Catalogue of Spacetimes
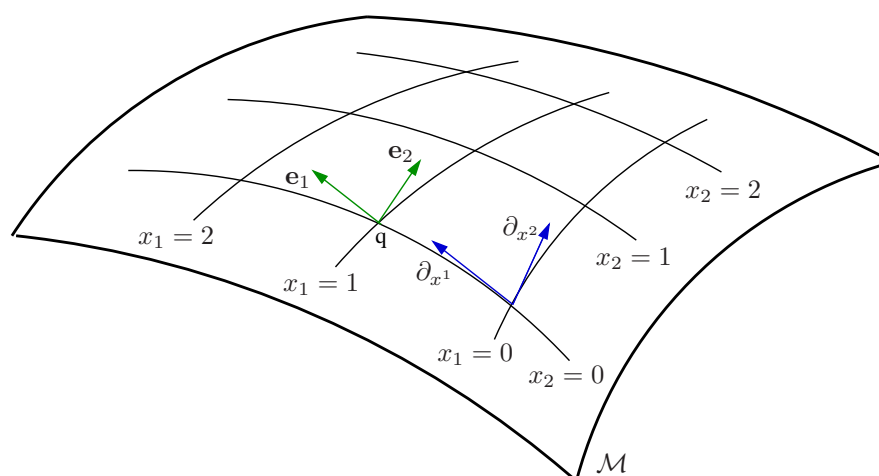
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Contents

Chapter 1

Introduction and Notation

The *Catalogue of Spacetimes* is a collection of four-dimensional Lorentzian spacetimes in the context of the General Theory of Relativity (GR). The aim of the catalogue is to give a quick reference for students who need some basic facts of the most well-known spacetimes in GR. For a detailed discussion of a metric, the reader is referred to the standard literature or the original articles. Important resources for exact solutions are the book by Stephani et al[SKM$^+$03] and the book by Griffiths and Podolský[GP09].

Most of the metrics in this catalogue are implemented in the Motion4D-library[MG09] and can be visualized using the GeodesicViewer[MG10]. Except for the Minkowski and Schwarzschild spacetimes, the metrics are sorted by their names.

1.1 Notation

The notation we use in this catalogue is as follows:

Indices: Coordinate indices are represented either by Greek letters or by coordinate names. Tetrad indices are indicated by Latin letters or coordinate names in brackets.

Einstein sum convention: When an index appears twice in a single term, once as lower index and once as upper index, we build the sum over all indices:

$$\zeta_\mu \zeta^\mu \equiv \sum_{\mu=0}^{3} \zeta_\mu \zeta^\mu. \tag{1.1.1}$$

Vectors: A coordinate vector in x^μ direction is represented as $\partial_{x^\mu} \equiv \partial_\mu$. For arbitrary vectors, we use boldface symbols. Hence, a vector **a** in coordinate representation reads $\mathbf{a} = a^\mu \partial_\mu$.

Derivatives: Partial derivatives are indicated by a comma, $\partial \psi / \partial x^\mu \equiv \partial_\mu \psi \equiv \psi_{,\mu}$, whereas covariant derivatives are indicated by a semicolon, $\nabla \psi = \psi_{;\mu}$.

Symmetrization and Antisymmetrization brackets:

$$a_{(\mu} b_{\nu)} = \frac{1}{2} \left(a_\mu b_\nu + a_\nu b_\mu \right), \qquad a_{[\mu} b_{\nu]} = \frac{1}{2} \left(a_\mu b_\nu - a_\nu b_\mu \right) \tag{1.1.2}$$

1.2 General remarks

The Einstein field equation in the most general form reads[MTW73]

$$G_{\mu\nu} = \varkappa T_{\mu\nu} - \Lambda g_{\mu\nu}, \qquad \varkappa = \frac{8\pi G}{c^4}, \tag{1.2.1}$$

with the symmetric and divergence-free Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R, the metric tensor $g_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$, the cosmological constant Λ, Newton's gravitational constant G, and the speed of light c. Because the Einstein tensor is divergence-free, the conservation equation $T^{\mu\nu}_{\ ;\nu} = 0$ is automatically fulfilled.

A solution to the field equation is given by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1.2.2}$$

with the symmetric, covariant metric tensor $g_{\mu\nu}$. The contravariant metric tensor $g^{\mu\nu}$ is related to the covariant tensor via $g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu$ with the Kronecker-δ. Even though $g_{\mu\nu}$ is only a component of the metric tensor $\mathbf{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu$, we will also call $g_{\mu\nu}$ the metric tensor.

Note that, in this catalogue, we mostly use the convention that the signature of the metric is $+2$. In general, we will also keep the physical constants c and G within the metrics.

1.3 Basic objects of a metric

The basic objects of a metric are the Christoffel symbols, the Riemann and Ricci tensors as well as the Ricci and Kretschmann scalars which are defined as follows:

Christoffel symbols of the first kind:[1]

$$\Gamma_{\nu\lambda\mu} = \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right) \tag{1.3.1}$$

with the relation

$$g_{\nu\lambda,\mu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} \tag{1.3.2}$$

Christoffel symbols of the second kind:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \tag{1.3.3}$$

which are related to the Christoffel symbols of the first kind via

$$\Gamma^\mu_{\nu\lambda} = g^{\mu\rho}\Gamma_{\nu\lambda\rho} \tag{1.3.4}$$

Riemann tensor:

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho} \tag{1.3.5}$$

or

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^\lambda{}_{\nu\rho\sigma} = \Gamma_{\nu\sigma\mu,\rho} - \Gamma_{\nu\rho\mu,\sigma} + \Gamma^\lambda_{\nu\rho}\Gamma_{\mu\sigma\lambda} - \Gamma^\lambda_{\nu\sigma}\Gamma_{\mu\sigma\lambda} \tag{1.3.6}$$

with symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \qquad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \qquad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \tag{1.3.7}$$

and

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \tag{1.3.8}$$

Ricci tensor:

$$R_{\mu\nu} = g^{\rho\sigma}R_{\rho\mu\sigma\nu} = R^\rho{}_{\mu\rho\nu} \tag{1.3.9}$$

Ricci and Kretschmann scalar:

$$\mathscr{R} = g^{\mu\nu}R_{\mu\nu} = R^\mu{}_\mu, \qquad \mathscr{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = R^{\gamma\delta}{}_{\alpha\beta}R^{\alpha\beta}{}_{\gamma\delta} \tag{1.3.10}$$

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[1]The notation of the Christoffel symbols of the first kind differs from the one used by Rindler[Rin01], $\Gamma^{\text{Rindler}}_{\mu\nu\lambda} = \Gamma^{\text{CoS}}_{\nu\lambda\mu}$.

