Facultad
de
Ciencias

Campos de Killing

Trabajo de Fin de Grado
para acceder al

GRADO EN FÍSICA

Autor: Gustavo Gancedo Crespo

Director: Diego Herranz Muñoz

Abstract

The purpose of this thesis is to make a rigorous and approachable introduction to Killing fields and their applications to any undergrad that is or has studied a general relativity course. Applications range from direct treatment of geodesics in usual metrics like the Schwarzschild, Kerr or FLRW metrics to applications to optics as an effective theory passing through the more geometrical sides of general relativity and maximally symmetric spaces.

Additionally, as part of the agreement with Brown university, there is a dedicated section talking about the internship results on the study of a certain quantum inflationary model and the validity and asymptotic behaviors of the model.

Contents

1 Motivation and introduction to symmetry in physics

The cornerstone of modern physics is, without a shadow of a doubt, symmetry. This is thanks to one of the most beautiful theorems of physics. **The Noether theorem**.

The Noether theorem establishes a one to one connection between symmetries and conservation laws, thus explaining the origin of conservation of energy, charge and all other conservation laws widely used in physics.

As an example, in classical mechanics, the action is defined as the integral of the lagrangian

$$S[q^i(t)] = \int \mathrm{d}t\, L(t, q^i, \dot{q}^i) \tag{1.1}$$

and equations of motion are obtained by enforcing that the action is constant for any perturbations to the trajectory that don't change the boundary conditions

$$\delta S[q^i_s, \delta q^i] = S[q^i_s + \delta q^i] - S[q^i_s] = 0 \tag{1.2}$$

on the other hand, a symmetry is defined as an infinitesimal transformation, meaning it can be seen as a perturbation to a trajectory, that regardless of the path (even for non physical paths) the action only changes by boundary terms

$$\delta S[q^i, \delta_s q^i] = \int \mathrm{d}t\, \frac{\mathrm{d}K}{\mathrm{d}t} \tag{1.3}$$

Noether's theorem allows to obtain a conserved quantity for every symmetry of a system defined as

$$Q = K - \sum_i \frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \tag{1.4}$$

In our case we are going to work with general relativity. A widely different theory from classical mechanics. The first question that might arise is, What constitutes a symmetry? How do you define it in this context? A symmetry is a transformation, such as translations, rotations or even of the internal degrees of freedom a theory might have, that maintains some aspect of the theory invariant.

In the case of classical mechanics the main object that encapsulates the behavior of the system allowing for the computation of the equations of motion. Therefore a symmetry in classical mechanics is any transformation to the lagrangian that doesn't change the equations of motion.

In our case we will find that the corresponding symmetries en general relativity are those that preserve the geometry of spacetime, meaning, the metric. The so called isometries. Killing fields are nothing more than the generators these transformations.

2 Formalism of general relativity

In order to understand symmetry and motivate the definition of Killing fields first it is required to understand is, in the mathematical sense, spacetime and define flows and Lie derivatives.

To do this we will introduce little by little mathematical structure based on the qualities that a spacetime should have

2.1. Spacetimes

A spacetime in the formalism of general relativity is defined as a pseudo-Riemannian manifold. We will start by understanding this ideas.

2.2. Continuity

First of all, a spacetime has a notion that it is continuous, further than that, it is path connected, meaning one can connect any point to any other point by a continuous path[1].

The notion of continuity is defined in the mathematical field of topology A topological space is a pair of sets (M, τ), the first of these is the set of all the points in the space, the second is called the topology of the space and represents all of the open sets. The core idea behind having a topology is introducing a notion of 'closeness' without introducing a metric, in our case there will be an additional notion of closeness defined because of the metric but this idea has to be introduced later. Any topology obeys the following relations of closure

$$\begin{gathered}\emptyset, M \in \tau \\ x_i \in \tau \Rightarrow \bigcup_{i=0}^{\infty} x_i \in \tau \\ x_i \in \tau \Rightarrow \bigcap_{i=0}^{n} x_i \in \tau\end{gathered} \tag{2.1}$$

This allows to define what a continuous function is, the idea of continuity is that any two 'close' points in the input of the function will be 'close' in the output. On topological spaces the definition is related to how open sets transform, here a function between topological spaces $f : (M, \tau_M) \to (N, \tau_N)$ is continuous if $\forall V \in \tau_N, f^{-1}(V) \in \tau_M$ meaning all open sets in the output are open sets in the input. This definition is inspired by the $\varepsilon - \delta$ definition usually defined for metric spaces[2], in fact if one uses the topology defined by the open balls (sets of points closer than some distance) the definitions are equivalent.

2.3. Coordinates

Whenever one talks about any kind of state in physics it is talked about in a coordinate system. It would be expected that in spacetimes one can do the same thing and label the points in spacetime. This is covered in the mathematical field of manifolds. A manifold is a topological space that additionally can

[1] This path is not required to be physical, it could be superluminic.

[2] Spaces with the notion of distance

be locally mapped to a cartesian coordinate system, meaning for any open set V there is a continuous bijection φ from V to $\mathbb{R}^n$ such that φ^{-1} is continuous too.

Additionally it is required that for any two mappings $\varphi_1 : V_1 \to \mathbb{R}^n$ and $\varphi_2 : V_2 \to \mathbb{R}^n$ such that $V_1 \cap V_2 \neq \emptyset$ there has to be a function from $\psi : \varphi_1(V_1 \cap V_2) \to \varphi_2(V_1 \cap V_2)$ that is a bijection, continuous and has a continuous inverse. This means that one can 'translate' one coordinate system to another if they map the same region.

In the case of physics it is additionally required that ψ is infinitely differentiable, this is the definition for smooth manifolds. This is necessary because otherwise a smooth function would be smooth on one coordinate system but it would not be smooth on a different coordinate system because of the chain rule.

The set of all coordinate systems with a smooth coordinate change is called an atlas or $\mathcal{A}$.

Another representation for a coordinate system is a collection of n functions $x^\mu : \mathbb{E} \longrightarrow \mathbb{R}$ such that x^0 gives the 0-th component of a coordinate system φ, x^1 the first component and so on. This representation is more common in physics and will be widely used in this thesis.

2.4. Fields on the spacetime

Now it is time to start talking about what can we 'place' on spacetime.

Scalar fields on spacetimes

A scalar field assigns a number to each point of our spacetime $\mathbb{E}$. So it will be any function of the form

$$\phi : \mathbb{E} \longrightarrow \mathbb{R} \tag{2.2}$$

This function can be 'placed' in a coordinate system by defining $\phi_\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by taking a coordinate system from the atlas $\varphi \in \mathcal{A}$ and applying the inverse to the input $\phi_\varphi = \phi \circ \varphi$. From now on ϕ_φ will be denoted just ϕ whenever the coordinate system is clear.

The set of all infinitely differentiable scalar fields on a manifold will be denoted $\mathcal{C}^\infty(M)$

Parametric curves

If one wishes to keep track of the path of a particle on a spacetime one would naturally use this kind of object. A parametric curve may be defined as a function

$$\gamma : \mathbb{R} \longrightarrow \mathbb{E} \tag{2.3}$$

Again this path can be represented in a coordinate system by composing it with a map $\varphi \in \mathcal{A}$, $\gamma_\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n, \gamma_\varphi = \varphi \circ \gamma$.

Vector fields on the spacetime

Motivated from the 'classical' version of a vector field defined as $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ it might be tempting to define $X : \mathbb{E} \longrightarrow \mathbb{R}^n$ as a vector field on an n dimensional smooth manifold. This definition has one big problem, it is not coordinate independent.

Imagine one has a 3-dimensional manifold with a coordinate system $\{x^\mu\}$ and a primed coordinate system $\{x'^\mu\}$ such that $x'^0 = x^1, x'^1 = x^0, x'^2 = x^2$. Now lets define a constant vector field on the 'x' direction $X(p) = (1, 0, 0)$. In the x^μ coordinate system this field points in the x^0 direction while on the x'^μ coordinate system points in the x'^0 direction, this would correspond to the x^0 direction by the coordinate transformations defined. Therefore this definition of a vector field is not independent of coordinate choice.

There are two equivalent definitions for vectors on a manifold at a point p that are coordinate independent.

The first of these is in terms of tangent vectors of curves, since a curve on the manifold is defined independently of the coordinate system it would be expected that the tangent vector is coordinate independent too. In this way the set of all vectors at a point $p \in \mathbb{E}$ is defined as the set of curves γ such that $\gamma(0) = p$. Here we will have to add an equivalence relation, similarly to how rational numbers are not all of the fractions but the fractions with the fact that two fractions are equal when they follow the relation $\frac{a}{b} \sim \frac{c}{d} \Leftrightarrow ad = bc$ here two of our vectors will be 'equal' if for any $\varphi \in \mathcal{A}$

$$\gamma \sim \hat{\gamma} \Leftrightarrow (\varphi \circ \gamma)' = (\varphi \circ \hat{\gamma})' \tag{2.4}$$

where $'$ is the usual derivative.

The second definitions is via derivations. A derivation at a point p is defined as a linear functional

$$D : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R} \tag{2.5}$$

that also obeys the product rule

$$\begin{gathered} f, g \in \mathcal{C}^\infty \\ D(f \cdot g) = f(p)D(g) + D(f)g(p) \end{gathered} \tag{2.6}$$

Any curve can be assigned a derivation via the following definition

$$D_\gamma f = (f \circ \gamma)'(0) \tag{2.7}$$

The equivalence of definitions may be proben by first proving both spaces have the same dimension. After that Eq. (2.7) gives a one to one correspondence on both spaces. When given a coordinate system the space of derivations has a basis defined by

$$\partial_\mu(p) = \frac{\partial}{\partial x^\mu}(p) \tag{2.8}$$

Where (p) denotes evaluation of the partial derivative at p

With any of the two definitions the vector space of all vectors at a point p of a manifold M is denoted T_pM.

By defining the set of all vectors tangent to the manifold $TM = \bigcup_{p \in M} T_pM$ a vector field may be defined as

$$\begin{aligned} X : M &\longrightarrow TM \\ p &\longrightarrow X(p) \in T_pM \end{aligned} \tag{2.9}$$

When given a coordinate system a vector field may be written as

$$X(p) = X^\mu(p)\frac{\partial}{\partial x^\mu}(p) \tag{2.10}$$

So a **smooth vector field** is defined as a vector field whose component functions, X^μ, are smooth. The set of all smooth vector fields is denoted as $\mathfrak{X}(M)$

For some proofs the notation $X(p, f) = X(f)(p) = X^\mu(p)\frac{\partial f}{\partial x^\mu}(p)$ will be useful

Covectors

It is easy now to define covectors. A covector at a point p is defined as a linear function

$$\omega : T_p M \longrightarrow \mathbb{R} \tag{2.11}$$

so the cotangent space $T_p^* M$ is the space of all covectors at a point p and the set of all covectors $T^* M = \bigcup_{p \in M} T_p^* M$ a covector field is

$$\begin{aligned} \omega : M &\longrightarrow T^* M \\ p &\longrightarrow \omega(p) \in T_p^* M \end{aligned} \tag{2.12}$$

for any basis ∂_μ the canonical basis for the covector space can be defined as a covector collection such that $\mathrm{d}x^\mu(\partial_\nu) = \delta_\nu^\mu$ where δ_ν^μ is the Kronecker delta.

A covector will be smooth if for a coordinate system the covector has components ω_μ defined by

$$\omega = \omega_\mu(p)\, \mathrm{d}x^\mu \tag{2.13}$$

are $\mathcal{C}^\infty(M)$ functions

Again the notation

$$\omega(p, X) = \omega_\mu(p)\, \mathrm{d}x^\mu(X)(p) \tag{2.14}$$

will be useful

Tensors

A tensor represents a multilinear map, meaning that for any input slot

$$T(a, b, ..., \alpha c + \beta d, ..., z) = \alpha T(a, b, ..., c, ..., z) + \beta T(a, b, ..., d, ..., z) \tag{2.15}$$

The most basic definition of a tensor one can come up with is

$$T : V_1 \times V_2 \times ... \times V_n \to \mathbb{R} \tag{2.16}$$

This is a tensor that takes n vectors as input and as output gives a number

It could be also output more vectors defining

$$\hat{T} : V_1 \times V_2 \times ... \times V_n \longrightarrow V_{n+1} \tag{2.17}$$

however by evaluating the output of $\hat{T}$ with a covector the result is a number representing some component. So it is common to represent this kind of tensors by

$$\hat{T} : V_1 \times V_2 \times ... \times V_n \times V_{n+1}^* \longrightarrow \mathbb{R} \tag{2.18}$$

Therefore the definition of a tensor over a vector space V of kind (q, p) or q times contravariant, p times covariant is defined as

$$T : \underbrace{V^* \times ... \times V^*}_{q \text{ copies}} \times \underbrace{V \times ... \times V}_{p \text{ copies}} \longrightarrow \mathbb{R} \tag{2.19}$$

In our case the corresponding vector spaces are the $T_p M$ and a tensor field will be a map

$$T : \underbrace{T^* M \times ... \times T^* M}_{q \text{ copies}} \times \underbrace{TM \times ... \times TM}_{p \text{ copies}} \longrightarrow \mathcal{C}^\infty(M) \tag{2.20}$$

The components of a tensor can be obtained by feeding it some vectors and applying Eq. (2.10) and Eq. (2.13)

$$T(\omega, ..., X, ...) = T\big(\omega_\mu\, \mathrm{d}x^\mu, ..., X^\nu \partial_\nu, ...\big) = \omega_\mu X^\nu ... T(\mathrm{d}x^\mu, ..., \partial_\nu, ...) =: \omega_\mu X^\nu ... T^{\mu ...}_{\nu ...} \tag{2.21}$$

So a tensor field is called smooth if the component functions $T^{\mu\dots}_{\nu\dots}$ are $\mathcal{C}^\infty(M)$

Another notation that will be useful is

$$T(p,\omega,\dots,X,\dots) = \omega_{\mu_1}(p)\dots X^{\nu_1}(p)\dots T^{\mu_1\dots}_{\nu_1\dots}(p) \tag{2.22}$$

2.5. Metrics

The last piece for constructing a spacetime is adding a notion of magnitude to our vectors and distance. This is constructed by adding a tensor field to the spacetime Manifold which we will call the metic.

The metric defines a dot product between vectors

$$X \cdot Y = g(X,Y) = X^\mu Y^\nu g_{\mu\nu} \tag{2.23}$$

also allowing to lower the indices of vectors and tensors by contracting with the metric

$$X_\mu = g_{\mu\nu} X^\nu \tag{2.24}$$

since we would like to be able to invert the relation it is defined $g^{\mu\nu}$ such that

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \tag{2.25}$$

so $X^\mu = g^{\mu\nu} X_\nu$

A Manifold equipped with a metric is called Riemannian if a metric can be diagonalized with all positive eigenvalues and pseudo-Riemannian if it can have both positive and negative.

In general relativity the equivalence principle can be stated in terms of the metric so that for any point there is a coordinate system such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2) \tag{2.26}$$

where η is the Minkowski metric.

This allows to define a distance[1] function between two points of the manifold by denoting $\Gamma(p,q)$ the set of all curves starting at p and ending at q

$$d(p,q) = \min_{\gamma\in\Gamma(p,q)} \int_\gamma (g(\gamma'(\tau),\gamma'(\tau)))\,\mathrm{d}\tau \tag{2.27}$$

where γ' is the tangent vector to γ

[1]This will only be a distance function whenever the manifold is Riemannian, for pseudo-Riemannian it is not since it can be negative, in general relativity the sign will be a way to encode when a path moves in the 'time direction', in the 'space direction' or like light.

3 Transformations of manifolds

In order to study the symmetries of our spacetime one has to first understand how to make a transformation.

First will start by defining a smooth transformation between manifolds M, N as a function $F : M \to N$ such that for any coordinate system of M, φ, that contains p, and any coordinate system of N, φ', that contains $F(p)$, the function $\varphi' \circ F \circ \varphi^{-1}$ is smooth.

A **diffeomorphism** is a smooth map that is also bijective and with a smooth inverse. Any pair of manifolds that have a diffeomorphism relating them will be called diffeomorphic manifolds.

Diffeomorphic manifolds are equivalent in the sense that any field, may it be scalar, vectorial or tensorial defined on one of the manifolds. Has an equivalent definition on the other. The operations that map a field on one of the manifolds to the other are called pullback and pushforward.

