symmetries of the system to be preserved by the approximation we require:

- The dictionary $\mathcal{D}_\Psi$ respects the symmetries of the system, i.e. if $\psi \in \text{span}(\mathcal{D}_\Psi)$, $\gamma_\rho \psi \in \text{span}(\mathcal{D}_\Psi)$ for any $\gamma \in \Gamma$
- The set of snapshot pairs $\{(x_i, y_i)\}$ respects the symmetries of the system, i.e. $\{(\gamma_\rho x_i, \gamma_\rho y_i)\} = \{(x_i, y_i)\}$ for any $\gamma \in \Gamma$

We show that the Koopman operator approximation K commutes with Γ_ρ if the requirements above are satisfied. That means if $\psi_p \in \mathcal{F}_p$, then $\mathcal{K}\psi_p \in \mathcal{F}_p$. Suppose we start with a dictionary $\mathcal{D}_\Psi$. We can form a symmetry adapted dictionary by projecting every basis function along isotypic components:

$$\psi_{qr,p}(x) = \frac{1}{c_{rp}} \mathcal{P}^p_{qr} \circ \psi(x), \text{ where } \mathcal{P}^p_{qr} = \frac{d_p}{|\Gamma|} \sum_{\gamma \in \Gamma} [R_p(\gamma)]^*_{qr} \gamma_\rho$$

This basis is advantageous since it block diagonalizes K into $\sum_p d_p$ blocks, where each d_p dimensional representation results in d_p identical blocks, which, for instance, simplifies computation of K. We compare the standard EDMD algorithm to EDMD in symmetry adapted basis below.

EDMD for Γ-equivariant systems

- Pick a dictionary of N observables
- Identify the symmetries Γ of the system, find the irreducible representations of Γ
- Change the basis to a Γ-symmetric basis: multiplying at most $N/|\Gamma|$ $|\Gamma| \times |\Gamma|$ matrices by vectors $|\Gamma| \times 1$. Let N_p be the number of functions obtained from applying a projection operator $\mathcal{P}^p_{qr}$ corresponding to the q^{th} row of the p^{th} irreducible representation of Γ.
- Evaluate the observables at data points x_i and y_i, add trajectories to reflect the symmetries if necessary
- To obtain the blocks K_{pq} of K (each isotypic component corresponds to d_p blocks), for each p:
  - Evaluate the entries of G_p, A_p: N_p^2 elements
  - Obtain G^+, $K = G^+ A$, find the eigendecomposition of K: $N_p \times N_p$ matrices
- $K = \bigoplus_p \bigoplus_{q=1}^{d_p} K_p$. Its eigenvalues are the eigenvalues of K_p, and its eigenvectors only have N_p nonzero elements. Mathematically, eigenvectors v_{kl} of K are of the form: $(v_{kl})_i = \bigoplus_p \delta_{pk} v_{pl}$.

[3] Williams, M. O., Kevrekidis, I. G., & Rowley, C. W. (2015). A data driven approximation of the Koopman operator: Extending dynamic mode decomposition. Journal of Nonlinear Science, 25(6), 1307-1346.

[4] Stiefel, E., & Fässler, A. (2012). Group theoretical methods and their applications. Springer Science & Business Media.