Weyl tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3}R g_{\mu[\rho}g_{\sigma]\nu} \tag{1.3.11}$$

If we change the signature of a metric, these basic objects transform as follows:

$$\Gamma^{\mu}_{\nu\lambda} \mapsto \Gamma^{\mu}_{\nu\lambda}, \qquad R_{\mu\nu\rho\sigma} \mapsto -R_{\mu\nu\rho\sigma}, \qquad C_{\mu\nu\rho\sigma} \mapsto -C_{\mu\nu\rho\sigma}, \tag{1.3.12a}$$

$$R_{\mu\nu} \mapsto R_{\mu\nu}, \qquad \mathcal{R} \mapsto -\mathcal{R}, \qquad \mathcal{K} \mapsto \mathcal{K}. \tag{1.3.12b}$$

Covariant derivative

$$\nabla_{\lambda}g_{\mu\nu} = g_{\mu\nu;\lambda} = 0. \tag{1.3.13}$$

Covariant derivative of the vector field ψ^{μ}:

$$\nabla_{\nu}\psi^{\mu} = \psi^{\mu}_{;\nu} = \partial_{\nu}\psi^{\mu} + \Gamma^{\mu}_{\nu\lambda}\psi^{\lambda} \tag{1.3.14}$$

Covariant derivative of a r-s-tensor field:

$$\nabla_c T^{a_1...a_r}{}_{b_1...b_s} = \partial_c T^{a_1...a_r}{}_{b_1...b_s} + \Gamma^{a_1}_{dc}T^{d...a_r}{}_{b_1...b_s} + ... + \Gamma^{a_r}_{dc}T^{a_1...a_{r-1}d}{}_{b_1...b_s}$$
$$- \Gamma^{d}_{b_1 c}T^{a_1...a_r}{}_{d...b_s} - ... - \Gamma^{d}_{b_s c}T^{a_1...a_r}{}_{b_1...b_{s-1}d} \tag{1.3.15}$$

Killing equation:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{1.3.16}$$

1.4 Natural local tetrad and initial conditions for geodesics

We will call a local tetrad natural if it is adapted to the symmetries or the coordinates of the spacetime. The four base vectors $\mathbf{e}_{(i)} = e^{\mu}_{(i)}\partial_{\mu}$ are given with respect to coordinate directions $\partial/\partial x^{\mu} = \partial_{\mu}$, compare Nakahara[Nak90] or Chandrasekhar[Cha06] for an introduction to the tetrad formalism. The inverse or dual tetrad is given by $\theta^{(i)} = \theta^{(i)}_{\mu}dx^{\mu}$ with

$$\theta^{(i)}_{\mu}e^{\mu}_{(j)} = \delta^{(i)}_{(j)} \qquad \text{and} \qquad \theta^{(i)}_{\mu}e^{\nu}_{(i)} = \delta^{\nu}_{\mu}. \tag{1.4.1}$$

Note that we us Latin indices in brackets for tetrads and Greek indices for coordinates.

1.4.1 Orthonormality condition

To be applicable as a local reference frame (Minkowski frame), a local tetrad $\mathbf{e}_{(i)}$ has to fulfill the orthonormality condition

$$\langle \mathbf{e}_{(i)}, \mathbf{e}_{(j)} \rangle_{\mathbf{g}} = \mathbf{g}\left(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right) = g_{\mu\nu}e^{\mu}_{(i)}e^{\nu}_{(j)} \stackrel{!}{=} \eta_{(i)(j)}, \tag{1.4.2}$$

where $\eta_{(i)(j)} = \text{diag}(\mp 1, \pm 1, \pm 1, \pm 1)$ depending on the signature $\text{sign}(\mathbf{g}) = \pm 2$ of the metric. Thus, the line element of a metric can be written as

$$ds^2 = \eta_{(i)(j)}\theta^{(i)}\theta^{(j)} = \eta_{(i)(j)}\theta^{(i)}_{\mu}\theta^{(j)}_{\nu}dx^{\mu}dx^{\nu}. \tag{1.4.3}$$

To obtain a local tetrad $\mathbf{e}_{(i)}$, we could first determine the dual tetrad $\theta^{(i)}$ via Eq. (1.4.3). If we combine all four dual tetrad vectors into one matrix Θ, we only have to determine its inverse Θ^{-1} to find the tetrad vectors,

$$\Theta = \begin{pmatrix} \theta^{(0)}_0 & \theta^{(0)}_1 & \theta^{(0)}_2 & \theta^{(0)}_3 \\ \theta^{(1)}_0 & \theta^{(1)}_1 & \theta^{(1)}_2 & \theta^{(1)}_3 \\ \theta^{(2)}_0 & \theta^{(2)}_1 & \theta^{(2)}_2 & \theta^{(2)}_3 \\ \theta^{(3)}_0 & \theta^{(3)}_1 & \theta^{(3)}_2 & \theta^{(3)}_3 \end{pmatrix} \quad \Rightarrow \quad \Theta^{-1} = \begin{pmatrix} e^0_{(0)} & e^0_{(1)} & e^0_{(2)} & e^0_{(3)} \\ e^1_{(0)} & e^1_{(1)} & e^1_{(2)} & e^1_{(3)} \\ e^2_{(0)} & e^2_{(1)} & e^2_{(2)} & e^2_{(3)} \\ e^3_{(0)} & e^3_{(1)} & e^3_{(2)} & e^3_{(3)} \end{pmatrix}. \tag{1.4.4}$$

There are also several useful relations:

$$e_{(a)\mu} = g_{\mu\nu}e^{\nu}_{(a)}, \qquad \eta_{(a)(b)} = e^{\mu}_{(a)}e_{(b)\mu}, \qquad e_{(b)\mu} = \eta_{(a)(b)}\theta^{(a)}_{\mu}, \tag{1.4.5a}$$

$$\theta^{(b)}_{\mu} = \eta^{(a)(b)}e_{(a)\mu}, \qquad g_{\mu\nu} = e_{(a)\mu}\theta^{(a)}_{\nu}, \qquad \eta^{(a)(b)} = \theta^{(a)}_{\mu}\theta^{(b)}_{\nu}g^{\mu\nu}. \tag{1.4.5b}$$

1.4.2 Tetrad transformations

Instead of the above found local tetrad that was directly constructed from the spacetime metric, we can also use any other local tetrad

$$\hat{\mathbf{e}}_{(i)} = A^{k}_{i}\mathbf{e}_{(k)}, \tag{1.4.6}$$

where $\mathbf{A}$ is an element of the Lorentz group $O(1,3)$. Hence $\mathbf{A}^{T}\eta\mathbf{A} = \eta$ and $(\det\mathbf{A})^{2} = 1$.
Lorentz-transformation in the direction $n^{a} = (\sin\chi\cos\xi, \sin\chi\sin\xi, \cos\xi)^{T} = n_{a}$ with $\gamma = 1/\sqrt{1-\beta^{2}}$,

$$\Lambda^{0}_{0} = \gamma, \qquad \Lambda^{0}_{a} = -\beta\gamma n_{a}, \qquad \Lambda^{a}_{0} = -\beta\gamma n^{a}, \qquad \Lambda^{a}_{b} = (\gamma-1)n^{a}n_{b} + \delta^{a}_{b}. \tag{1.4.7}$$

1.4.3 Ricci rotation-, connection-, and structure coefficients

The Ricci rotation coefficients $\gamma_{(i)(j)(k)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\gamma_{(i)(j)(k)} := g_{\mu\lambda}e^{\mu}_{(i)}\nabla_{\mathbf{e}_{(k)}}e^{\lambda}_{(j)} = g_{\mu\lambda}e^{\mu}_{(i)}e^{\nu}_{(k)}\nabla_{\nu}e^{\lambda}_{(j)} = g_{\mu\lambda}e^{\mu}_{(i)}e^{\nu}_{(k)}\left(\partial_{\nu}e^{\lambda}_{(j)} + \Gamma^{\lambda}_{\nu\beta}e^{\beta}_{(j)}\right). \tag{1.4.8}$$