3.1. Pullback and pushforward of scalar fields

Given a function $F : M \to N$. A pullback will map fields defined on N to fields defined on M. The simplest case is for scalar fields. The pullback of a scalar field $f \in \mathcal{C}^\infty(N)$ is defined as

$$F^* f = f \circ F \tag{3.1}$$

so that F maps points of M to N and then f maps it to $\mathbb{R}$ so the complete map is $M \to \mathbb{R}$.

The pushforward is the opposite transformation to the pullback, mapping fields from M to N. In the case of diffeomorphisms it can be defined as the pullback by the inverse function. So if one has a function $f \in \mathcal{C}^\infty(M)$ the pushforward by F is defined as

$$F_* f = \left(F^{-1}\right)^* f = f \circ F^{-1} \tag{3.2}$$

The motivation behind this definition is that, if one pushes forward a function and then pulls it back, it would be reasonable for the function to remain unchanged therefore $F^* F_* f = f$

It is easy to see that the pullback and pushforward are linear since composition is linear so that

$$\begin{aligned} F_*(\alpha f + \beta g) &= \alpha F_* f + \beta F_* g \\ F^*(\alpha f + \beta g) &= \alpha F^* f + \beta F^* g \end{aligned} \tag{3.3}$$

Under this definitions diffeomorphisms may be thought rather than as mappings between manifolds, as coordinate changes, since for any coordinate system on N, x'^μ, it can be thought of as a coordinate system on M defined as $x^\mu = F^* x'^\mu$. This will come up later in the chapter in the notion of passive vs active transformations.

3.2. Pullback and pushforward of vector fields

Since a vector field was defined as a collection of derivations, it may be thought of as a function $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, that maps a function to the directional derivative of the function at that point.

The pushforward of vector fields may be thought of as first pulling back the vector field the corresponding function then pushing forward the result, for clarity $X(p, f)$ denotes evaluating $X(f)$ at p, $p \in M$, $\hat{P} \in N$ $g \in \mathcal{C}^\infty(N)$

$$(F_* X)(g) = X(F^{-1}(\hat{p}), F^* g) \tag{3.4}$$

The pullback is defined as the pushforward by the inverse function

$$F^* X = \left(F^{-1}\right)_* X \tag{3.5}$$

Again pushforward and pullback of vector fields is linear

$$\begin{aligned} F_*(\alpha X + \beta Y)(f) &= (\alpha X + \beta Y)(f \circ F) = \alpha X(f \circ F) + \beta Y(f \circ F) \\ &= \alpha F_*(X)(f) + \beta F_*(Y)(f) \end{aligned} \tag{3.6}$$

Also by defining multiplication of vector fields and scalar fields fX as

$$(fX)(p, g) = f(p) X(p, g) \tag{3.7}$$

the pushforward of this composition is linear in the following sense

$$\begin{aligned} F_*(fX)(\hat{p}, g) &= (f, X)(F^{-1}(\hat{p}), F^* g) = f(F^{-1}(\hat{p})) X(F^{-1}(\hat{p}), F^* g) \\ &= (F_* f)(\hat{p}) \cdot (F_* X)(\hat{p}, g) \end{aligned} \tag{3.8}$$

where $\cdot$ denotes the product of real numbers

equivalently for the pullback

$$F^*(fX) = (F^* f)(F^* X) \tag{3.9}$$

This equations are coordinate independent, however for computations it is easier to obtain the transformations by coordinate systems, in order to obtain the coordinate transformation we will write the coordinate system of M as x^μ and the coordinate system of N obtained as the pushforward of x^μ, x'^μ

Now our vector field $X \in \mathfrak{X}(M)$ can be written

$$X = X^\mu(x^\mu) \frac{\partial}{\partial x^\mu} \tag{3.10}$$

by defining a vector field on N, X' as the pushforward of X

$$X' = F_* X = X'(x'^\mu) \frac{\partial}{\partial x'^\mu} \tag{3.11}$$

Since $x'^\mu = F(x^\mu)$ for a function in $f \in \mathcal{C}^\infty(N)$ and defining $p \in M, \hat{p} \in N$ so that $\hat{p} = F(p)$

$$\begin{aligned} F_* X(\hat{p}, f) &= F_* \left(X^\mu \frac{\partial}{\partial x^\mu} \right)(\hat{p}, f) = X^\mu(F^{-1}(\hat{p})) \frac{\partial f \circ F}{\partial x^\mu}(F^{-1}(\hat{p})) \\ &= X^\mu(p) \frac{\partial f \circ F}{\partial x^\mu}(p) = X^\mu(p) \frac{\partial f}{\partial x'^\nu}(\hat{p}) \frac{\partial x'^\nu}{\partial x^\mu}(p) \end{aligned} \tag{3.12}$$

Therefore by comparing the Eq. (3.11) and Eq. (3.12) the resulting transformation on a coordinate system is

$$X'^\mu(\hat{p}) = \frac{\partial x'^\mu}{\partial x^\nu}(F^{-1}(\hat{p})) X^\nu(F^{-1}(\hat{p})) \tag{3.13}$$

<

This equation might seem purely mathematical but it explains the physical transformations that we will find. These have a translation component, encoded on the term of $X^\nu(F^{-1}(\hat{p}))$, that because of the F^{-1} term shifts the position of the $X^{\mu(p)}$ vector. The other component are rotations, or expansions, encoded on the $\frac{\partial x'^\mu}{\partial x^\nu}(F^{-1}(\hat{p}))$, this is because this term mixes the components and allows for changing the direction of the vector or length of the vector.

For the pullback the result is equivalent by changing $x'^\mu \to x^\mu$, $x^\mu \to x'^\mu$ $\hat{p} \to p$ and $F \to F^{-1}$

so

$$X'^\mu(p) = \frac{\partial x^\mu}{\partial x'^\nu}(F(p))X^\nu(F(p)) \tag{3.14}$$

3.3. Pullback and pushforward of covector fields

Just as we did with vector fields, covector fields map vector fields to scalar fields the definitions and results are equivalent so for a covector ω

$$F_*\omega(\hat{p}, X) = \omega(F^{-1}(\hat{p}), F^*X) \tag{3.15}$$

and for the pullback

$$F^*\omega(p, X) = \omega(F(p), F_*X) \tag{3.16}$$

Again these are linear over addition of covectors and products by real numbers, and by defining the product of covectors by

$$(f\omega)(p, X) = f(p)\omega(p, X) \tag{3.17}$$

the pushforward is 'linear' over these in the sense that

$$\begin{aligned} F_*(f\omega)(\hat{p}, X) &= (f\omega)(F^{-1}(\hat{p}), F^*X) = f(F^{-1}(\hat{p}))\omega(F^{-1}(\hat{p}), F^*X) \\ &= (F_*f)(\hat{p})(F_*\omega)(\hat{p}, X) \end{aligned} \tag{3.18}$$

and equivalently for the pullback

$$F^*(f\omega) = (F^*f)(F^*\omega) \tag{3.19}$$

When given a coordinate system for M and N, x^μ and x'^μ respectively, then the covector on M may be written as

$$\omega = \omega_\mu(p)\,\mathrm{d}x^\mu \tag{3.20}$$

and the pushforward

$$\omega' = F_*\omega = \omega_\mu(\hat{p})\,\mathrm{d}x'^\mu \tag{3.21}$$

so by applying the definition of pushforward of a covector field, Eq. (3.15), one finds by setting $X' \in \mathfrak{X}(N)$

$$\begin{aligned} F_*\omega(\hat{p}, X') &= F_*\big(\omega_\mu\,\mathrm{d}x^\mu\big)(\hat{p}, X') = F_*\big(\omega_\mu\big)(\hat{p})F_*(\mathrm{d}x^\mu)(\hat{p}, X') = \omega_\mu(p)\,\mathrm{d}x^\mu(p, F^*X') \\ &= \omega_\mu(p)\,\mathrm{d}x^\mu\left(p, X'^\nu(\hat{p})\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\right) = \omega_\mu(p)X'^\nu(\hat{p})\frac{\partial x^\alpha}{\partial x'^\nu}\underbrace{\mathrm{d}x^\mu\left(p, \frac{\partial}{\partial x^\alpha}\right)}_{\delta^\mu_\alpha} \\ &= \omega_\mu(p)X'^\nu(\hat{p})\frac{\partial x^\alpha}{\partial x'^\nu}(\hat{p})\delta^\mu_\alpha = \frac{\partial x^\mu}{\partial x'^\nu}(\hat{p})\omega_\mu(p)X'^\nu(\hat{p}) \end{aligned} \tag{3.22}$$

now, by feeding the same input to the Eq. (3.21) one obtains

$$\begin{aligned}\omega'(\hat{p}, X') &= \omega'_\mu(\hat{p})\,\mathrm{d}x'^\mu(X') = \omega'_\mu(\hat{p})X'^\nu(\hat{p})\,\mathrm{d}x'^\mu\left(\frac{\partial}{\partial x'^\nu}\right) \\ &= \omega'_\mu(\hat{p})X'^\nu(\hat{p})\delta^\mu_\nu = \omega'_\mu(\hat{p})X'^\nu(\hat{p})\end{aligned} \tag{3.23}$$

by comparing Eq. (3.22) and Eq. (3.23)

one obtains

$$\omega'_\mu(\hat{p}) = \frac{\partial x^\nu}{\partial x'^\mu}(\hat{p})\omega_\nu(F^{-1}(\hat{p})) \tag{3.24}$$

The equivalent reasoning for the pullback gives

$$\omega_\mu(p) = \frac{\partial x'^\nu}{\partial x^\mu}(p)\omega'_\nu(F(p)) \tag{3.25}$$

Again here one can identify a translation and a rotation or expansion term, however here the rotation is inverted.

3.4. Tensor pullbacks and pushforwards

The pushforward of a tensor field, just as we did before with vectors and covectors is defined by pulling back the vector and covector fields and then pushing forward the results

$$F_* T(p, \omega, ..., X, ...) = T(F^{-1}(p), F^*\omega, ..., F^*X, ...) \tag{3.26}$$

and equivalently for the pullback

$$F^* T(p, \omega, ..., X, ...) = T(F(p), F_*\omega, ..., F_*X, ...) \tag{3.27}$$

Just as proven with the method in the Eq. (3.22) it can be proben that if T is a tensor in M and T' is the pushforward on N, and by choosing a coordinate system x^μ on M and the pushforward of this coordinate system to N, x'^μ, one obtains the relationship between the coordinate systems of both as

$$T'^{\nu_1 \cdots}_{\mu_1 \cdots}(\hat{p}) = \frac{\partial x'^{\nu_1}}{\partial x^{\alpha_1}}(F^{-1}(\hat{p}))...\frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}}(\hat{p})T^{\alpha_1 \cdots}_{\beta_1 \cdots}(F^{-1}(\hat{p})) \tag{3.28}$$

3.5. Isometries

An isometry, is a diffeomorphism between Riemannian or pseudo-Riemannian manifolds, $F : M \to N$, where g_M is the metric on M and g_N is the metric on N then F is an isometry if

$$g_N = F_* g_M \tag{3.29}$$

thus preserving the metric.

Any object that only depends on the metric is called **intrinsic** and is preserved under isometries in the same sense that the metric is preserved.

A few examples are:
- The Levi-Civita connection (∇_μ)
- The Riemann tensor (${R^\mu}_{\nu\gamma\sigma}$)
- The length of a curve ($\int_\gamma \sqrt{g_{\mu\nu}\gamma'^\mu\gamma'^\nu}\,\mathrm{d}\tau$)

4 Flows

A flow, intuitively, is described as the movement of a liquid or a gas that at each point moves in one particular direction.

Mathematically this can be described by a velocity field, that describes the movement of the fluid.

This might not seem relevant to the study of transformations in general relativity, however this concept is the definition we will use to build all of the transformations.

First we will start by defining a flow as a curve that solves the following differential equation

$$\begin{cases} \frac{\partial \phi}{\partial \tau}(\tau, x_0) = V(\phi(\tau, x_0)) \\ \phi(0, x_0) = x_0 \end{cases} \tag{4.1}$$

where V is the velocity field and ϕ is a curve on the manifold. There are a few interesting properties of flows that will be important later.

First of all, since V is a smooth vector field, the solutions to $\phi(\tau, x_0)$ are unique, this also means that for any fixed τ the transformation $\phi_\tau : M \to M$ defined as $\phi_\tau(p) = \phi(\tau, p)$ is a diffeomorphism since the solutions are unique and since the function is differentiable with respect to τ it has to be smooth.

Another property that flow has is that these are defined except by a constant translation on the parameter τ. Meaning if $\phi(\tau, x_0)$ is a flow of a field V then $\phi(\tau + s, x_0)$ is also a flow of the field V. Unless stated otherwise the convention we will take is such that

$$\phi(0, x_0) = x_0 \tag{4.2}$$

There is an interesting property of flows that can be stated as follows

$$\phi(t + s, x_0) = \phi(t, \phi(s, x_0)) \tag{4.3}$$

this is easy to check since by uniqueness both $\phi(t + s, x_0)$ and $\phi(t, p_0)$ where $p_0 = \phi(s, x_0)$ solve the same initial value problem therefore the equality is true. As a lemma we have that the inverse diffeomorphism $\phi_\tau^{-1}(p)$ is equivalent to $\phi_{-\tau}(p)$

5 Lie derivatives

Finally after all of the mathematical conundrum we are finally ready to define the Lie derivative. The Lie derivative is an object that takes in a vector field V and some geometrical object, such as scalar fields, vector fields or tensor fields. And what the Lie derivative represents is, if someone following the flow of V, made a function of how they see these objects change as a function of τ and then took a derivative at $\tau = 0$, what would be the value of that derivative.

This intuitive image of what a Lie derivative wants to answer can be stated in two equivalent ways, and active way and a passive way. The active way models the path of the observer by keeping coordinates stationary and transforming the fields so that $\varphi \to \varphi'$ and the Lie derivative would be something like $(\varphi' - \varphi)/\varepsilon$. The passive approach transforms the coordinates so that the fields stay still and the observer is the one moving from this coordinate system. In this case $x^\mu \to x'^\mu$ and the Lie derivative would take the form of $(\varphi(x'^\mu) - \varphi(x^\mu))/\varepsilon$.

Here we will take the active approach and model the derivative by transforming the fields.

First we will motivate the equation with an example and then will start computing the Lie derivative in component form of multiple kinds of objects.

To obtain a Lie derivative of some field φ with respect to some vector field X, first of all the flow of X is computed obtaining $\phi_\tau(p)$. Now the manifold M is mapped by ϕ_τ to M' which is nothing else than the same manifold but with a different coordinate system. Here the interpretation is not that the coordinate system changed, when we interpret is that the coordinate systems of M and M' are the same but the fields changed. Now φ' is a field on M' so what we can do to compare it with φ is to pull it back to M and by taking the limit as $\tau \to 0$ one obtains the Lie derivative

$$\mathscr{L}_X \varphi = \lim_{\tau \to 0} \frac{\phi_\tau^* \varphi - \varphi}{\tau} = \lim_{\tau \to 0} \frac{\phi_{-\tau *} \varphi - \varphi}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_\tau^* \varphi \tag{5.1}$$

Now it is possible to compute Lie derivatives of several objects in a coordinate system.

By defining x^μ as a coordinate system on M and $x'^\mu = \phi_\tau(x^\mu)$ then it follows that $x'^\mu = x^\mu + X^\mu \tau + \mathcal{O}(\tau^2)$.