They are antisymmetric in the first two indices, $\gamma_{(i)(j)(k)} = -\gamma_{(j)(i)(k)}$, which follows from the definition, Eq. (1.4.8), and the relation

$$0 = \partial_{\mu}\eta_{(i)(j)} = \nabla_{\mu}\left(g_{\beta\nu}e^{\beta}_{(i)}e^{\nu}_{(j)}\right), \tag{1.4.9}$$

where $\nabla_{\mu}g_{\beta\nu} = 0$, compare [Cha06]. Otherwise, we have

$$\gamma^{(i)}{}_{(j)(k)} = \theta^{(i)}_{\lambda}e^{\nu}_{(k)}\nabla_{\nu}e^{\lambda}_{(j)} = -e^{\lambda}_{(j)}e^{\nu}_{(k)}\nabla_{\nu}\theta^{(i)}_{\lambda}. \tag{1.4.10}$$

The contraction of the first and the last index is given by

$$\gamma_{(j)} = \gamma^{(k)}{}_{(j)(k)} = \eta^{(k)(i)}\gamma_{(i)(j)(k)} = -\gamma_{(0)(j)(0)} + \gamma_{(1)(j)(1)} + \gamma_{(2)(j)(2)} + \gamma_{(3)(j)(3)} = \nabla_{\nu}e^{\nu}_{(j)}. \tag{1.4.11}$$

The connection coefficients $\omega^{(m)}_{(j)(n)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\omega^{(m)}_{(j)(n)} := \theta^{(m)}_{\mu}\nabla_{\mathbf{e}_{(j)}}e^{\mu}_{(n)} = \theta^{(m)}_{\mu}e^{\alpha}_{(j)}\nabla_{\alpha}e^{\mu}_{(n)} = \theta^{(m)}_{\mu}e^{\alpha}_{(j)}\left(\partial_{\alpha}e^{\mu}_{(n)} + \Gamma^{\mu}_{\alpha\beta}e^{\beta}_{(n)}\right), \tag{1.4.12}$$

compare Nakahara[Nak90]. They are related to the Ricci rotation coefficients via

$$\gamma_{(i)(j)(k)} = \eta_{(i)(m)}\omega^{(m)}_{(k)(j)}. \tag{1.4.13}$$

Furthermore, the local tetrad has a non-vanishing Lie-bracket $[X,Y]^{\nu} = X^{\mu}\partial_{\mu}Y^{\nu} - Y^{\mu}\partial_{\mu}X^{\nu}$. Thus,

$$\left[\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right] = c^{(k)}_{(i)(j)}\mathbf{e}_{(k)} \qquad \text{or} \qquad c^{(k)}_{(i)(j)} = \theta^{(k)}\left[\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right]. \tag{1.4.14}$$

The structure coefficients $c^{(k)}_{(i)(j)}$ are related to the connection coefficients or the Ricci rotation coefficients via

$$c^{(k)}_{(i)(j)} = \omega^{(k)}_{(i)(j)} - \omega^{(k)}_{(j)(i)} = \eta^{(k)(m)}\left(\gamma_{(m)(j)(i)} - \gamma_{(m)(i)(j)}\right) = \gamma^{(k)}{}_{(j)(i)} - \gamma^{(k)}{}_{(i)(j)}. \tag{1.4.15}$$

1.4.4 Riemann-, Ricci-, and Weyl-tensor with respect to a local tetrad

The transformations between the coordinate representations of the Riemann-, Ricci-, and Weyl-tensors and their representation with respect to a local tetrad $\mathbf{e}_{(i)}$ are given by

$$R_{(a)(b)(c)(d)} = R_{\mu\nu\rho\sigma} e^{\mu}_{(a)} e^{\nu}_{(b)} e^{\rho}_{(c)} e^{\sigma}_{(d)}, \tag{1.4.16a}$$

$$R_{(a)(b)} = R_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}, \tag{1.4.16b}$$

$$C_{(a)(b)(c)(d)} = C_{\mu\nu\rho\sigma} e^{\mu}_{(a)} e^{\nu}_{(b)} e^{\rho}_{(c)} e^{\sigma}_{(d)}$$
$$= R_{(a)(b)(c)(d)} - \frac{1}{2} \left(\eta_{(a)[(c)} R_{(d)](b)} - \eta_{(b)[(c)} R_{(d)](a)} \right) + \frac{R}{3} \eta_{(a)[(c)} \eta_{(d)](b)}. \tag{1.4.16c}$$

1.4.5 Null or timelike directions

A null or timelike direction $\upsilon = \upsilon^{(i)} \mathbf{e}_{(i)}$ with respect to a local tetrad $\mathbf{e}_{(i)}$ can be written as

$$\upsilon = \upsilon^{(0)} \mathbf{e}_{(0)} + \psi \left(\sin\chi \cos\xi \, \mathbf{e}_{(1)} + \sin\chi \sin\xi \, \mathbf{e}_{(2)} + \cos\chi \, \mathbf{e}_{(3)} \right) = \upsilon^{(0)} \mathbf{e}_{(0)} + \psi \mathbf{n}. \tag{1.4.17}$$

In the case of a null direction we have $\psi = 1$ and $\upsilon^{(0)} = \pm 1$. A timelike direction can be identified with an initial four-velocity $\mathbf{u} = c\gamma(\mathbf{e}_0 + \beta\mathbf{n})$, where

$$\mathbf{u}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}} = c^2\gamma^2 \langle \mathbf{e}_{(0)} + \beta\mathbf{n}, \mathbf{e}_{(0)} + \beta\mathbf{n} \rangle = c^2\gamma^2 \left(-1 + \beta^2 \right) = \mp c^2, \qquad \text{sign}(\mathbf{g}) = \pm 2. \tag{1.4.18}$$

Thus, $\psi = c\beta\gamma$ and $\upsilon^0 = \pm c\gamma$. The sign of $\upsilon^{(0)}$ determines the time direction.
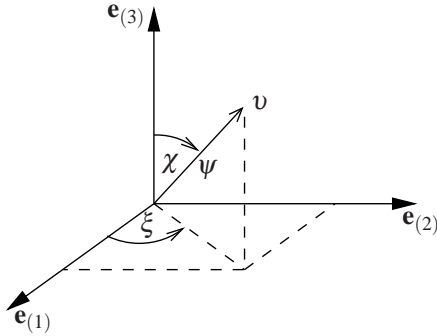
Figure 1.1: Null or timelike direction υ with respect to the local tetrad $\mathbf{e}_{(i)}$.

The transformations between a local direction $\upsilon^{(i)}$ and its coordinate representation υ^{μ} read

$$\upsilon^{\mu} = \upsilon^{(i)} e^{\mu}_{(i)} \qquad \text{and} \qquad \upsilon^{(i)} = \theta^{(i)}_{\mu} \upsilon^{\mu}. \tag{1.4.19}$$

1.4.6 Local tetrad for diagonal metrics

If a spacetime is represented by a diagonal metric

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \tag{1.4.20}$$

the natural local tetrad reads

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{g_{00}}} \partial_0, \quad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{11}}} \partial_1, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{22}}} \partial_2, \quad \mathbf{e}_{(3)} = \frac{1}{\sqrt{g_{33}}} \partial_3, \tag{1.4.21}$$

given that the metric coefficients are well behaved. Analogously, the dual tetrad reads

$$\theta^{(0)} = \sqrt{g_{00}} \, dx^0, \quad \theta^{(1)} = \sqrt{g_{11}} \, dx^1, \quad \theta^{(2)} = \sqrt{g_{22}} \, dx^2, \quad \theta^{(3)} = \sqrt{g_{33}} \, dx^3. \tag{1.4.22}$$

1.4.7 Local tetrad for stationary axisymmetric spacetimes

The line element of a stationary axisymmetric spacetime is given by

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dt\,d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2, \tag{1.4.23}$$

where the metric components are functions of r and ϑ only.
The local tetrad for an observer on a stationary circular orbit, $(r = \text{const}, \vartheta = \text{const})$, with four velocity $\mathbf{u} = c\Gamma\left(\partial_t + \zeta\partial_\varphi\right)$ can be defined as, compare Bini[BJ00],

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{rr}}}\partial_r, \qquad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{\vartheta\vartheta}}}\partial_\vartheta, \tag{1.4.24a}$$

$$\mathbf{e}_{(3)} = \Delta\Gamma\left[\pm(g_{t\varphi} + \zeta g_{\varphi\varphi})\partial_t \mp (g_{tt} + \zeta g_{t\varphi})\partial_\varphi\right], \tag{1.4.24b}$$

where

$$\Gamma = \frac{1}{\sqrt{-\left(g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}\right)}} \qquad \text{and} \qquad \Delta = \frac{1}{\sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}. \tag{1.4.25}$$

The angular velocity ζ is limited due to $g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi} < 0$

$$\zeta_{\min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \qquad \text{and} \qquad \zeta_{\max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{1.4.26}$$

with $\omega = -g_{t\varphi}/g_{\varphi\varphi}$.
For $\zeta = 0$, the observer is static with respect to spatial infinity. The locally non-rotating frame (LNRF) has angular velocity $\zeta = \omega$, see also MTW[MTW73], exercise 33.3.
Static limit: $\zeta_{\min} = 0 \quad \Rightarrow \quad g_{tt} = 0$.
The transformation between the local direction $v^{(i)}$ and the coordinate direction v^μ reads

$$v^0 = \Gamma\left(v^{(0)} \pm v^{(3)}\Delta w_1\right), \qquad v^1 = \frac{v^{(1)}}{\sqrt{g_{rr}}}, \qquad v^2 = \frac{v^{(2)}}{\sqrt{g_{\vartheta\vartheta}}}, \qquad v^3 = \Gamma\left(v^{(0)}\zeta \mp v^{(3)}\Delta w_2\right), \tag{1.4.27}$$

with

$$w_1 = g_{t\varphi} + \zeta g_{\varphi\varphi} \qquad \text{and} \qquad w_2 = g_{tt} + \zeta g_{t\varphi}. \tag{1.4.28}$$

The back transformation reads

$$v^{(0)} = \frac{1}{\Gamma}\frac{v^0 w_2 + v^3 w_1}{\zeta w_1 + w_2}, \qquad v^{(1)} = \sqrt{g_{rr}}\,v^1, \quad v^{(2)} = \sqrt{g_{\vartheta\vartheta}}\,v^2, \qquad v^{(3)} = \pm\frac{1}{\Delta\Gamma}\frac{\zeta v^0 - v^3}{\zeta w_1 + w_2}. \tag{1.4.29}$$

Note, to obtain a right-handed local tetrad, $\det\left(e_{(i)}^\mu\right) > 0$, the upper sign has to be used.

1.5 Newman-Penrose tetrad and spin-coefficients

The Newman-Penrose tetrad consists of four null vectors $\mathbf{e}_{(i)}^\star = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, where $\mathbf{l}$ and $\mathbf{n}$ are real and $\mathbf{m}$ and $\bar{\mathbf{m}}$ are complex conjugates; see Penrose and Rindler[PR84] or Chandrasekhar[Cha06] for a thorough discussion. The Newman-Penrose (NP) tetrad has to fulfill the orthonormality relation

$$\left\langle \mathbf{e}_{(i)}^\star, \mathbf{e}_{(j)}^\star \right\rangle = \eta_{(i)(j)}^\star \qquad \text{with} \qquad \eta_{(i)(j)}^\star = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1.5.1}$$

A straightforward relation between the NP tetrad and the natural local tetrad, as discussed in Sec. 1.4, is given by

$$\mathbf{l} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_{(0)} + \mathbf{e}_{(1)}\right), \quad \mathbf{n} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_{(0)} - \mathbf{e}_{(1)}\right), \quad \mathbf{m} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_{(2)} + i\mathbf{e}_{(3)}\right), \tag{1.5.2}$$

where the upper/lower sign has to be used for metrics with positive/negative signature. The Ricci rotation-coefficients of a NP tetrad are now called *spin coefficients* and are designated by specific symbols:

$$\kappa = \gamma_{(2)(1)(1)}, \qquad \rho = \gamma_{(2)(0)(3)}, \qquad \varepsilon = \frac{1}{2}\left(\gamma_{(1)(0)(0)} + \gamma_{(2)(3)(0)}\right), \tag{1.5.3a}$$

$$\sigma = \gamma_{(2)(0)(2)}, \qquad \mu = \gamma_{(1)(3)(2)}, \qquad \gamma = \frac{1}{2}\left(\gamma_{(1)(0)(1)} + \gamma_{(2)(3)(1)}\right), \tag{1.5.3b}$$

$$\lambda = \gamma_{(1)(3)(3)}, \qquad \tau = \gamma_{(2)(0)(1)}, \qquad \alpha = \frac{1}{2}\left(\gamma_{(1)(0)(3)} + \gamma_{(2)(3)(3)}\right), \tag{1.5.3c}$$

$$\nu = \gamma_{(1)(3)(1)}, \qquad \pi = \gamma_{(1)(3)(0)}, \qquad \beta = \frac{1}{2}\left(\gamma_{(1)(0)(2)} + \gamma_{(2)(3)(2)}\right). \tag{1.5.3d}$$

1.6 Coordinate relations

1.6.1 Spherical and Cartesian coordinates

The well-known relation between the spherical coordinates (r, ϑ, φ) and the Cartesian coordinates (x, y, z), compare Fig. 1.2, are

$$x = r\sin\vartheta\cos\varphi, \qquad y = r\sin\vartheta\sin\varphi, \qquad z = r\cos\vartheta, \tag{1.6.1}$$

and

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \vartheta = \arctan 2(\sqrt{x^2 + y^2}, z), \qquad \varphi = \arctan 2(y, x), \tag{1.6.2}$$

where $\arctan 2()$ ensures that $\varphi \in [0, 2\pi)$ and $\vartheta \in (0, \pi)$.

Figure 1.2: Relation between spherical and Cartesian coordinates.