5.1. Lie derivative of a scalar field

The Lie derivative of a scalar field is the simplest Lie derivative to compute. This is because by Eq. (3.1) the Lie derivative may be computed as

$$\begin{aligned} \mathscr{L}_X f = \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_\tau^* f \bigg|_{\tau=0} &= \frac{\mathrm{d}}{\mathrm{d}\tau} f(\phi(\tau, x_0)) \bigg|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau} f(x'^\mu) \bigg|_{\tau=0} = \frac{\partial f}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial \tau} \bigg|_{\tau=0} \\ &= \frac{\partial f}{\partial x^\nu} \left(\delta^\nu_\mu + \mathcal{O}(\tau)\right)\left(X^\mu + \mathcal{O}(\tau)\right) \bigg|_{\tau=0} = X^\mu \frac{\partial f}{\partial x^\mu} \end{aligned} \tag{5.2}$$

Here one can see that the Lie derivative of a scalar field with respect to some vector field is nothing else than the directional derivative of f in the direction of X. In general this is not the case, the reason for this is that scalar fields have no sense of 'direction' so they are not affected by rotations, as we will see now this is not the case for vector fields

5.2. Lie derivatives of vector fields

The same procedure as done in Eq. (5.2) to find the Lie derivative of a vector field. However here the corresponding pullback equation is Eq. (3.14)

$$\begin{aligned}(\mathscr{L}_X Y)^\mu &= \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_\tau^* Y\Big|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x^\mu}{\partial x'^\nu}Y^\nu(x')\right)\Big|_{\tau=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)Y^\nu(x')\Big|_{\tau=0} + \frac{\partial x^\mu}{\partial x'^\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}Y^{\nu(x')}\Big|_{\tau=0} \\ &= \left(-\frac{\partial X^\mu}{\partial x'^\nu} + \mathcal{O}(\tau)\right)Y^\nu(x')\Big|_{\tau=0} + (\delta^\mu_\nu + \mathcal{O}(\tau))\frac{\mathrm{d}}{\mathrm{d}\tau}Y^\nu(x')\Big|_{\tau=0} \\ &= -Y^\nu(x)\frac{\partial X^\mu}{\partial x^\nu} + (\delta^\mu_\nu + \mathcal{O}(\tau))\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau}\frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial Y^\nu}{\partial x^\beta}\Big|_{\tau=0} \\ &= -Y^\nu(x)\frac{\partial X^\mu}{\partial x^\nu} + (\delta^\mu_\nu + \mathcal{O}(\tau))X^\alpha\frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial Y^\nu}{\partial x^\beta}\Big|_{\tau=0} \\ &= -Y^\nu(x)\frac{\partial X^\mu}{\partial x^\nu} + X^\alpha\frac{\partial Y^\mu}{\partial x^\alpha}\end{aligned} \tag{5.3}$$

Here there are two terms, the first one is associated with translations. This term is the $X^\alpha\frac{\partial Y^\mu}{\partial x^\alpha}$. This is essentially what was obtained in the scalar field case and represents a directional derivative. The second term, $-Y^\alpha\frac{\partial X^\mu}{\partial x^\alpha}$, represents rotations and dilations. This is because it 'mixes' components of the vector Y and allows for changing the norm of the vector.

This means that for any two vector fields X, Y their Lie derivative applied to a scalar field is

$$\mathscr{L}_X Y(f) = X(Y(f)) - Y(X(f)) = [X,Y](f) \tag{5.4}$$

5.3. Lie derivatives of tensor fields

Once again, by using these methods one finds the general equation

$$\begin{aligned}(\mathscr{L}_X T)^{\alpha_1\cdots}{}_{\beta_1\cdots} = X^\alpha\frac{\partial T^{\alpha_1\cdots}{}_{\beta_1\cdots}}{\partial x^\alpha} \\ +T^{\alpha_1\cdots\alpha_{i-1}\sigma\alpha_{i+1}\cdots}{}_{\beta_1\cdots}\frac{\partial X_i^\alpha}{\partial x^\sigma} + \ldots \\ -T^{\alpha_1\cdots}{}_{\beta_1\cdots\beta_{i-1}\sigma\beta_{i+1}\cdots}\frac{\partial X^\sigma}{\partial x^{\beta_i}} - \ldots\end{aligned} \tag{5.5}$$

This equation also can be written in the following form

$$(\mathscr{L}_X T)(A,B,\ldots) = \mathscr{L}_X(T(A,B,\ldots)) + T(\mathscr{L}_X A, B,\ldots) + T(A,\mathscr{L}_X B,\ldots) + \ldots \tag{5.6}$$

6 Killing fields

Finally we can write the definition of Killing fields. A Killing field, K, is a vector field such that the flow it generates is an isometry. In physical terms, moving along the 'velocity field' K doesn't change the metric tensor

$$\phi_\tau^* g = g \tag{6.1}$$

that can be written as

$$(\mathscr{L}_K g)_{\mu\nu} = 0 \tag{6.2}$$

In component form the Killing equation may be written in contravariant form as

$$K^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu} \frac{\partial K^\alpha}{\partial x^\mu} + g_{\mu\alpha} \frac{\partial K^\alpha}{\partial x^\nu} = 0 \tag{6.3}$$

Additionally, whenever a symmetric connection is used, such as the Levi-Civita connection used in general relativity, the following covariant from is equivalent

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{6.4}$$

6.1. Killing tensors

By analogy to Eq. (6.4) a Killing tensor is defined as a tensor ${T_{\alpha\beta}}^{\gamma\ldots}$ such that

$$\nabla_{(\mu} T_{\alpha\beta\gamma)} = 0 \tag{6.5}$$

where the parenthesis is the average over all of the permutations of the indices.

6.2. Properties of Killing fields

Killing fields form vector spaces, this is because the Lie derivative is linear on the vector field wrt which it differentiates

$$\mathscr{L}_{\alpha X + \beta Y} g = \alpha\ \mathscr{L}_X g + \beta\ \mathscr{L}_Y g = 0 \tag{6.6}$$

This can easily be seen on the component definitions of Lie derivatives.

Additionally these form a Lie algebra, this is because Lie derivatives have the following property

$$\mathscr{L}_{[X,Y]} T = \mathscr{L}_X\ \mathscr{L}_Y T - \mathscr{L}_Y\ \mathscr{L}_X T \tag{6.7}$$

So that if X and Y are Killing fields, then $[X, Y]$ form a Killing field too.

The norm of a Killing vector field is constant along its own flow, this is easily probed by

$$\mathscr{L}_K K^2 = K^\nu \nabla_\nu (K^\mu K_\mu) = 2K^\nu K^\mu \nabla_\nu K_\mu = K^\nu K^\mu \underbrace{(\nabla_\nu K_\mu + \nabla_\mu K_\nu)}_{\text{Killing equation}} = 0 \tag{6.8}$$

These are also divergenceless tensors. Intuitively one can think of Killing fields as flows of an incompresible fluid. One can prove this by multiplying Eq. (6.4) by the metric so that

$$0 = g^{\mu\nu}\nabla_\mu K_\nu + g^{\mu\nu}\nabla_\mu K_\nu = \nabla_\mu K^\mu + \nabla_\nu K^\nu = 2\nabla_\mu K^\mu \Rightarrow \nabla_\mu K^\mu = 0 \tag{6.9}$$

The main property of Killing fields is that it allows us to build conserved quantities on geodesics, meaning, if $x^\mu(\tau)$ is a geodesic and K^μ a Killing field, then

$$\frac{\mathrm{d}(\dot{x}^\mu K_\mu)}{\mathrm{d}\tau} = \underbrace{\frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\tau}}_{\text{zero by geodesic}} K_\mu + \dot{x}^\mu \frac{\mathrm{d}K_\mu}{\mathrm{d}\tau} = \dot{x}^\mu \dot{x}^\nu \nabla_\nu K_\mu = \frac{1}{2}\dot{x}^\mu \dot{x}^\nu \underbrace{(\nabla_\nu K_\mu + \nabla_\mu K_\nu)}_{\text{Killing equation}} = 0 \tag{6.10}$$

It also allows for defining conserved currents for any divergenceless rank two symmetric tensor because

$$\nabla_\mu(T^{\mu\nu}K_\nu) = \cancel{\nabla_\mu T^{\mu\nu} K_\nu} + T^{\mu\nu}\nabla_\mu K_\nu = \frac{1}{2}T^{\mu\nu}(\nabla_\nu K_\mu + \nabla_\mu K_\nu) = 0 \tag{6.11}$$

as an example of such a tensor the Stress-energy tensor, allowing to define energy and momentum densities in curved spacetimes.

This same idea can be extended to Killing tensors defined by Eq. (6.5) so that if $A_{\alpha\beta\gamma\ldots}$ is a Killing tensor

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(A_{\alpha\beta\gamma\ldots}u^\alpha u^\beta u^\gamma\right) = 0 \tag{6.12}$$

is also a conserved quantities.

Number of Killing fields

A good question now is to ask "How many Killing fields does our space have?", since this will lead to n conserved quantities, simplifying the resulting equations.

As it turns out it is not possible, in general, to know exactly to know how many Killing fields there are without solving the equations, however, it is possible to place an upper bound on the number of Killing fields, and giving an interpretation of these.

To prove this the starting point is the Riemann tensor

$$R^\delta{}_{\alpha\beta\gamma}K_\delta = \nabla_\alpha\nabla_\beta K_\gamma - \nabla_\beta\nabla_\alpha K_\gamma \tag{6.13}$$

and the Bianchi identities

$$R^\delta{}_{\alpha\beta\gamma} + R^\delta{}_{\gamma\alpha\beta} + R^\delta{}_{\beta\gamma\alpha} = 0 \tag{6.14}$$

By multiplying the Bianchi identities by K_δ and applying Eq. (6.13) the result is

$$\begin{gathered}
0 = \nabla_\alpha\nabla_\beta K_\gamma - \nabla_\beta\nabla_\alpha K_\gamma + \nabla_\gamma\nabla_\alpha K_\beta - \nabla_\alpha\nabla_\gamma K_\beta + \nabla_\beta\nabla_\gamma K_\alpha - \nabla_\gamma\nabla_\beta K_\alpha = \\
\nabla_\alpha\nabla_\beta K_\gamma - \nabla_\alpha\nabla_\gamma K_\beta + \nabla_\beta\nabla_\gamma K_\alpha - \nabla_\beta\nabla_\alpha K_\gamma + \nabla_\gamma\nabla_\alpha K_\beta - \nabla_\gamma\nabla_\beta K_\alpha = \\
\nabla_\alpha(\nabla_\beta K_\gamma - \nabla_\gamma K_\beta) + \nabla_\beta(\nabla_\gamma K_\alpha - \nabla_\alpha K_\gamma) + \nabla_\gamma(\nabla_\alpha K_\beta - \nabla_\beta K_\alpha) \underset{\text{Killing equation}}{=} \\
2(\nabla_\alpha\nabla_\beta K_\gamma - \nabla_\gamma\nabla_\beta K_\alpha + \nabla_\gamma\nabla_\alpha K_\beta) = 2\left(R^\delta{}_{\alpha\beta\gamma}K_\delta + \nabla_\gamma\nabla_\alpha K_\beta\right) \Rightarrow \\
\Rightarrow R^\delta{}_{\alpha\beta\gamma}K_\delta = -\nabla_\gamma\nabla_\alpha K_\beta
\end{gathered} \tag{6.15}$$

This allows, by substituting into the Taylor series to obtain an expression of the solution to the Killing field equation.

By using the following multi-index notation

$$
\begin{aligned}
\alpha &= (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n) \in \mathbb{N}^n \\
|\alpha| &= \sum_i \alpha_i \\
D^\alpha f &= \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} ... (\partial x^n)^{\alpha_n}} \\
\alpha! &= \prod_i \alpha_i! \\
x^\alpha &= \prod_i (x^i)^{\alpha_i}
\end{aligned}
\tag{6.16}
$$

The Taylor series in multiple variables is

$$
K_\delta(x) = \sum_{|\alpha|=0}^{\infty} \frac{D^\alpha K_\delta(p)}{\alpha!} (x-p)^\alpha \tag{6.17}
$$

Now, since Eq. (6.15) gives a linear relationship between the second and zeroth order derivatives, one can also obtain the third derivative as a linear combination of the first derivatives and the zeroth order by deriving the equation[1], and so on with higher order derivatives. By defining two linear objects $\hat{A}$ and $\hat{B}$ so that

$$
D^\alpha K_\delta = \hat{A}_\delta{}^\gamma(p, \alpha) K_\gamma(p) + \hat{B}_\delta{}^{\gamma\beta}(p, \alpha) \partial_\gamma K_\beta(p) \tag{6.18}
$$

It is important to note that these objects are, in general, not tensors, since the left hand side is not a tensor.

By substituting into the Taylor series one obtains

$$
\begin{aligned}
K_\delta(x) &= \sum_{|\alpha|\,=0}^{\infty} \frac{D^\alpha K_\delta(p)}{\alpha!} (x-p)^\alpha \\
&= \sum_{|\alpha|\,=0}^{\infty} \frac{1}{\alpha!} (x-p)^\alpha \left(\hat{A}_\delta{}^\gamma(p,\alpha) K_\gamma(p) + \hat{B}_\delta{}^{\gamma\beta}(p,\alpha) \partial_\gamma K_\beta(p) \right) \\
&= \left(\sum_{|\alpha|\,=0}^{\infty} \frac{1}{\alpha!} (x-p)^\alpha \hat{A}_\delta{}^\gamma(p,\alpha) \right) K_\gamma(p) + \left(\sum_{|\alpha|\,=0}^{\infty} \frac{1}{\alpha!} (x-p)^\alpha \hat{B}_\delta{}^{\gamma\beta}(p,\alpha) \right) \partial_\gamma K_\beta(p) \\
&= \tilde{A}_\delta{}^\gamma(x,p) K_\gamma(p) + \tilde{B}_\delta{}^{\gamma\beta}(x,p) \partial_\gamma K_\beta(p)
\end{aligned}
\tag{6.19}
$$

Now, from this expression it looks like there should be $n + n^2$ Killing fields since there are that many free parameters as initial conditions, however there is an additional restriction, the Killing equation.

$$
\partial_\alpha K_\beta = -\partial_\beta K_\alpha + 2\Gamma^\sigma{}_{\alpha\beta} K_\sigma \tag{6.20}
$$

so that

[1] As long as the Riemann tensor is smooth

$$\begin{aligned}
K_\delta(x) &= \hat{A}_\delta{}^\gamma(x,p)K_\gamma(p) + \frac{1}{2}\hat{B}_\delta{}^{\gamma\sigma}(x,p)\big(\partial_\gamma K_\sigma(p) + \partial_\gamma K_\sigma(p)\big) \\
&= \hat{A}_\delta{}^\gamma(x,p)K_\gamma(p) + \frac{1}{2}\hat{B}_\delta{}^{\gamma\sigma}(x,p)\big(\partial_\gamma K_\sigma(p) - \partial_\sigma K_\gamma(p) + 2\Gamma^\alpha{}_{\gamma\sigma}K_\alpha(p)\big) \\
&= \underbrace{\Big(\hat{A}_\delta{}^\alpha(x,p) + \hat{B}_\delta{}^{\gamma\sigma}(x,p)\Gamma^\alpha{}_{\gamma\sigma}\Big)}_{A_\delta{}^\alpha(x,p)}K_\alpha + \underbrace{\frac{1}{2}\hat{B}_\delta{}^{\gamma\sigma}(x,p)}_{B_\delta{}^{\gamma\sigma}}\big(\partial_\gamma K_\sigma(p) - \partial_\sigma K_\gamma(p)\big) \\
&= A_\delta{}^\gamma(x,p)K_\gamma(p) + B_\delta{}^{\gamma\sigma}(x,p)\big(\partial_\gamma K_\sigma(p) - \partial_\sigma K_\gamma(p)\big)
\end{aligned} \tag{6.21}$$

So now we only care about the antisymmetric part of B on the upper indices, meaning now there are only $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ linearly independent Killing fields. Any space that has all of the Killing fields it is allowed by its dimension will be called **maximally symmetric**.

This result is important on its own, however, by categorizing the basis of Killing fields one gains more insight on the kinds of allowed transformations.

Killing fields of the form

$$\begin{cases} K_\delta^{(\alpha)}(p) = \delta_\delta^\alpha \\ \partial_\sigma K_\delta^{(\alpha)}(p) = 0 \end{cases} \tag{6.22}$$

where (α) acts as a label will be denoted "translations". The motivation for this definition has two origins, first, we have the same number of translations as dimensions, so one can assign one translation to each direction. Secondly, from the way objects transform under isomorphisms one can see that the components of vector and tensors don't change under this kind of transformation, but their position does.

The second family of transformations are the rotations, these have two labels that are antisymmetric, $\alpha\beta$ and are defined by

$$\begin{cases} K_\delta^{(\alpha,\beta)}(p) = 0 \\ \partial_\sigma K_\delta^{(\alpha,\beta)}(p) = \delta_\sigma^\alpha\delta_\delta^\beta - \delta_\delta^\alpha\delta_\sigma^\beta \end{cases} \tag{6.23}$$

There are $\frac{n(n-1)}{2}$ distinct rotations, one per unique combination of axis, Since rotations are on a plane defined by two axis. These rotations are centered at p because objects in this point are not translated, however, objects like vectors are rotated.

Generating geodesics

Since the covariant derivative is an intrinsic object it is preserved under isometries, meaning for an isometry $F : M \to \widetilde{M}$

$$\tilde{\nabla}_\mu(F_*X) = F_*\big(\nabla_\mu X\big) \tag{6.24}$$

where X is any geometrical object. This means that transforming a geodesic by an isometry (for example those generated by Killing vectors) results in a different geodesic. In fact, this condition can be relaxed and it is not necessary for the transformation to be an isometry, it is enough for the transformation to be conformal. A transformation $F : M \to \widetilde{M}$, is said to be conformal if

$$F_*g = \kappa\tilde{g} \tag{6.25}$$

where g is the metric on M, $\tilde{g}$ the metric on $\tilde{g}$ and κ is some constant. One can see this by looking at the action of a free particle

$$S = \int g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \, \mathrm{d}\tau \tag{6.26}$$

by applying a conformal transformation the resulting action is

$$S' = \kappa S \tag{6.27}$$

so $\delta S' = 0 \Leftrightarrow \delta S = 0$

One can generate these transformations via **conformal Killing fields** defined in an analogous way to Killing fields as

$$\mathscr{L}_K g = \kappa g \tag{6.28}$$

where K is a conformal Killing field.

This allows to generate new geodesics from existing ones and will simplify the computations of redshifts.

7 Schwarzschild geodesics

The Schwarzschild metric is one of the most important metrics in general relativity. By Birkhoff's theorem it is known to be the unique spherically symmetric vacuum solution to Einstein field equations. This means that any spherically symmetric distribution of energy will produce this metric on the outside (up to gravitational waves). It also is the first prediction of black holes.

By using spherical coordinates (t, r, θ, φ) so that t is the time coordinate, r is a radial coordinate such that the "sphere" defined by fixing t and r has area of $A = 4\pi r^2$ and θ, φ are the polar and azimuthal angles respectively, defined so that the submanifolds of constant time and radius have the induced metric of a sphere.

The metric on this coordinates can be written as

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \,\mathrm{d}\theta^2 - r^2 \sin^2(\theta) \,\mathrm{d}\varphi^2 \tag{7.1}$$

This metric has 4 Killing vectors. One is associated with time translations, the other three are associated with rotations and obey the SO(3) Lie algebra

$$\begin{aligned} [X, Y] &= Z \\ [Y, Z] &= X \\ [Z, X] &= Y \end{aligned} \tag{7.2}$$

In spherical coordinates these can be written as

$$\begin{aligned} X &= -\sin\varphi \partial_\theta - \cot\theta \cos\varphi \partial_\varphi \\ Y &= \cos\varphi \partial_\theta - \cot\theta \sin\varphi \partial_\varphi \\ Z &= \partial_\varphi \end{aligned} \tag{7.3}$$

The Killing vector associated to time translation is $T = \partial_t$

Since this metric is rotation invariant we are allowed to choose the coordinate system so that both the initial position and spacial part of the four-velocity lie on the $\theta = \frac{\pi}{2}$ plane. Reducing the complexity of the system. In this plane with the restriction $u^\theta = 0$, this restriction is valid for all time, this can be proved by looking at the conserved quantities of X and Y

$$\begin{cases} X_\mu u^\mu = \sin\varphi r^2 u^\theta + \cancel{r^2 \cot\theta \cos\varphi \sin^2\theta u^\varphi} = 0 \\ Y_\mu u^\mu = -\cos\varphi r^2 u^\theta + \cancel{r^2 \cot\theta \sin\varphi \sin^2\theta u^\varphi} = 0 \end{cases} \tag{7.4}$$

Where the equality with 0 comes from the initial condition. Conservation of both quantities leads to either $r = 0$ or $u^\theta = 0$, since $u^\theta = 0$ for all τ $x^\theta = \frac{\pi}{2}$ is valid for all τ

The other two conserved quantities are related to classical quantities. The conserved quantity associated to T will be denoted energy

$$E = T_\mu u^\mu = g_{\mu\alpha} T^\alpha u^\mu = g_{\mu\alpha} \delta^{\alpha 0} u^\mu = g_{00} u^0 = g_{tt} u^t \tag{7.5}$$

this quantity agrees with energy per unit of mass of a free particle in the limit $r \to \infty$

The conserved quantity associated with Z is denoted angular momentum

$$L = Z_\mu u^\mu = g_{\mu\alpha} Z^\alpha u^\mu = g_{\mu\alpha} \delta^{\alpha 3} u^\mu = g_{33} u^3 = g_{\varphi\varphi} u^\varphi \tag{7.6}$$

this quantity can be recognized as the angular momentum per unit mass of a free particle *if r is the radius*, however, here r is only the "classical radius" in the limit $r \to \infty$, so these quantities don't exactly agree.

These two quantities allow to know u^φ and u^t at any point of the geodesic if the initial conditions are known. Additionally, since $\|u\|^2 = \begin{cases} 1 \text{ for matter} \\ 0 \text{ for light} \end{cases}$ one can obtain u^r as

$$u^r = \pm \sqrt{\frac{\|u\|^2}{g_{rr}} - \frac{L^2}{g_{\varphi\varphi} g_{rr}} + E^2} \tag{7.7}$$

Importantly this allows for computing quantities such as the three velocity as seen by an observer at ∞

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r \,/\, \mathrm{d}\tau}{\mathrm{d}t \,/\, \mathrm{d}\tau} = \frac{u^r}{u^t} = \pm\, g_{tt} \sqrt{\frac{\|u\|^2}{g_{rr} E^2} + \frac{L^2}{g_{\varphi\varphi} g_{rr} E^2} + 1} \tag{7.8}$$

since in the limit $r \to r_s$ $g_{tt} \to 0$ and $g_{rr} \to -\infty$ objects stop moving when close to the limit horizon.

Also one can compute how fast things rotate around a black hole as seen at ∞

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\varphi \,/\, \mathrm{d}\tau}{\mathrm{d}t \,/\, \mathrm{d}\tau} = \frac{u^\varphi}{u^t} = \frac{L g_{tt}}{E g_{\varphi\varphi}} \tag{7.9}$$

Once again since $g_{tt} \to 0$ when $r \to r_s$ things stop rotating when close to the Schwarzschild radius.

This method allows us to obtain the equations of motion of the particles without having to compute the Christoffel symbols.

8 Kerr back holes

The Kerr metric represents a spinning black hole. From this one can already think that spherical symmetry and time reversal symmetry are going to be broken. The first one because there will be an “special axis” around which the black hole spins. The second cone because by reversing time the direction of rotation of the black hole is also reversed breaking the symmetry. To recover the time reversal symmetry it is necessary to also do an inversion around the axis of symmetry of the solution. The result is that in the metric there is at least one cross term allowed between time and space.

By using the Boyer-Lindquist coordinates, (t, r, θ, φ) related to cartesian coordinates by

$$\begin{cases} x = \sqrt{r^2 + a^2} \sin\theta \cos\varphi \\ y = \sqrt{r^2 + a^2} \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases} \tag{8.1}$$

so that in the limit $r \to \infty$ they reduce to usual spherical coordinates.

In this coordinate system the black hole spins around $\theta = 0$ and the metric can be written as

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{r_s r}{\Sigma}\right) \mathrm{d}t^2 + \frac{2 r_s r a \sin^2\theta}{\Sigma} \,\mathrm{d}t\,\mathrm{d}\varphi - \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 \\ & - \Sigma\,\mathrm{d}\theta^2 - \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma} \sin^2\theta\right) \sin^2(\theta)\,\mathrm{d}\varphi^2 \end{aligned} \tag{8.2}$$

Where $a = \frac{J}{M}$ where J is the angular momentum of the black hole, $\Sigma = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 - r_s r + a^2$

There are two Killing fields in this metric, one associated the axial symmetry $R = \partial_\varphi$ and the other with time symmetry $K = \partial_t$.

Just as with the schwarzschild geodesics we will call angular momentum and energy to the conserved quantities

$$E = K_\mu u^\mu = g_{t\mu} u^\mu = g_{tt} u^t + g_{t\varphi} u^\varphi = u_t \tag{8.3}$$

and angular momentum

$$L = R_\mu u^\mu = g_{\varphi t} u^t + g_{\varphi\varphi} u^\varphi = u_\varphi \tag{8.4}$$

Immediatelly there is an interesting result, the system of equations above can be solved for either u^t or u^φ to give

$$\begin{aligned} u^t &= \frac{L g_{\varphi t} - E g_{\varphi\varphi}}{\left(g_{\varphi t}\right)^2 - g_{tt} g_{\varphi\varphi}} \\ u^\varphi &= \frac{E g_{\varphi t} - L g_{tt}}{\left(g_{\varphi t}\right)^2 - g_{tt} g_{\varphi\varphi}} \end{aligned} \tag{8.5}$$

As one can see u^φ is not necesarily zero even if it is at one point in time. This can be interpreted as the black hole forcing particles to rotate with it. This effect, called frame dragging, is only dependent on the spacetime being axially symmetric with the time-axial inversion symmetry.

In the case of the Kerr black hole there is a region where the tt component of the metric becomes spacelike. This region is called the ergosphere of the Kerr black hole. This region is not enclosed in an event horizon and particles and light are allowed to go in and out of the ergosphere. This is because even if g_{tt} is zero, the $g_{\varphi t}$ is positive, so that the particle can have radial component of four velocity provided u^t and u^φ are large enough.

For particles outside the ergosphere the energy is always positive (using $(+---)$ signature). This is because K is a timelike vector, and, since four velocities are either timelike or null, depending on if the particle has mass or not, the product has to be positive.

To see this we can use the equivalence principle and prove this in Minkowski spacetime. For all non spacelike vectors in Minkowski spacetime we have that

$$\left(u^0\right)^2 \geq \|\vec{u}\|^2 \tag{8.6}$$

where equality happens for null vectors. Therefore if we have the product of two four vectors and one of them is timelike we have that

$$\begin{aligned} u^0 v^0 > \|\vec{u}\|\|\vec{v}\| \geq \langle \vec{u}, \vec{v} \rangle \Rightarrow \\ \Rightarrow u^0 v^0 - \langle \vec{u}, \vec{v} \rangle = g_{\mu\nu} u^\mu v^\nu > 0 \end{aligned} \tag{8.7}$$

so the product is positive.

Inside the ergosphere we can solve for $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ obtaining

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{g_{\varphi t}} \left(\frac{E}{u^t} - g_{tt} \right) \tag{8.8}$$

and, since E, u^t and $-g_{tt}$ are positive and $g_{\varphi t}$ has the sign of the angular momentum of the black hole, this means that any particle that goes into the ergosphere has to rotate with the black hole.

If we wanted to obtain the equations of motion like how we did in the Schwarzschild metric we would find a problem. In the Schwarzschild metric we had 3 Killing fields that, when combined with the norm of the four velocity allowed to solve for the derivatives. Here we only have two Killing fields. Fortunately there is an additional Killing rank 2 tensor σ defined as

$$\begin{aligned} \sigma_{\mu\nu} &= \Sigma^2 \left(l_\mu n_\nu + l_\nu n_\mu \right) + r^2 g_{\mu\nu} \\ l^\mu &= \frac{1}{\Delta} \left(r^2 + a^2, \Delta, 0, a \right) \\ n^\mu &= \frac{1}{2\Sigma^2} \left(r^2 + a^2, -\Delta, 0, a \right) \end{aligned} \tag{8.9}$$

the associated conserved quantity is called the Carter constant and is usually denoted C. The system of equations Eq. (8.9) together with Eq. (8.5) allows to solve for u^r and with these three constants it is possible to finally obtain u^θ from the norm of the four-velocity and obtain the equations of motion. However the equations obtained, although straightforward to obtain, are quite lengthy and don't have any really give any insights on the geodesics.

9 FLRW Metric

In cosmology the most widely used metric to describe spacetime at big scales is is the FLRW metric. Even in contexts where general relativity is modified this is the case, this is because the principles that lead towards this metric are, to some degree, independent of general relativity.

The fundamental principle behind this metric is the observed homogeneity and isotropy of the universe at big scales. In more formal terms this means that there have to be a family of spacelike surfaces that are isometric under translations in all spacial directions and rotations around any point. This is, there are a family of spacial slices that are maximally symmetric so that for any point in the manifold is part of one of the slices.

The usual ways to write this metric are

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{\sqrt{1-kr^2}} + r^2\,\mathrm{d}\Omega^2\right) \\ &= -\mathrm{d}t^2 + a^2(t)(\mathrm{d}\chi^2 + S_k^2(\chi)\,\mathrm{d}\Omega^2) \end{aligned} \tag{9.1}$$

Where χ is the comoving distance, $\mathrm{d}\Omega$ represents the angular part of the metric and $a(t)$ is the so called scale factor and whose value is obtained via the Einstein field equations in general relativity or the equations of motion of the metric in other gravitational theories.

Since this is going to be a math intensive chapter here an outline of how to prove the form of the metric is outlined here.

1. First prove that the metric is separable, meaning $\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ and then prove that $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)h_{ij}$ where h is independent of time
2. After this prove that since $t = \mathrm{const}$ are maximally submanifolds g_{ij} is a constant sectional curvature space
3. By applying the Killing-Hopf theorem the spacial submanifolds have to be isometric to a 4-sphere, 4-hyperboloid or a flat space.

9.1. Separability of the metric

The separability of the metric at any given point is trivial, if our spacetime is the manifold $\mathbb{M}$, by picking a coordinate system x^i on a maximally symmetric submanifold $\mathbb{E}$, this coordinate system can be extended to a global coordinate system by first picking a normal vector to the surface n. This can be done by taking the basis of $T_p\mathbb{E}$ generated by our coordinate system and then by Gram-Schmidt algorithm extend it to basis of $T_p\mathbb{M}$ and picking n to be normal to $\mathbb{E}$, this normal vector can be extended to a normal vector field by doing this process at every point and making the field smooth forces an orientation. After this the coordinate system x^i can be extended by adding an additional coordinate x^0, that we will denote t, defined as the parameter of the geodesics starting at x^i with initial four velocity n[1]. In this way the set of points of constant x^i is defined as the points that the geodesic goes through

[1]Here we are assuming geodesic completeness, this is fine because it is assumed that our space is complete and the Hopf-Rinow theorem, gives an equivalence between both statements.

and as a time variable the arclength of the geodesic is chosen. An intuitive way to see this coordinate system is to imagine that we place an observer at each point of our surface and assign a coordinate to each of them, after this we allow all of the observers to evolve and take as a time coordinate the time they measure with their clocks.

To check that the coordinate system we have actually has $g_{0i} = 0$. A way to check this would be by using that the Christoffel symbol Γ_{i00} is related to the metric by

$$\Gamma_{i00} = \frac{1}{2}(2\partial_0 g_{0i} - \partial_i g_{00}) \tag{9.2}$$

and since $g_{00} = \|n\|^2 = -1$ that means that

$$\Gamma_{i00} = \partial_0 g_{0i} \tag{9.3}$$

so that by proving that $\Gamma_{i00} = 0$, $g_{0i} = 0$ is also proven. To compute Γ_{i00} we can use the geodesic equation and remember that our coordinate system is defined by geodesics so that

$$\begin{gathered} \ddot{x}^i = 0 = -{\Gamma^i}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -{\Gamma^i}_{\mu\nu}\delta^\mu_0 \delta^\nu_0 = -{\Gamma^i}_{00} \Rightarrow \\ \Rightarrow \Gamma_{i00} = 0 \end{gathered} \tag{9.4}$$

.

Now that have a coordinate system that has the initial form we wanted. Next, it is important to check that in this coordinate system all spacial slices are maximally symmetric. Right now we have only assumed that the initial slice is maximally symmetric, however that doesn't tell us anything about the structure of all of the spacial slices. This is an interesting question because it could be the case where some spaces start maximally symmetric but at some point they stop being symmetric. As we will see this is not the case.

To do this first we extend the Killing vectors on the slice to Killing vectors of all of space by the insertion map of the slice defined as

$$i(v^i) = (0, v^i) = v^\mu \tag{9.5}$$

So that the tangent vectors on the slice are tangent when mapped to the manifold. Therefore in this coordinate system the Killing fields on the initial surface take the following form $K^0(t = 0, x^i) = 0$. To check that the Killing fields are Killing fields in all of the constant time surfaces first we use the $(0, 0)$ component of the Killing equation

$$2g_{00}\partial_0 K^0 + \cancel{K^0 \partial_0 g_{00}} = 0 \Rightarrow \partial_0 K^0 = 0 \tag{9.6}$$

This with the initial condition of $K^0(t = 0) = 0$ shows that the Killing fields are tangent to constant time slices.