The total differentials of the spherical coordinates read

$$dr = \frac{x\,dx + y\,dy + z\,dz}{r}, \qquad d\vartheta = \frac{xz\,dx + yz\,dy - (x^2 + y^2)dz}{r^2\sqrt{x^2 + y^2}}, \qquad d\varphi = \frac{-y\,dx + x\,dy}{x^2 + y^2}, \tag{1.6.3}$$

whereas the coordinate derivatives read

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z \quad = \sin\vartheta\cos\varphi\,\partial_x + \sin\vartheta\sin\varphi\,\partial_y + \cos\vartheta\,\partial_z, \tag{1.6.4a}$$

$$\partial_\vartheta = \frac{\partial x}{\partial\vartheta}\partial_x + \frac{\partial y}{\partial\vartheta}\partial_y + \frac{\partial z}{\partial\vartheta}\partial_z = r\cos\vartheta\cos\varphi\,\partial_x + r\cos\vartheta\sin\varphi\,\partial_y - r\sin\vartheta\,\partial_z, \tag{1.6.4b}$$

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y + \frac{\partial z}{\partial\varphi}\partial_z \quad = -r\sin\vartheta\sin\varphi\,\partial_x + r\sin\vartheta\cos\varphi\,\partial_y, \tag{1.6.4c}$$

and

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \vartheta}{\partial x}\partial_\vartheta + \frac{\partial \varphi}{\partial x}\partial_\varphi = \sin\vartheta\cos\varphi\,\partial_r + \frac{\cos\vartheta\cos\varphi}{r}\partial_\vartheta - \frac{\sin\varphi}{r\sin\vartheta}\partial_\varphi, \tag{1.6.5a}$$

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial \vartheta}{\partial y}\partial_\vartheta + \frac{\partial \varphi}{\partial y}\partial_\varphi = \sin\vartheta\sin\varphi\,\partial_r + \frac{\cos\vartheta\sin\varphi}{r}\partial_\vartheta + \frac{\cos\varphi}{r\sin\vartheta}\partial_\varphi, \tag{1.6.5b}$$

$$\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial \vartheta}{\partial z}\partial_\vartheta + \frac{\partial \varphi}{\partial z}\partial_\varphi = \cos\vartheta\,\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta. \tag{1.6.5c}$$

1.6.2 Cylindrical and Cartesian coordinates

The relation between cylindrical coordinates (r,φ,z) and Cartesian coordinates (x,y,z) is given by

$$x = r\cos\varphi, \quad y = r\sin\varphi, \qquad \text{and} \qquad r = \sqrt{x^2+y^2}, \quad \varphi = \arctan 2(y,x), \tag{1.6.6}$$

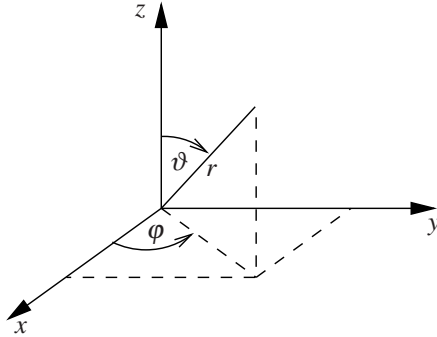where $\arctan 2()$ again ensures that the angle $\varphi \in [0,2\pi)$.
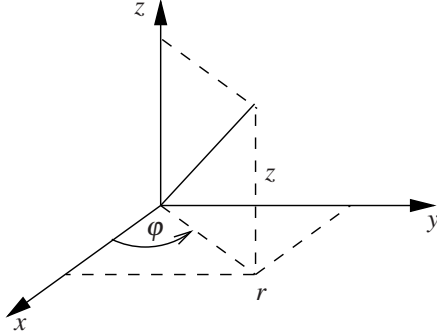
Figure 1.3: Relation between cylindrical and Cartesian coordinates.

The total differentials of the spherical coordinates are given by

$$dr = \frac{x\,dx + y\,dy}{r}, \qquad d\varphi = \frac{-y\,dx + x\,dy}{r^2}, \tag{1.6.7}$$

and

$$dx = \cos\varphi\,dr - r\sin\varphi\,d\varphi, \qquad dy = \sin\varphi\,dr + r\cos\varphi\,d\varphi. \tag{1.6.8}$$

The coordinate derivatives are

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\,\partial_x + \sin\varphi\,\partial_y, \tag{1.6.9a}$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y = -r\sin\varphi\,\partial_x + r\cos\varphi\,\partial_y m \tag{1.6.9b}$$

and

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \varphi}{\partial x}\partial_\varphi = \cos\varphi\,\partial_r - \frac{\sin\varphi}{r}\partial_y, \tag{1.6.10a}$$

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial \varphi}{\partial y}\partial_\varphi = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\partial_y. \tag{1.6.10b}$$

1.7 Embedding diagram

A two-dimensional hypersurface with line segment

$$d\sigma^2 = g_{rr}(r)dr^2 + g_{\varphi\varphi}(r)d\varphi^2 \tag{1.7.1}$$

can be embedded in a three-dimensional Euclidean space with cylindrical coordinates,

$$d\sigma^2 = \left[1 + \left(\frac{dz}{d\rho} \right)^2 \right] d\rho^2 + \rho^2 d\varphi^2. \tag{1.7.2}$$

With $\rho(r)^2 = g_{\varphi\varphi}(r)$ and $dr = (dr/d\rho)d\rho$, we obtain for the embedding function $z = z(r)$,

$$\frac{dz}{dr} = \pm \sqrt{ g_{rr} - \left(\frac{d\sqrt{g_{\varphi\varphi}}}{dr} \right)^2 }. \tag{1.7.3}$$

If $g_{\varphi\varphi}(r) = r^2$, then $d\sqrt{g_{\varphi\varphi}}/dr = 1$.

1.8 Equations of motion and transport equations

1.8.1 Geodesic equation

The geodesic equation reads

$$\frac{D^2 x^\mu}{d\lambda^2} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \tag{1.8.1}$$

with the affine parameter λ. For timelike geodesics, however, we replace the affine parameter by the proper time τ.

The geodesic equation (1.8.1) is a system of ordinary differential equations of second order. Hence, to solve these differential equations, we need an initial position $x^\mu(\lambda = 0)$ as well as an initial direction $(dx^\mu/d\lambda)(\lambda = 0)$. This initial direction has to fulfill the constraint equation

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \kappa c^2, \tag{1.8.2}$$

where $\kappa = 0$ for lightlike and $\kappa = \mp 1$, $(\text{sign}(\mathbf{g}) = \pm 2)$, for timelike geodesics.

The initial direction can also be determined by means of a local reference frame, compare sec. 1.4.5, that automatically fulfills the constraint equation (1.8.2). If we use the natural local tetrad as local reference frame, we have

$$\left. \frac{dx^\mu}{d\lambda} \right|_{\lambda=0} = \upsilon^\mu = \upsilon^{(i)} e^\mu_{(i)}. \tag{1.8.3}$$

1.8.2 Fermi-Walker transport

The Fermi-Walker transport, see e.g. Stephani[SS90], of a vector $\mathbf{X} = X^\mu \partial_\mu$ along the worldline $x^\mu(\tau)$ with four-velocity $\mathbf{u} = u^\mu(\tau)\partial_\mu$ is given by $\mathbb{F}_\mathbf{u} X^\mu = 0$ with

$$\mathbb{F}_\mathbf{u} X^\mu := \frac{dX^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho X^\sigma + \frac{1}{c^2} \left(u^\sigma a^\mu - a^\sigma u^\mu \right) g_{\rho\sigma} X^\rho. \tag{1.8.4}$$

The four-acceleration follows from the four-velocity via

$$a^\mu = \frac{D^2 x^\mu}{d\tau^2} = \frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma. \tag{1.8.5}$$

1.8.3 Parallel transport

If the four-acceleration vanishes, the Fermi-Walker transport simplifies to the parallel transport $\mathbb{P}_\mathbf{u} X^\mu = 0$ with

$$\mathbb{P}_\mathbf{u} X^\mu := \frac{DX^\mu}{d\tau} = \frac{dX^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho X^\sigma. \tag{1.8.6}$$

1.8.4   Euler-Lagrange formalism

A detailed discussion of the Euler-Lagrange formalism can be found, e.g., in Rindler[Rin01]. The Lagrangian $\mathscr{L}$ is defined as

$$\mathscr{L} := g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \qquad \mathscr{L} \overset{!}{=} \kappa c^2, \tag{1.8.7}$$

where x^μ are the coordinates of the metric, and the dot means differentiation with respect to the affine parameter λ. For timelike geodesics, $\kappa = \mp 1$ depending on the signature of the metric, $\mathrm{sign}(\mathbf{g}) = \pm 2$. For lightlike geodesics, $\kappa = 0$.