Now we are close to proving that all spacial slices are maximally symmetric, since the differential equation has unique solutions as long as the metric is enough well behaved this looks like is enough to prove that the slices have the same number of Killing fields as the initial one, so they are maximally symmetric too. However there is an edge case, that being whenever a Killing field becomes zero. To prove this is not the case it is easy to use the $(0, i)$ components of the Killing equation.

$$\cancel{K^\alpha \partial_\alpha g_{0i}} + \cancel{g_{0\alpha}\partial_i K^\alpha} + g_{\alpha i}\partial_0 K^\alpha = g_{ji}\partial_0 K^j = 0 \Rightarrow \partial_0 K^j = 0 \tag{9.7}$$

The equality between the first and second expression might not be completely clear unless one remembers that the spacial part of the metric is positive definite so that in matrix it is a non-singular matrix.

Just as before this implies that the spacial parts of the Killing vectors are constant in time. Therefore all slices are maximally symmetric.

To show that the spacial part has the form of $g_{ij} = a^2(T)h_{ij}$ where $h_{i,j}$ is independent of time.

The path it to use two properties of the Lie derivative with respect to a Killing field.

These are:
1. For any symmetric tensor $T_{\alpha\beta}$ in a maximally symmetric manifold

$$\mathscr{L}_K T_{\alpha\beta} = 0 \Rightarrow T_{\alpha\beta} = \frac{T}{n} g_{\alpha\beta} \tag{9.8}$$

where $T := T_\alpha{}^\alpha$ is the trace of the tensor and n is the dimension of the space

2. In this coordinate system

$$\mathscr{L}_K \dot{g}_{ij} = 0 \tag{9.9}$$

and with these two facts the resulting differential equation has as a solution

$$g_{ij} = e^{f(t)} g_{ij}(t=0) \tag{9.10}$$

proving the form of the metric we expected.

Proving Eq. (9.9) is easy by direct computation

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{L}_K g_{ij}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(K^\alpha \partial_\alpha g_{ij} + g_{\alpha j}\partial_i K^\alpha + g_{i\alpha}\partial_j K^\alpha\right) \\ &= K^\alpha \partial_\alpha \dot{g}_{ij} + \dot{g}_{\alpha j}\partial_i K^\alpha + \dot{g}_{i\alpha}\partial_j K^\alpha \\ &\cancel{+\dot{K}^\alpha \partial_\alpha g_{ij} + g_{\alpha j}\partial_i \dot{K}^\alpha + g_{i\alpha}\partial_j \dot{K}^\alpha} \\ &= \mathscr{L}_K \dot{g}_{ij} \end{aligned} \tag{9.11}$$

The Eq. (9.8) is proven by proving the stronger statement restricted to only isotropic manifolds. By choosing a rotation around a point K one has:

$$\mathscr{L}_K T = T_{\gamma\beta}\partial_\alpha K^\gamma + T_{\alpha\gamma}\partial_\beta K^\gamma + \cancel{K^\gamma \partial_\gamma T_{\alpha\beta}} \overset{\partial_\alpha K^\gamma = \nabla_\alpha K^\gamma}{=} T^\gamma{}_\beta \partial_\alpha K_\gamma + T_\alpha{}^\gamma \partial_\beta K_\gamma = 0 \tag{9.12}$$

The last expression can be rewritten as

$$T^\gamma{}_\beta \partial_\alpha K_\gamma + T_\alpha{}^\gamma \partial_\beta K_\gamma = \left(T^\gamma{}_\beta \delta^\sigma_\alpha + T_\alpha{}^\gamma \delta^\sigma_\beta\right)\partial_\sigma K_\gamma = 0 \tag{9.13}$$

since rotations are antisymmetric on σ and γ that means that

$$T^\gamma{}_\beta \delta^\sigma_\alpha + T_\alpha{}^\gamma \delta^\sigma_\beta = T^\sigma{}_\beta \delta^\gamma_\alpha + T_\alpha{}^\sigma \delta^\gamma_\beta \tag{9.14}$$

and by contracting β and γ the resulting expression is

$$\begin{aligned} &nT^\gamma{}_\beta + T_\beta{}^\gamma = T^\gamma{}_\beta + \delta^\gamma_\beta T \Rightarrow \\ &\Rightarrow (n-1)T^\gamma{}_\beta + T_\beta{}^\gamma = \delta^\gamma_\beta T \end{aligned} \tag{9.15}$$

and finally by multiplying both sides by $g_{\alpha\gamma}$ the theorem is proven

$$(n-1)T_{\beta\alpha} + T_{\alpha\beta} = nT_{\alpha\beta} = g_{\alpha\beta}T \tag{9.16}$$

Now that we know that the metric has the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{9.17}$$

the next step is to prove that maximally symmetric manifolds are constant curvature spaces. By proving that they are we can apply the Killing-Hopf theorem and now the submanifolds are isometric to either flat space, a 3 hyperboloid or a 3 sphere. Allowing to get the components h_{ij} and obtaining the final form of the metric.

But first we will introduce the notion of the sectional curvature.

9.2. Sectional curvature and maximally symmetric spaces

Sectional curvature is defined as a generalization of Gaussian curvature. In $\mathbb{R}^3$ surfaces can be assigned a curvature that is related to the Riemann tensor by

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{9.18}$$

by choosing two linearly independent vectors tangent to the surface u and v and multiplying both sides by $u^a v^b v^c u^d$ (the only non-zero combination of two vectors) one obtains

$$K = \frac{R_{abcd}u^a v^b v^c u^d}{\langle u, v\rangle^2 - \langle u, u\rangle\langle v, v\rangle} \tag{9.19}$$

generalizing this formula to any other space is easy just by making K a 2-covariant tensor so that it takes in two vectors and plugs them into Eq. (9.19). The Killing-Hopf theorem states that any pair of manifolds with constant sectional curvature are isometric if their sectional curvatures are equal. So by proving that maximally symmetric spaces have a Riemann tensor of the form Eq. (9.18) with constant K will give us the metric on the spacial slices.

9.3. Riemann tensor on maximally symmetric tensor

To obtain the form of the Riemann tensor we start from the fact that the Riemann tensor is intrinsic, therefore, the Lie derivative of the Riemann tensor with respect to one of the rotations at any given point must be zero. This gives the following equation

$$\begin{aligned}
\mathscr{L}_K R &= \cancel{K^\alpha\partial_\alpha R_{\mu\nu\gamma\sigma}} + R_{\alpha\nu\gamma\sigma}\partial_\mu K^\alpha + R_{\mu\alpha\gamma\sigma}\partial_\nu K^\alpha + R_{\mu\nu\alpha\sigma}\partial_\gamma K^\alpha + R_{\mu\nu\gamma\alpha}\partial_\delta K^\alpha = \\
&= R_{\alpha\nu\gamma\sigma}\partial_\mu K^\alpha - R_{\alpha\mu\gamma\sigma}\partial_\nu K^\alpha + R_{\alpha\sigma\mu\nu}\partial_\gamma K^\alpha - R_{\alpha\gamma\mu\nu}\partial_\sigma K^\alpha = \\
\underset{\text{for rotations}}{\overset{\nabla_\alpha K = \partial_\alpha K}{}} &= R^\alpha{}_{\nu\gamma\sigma}\nabla_\mu K_\alpha - R^\alpha{}_{\mu\gamma\sigma}\nabla_\nu K_\alpha + R^\alpha{}_{\sigma\mu\nu}\nabla_\gamma K_\alpha - R^\alpha{}_{\gamma\mu\nu}\nabla_\sigma K_\alpha = \\
&= \left(R^\alpha{}_{\nu\gamma\sigma}\delta^\varepsilon_\mu - R^\alpha{}_{\mu\gamma\sigma}\delta^\varepsilon_\nu + R^\alpha{}_{\sigma\mu\nu}\delta^\varepsilon_\gamma - R^\alpha{}_{\gamma\mu\nu}\delta^\varepsilon_\sigma\right)\nabla_\varepsilon K_\alpha = 0
\end{aligned} \tag{9.20}$$

since the covariant derivative of a Killing field is antisymmetric this equation is solved for all of the rotations if and only if

$$\begin{aligned}
R^\alpha{}_{\nu\gamma\sigma}\delta^\varepsilon_\mu - R^\alpha{}_{\mu\gamma\sigma}\delta^\varepsilon_\nu + R^\alpha{}_{\sigma\mu\nu}\delta^\varepsilon_\gamma - R^\alpha{}_{\gamma\mu\nu}\delta^\varepsilon_\sigma &= \\
= R^\varepsilon{}_{\nu\gamma\sigma}\delta^\alpha_\mu - R^\varepsilon{}_{\mu\gamma\sigma}\delta^\alpha_\nu + R^\varepsilon{}_{\sigma\mu\nu}\delta^\alpha_\gamma - R^\varepsilon{}_{\gamma\mu\nu}\delta^\alpha_\sigma
\end{aligned} \tag{9.21}$$

The easiest way to obtain a relationship between the Riemann tensor and the metric from here is to make contractions that give Ricci tensors. This is because since the Ricci tensor is intrinsic and symmetric by Eq. (9.8)

$$R_{\alpha\beta} = \frac{R}{n}g_{\alpha\beta} \tag{9.22}$$

Contracting ε and μ gives

$$\begin{aligned}
\text{LHS} &= R^{\alpha}{}_{\nu\gamma\sigma}\delta^{\mu}_{\mu} - R^{\alpha}{}_{\mu\gamma\sigma}\delta^{\mu}_{\nu} + R^{\alpha}{}_{\sigma\mu\nu}\delta^{\mu}_{\gamma} - R^{\alpha}{}_{\gamma\mu\nu}\delta^{\mu}_{\sigma} \\
&= nR^{\alpha}{}_{\nu\gamma\sigma} - R^{\alpha}{}_{\nu\gamma\sigma} + R^{\alpha}{}_{\sigma\gamma\nu} - R^{\alpha}{}_{\gamma\sigma\nu} \\
&= (n-1)R^{\alpha}{}_{\nu\gamma\sigma} + \underbrace{R^{\alpha}{}_{\sigma\gamma\nu} + R^{\alpha}{}_{\gamma\nu\sigma}}_{\text{Bianchi identity}} \\
&= (n-1)R^{\alpha}{}_{\nu\gamma\sigma} - R^{\alpha}{}_{\nu\sigma\gamma} \\
\text{RHS} &= R^{\mu}{}_{\nu\gamma\sigma}\delta^{\alpha}_{\mu} - \underbrace{\left(R^{\mu}{}_{\mu\gamma\sigma}\delta^{\alpha}_{\nu}\right)}_{R_{\alpha\beta\gamma\sigma}=-R_{\beta\alpha\gamma\sigma}} + R^{\mu}{}_{\sigma\mu\nu}\delta^{\alpha}_{\gamma} - R^{\mu}{}_{\gamma\mu\nu}\delta^{\alpha}_{\sigma} \\
&= R^{\alpha}{}_{\nu\gamma\sigma} + R_{\sigma\nu}\delta^{\alpha}_{\gamma} - R_{\gamma\nu}\delta^{\alpha}_{\sigma} \\
&= -R^{\alpha}{}_{\nu\sigma\gamma} + R_{\sigma\nu}\delta^{\alpha}_{\gamma} - R_{\gamma\nu}\delta^{\alpha}_{\sigma} \\
\text{LHS} &= \text{RHS} + \text{Eq. (9.22)} \Rightarrow \\
&\Rightarrow R^{\alpha}{}_{\nu\gamma\sigma} = \frac{R}{n(n-1)}\left(g_{\sigma\nu}\delta^{\alpha}_{\gamma} - g_{\gamma\nu}\delta^{\alpha}_{\sigma}\right) \Rightarrow \\
\cdot\, g_{\alpha\beta} &\Rightarrow R_{\beta\nu\gamma\sigma} = \frac{R}{n(n-1)}\left(g_{\sigma\nu}g_{\beta\gamma} - g_{\gamma\nu}g_{\beta\sigma}\right)
\end{aligned} \tag{9.23}$$

Our sectional curvature is $K = \frac{R}{n(n-1)}$ and is independent of the orientation of the surface. The only remaining fact to prove is that this is constant through space. To do this we use that the Ricci scalar is intrinsic and by taking the Lie derivative with respect to a translation

$$\mathscr{L}_K R = K^{\alpha}\partial_{\alpha}R = 0 \tag{9.24}$$

therefore the sectional curvature is constant.

9.4. Geodesics on the FLRW metric

The Killing fields on this space are defined by 6 parameters $\delta a_x, \delta a_y, \delta a_z, \delta b_x, \delta b_y$ and δb_z and their components are

$$\begin{aligned}
\xi^t &= 0 \\
\xi^r &= \sqrt{1-kr^2}\left(\sin\theta\left(\cos\phi\,\delta a_x + \sin\phi\,\delta a_y\right) + \cos\theta\,\delta a_z\right) \\
\xi^\theta &= \frac{\sqrt{1-kr^2}}{r}\left[\cos\theta\left(\cos\phi\,\delta a_x + \sin\phi\,\delta a_y\right) - \sin\theta\,\delta a_z\right] + \left(\sin\phi\,\delta b_x - \cos\phi\,\delta b_y\right) \\
\xi^\phi &= \frac{\sqrt{1-kr^2}}{r}\left[\frac{1}{\sin(\theta)}\left(\cos\phi\,\delta a_y - \sin\phi\,\delta a_x\right)\right] + \cot\theta\left(\cos\phi\,\delta b_x + \sin\phi\,\delta b_y\right) - \delta b_z
\end{aligned} \tag{9.25}$$

Since the spacial slices are maximally symmetric we can work without the loose of generality in a coordinate system where the initial position is in the ray defined by $\theta = \frac{\pi}{2}$ and $\phi = 0$ and the four-velocity has only a time and radial component.

By choosing the Killing field generated by $\delta\vec{a} = 0$ and $\delta b_x = \delta b_z = 0$ and $\delta b_y = 1$, that we will denote K the components of the Killing field are

$$K = -\cos(\phi)\partial_\theta \tag{9.26}$$

therefore the conserved quantity

$$K_\mu u^\mu = -a^2(t) r \sqrt{1-kr^2} \cos\phi u^\theta = 0 \tag{9.27}$$

enforces $u^\theta = 0$ for all time as long as $\phi \neq \pm \frac{\pi}{2}$.

Similarly by taking $\delta\vec{a} = 0$ and $\delta b_x = \delta b_y = 0$ $\delta b_z = -1$ and calling this Killing field ω we find that

$$\omega = \frac{\sqrt{1-kr^2}}{r} \partial_\phi \tag{9.28}$$

giving as a conserved quantity

$$\omega_\mu u^\mu = a^2(t) r \sqrt{1-kr^2} \sin\theta u^\phi = 0 \tag{9.29}$$

therefore once again $u^\phi = 0$ as long as $\theta \neq n\pi$ with $n \in \mathbb{Z}$.

With these two conditions we can see that the θ and ϕ components will be constants. This leaves us with the choice of $\delta a_x = 1$, $\delta a_y = \delta a_z = \delta\vec{b} = 0$ giving the Killing field we will denote Ξ that can be written as

$$\Xi = \sqrt{1-kr^2} \partial_r \tag{9.30}$$

and the associated conserved quantity is

$$P = \Xi_\mu u^\mu = a^2(t) u^r \tag{9.31}$$

This, already gives an interesting result. By inspecting the metric it might seem like for $k = 1$ (spherical universes) the surface at $r = 1$ is some kind of event horizon and that $u^r \to 0$ preventing matter or light from crossing it. However as we see in Eq. (9.31) this is not the case. The reason for this is that the r coordinate chosen here would be the equivalent to the distance from some point on a sphere to the z axis. And the region at $r = 1$ would be the equator. The apparent singularity appears because close to the equator the distance to the z axis is constant, not allowing r to be a valid variable.

By also using the condition of the normalization of u we can obtain following differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{u^r}{u^t} = \frac{u^r}{\sqrt{1 - \frac{a^2(t)}{\sqrt{1-kr^2}} (u^r)^2}} = \frac{Pa^{-2}}{\sqrt{1 - P^2 \frac{a^{-2}(t)}{\sqrt{1-kr^2}}}} = \frac{P}{\sqrt{a^4(t) - \frac{P^2 a^2(t)}{\sqrt{1-kr^2}}}} \tag{9.32}$$

as we can see as the universe increases in size matter slows down since as a increases $\frac{\mathrm{d}r}{\mathrm{d}t}$ decreases.