The Euler-Lagrange equations read

$$\frac{d}{d\lambda}\frac{\partial\mathscr{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathscr{L}}{\partial x^\mu} = 0. \tag{1.8.8}$$

If $\mathscr{L}$ is independent of x^ρ, then x^ρ is a cyclic variable and

$$p_\rho = g_{\rho\nu}\dot{x}^\nu = \text{const.} \tag{1.8.9}$$

Note that $[\mathscr{L}]_{\mathrm{U}} = \frac{\text{length}^2}{\text{time}^2}$ for timelike and $[\mathscr{L}]_{\mathrm{U}} = 1$ for lightlike geodesics, see Sec. 1.9.

1.8.5   Hamilton formalism

The super-Hamiltonian $\mathscr{H}$ is defined as

$$\mathscr{H} := \frac{1}{2}g^{\mu\nu}p_\mu p_\nu, \qquad \mathscr{H} \overset{!}{=} \frac{1}{2}\kappa c^2, \tag{1.8.10}$$

where $p_\mu = g_{\mu\nu}\dot{x}^\nu$ are the canonical momenta, see e.g. MTW[MTW73], para. 21.1. As in classical mechanics, we have

$$\frac{dx^\mu}{d\lambda} = \frac{\partial\mathscr{H}}{\partial p_\mu} \qquad \text{and} \qquad \frac{dp_\mu}{d\lambda} = -\frac{\partial\mathscr{H}}{\partial x^\mu}. \tag{1.8.11}$$

1.9   Units

A first test in analyzing whether an equation is correct is to check the units. Newton's gravitational constant G, for example, has the following units

$$[G]_{\mathrm{U}} = \frac{\text{length}^3}{\text{mass}\cdot\text{time}^2}, \tag{1.9.1}$$

where $[\cdot]_{\mathrm{U}}$ indicates that we evaluate the units of the enclosed expression. Further examples are

$$[ds]_{\mathrm{U}} = \text{length}, \qquad [\mathbf{u}]_{\mathrm{U}} = \frac{\text{length}}{\text{time}}, \qquad [R^{\text{Schwarzschild}}_{trtr}]_{\mathrm{U}} = \frac{1}{\text{time}^2}, \qquad \left[R^{\text{Schwarzschild}}_{\vartheta\varphi\vartheta\varphi}\right]_{\mathrm{U}} = \text{length}^2. \tag{1.9.2}$$

1.10   Tools

1.10.1   Maple/GRTensorII

The Christoffel symbols, the Riemann- and Ricci-tensors as well as the Ricci and Kretschmann scalars in this catalogue were determined by means of the software Maple together with the GRTensorII package by Musgrave, Pollney, and Lake.[2]

A typical worksheet to enter a new metric may look like this:

---

[2]The commercial software Maple can be found here: http://www.maplesoft.com. The GRTensorII-package is free: http://grtensor.phy.queensu.ca.

```
> grtw();
> makeg(Schwarzschild);

Makeg 2.0: GRTensor metric/basis entry utility
To quit makeg, type 'exit' at any prompt.
Do you wish to enter a 1) metric [g(dn,dn)],
                        2) line element [ds],
                        3) non-holonomic basis [e(1)...e(n)], or
                        4) NP tetrad [l,n,m,mbar]?
> 2:

Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi]:

Enter the line element using d[coord] to indicate differentials.
(for example,  r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
ds^2 =

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
> {}:

You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Correct an element of the metric,
                  3) Re-enter the metric,
                  4) Add/change constraint equations,
                  5) Add a text description, or
                  6) Abandon this metric and return to Maple.
> 0:
```

The worksheets for some of the metrics in this catalogue can be found on the authors homepage. To determine the objects that are defined with respect to a local tetrad, the metric must be given as non-holonomic basis.
The various basic objects can be determined via

Christoffel symbols $\Gamma^{\mu}_{\nu\rho}$	`grcalc(Chr2);`	`grcalc(Chr(dn,dn,up));`
partial derivatives $\Gamma^{\mu}_{\nu\rho,\sigma}$		`grcalc(Chr(dn,dn,up,pdn));`
Riemann tensor $R_{\mu\nu\rho\sigma}$	`grcalc(Riemman);`	`grcalc(R(dn,dn,dn,dn));`
Ricci tensor $R_{\mu\nu}$	`grcalc(Ricci);`	`grcalc(R(dn,dn));`
Ricci scalar $\mathscr{R}$	`grcalc(Ricciscalar);`	
Kretschmann scalar $\mathscr{K}$	`grcalc(RiemSq);`	

1.10.2 Mathematica

The calculation of the Christoffel symbols, the Riemann- or Ricci-tensor within *Mathematica* could read like this:

```
Clearing the values of symbols:
    In[1]:=  Clear[coord, metric, inversemetric, affine,
                   t, r, Theta, Phi]

Setting the dimension:
    In[2]:=  n := 4

Defining a list of coordinates:
    In[3]:=  coord := {t, r, Theta, Phi}

Defining the metric:
    In[4]:=  metric := {{-(1 - rs/r) c^2, 0, 0, 0},
                        {0, 1/(1 - rs/r), 0, 0},
                        {0, 0, r^2, 0},
                        {0, 0, 0, r ^2 Sin[Theta]^2}}
    In[5]:=  metric // MatrixForm
```

```
Calculating the inverse metric:
    In[6]:=  inversemetric := Simplify[Inverse[metric]]

    In[7]:=  inversemetric // MatrixForm

Calculating the Christoffel symbols of the second kind:
    In[8]:=  affine :=  affine = Simplify[
      Table[(1/2) Sum[inversemetric[[Mu, Rho]] (
          D[metric[[Rho, Nu]], coord[[Lambda]]] +
          D[metric[[Rho, Lambda]], coord[[Nu]]] -
          D[metric[[Nu, Lambda]], coord[[Rho]]]),
      {Rho, 1, n}], {Nu, 1, n}, {Lambda, 1, n}, {Mu, 1, n}]]

Displaying the Christoffel symbols of the second kind:
    In[9]:=  listaffine :=
      Table[If[UnsameQ[affine[[Nu, Lambda, Mu]], 0],
        {Style[ Subsuperscript[\[CapitalGamma],
          Row[{coord[[Nu]], coord[[Lambda]]}], coord[[Mu]]], 18],
          "=",
          Style[affine[[Nu, Lambda, Mu]], 14]}],
        {Lambda, 1, n}, {Nu, 1, Lambda}, {Mu, 1, n}]

    In[10]:=  TableForm[Partition[DeleteCases[Flatten[listaffine],
                                            Null], 3],
                      TableSpacing -> {1, 2}]

Defining the Riemann tensor:
    In[11]:=  riemann :=  riemann =
      Table[D[affine[[Nu, Sigma, Mu]], coord[[Rho]]] -
          D[affine[[Nu, Rho, Mu]], coord[[Sigma]]] +
          Sum[affine[[Rho, Lambda, Mu]]
              affine[[Nu, Sigma, Lambda]] -
              affine[[Sigma, Lambda, Mu]]
              affine[[Nu, Rho, Lambda]],
          {Lambda, 1, n}],
      {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

Defining the Riemann tensor with lower indices:
    In[12]:= riemannDn :=  riemannDn =
      Table[Simplify[
        Sum[metric[[Mu, Kappa]] riemann[[Kappa, Nu, Rho, Sigma]],
        {Kappa, 1, n}]],
      {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

    In[13]:= listRiemann :=
      Table[If[UnsameQ[riemannDn[[Mu, Nu, Rho, Sigma]], 0],
      {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]], coord[[Rho]],
        coord[[Sigma]]}]], 16], "=",
        riemannDn[[Mu, Nu, Rho, Sigma]]}],
      {Nu, 1, n}, {Mu, 1, Nu}, {Sigma, 1, n}, {Rho, 1, Sigma}]

    In[14]:= TableForm[Partition[DeleteCases[Flatten[listRiemann],
                                            Null], 3],
                    TableSpacing -> {2, 2}]

Defining the Ricci tensor:
    In[15]:= ricci := ricci =
      Table[Simplify[
        Sum[riemann[[Rho, Mu, Rho, Nu]], {Rho, 1, n}]],
      {Mu, 1, n}, {Nu, 1, n}]

    In[16]:= listRicci :=
      Table[If[UnsameQ[ricci[[Mu, Nu]], 0],
        {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]]}]], 16],
        "=",
        Style[ricci[[Mu, Nu]], 16]}], {Nu, 1, 4}, {Mu, 1, Nu}]

    In[17]:= TableForm[Partition[DeleteCases[Flatten[listRicci],
                                            Null], 3],
                    TableSpacing -> {1, 2}]

Defining the Ricci scalar:
    In[18]:=  ricciscalar :=  ricciscalar =
      Simplify[Sum[
```

```
      Sum[inversemetric[[Mu, Nu]] ricci[[Nu, Mu]],
      {Mu, 1, n}], {Nu, 1, n}]]

Defining the Kretschmann scalar:
  In[19]:= riemannUp :=  riemannUp =
    Table[Simplify[
     Sum[inversemetric[[Nu, Kappa]]
          riemann[[Mu, Kappa, Rho, Sigma]], {Kappa, 1, n}]],
     {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

  In[20]:= kretschmann := kretschmann =
    Simplify[Sum[ Sum[Sum[Sum[
      riemannUp[[Mu, Nu, Rho, Sigma]]
      riemannUp[[Rho, Sigma, Mu, Nu]],
      {Mu, 1, n}], {Nu, 1, n}], {Rho, 1, n}], {Sigma, 1, n}]]
```

Some example notebooks can be found on the authors homepage.

1.10.3 Maxima

Instead of using commercial software like *Maple* or *Mathematica,* Maxima also offers a tensor package that helps to calculate the Christoffel symbols etc. The above example for the Schwarzschild metric can be written as a maxima worksheet as follows:

```
/* load ctensor package */
load(ctensor);

/* define coordinates to use */
ct_coords:[t,r,theta,phi];

/* start with the identity metric */
lg:ident(4);
lg[1,1]:c^2*(1-rs/r);
lg[2,2]:-1/(1-rs/r);
lg[3,3]:-r^2;
lg[4,4]:-r^2*sin(theta)^2;
cmetric();

/* calculate the christoffel symbols of the second kind */
christof(mcs);

/* calculate the riemann tensor */
lriemann(mcs);

/* calculate the ricci tensor */
ricci(mcs);

/* calculate the ricci scalar */
scurvature();

/* calculate the Kretschmann scalar */
uriemann(mcs);
rinvariant();
ratsimp(%);
```

As you may have noticed, the Schwarzschild metric must be given with negative signature.

Chapter 2

Spacetimes

2.1 Minkowski

2.1.1 Cartesian coordinates

The Minkowski metric in Cartesian coordinates $\{t, x, y, z \in \mathbb{R}\}$ reads

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1.1}$$

All Christoffel symbols as well as the Riemann- and Ricci-tensor vanish identically. The natural local tetrad is trivial,

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(x)} = \partial_x, \qquad \mathbf{e}_{(y)} = \partial_y, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.2}$$

with dual

$$\theta^{(t)} = c\,dt, \qquad \theta^{(x)} = dx, \qquad \theta^{(y)} = dy, \qquad \theta^{(z)} = dz. \tag{2.1.3}$$

2.1.2 Cylindrical coordinates

The Minkowski metric in cylindrical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\varphi^2 + dz^2, \tag{2.1.4}$$

has the natural local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.5}$$

Christoffel symbols:

$$\Gamma^r_{\varphi\varphi} = -r, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}. \tag{2.1.6}$$

Partial derivatives

$$\Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\varphi\varphi,r} = -1. \tag{2.1.7}$$

Ricci rotation coefficients:

$$\gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \qquad \text{and} \qquad \gamma_{(r)} = \frac{1}{r}. \tag{2.1.8}$$

2.1.3 Spherical coordinates

In spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$, the Minkowski metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right). \tag{2.1.9}$$

Christoffel symbols:

$$\Gamma^r_{\vartheta\vartheta} = -r, \qquad \Gamma^r_{\varphi\varphi} = -r \sin^2 \vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.1.10a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{2.1.10b}$$

Partial derivatives

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -1, \tag{2.1.11a}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.1.11b}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -\sin(2\vartheta). \tag{2.1.11c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.1.12}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.1.13}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.1.14}$$

2.1.4 Conform-compactified coordinates

The Minkowski metric in conform-compactified coordinates $\{\psi \in [-\pi, \pi], \xi \in (0, \pi), \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads[HE99]

$$ds^2 = -d\psi^2 + d\xi^2 + \sin^2\xi \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right). \tag{2.1.15}$$

This form follows from the spherical Minkowski metric (2.1.9) by means of the coordinate transformation

$$ct + r = \tan\frac{\psi + \xi}{2}, \qquad ct - r = \tan\frac{\psi - \xi}{2}, \tag{2.1.16}$$

resulting in the metric

$$d\tilde{s}^2 = \frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}} + \frac{\sin^2\xi}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{2.1.17}$$

and by the conformal transformation $ds^2 = \Omega^2 d\tilde{s}^2$ with $\Omega^2 = 4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}$.