10 Computation of redshifts

The redshift is the quantity that relates the frequency of a light wave when it was emitted at a point A and when it is measured at a point B. The redshift, denoted z is usually defined as

$$1 + z = \frac{\lambda_B}{\lambda_A} = \frac{f_A}{f_B} = \frac{T_B}{T_A} = \frac{E_A}{E_B} \tag{10.1}$$

where λ is the wavelength, f the frequency, T the period and E the energy of each photon.

So far we have been talking about the geodesics and how Killing fields generate conserved quantities, in particular we have seen that when there is a time-like Killing field we can define energy for particles. It would be nice if we could extend this idea to photons, and since photon energy is related to the frequency obtain gravitational redshifts with it. This direct argument has a problem. Energy is not a scalar quantity in relativity since it is the 0th component of the four-momentum. The energy we defined with the Killing field is a scalar because is the energy someone at infinity measures and thats why it doesn't depend on the observer. However, here we actually want to know how different observers measure different energies and how that gives a redshift.

This can be achieved by using Killing fields in a different way. The Killing fields are used to define who is an stationary observer.

Imagine we have two observers, Alice and Bob. Alice is emitting light from the curve[1] $x_A^\mu(\tau)$ to Bob, who is in curve $x_B^\mu(\tau)$. A way to compute the redshift is to take two geodesics that start at x_A^μ and end x_B^μ. To compute T_A and T_B would therefore be the proper time differences between the proper times of the intersection points of the light geodesics on the observer geodesics.

This approach works but requires to know both the geodesics of the emitter, the observer and the light. This requirement can be dropped by using Killing fields.

Assume we have a time-like Killing field K. Now we define stationary observers as those whose four-velocity is proportional to K.

If Alice and Bob are stationary observers, lets call A_0 and B_0 the initial two points connected by a null geodesic. Now, since their four-velocities are proportional to the Killing field, they move through the flow of the Killing field. Therefore their geodesics for some small $\Delta\tau$ is

$$x_A^\mu(\Delta\tau) = \phi_{\Delta\tau}(A_0) = A_0 + K^\mu \Delta\tau \tag{10.2}$$

more importantly, since the Killing field generates an isometry it preserves geodesics. Therefore since A and B are connected by a null geodesic so are $x_A^\mu(\Delta\tau)$ and $x_B^\mu(\Delta\tau)$ for all $\Delta\tau$. Therefore the period as measured by A would be

$$T_A = \int_0^{\Delta\tau} \sqrt{\pm\, g_{\mu\nu}\dot{x}_A^\mu(s)\dot{x}_A^\nu(s)}\,\mathrm{d}s = \int_0^{\Delta\tau} \sqrt{\pm\, K_\mu K^\mu}\,\mathrm{d}s \approx \sqrt{\pm\, K_\mu K^\nu}\Delta\tau \tag{10.3}$$

[1]Since observers move through time.

where the $\pm$ depends on the chosen signature and ensures the quantity in the square root is positive as long as K is time-like. The same expression can be found for B. By substituting this expression on the redshift formula we obtain the following relationship between redshift and Killing fields.

$$1 + z = \sqrt{\frac{K_\mu(B_0)K^\mu(B_0)}{K_\mu(A_0)K^\mu(A_0)}} \tag{10.4}$$

In the case of the of stationary metrics, using the Killing field $K^\mu = \delta^\mu_t$ we obtain the well known formula

$$1 + z = \sqrt{\frac{g_{00}(B)}{g_{00}(A)}} \tag{10.5}$$

This also applies for the Kerr black hole, however an interesting result comes out. With this formula the border of the ergosphere is an infinite redshift surface. This is surprising because we saw that this surface is not an event horizon and we should be able to see through it. The problem here is that the stationary observers we picked doesn't take into account that our metric is not static, we are picking as a stationary observer, someone that doesn't rotate with the black hole. Inside the ergosphere that means someone moving faster than light, thus by a Lorentz doppler effect we get an infinite redshift. The trick here is to use a combination of the time and axial Killing fields $K = \partial_t + \Omega_H \partial_\varphi$ where Ω_H makes it so that the Killing field is null at the exterior event horizon. This way stationary observers do rotate with the black hole giving a finite redshift between the interior and exterior regions of the ergosphere.

This shows that one has to be careful when picking the Killing field used to compute redshifts since it doesn't tell the hole story and might give infinite redshifts on surfaces that are not infinite redshift surfaces and not give infinite redshifts on surfaces that actually are infinite redshift surfaces.

Some metrics are not stationary. As an example, for the FLRW metric for an arbitrary $a(t)$ there aren't any time-like Killing fields.

However there is a way to extend this kind of method to some of these spaces by using conformal Killing fields defined in Eq. (6.28).

Since these kind of vector fields also preserve geodesics under the transformations they generate the same argument applies by using these. For example, in the case of the FLRW metric there is one Homothetic Killing field

$$H = a(t)\partial_t \tag{10.6}$$

that after applying Eq. (10.5) the resulting redshift is

$$1 + z = \frac{a(B)}{a(A)} \tag{10.7}$$

11 Optics

Classical geometrical optics treat light as rays that move at a certain speed and bend by Snell law. General relativity treats light rays in a similar way but uses geodesics to shape their path. Treating geometrical optics as an effective theory of general relativity gives an insight on the origin of the Snell law.

In some medium, because of the light matter interactions, light moves at a speed different from the one it would have when left unperturbed. This is usually modelled by assigning a number to each material called index of refraction defined as

$$n = \frac{c}{v} \tag{11.1}$$

where v is the speed of light on the medium.

In general relativity the speed of light is obtained by the condition of the four-velocity of the light being a null vector. This allows to give an index of refraction to each point in spacetime by using the following metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - n^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{11.2}$$

as it can be seen the speed of light in any of the three cartesian directions is

$$c^2\,\mathrm{d}t^2 - n^2\,\mathrm{d}x^2 = 0 \Rightarrow \frac{\mathrm{d}x^2}{\mathrm{d}t^2} = \frac{c^2}{n^2} = v^2 \tag{11.3}$$

as expected.

First lets start with a simple case where we have some material on one side and vacuum on the other. To do this we will pick

$$n(x) = n\theta(x) \tag{11.4}$$

where θ is the Heaviside function.

It is easy to see that for this space there are four Killing fields.

$$\begin{aligned} T &= \partial_t \\ Y &= -\partial_y \\ Z &= -\partial_z \\ \omega &= -z\partial_y + y\partial_z \end{aligned} \tag{11.5}$$

Y and Z allow to shift both the y and z coordinates and ω generates rotations on y and z coordinates, thus without loosing generality we will pick the light ray to start with $u^z = 0$. This condition remains valid for all of the path of the light ray since the conserved quantity generated by Z is nu^z and since it starts being zero it requires $u^z = 0$ for all time.

The conserved quantities generated by T and Y are

$$
\begin{aligned}
T_\mu u^\mu &= c^2 u^0 \\
Y_\mu u^\mu &= n^2 u^y
\end{aligned}
\tag{11.6}
$$

Since combinations of conserved quantities is also conserved we can see that the quantity

$$
\mathcal{S} = \frac{n^2}{c^2}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{v^2}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{11.7}
$$

is conserved. Since the spacial part of the metric is euclidean $\frac{\mathrm{d}y}{\mathrm{d}t}$ is equivalent to the classical y component of the velocity of the light ray. Therefore

$$
\frac{\mathrm{d}y}{\mathrm{d}t} = v\sin(\theta) \tag{11.8}
$$

where θ is the classical angle between the light ray direction and the x axis. After making this substitution we get

$$
\mathcal{S} = \frac{\sin(\theta)}{v} \tag{11.9}
$$

Since this is conserved, it is equal on the vacuum and the material. Therefore by making the equality on both sides and multiplying by c we get Snell's law

$$
n_1\sin(\theta_1) = n_2\sin(\theta_2) \tag{11.10}
$$

This argument can also be extended to other geometries since, close to the point of contact of the light ray with the boundary, by taking a small enough neighborhood of the contact point the surface will be flat and the same three Killing fields will be valid giving the same law.

This also allows us to work on more complex results.

As an example for moving matter along the x axis with with speed $\beta = \frac{v}{c}$. By using special relativity we can do a Lorentz transformation on the stationary reference frame to obtain the equations when the block moves. By calling the frame where block moves x'^μ and the frame where the matter doesn't move.

The Lorentz transformation in this case is

$$
\begin{cases}
ct = \gamma(ct' - \beta x') \\
x \;= \gamma(x' - \beta ct') \\
y \;= y' \\
z \;= z'
\end{cases}
\tag{11.11}
$$

The first quantity we are interested in is the norm of the four-velocity witch can be Lorentz transformed to give

$$
\begin{gathered}
\gamma^2(cu'^t - \beta u'^x)^2 - n^2\left(\gamma^2(u'^x - \beta cu'^t)^2 + (u'^y)^2\right) = \\
\gamma^2c^2(1-\beta^2n^2)(u'^t)^2 + 2\beta c\gamma^2(n^2-1)u'^t u'^x + \gamma^2(\beta^2-n^2)(u'^x)^2 - n^2(u'^y)^2 = 0
\end{gathered}
\tag{11.12}
$$

As we can see the resulting material behaves as if it was anisotropic, by defining

$$
\begin{aligned}
\frac{\mathrm{d}x'}{\mathrm{d}t'} &= v'(\theta')\cos(\theta') \\
\frac{\mathrm{d}y'}{\mathrm{d}t'} &= v'(\theta')\sin(\theta')
\end{aligned}
\tag{11.13}
$$

We can obtain the speed of light depending on the direction, the resulting expression is

$$
\begin{aligned}
v'(\theta') &= \frac{2\beta c(n^2-1)\cos(\theta') \pm \sqrt{\Delta(\theta')}}{2(\gamma^2(n^2-\beta^2)\cos^2(\theta')+n^2\sin^2(\theta'))} \\
\Delta(\theta') &= 2\beta c(n^2-1)\cos^2(\theta') + 4c^2\gamma^2(1-n^2\beta^2)(\gamma^2(n^2-\beta^2)\cos^2(\theta')+n^2\sin^2(\theta')
\end{aligned}
\tag{11.14}
$$

Restricting to only positive solutions of v we see that there are two cases.

The range where $\beta^2 > \frac{1}{n^2}$ and the range where $\beta^2 < \frac{1}{n^2}$.

If the material moves more slowly than the light rays all angles of movement are allowed. However light tends to go faster in the direction of the material. On the other hand when the material goes faster than the light inside of it only certain θ' are allowed. This is because the material is dragged with the material similarly how black holes drag light when moving or when rotating.

To obtain the equivalent to Snell law in this frame of reference we can Lorentz boost the Killing fields obtaining

$$
\begin{aligned}
[T'_\mu] &= (c^2\gamma, -\beta c\gamma, 0, 0) \\
[Y'_\mu] &= (0, 0, n^2, 0)
\end{aligned}
\tag{11.15}
$$

And by the same process as the one used to find $\mathcal{S}$

$$
\mathcal{S}' = \frac{n^2 u'^y}{c^2\gamma\left(u'^t - \frac{\beta}{c}u'^x\right)} = \frac{n^2 u'^y}{c^2\gamma u^t\left(1-\frac{\beta}{c}v'(\theta')\cos(\theta')\right)} = \frac{\sin(\theta')}{v'(\theta')\gamma\left(1-\frac{\beta}{c}v'(\theta')\cos(\theta')\right)} \tag{11.16}
$$

12 Quantum inflation and cosmology

As part of the agreement of the University of Cantabria and University of Brown this project was developed the summer of 2025.

12.1. Theory

Inflation is a proposed mechanism where at the big bang there was a big expansion before the observed big bang explaining the observed flatness and isotropy of the universe. Multiple mechanisms for inflation have been proposed, and usually involve adding a field coupled to general relativity that drives inflation.

One of these proposals is by Stephon Alexander in his 2003 paper [1] where the cosmological constant is replaced by the potential of a scalar field in the action.

The resulting action would look something like the following

$$S = \int \left[\frac{1}{2k}(R - 2V(\phi)) + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \right] \sqrt{g}\, \mathrm{d}^4 x \tag{12.1}$$

In the paper Ashtekar variables are used. To understand what these are first is necessary to introduce ADM and tetrad formalism. From now on natural units with $l_p = c = \hbar = 1$ are used.

In tetrad formalism is based on the idea of using the equivalence principle to work on a "Minkowski like space". What is meant by this is that the metric becomes the Minkowski metric but the coordinate system is no longer holonomic, meaning the basis for the tangent space is no longer the partial derivatives with respect to the coordinate system. Formally a tetrad are four vectors of the tangent space labeled by what we will call internal indices $i, j, k...$ (not to be confused with indices that only involve spacial components those will be called $a, b, c...$ from now on) so that

$$e_i = e^\mu{}_i \partial_\mu \tag{12.2}$$

is a basis for the tangent space. That additionally has the following property

$$e_i \cdot e_j = \eta_{ij} \tag{12.3}$$

so that the metric of the internal space is the Minkowski metric.

The tensor components of the tetrad allow to define a map between "normal" and "internal" spaces by defining

$$\begin{aligned} v_i &= e^\mu{}_i v_\mu \\ v^i &= e_\mu{}^i v^\mu \end{aligned} \tag{12.4}$$

The transformation tensors with upper and lower indices are defined as inverses so that

$$\begin{aligned} e^{\mu}{}_{i} e_{\mu}{}^{j} &= \delta^{j}_{i} \\ e^{\mu}{}_{i} e_{\nu}{}^{i} &= \delta^{\mu}_{\nu} \\ e_{\mu}{}^{i} e_{\nu i} &= g_{\mu\nu} \\ e^{\mu}{}_{i} e_{\mu j} &= \eta_{ij} \end{aligned} \tag{12.5}$$

Internal indices work under the Minkowski metric and normal indices under the usual metric. Therefore, while the usual metric will be used to raise and lower normal indices internal indices will be raised and lowered by a Minkowski metric.

On the other hand ADM formalism takes spacetime and slices it into a family of space-like surfaces that represent the state of spacetime at a given time. These space-like surfaces will have an induced metric q_{ij}, in ADM formalism this metric is treated as a dynamical variable that evolves in time.

Both formalism can be combined by taking a tetrad such that e_0 is orthonormal to the surface and e_a are tangent to each slice. Thus we will call

$$n = e_0 \tag{12.6}$$

As the time coordinate evolves not only we change from one slice to another we also could be shifted on the coordinates.

Therefore we have

$$\partial_t = Nn + N^a e_a \tag{12.7}$$

where N is the lapse function and N^a are the shift functions.

Now we have a gauge choice, we pick the coordinate system so that n only has a time component and each of the e_a only have spacial components. Therefore internal indices are only spacial from now on. This also has the effect, since internal indices are spacial, that the metric on internal indices is the Kronecker delta. Therefore upper and lower internal indices are equivalent.

To be able to take covariant derivatives of tensors expressed in terms of internal coordinates the spin connection $\omega_{\mu}{}^{i}{}_{j}$ is defined so that the connection is compatible with the internal space so that

$$\nabla_{\nu} \boldsymbol{V} = (\nabla_{\mu} V^{\nu}) \partial_{\nu} = (\nabla_{\mu} V^{i}) e_{i} \tag{12.8}$$

so the required condition is

$$\nabla_{\nu} e^{\mu}{}_{i} = \nabla_{\nu} e_{\mu}{}^{i} = 0 \tag{12.9}$$

giving as a solution

$$\omega_{\mu}{}^{i}{}_{j} = e_{\rho}{}^{i} \Gamma^{\rho}{}_{\mu\nu} e^{\nu}{}_{j} - e^{\nu}{}_{j} \partial_{\mu} e_{\nu}{}^{i} \tag{12.10}$$

We also have to define the extrinsic curvature of the spacial slices similarly to how gaussian curvature is defined giving

$K_{ab} = \nabla_a n_b$

From these quantities we can define the Ashtekar variables. The first one is the densitized triad defined as

$$E^{a}{}_{i} = \det(e) e^{a}{}_{i} \tag{12.11}$$

where $\det(e)$ is the determinant of the matrix whose indices are $e_{a}{}^{i}$

so that

$$\det(g_{ab})g^{ab} = E^{ai}E^{b}{}_{i} \tag{12.12}$$

and the associated conjugated momentum

$$A_a^i = \varepsilon^{ikl}\omega_{akl} + iK_{ab}e^{bi} \tag{12.13}$$

.

Lastly we define the curvature of $A_a{}^{ij} := \frac{1}{2}\epsilon_{ijk}A_a^k$

$$F_{ab}^{ij} = 2\partial_{[a}A_{b]}{}^{ij} + A_a{}^{ik}A_{ak}{}^{j} - A_b{}^{ik}A_{ak}{}^{j} \tag{12.14}$$

And define

$$F_{ab}^{i} = \epsilon_{ijk}F_{ab}^{jk} \tag{12.15}$$

With these variables the usual Hamiltonian has the following form

$$H = \int \mathrm{d}^3x\, N\mathcal{H} + N^a\mathcal{H}_a + \lambda^i\mathcal{G}_i \tag{12.16}$$

where $\mathcal{H}$ is the Hamiltonian density and enforces the time diffeomorphism invariance, $\mathcal{H}_a$ are the spacial constraints and enforce the spacial diffeomorphism invariance and $\mathcal{G}$ are the Gaussian constraints and enforce the $SO(3)$ gauge invariance of the triad.

Now that we have defined all relevant quantities the Hamiltonian density used in [1] is the following

$$\mathcal{H} = \frac{1}{l_p^2}\epsilon_{ijk}E^{ai}E^{bj}\left(F_{ab}^k + \frac{GV(\phi)}{3}\epsilon_{abc}E^{ck}\right) + \frac{1}{2}p_\phi^2 + \frac{1}{2}E^{ai}E_i^b\partial_a\phi\partial_b\phi \tag{12.17}$$

where p_φ is the momentum of the scalar field. Notice that the first term is the usual general relativity gravitational Hamiltonian with the cosmological constant dependent on the field.

In the total Hamiltonian the Gaussian constraints are not included since these are boundary terms and in this model a boundary for the universe is not considered. The diffeomorphism constraints are not included by choosing a coordinate system so that $N^a = 0$. In this way the total Hamiltonian is

$$H = \int_{\mathcal{S}} N\mathcal{H} \tag{12.18}$$

where $\mathcal{S}$ is the corresponding spacial slice.

By taking the value of the scalar field as a time variable $T = \phi$ and defining $N = \frac{\mathrm{d}t}{\mathrm{d}\phi}$ where this enforces the lapse function to be

$$N = \frac{k}{p_\varphi} \tag{12.19}$$

where k is some constant. Without loosing generality $k = 1$ is picked.

By the equation of motion generated by N we have the Hamiltonian constraint that enforces

$$\mathcal{H} = 0 \tag{12.20}$$

with this equation we can find that

$$p_\phi = \pm\sqrt{-2\mathcal{H}} \tag{12.21}$$

and substituting into the lagrangian we get

$$H = \int_S p_\phi \tag{12.22}$$

To apply this to the deSitter spacetime homogeneity and isotropy together with a flat space give as a result

$$A_{ai} = i\delta_{ai}A(T) \qquad E_{ai} = \delta_{ai}E(T) \tag{12.23}$$

The resulting metric is

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}T^2 + E(T)(\mathrm{d}\Sigma^2) \tag{12.24}$$

where $\mathrm{d}\Sigma$ is the spatially flat metric and E takes the role of the scale factor, a^2 of cosmology.

Since the spacial slice is infinite the integral would diverge, to avoid this problem the theory is built over a compact region of the slice and take this region as a finite representation of the whole spacetime.

We define the "side length", R, so that the volume of the region to be integrated over is R^3. The resulting Hamiltonian is

$$H = R^3\sqrt{12E^2\left(A^2 - \frac{V(T)}{3}E\right)} \tag{12.25}$$

where G has been absorbed into V.

To obtain solutions Hamilton-Jacobi theory is used arriving to a Hamilton principal function

$$S(A, T, \alpha) = \frac{R^3A^3}{3V(T)}(1 + u(T)) \tag{12.26}$$

where u solves the initial value problem

$$\begin{cases} \dot{u} = (1+u)\left(\frac{\dot{V}}{V} + 18\sqrt{-3u}\right) \\ u(T_0) = \alpha \end{cases} \tag{12.27}$$

by using the second constant

$$\beta = \frac{\partial S}{\partial \alpha} \tag{12.28}$$

since the Hamilton principal function is related to E by

$$\frac{3}{R^3}\frac{\partial S}{\partial A} \tag{12.29}$$

one can obtain equations of motion for A and E by inverting Eq. (12.28) resulting in

$$E(T) = \frac{3^{5/3}\beta^{2/3}(1+u)}{R^2V^{1/3}}\left(\frac{\partial u}{\partial \alpha}\right)^{-2/3} \tag{12.30}$$

It is also possible to obtain the lapse function

$$N = \frac{R^3}{18\beta(1+u)\sqrt{-3u}}\left(\frac{\partial u}{\partial \alpha}\right) \tag{12.31}$$

This equation together with Eq. (12.27) and Eq. (12.30) enforce the conditions

$$-1 < u < 0 \tag{12.32}$$

since we need $1 + u > 0$ for the metric to be Lorentzian and the lapse function to be finite and $u < 0$ for Eq. (12.27) to be real.

From this model a quantum cosmological model is developed by a quantum mini-superspace. A mini-superspace in quantum gravity refers to an approximation where, instead of the full path integral of the metric, only certain components are quantized. In this case the spacial metric is quantized by making a quantum theory of the E and A parameters.

To do this A is upgraded to a multiplicative operator and E to an operator via

$$\hat{E} = -\frac{i}{R^3}\frac{\partial}{\partial A} \tag{12.33}$$

The evolution of the quantum state is defined by a schrödinger equation

$$i\frac{\partial \Psi}{\partial T} = \hat{H}\Psi \tag{12.34}$$

whose Hamiltonian is the quantized version of Eq. (12.25) with ordering

$$\hat{H} = i\sqrt{12}\frac{\partial}{\partial A}\left(A\sqrt{\hat{J}}\right) \tag{12.35}$$

where

$$\hat{J} = 1 - \frac{iV}{R^3 A^2}\frac{\partial}{\partial A} \tag{12.36}$$

The solutions are

$$\Psi_u(A, T) = e^{6\sqrt{3}\int_{T_0}^{T}\sqrt{-u(s)}\,\mathrm{d}s + i\frac{A^3 R^3}{3V}(1+u)} \tag{12.37}$$

12.2. Results

From this model the objective of the internship at Brown was proving the attractor behavior of Eq. (12.27), the asymptotic behavior of E in the limits $u \to -1$ and $u \to 0$ and check how well behaved the quantum model was.

The attractor behavior of Eq. (12.27) refers to the result that, for certain kinds of potentials, the solutions are almost independent of the potential of choice.

This can be seen by making a plot of $\dot{u}$ against u.

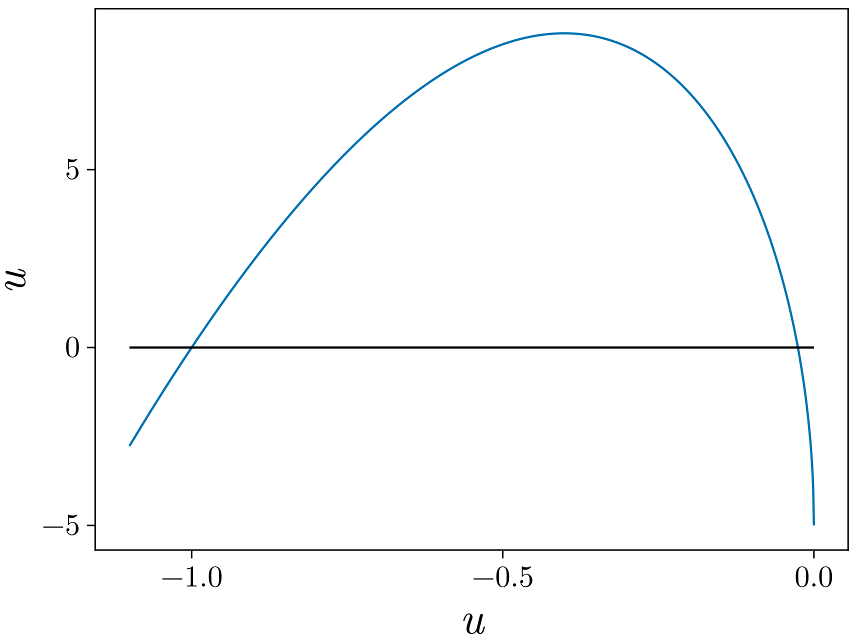

Figure 12.1: In this figure the value of $\dot{u}$ is represented as a function of u for $\frac{\dot{V}}{V} = -0.2$. The result is that solutions tend to follow $\frac{\dot{V}}{V}$

As can be seen $\dot{u}$ becomes zero when $u = -\left(-\frac{1}{\lambda}\frac{\dot{V}}{V}\right)^2$ where $\lambda = 18\sqrt{3}$. Additionally, solutions tend towards this value since solutions with higher values of u than this have a negative negative derivative and solutions with a smaller value have a positive derivative. This makes it so that the solutions have an attractor behavior towards $u = 0$ as long as $\frac{\dot{V}}{V} \ll \lambda$ and V is strictly decreasing. This condition is called the slow rolling condition. Solutions will be well behaved with the following requirements

- The potential follows the slow rolling condition and is strictly decreasing in a simply connected set of points $\mathcal{T}$
- The initial condition is chosen to be between $-1 < u(T_0) < -\left(\frac{1}{\lambda}\frac{\dot{V}}{V}\right)^2$ with $T_0 \in \mathcal{T}$
- The allowed time for the evolution of the solutions is $-\infty < T < \sup(\mathcal{T})$

The region where the potential follows the slow rolling condition prevents solutions from becoming complex when evolved forward in time. However nothing prevents us from evolving the solutions backwards in time, and since $u = -1$ is a stationary solution and the EDO of u are unique (since $\dot{u}$ is $\mathcal{C}^\infty$ on u in the valid range for u) the condition of $u > -1$ is secured.

Since we wanted to give an upper bound on the time where the solutions are valid the result we enforce $\mathcal{T}$ to be simply connected. Otherwise we could have "holes[1]" where solutions become complex. Therefore, just to be safe, we take a conservative upper bound for time.

To check this numerically we simulated Eq. (12.27) for some different potentials using Runge-Kutta 3/8 rule. The results can be seen in Fig. 12.2. The results are pretty similar even if the potentials are different.

[1]For example we could have $\mathcal{T} = (0, 1) \cup (3, 4)$

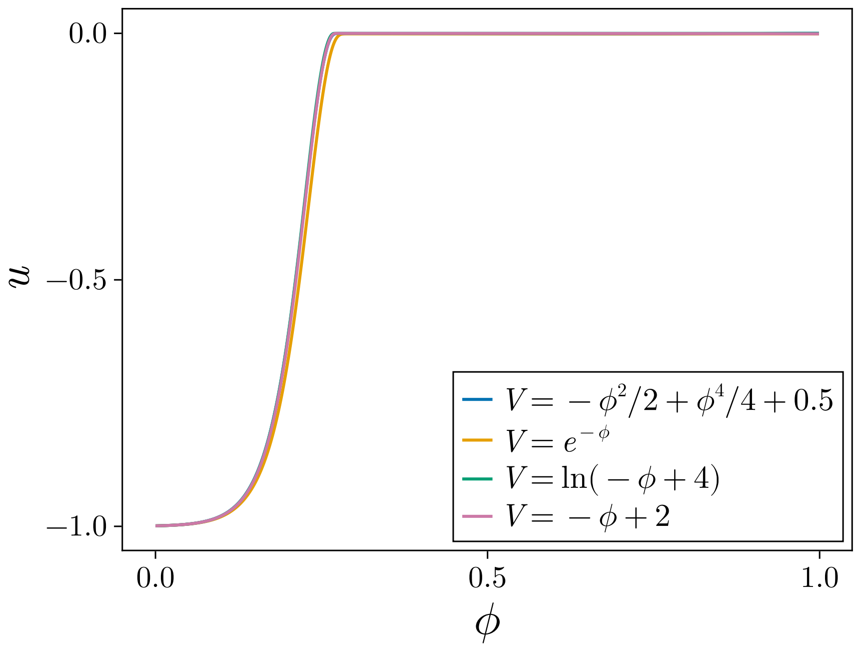

Figure 12.2: In this figure multiple simulations of u with different potentials are shown. As can be seen the result is really similar for all of them. The initial conditions for the differential equations are $u(-1) = -0.999$.

Now, from Eq. (12.30) E is well defined as long as both u and $\frac{\partial u}{\partial \alpha}$ are well defined since all of the other quantities involved are well defined. To do this first we define the density of solutions as

$$\rho = \left(\frac{\partial u}{\partial \alpha}\right)^{-1} \tag{12.38}$$

so that E depends on ρ. To obtain ρ we make use of Eq. (12.27) to obtain the following differential equation

$$\begin{cases} \dot{\rho}_u = -\rho_u \left(\frac{V'}{V} + \lambda\sqrt{-u} - \lambda\frac{(1+u)}{2\sqrt{-u}}\right) \\ \rho_u(T_0) = 1 \end{cases} \tag{12.39}$$

where u is previously computed from Eq. (12.27). Once again the equation has unique solutions, this ensures that ρ is positive for all time since $\rho = 0$ is a stationary solution. From this we can see that in the limit $u \to -1$ or equivalently $T \to -\infty$ the differential equation for ρ can be approximated as

$$\dot{\rho}_u = -\rho_u \left(\frac{V'}{V} + \lambda\right) \approx -\rho_u \lambda \Longrightarrow \rho_u \approx Ce^{-\lambda T} \tag{12.40}$$

when substituting this into Eq. (12.30) we obtain

$$E \propto \frac{1+u}{V} e^{\lambda T} \tag{12.41}$$

and differentiating wrt time we obtain

$$\dot{E} \propto E\left(\frac{V'}{V} + \frac{\lambda}{3}\right) \Rightarrow E \propto e^{\alpha T} \tag{12.42}$$

therefore in the limit $T \to -\infty$ we have $E \to 0$.

In the limit $u \to 0$ ρ can be approximated as

$$\dot{\rho}_u \approx \rho\lambda \frac{1}{2\sqrt{-u}} \implies \rho \approx Ce^{\frac{\lambda}{2}\int \frac{1}{\sqrt{-u(s)}}\mathrm{d}s} \tag{12.43}$$

so that

$$E \propto \frac{C}{V} e^{\frac{\lambda}{2}\int \frac{1}{\sqrt{-u(s)}}\mathrm{d}s} \tag{12.44}$$

so the evolution of E is exponential when $u \approx 0$ and faster than exponential when $u \to 0$. This last condition happens when $\frac{V'}{V} \to 0$ since the point where $\dot{u} = 0$ goes to zero, this moment is also where the gauge of ϕ as a time variable breaks down since for these models $N \to 0$ and after that we would expect for ϕ to remain at the minimum of the potential so that the Hamiltonian includes the cosmological constant.

To compute numerically the values of E, at first, I tried computing two similar solutions and tried to estimate ρ from finite differences. However, this method had one issue, since the density of solutions got so large eventually solutions reached the same value because of floating point precision. The solution for this problem was using the numerical results on Fig. 12.2 and integrating the right hand side of Eq. (12.39) with the trapezoidal rule since the solution is

$$\rho = \exp\left(\int_{T_0}^{T} \frac{V'(s)}{V(s)} + \lambda\sqrt{-u(s)} - \lambda\frac{(1+u(s))}{2\sqrt{-u(s)}}\,\mathrm{d}s\right) \tag{12.45}$$

With these values E could be computed and the results can be found in

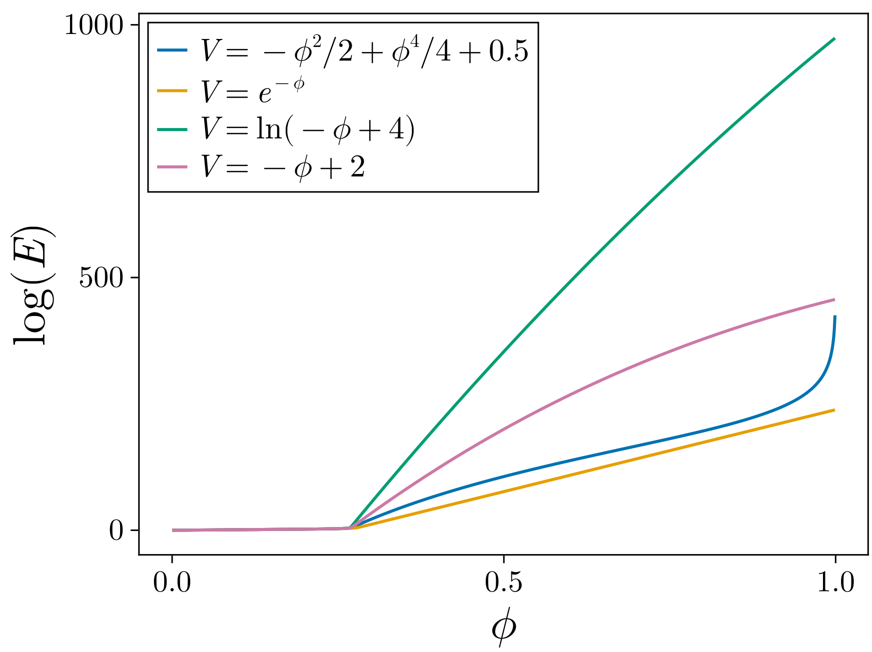

Figure 12.3: In this figure simulations of E using Eq. (12.30). As we can see the fourth order potential is the only one that has a faster than exponential behavior since it is the only one reaching the end of the inflationary epoch. The initial values for E are $E(0) = 1$ and the solutions for u are the ones in Fig. 12.2.

The quantum mechanical approximation solution is in terms of A as a variable. To change to E representation we have to take the fourier transform of the wavefunction.

$$\begin{aligned}\tilde{\Psi}_u(E,T) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}A\, e^{-iR^3EA}\Psi_u(A) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}A\, e^{-iR^3EA}e^{\alpha\mathcal{E}_u + i\gamma_u\frac{A^3}{3}} \\ &\underset{B=-\gamma^{1/3}A}{=} \gamma_u^{1/3}e^{\alpha\mathcal{E}_u}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}B\exp\left(-i\left(-\frac{R^3E}{\gamma^{1/3}}B + \frac{B^3}{3}\right)\right) \\ &\underset{[2]}{=} \gamma_u^{1/3}e^{\alpha\mathcal{E}_u}\ \mathrm{Ai}\left(-\frac{R^3E}{\gamma_u^{1/3}}\right)\end{aligned} \tag{12.46}$$

where $\alpha = 6\sqrt{3}$, $\gamma = \frac{R^3}{V}(1+u)$ y $\mathcal{E}_u = \int_{T_0}^{T}\sqrt{-u(s)}\,\mathrm{d}s$ and Ai is the Airy function of the first kind.

This result is similar to the quantum version of a parabolic trajectory of a particle in a linear potential since the solutions to that system are also in terms of Airy functions of the first kind. The main difference between the two is that in the linear potential case the solutions are of the form $\psi_u(x) = \mathrm{Ai}(x+\beta_x)$ while our solutions are of the form $\psi_u(x) = \mathrm{Ai}(\beta_u x)$. This difference prevents the solutions from being delta normalizable since the inner product of two distinct solutions can be written as

$$\begin{aligned}\langle\Psi_v|\Psi_u\rangle &= \int_{-\infty}^{\infty}\mathrm{d}E\, e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}\tfrac{1}{\sqrt{2\pi}}\int_C \mathrm{d}A\exp\left(-i\left(-R^3EA+\gamma_v\tfrac{A^3}{3}\right)\right)\tfrac{1}{\sqrt{2\pi}}\int_C\mathrm{d}B\exp\left(i\left(-R^3EB+\gamma_u\tfrac{B^3}{3}\right)\right) \\ &\underset{E\to R^3E}{=} \tfrac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3 2\pi}\iint_C \mathrm{d}A\,\mathrm{d}B\exp\left(i\left(\gamma_u\tfrac{B^3}{3}-\gamma_v\tfrac{A^3}{3}\right)\right)\int_{-\infty}^{\infty}\mathrm{d}E\exp(iEA-B) \\ &= \tfrac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}\iint_C\mathrm{d}A\,\mathrm{d}B\exp\left(i\left(\gamma_u\tfrac{B^3}{3}-\gamma_v\tfrac{A^3}{3}\right)\right)\delta(A-B) \\ &= \tfrac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}\int_C\mathrm{d}A\exp\left(i(\gamma_u-\gamma_v)\tfrac{A^3}{3}\right)\end{aligned}$$

where C is a path that starts at complex value $z_{-\infty}$ with $|z_{-\infty}|\to\infty$ and $\frac{2}{3}\pi \le \arg(z_{-\infty}) \le \pi$ and ends at z_∞ with $|z_\infty|\to\infty$ and $0\le\arg(z_\infty)\le\frac{\pi}{3}$. By choosing the path that goes through the real line we can see that the integral is equivalent to twice the real part of the integral over the positive negative real axis[1]. The result is that the integral reduces to an integral from 0 to ∞ with the argument of the curve stays between $\frac{2}{3}\pi$ and $\frac{5}{6}\pi$. Therefore we can write

$$\begin{aligned}\langle\Psi_v|\Psi_u\rangle &= \frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}2\,\mathrm{Re}\left(\int_{C^-}\mathrm{d}A\exp\left(i(\gamma_u-\gamma_v)\frac{A^3}{3}\right)\right) \\ \left(\frac{\text{Variable change}}{B=-i\frac{|\gamma_u-\gamma_v|}{3}A^3}\right) &= 2\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}\mathrm{Re}\left(\frac{\sqrt[3]{i}}{\sqrt[3]{3^2|\gamma_u-\gamma_v|}}\int_{\hat{C}^-}\mathrm{d}B\,B^{\frac{1}{3}-1}e^{-B}\right) \\ &= 2\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}\mathrm{Re}\left(\frac{\sqrt[3]{i}}{\sqrt[3]{3^2|\gamma_u-\gamma_v|}}\Gamma\left(\frac{1}{3},z\right)\Big|_{|z|\to\infty}^{z=0}\right) \\ &= 2\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}\mathrm{Re}\left(\frac{\sqrt[3]{i}}{\sqrt[3]{3^2|\gamma_u-\gamma_v|}}\Gamma\left(\frac{1}{3}\right)\right) \\ &= \frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_u)}}{R^3\sqrt{2\pi}}\frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt[3]{3}\sqrt[3]{|\gamma_u-\gamma_v|}}\end{aligned} \tag{12.47}$$

where $\Gamma\left(\frac{1}{3},z\right)$ is the lower incomplete gamma function, $\Gamma\left(\frac{1}{3}\right)$ is the gamma function and the absolute value on $\gamma_u-\gamma_v$ when defining B together with picking the integration path C^+ ensures that B has positive real part. This is necessary for the incomplete gamma function to converge to the gamma function.

[1] Since $z+z^* = 2\,\mathrm{Re}(z)$

Similarly, the $\langle E\rangle$ diverges towards positive infinity since the integral of the Airy function squared over the negative values of E is finite but the integral over the positive values diverges. This also tells something really interesting. Since the provability of a certain value of E is related to the area of the square norm of the function this means that the predicted provability of finding Euclidean, instead of Lorentzian, spaces is zero. This also implies that the provability of finding a singularity with zero scale factor is also zero.

However infinite values are most certainly not the desired result. Fortunately it is possible, by building wave packets of these plane waves, to find states with well defined values. To do this we define

$$\varphi(E,T) = \int_{-1}^{0} \mathrm{d}v\, f(v)\tilde{\Psi}_u(E,T) \tag{12.48}$$

where f is a function with support on $(-1,0)$. The expected value of E and norm of φ can be computed in terms of all of the defined quantities as

$$\begin{aligned}\langle\varphi|\varphi\rangle &= \int_{-1}^{0}\mathrm{d}v\int_{-1}^{0}\mathrm{d}w\, f^*(v)f^*(w)\langle\Psi_v|\Psi_w\rangle \\ &= \frac{\Gamma\left(\frac{1}{3}\right)}{R^3\sqrt[3]{3}\sqrt{2\pi}}\int_{-1}^{0}\int_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\, f^*(v)f(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{\sqrt[3]{|\gamma_u-\gamma_v|}}\end{aligned} \tag{12.49}$$

and

$$\begin{aligned}\langle\varphi|E|\varphi\rangle &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\iiint_{-\infty}^{\infty}\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}E\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{2\pi}Ee^{i\left(-R^3EA+R^3EB+\gamma_v\frac{A^3}{3}-\gamma_w\frac{B^3}{3}\right)} \\ &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\iint_{-\infty}^{\infty}\mathrm{d}A\,\mathrm{d}B\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{2\pi}e^{i\left(\gamma_v\frac{A^3}{3}-\gamma_w\frac{B^3}{3}\right)}\int_{-\infty}^{\infty}\mathrm{d}E\, Ee^{iR^3(B-A)E} \\ &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\iint_{-\infty}^{\infty}\mathrm{d}A\,\mathrm{d}B\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{R^6\sqrt{2\pi}}e^{i\left(\gamma_v\frac{A^3}{3}-\gamma_w\frac{B^3}{3}\right)}\delta'(B-A) \\ &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{R^6\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}A\,\gamma_w A^2e^{i(\gamma_v-\gamma_w)\frac{A^3}{3}} \\ \left(\begin{matrix}\text{Variable change}\\ C=\frac{1}{\langle\varphi|\varphi\rangle}\gamma_w\frac{A^3}{3}\end{matrix}\right) &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{R^6\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}C\, e^{i\left(\frac{\gamma_v}{\gamma_w}-1\right)C} \\ &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0}\mathrm{d}v\,\mathrm{d}w\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{R^6}\delta\left(\frac{\gamma_v}{\gamma_w}-1\right)\end{aligned} \tag{12.50}$$

Now, since

$$\frac{\partial}{\partial v}\left(\frac{\gamma_v}{\gamma_w}-1\right) = \frac{\partial_v\gamma_v}{\gamma_w} = \frac{1}{\gamma_w}\left(\frac{R^3}{V}\partial_v(1+u)\right) = \frac{R^3}{V\rho_v\gamma_w} \tag{12.51}$$

applying $\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$ to Eq. (12.50)

$$
\begin{aligned}
\langle\varphi|E|\varphi\rangle &= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0} \mathrm{d}v\,\mathrm{d}w\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{R^6}\rho_v(1+u_w)\delta(v-w) \\
&= \frac{1}{\langle\varphi|\varphi\rangle}\iint_{-1}^{0} \mathrm{d}w\,\mathrm{d}v\, f(v)f^*(w)\frac{e^{\alpha(\mathcal{E}_v+\mathcal{E}_w)}}{R^6}\rho_v(1+u_w)\delta(v-w) \\
&= \frac{1}{\langle\varphi|\varphi\rangle}\int_{-1}^{0} \mathrm{d}w\,|f(w)|^2\frac{e^{2\alpha\mathcal{E}_w}}{R^6}\rho_w(1+u_w)
\end{aligned}
\tag{12.52}
$$

By using a uniform distribution between -0.7 and -0.2 at time $T = 0.3$

$$
f(v) = \begin{cases} 2 \text{ if} -0.7 \leq v \leq -0.2 \\ 0 \text{ otherwise} \end{cases} \tag{12.53}
$$

and the fourth order potential from Fig. 12.3 we can simulate the results. The norm and expected values can be found in Fig. 12.4 and Fig. 12.5.

As we can see the norm of the wavefunction is not a constant. This can be traced to the chosen Hamiltonian, since the Hamiltonian is not Hermitian, the time evolution operator

$$
U(\Delta t) = \exp\left(-i\hat{H}\Delta t/\hbar\right) \tag{12.54}
$$

is not unitary. Therefore the norm is not a conserved quantity.

It is also interesting to note how Fig. 12.5 and the fourth order potential solution in Fig. 12.3. This can be understood in terms

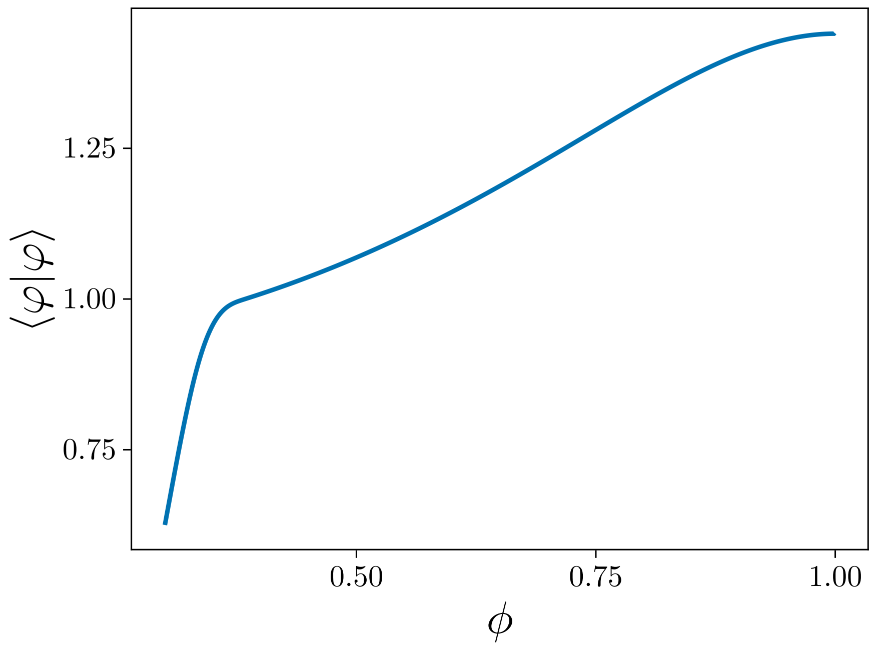

Figure 12.4: In this figure the norm of the wave packet described in Eq. (12.53) is computed by using Eq. (12.49). The number of steps in time was of 10000 and for the initial conditions in u was 100.

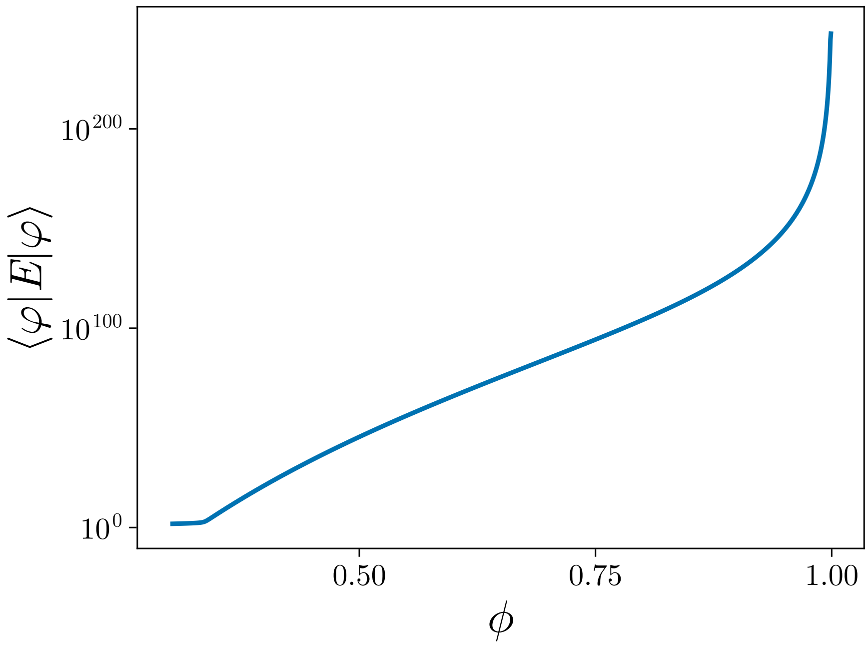

Figure 12.5: In this figure the expected value of E is computed from Eq. (12.52). using the distribution in Eq. (12.53). The number of steps in time for integration was of 10000 and in the initial conditions for u was 100.

Bibliography

[1] S. Alexander, J. Malecki, and L. Smolin, “Quantum gravity and inflation,” *Phys. Rev. D*, vol. 70, p. 44025, 2004, doi: 10.1103/PhysRevD.70.044025.

[2] O. Vallée and M. Soares, *Airy Functions and Applications to Physics*, 2nd ed. Hackensack, NJ: Imperial College Press, 2010.