Christoffel symbols:

$$\Gamma^\vartheta_{\xi\vartheta} = \cot\xi, \qquad \Gamma^\varphi_{\xi\varphi} = \cot\xi, \qquad \Gamma^\xi_{\vartheta\vartheta} = -\sin\xi\cos\xi, \tag{2.1.18a}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\xi_{\varphi\varphi} = -\sin\xi\cos\xi\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.1.18b}$$

Partial derivatives

$$\Gamma^{\vartheta}_{\xi\vartheta,\xi} = -\frac{1}{\sin^2\xi}, \qquad\qquad \Gamma^{\varphi}_{\xi\varphi,\xi} = -\frac{1}{\sin^2\xi}, \qquad\qquad \Gamma^{\xi}_{\vartheta\vartheta,\xi} = -\cos(2\xi), \qquad\qquad (2.1.19a)$$

$$\Gamma^{\varphi}_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad\qquad \Gamma^{\xi}_{\varphi\varphi,\xi} = -\cos(2\xi)\sin^2\vartheta, \quad \Gamma^{\vartheta}_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \qquad (2.1.19b)$$

$$\Gamma^{\xi}_{\varphi\varphi,\vartheta} = -\frac{1}{2}\sin(2\xi)\sin(2\vartheta). \qquad\qquad\qquad (2.1.19c)$$

Riemann-Tensor:

$$R_{\xi\vartheta\xi\vartheta} = \sin^2\xi, \qquad R_{\xi\varphi\xi\varphi} = \sin^2\xi\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = \sin^4\xi\sin^2\vartheta. \qquad (2.1.20)$$

Ricci-Tensor:

$$R_{\xi\xi} = 2, \qquad R_{\vartheta\vartheta} = 2\sin^2\xi, \qquad R_{\varphi\varphi} = 2\sin^2\xi\sin^2\vartheta. \qquad (2.1.21)$$

Ricci and Kretschmann scalars:

$$\mathscr{R} = 6, \qquad \mathscr{K} = 12. \qquad\qquad\qquad (2.1.22)$$

The Weyl tensor vanishs identically.
Local tetrad:

$$\mathbf{e}_{(\psi)} = \partial_{\psi}, \qquad \mathbf{e}_{(\xi)} = \partial_{\xi}, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{\sin\xi}\partial_{\vartheta}, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\sin\xi\sin\vartheta}\partial_{\varphi}. \qquad (2.1.23)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(\xi)(\vartheta)} = \gamma_{(\varphi)(\xi)(\varphi)} = \cot\xi, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sin\xi}. \qquad (2.1.24)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\xi)} = 2\cot\xi, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sin\xi}. \qquad\qquad (2.1.25)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\xi)(\vartheta)(\xi)(\vartheta)} = R_{(\xi)(\varphi)(\xi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = 1. \qquad (2.1.26)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\xi)(\xi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 2. \qquad\qquad (2.1.27)$$

2.1.5   Rotating coordinates

The transformation $d\varphi \mapsto d\varphi + \omega dt$ brings the Minkowski metric (2.1.4) into the rotating form[Rin01] with coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$ds^2 = -\left(1 - \frac{\omega^2 r^2}{c^2}\right)[c\,dt - \Omega(r)d\varphi]^2 + dr^2 + \frac{r^2}{1 - \omega^2 r^2/c^2}d\varphi^2 + dz^2 \qquad (2.1.28)$$

with $\Omega(r) = (r^2\omega/c)/(1 - \omega^2 r^2/c^2)$.
Metric-Tensor:

$$g_{tt} = -c^2 + \omega^2 r^2, \qquad g_{t\varphi} = \omega r^2, \qquad g_{rr} = g_{zz} = 1, \qquad g_{\varphi\varphi} = r^2. \qquad (2.1.29)$$

Christoffel symbols:

$$\Gamma^r_{tt} = -\omega^2 r, \qquad \Gamma^\varphi_{tr} = \frac{\omega}{r}, \qquad \Gamma^r_{t\varphi} = -\omega r, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\varphi\varphi} = -r. \tag{2.1.30}$$

Partial derivatives

$$\Gamma^r_{tt,r} = -\omega^2, \quad \Gamma^\varphi_{tr,r} = -\frac{\omega}{r^2}, \quad \Gamma^r_{t\varphi,r} = -\omega, \quad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \quad \Gamma^r_{\varphi\varphi,r} = -1. \tag{2.1.31}$$

The local tetrad of the comoving observer is

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t - \frac{\omega}{c}\partial_\varphi, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.32}$$

whereas the static observer has the local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - \omega^2 r^2/c^2}}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.33a}$$

$$\mathbf{e}_{(\varphi)} = \frac{\omega r}{c^2\sqrt{1 - \omega^2 r^2/c^2}}\partial_t + \frac{\sqrt{1 - \omega^2 r^2/c^2}}{r}\partial_\varphi. \tag{2.1.33b}$$

2.1.6 Rindler coordinates

The worldline of an observer in the Minkowski spacetime who moves with constant proper acceleration α along the x direction reads

$$x = \frac{c^2}{\alpha}\cosh\frac{\alpha t'}{c}, \qquad ct = \frac{c^2}{\alpha}\sinh\frac{\alpha t'}{c}, \tag{2.1.34}$$

where t' is the observer's proper time. The observer starts at $x = 1$ with zero velocity.
However, such an observer could also be described with Rindler coordinates. With the coordinate transformation

$$(ct, x) \mapsto (\tau, \rho): \qquad ct = \frac{1}{\rho}\sinh\tau, \qquad x = \frac{1}{\rho}\cosh\tau, \tag{2.1.35}$$

where $\rho = \alpha/c^2$, the Rindler metric reads

$$\boxed{ds^2 = -\frac{1}{\rho^2}d\tau^2 + \frac{1}{\rho^4}d\rho^2 + dy^2 + dz^2.} \tag{2.1.36}$$

Christoffel symbols:

$$\Gamma^\rho_{\tau\tau} = -\rho, \qquad \Gamma^\tau_{\tau\rho} = -\frac{1}{\rho}, \qquad \Gamma^\rho_{\rho\rho} = -\frac{2}{\rho}. \tag{2.1.37}$$

Partial derivatives

$$\Gamma^\rho_{\tau\tau,\rho} = -1, \qquad \Gamma^\tau_{\tau\rho,\rho} = \frac{1}{\rho^2}, \qquad \Gamma^\rho_{\rho\rho,\rho} = \frac{2}{\rho^2}. \tag{2.1.38}$$

The Riemann and Ricci tensors as well as the Ricci and Kretschmann scalar vanish identically.
Local tetrad:

$$\mathbf{e}_{(\tau)} = \rho\partial_\tau, \qquad \mathbf{e}_{(\rho)} = \rho^2\partial_\rho, \qquad \mathbf{e}_{(y)} = \partial_y, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.39}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(\rho)(\tau)} = \rho, \qquad \text{and} \qquad \gamma_{(\rho)} = -\rho. \tag{2.1.40}$$

2.2 Schwarzschild spacetime

2.2.1 Schwarzschild coordinates

In Schwarzschild coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$, the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2\right), \tag{2.2.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The critical point $r = 0$ is a real curvature singularity while the event horizon, $r = r_s$, is only a coordinate singularity, see e.g. the Kretschmann scalar.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s (r - r_s)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s}{2r(r - r_s)}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r(r - r_s)}, \tag{2.2.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r - r_s), \tag{2.2.2b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -(r - r_s)\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.2c}$$

Partial derivatives

$$\Gamma^r_{tt,r} = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \qquad \Gamma^t_{tr,r} = -\frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \qquad \Gamma^r_{rr,r} = \frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \tag{2.2.3a}$$

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -1, \tag{2.2.3b}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.2.3c}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -(r - r_s)\sin(2\vartheta). \tag{2.2.3d}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}\frac{c^2 (r - r_s) r_s}{r^2}, \qquad R_{t\varphi t\varphi} = \frac{1}{2}\frac{c^2 (r - r_s) r_s \sin^2\vartheta}{r^2}, \tag{2.2.4a}$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2}\frac{r_s}{r - r_s}, \qquad R_{r\varphi r\varphi} = -\frac{1}{2}\frac{r_s \sin^2\vartheta}{r - r_s}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2\vartheta. \tag{2.2.4b}$$

As aspected, the Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12\frac{r_s^2}{r^6}. \tag{2.2.5}$$

Here, it becomes clear that at $r = r_s$ there is no real singularity.

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.6}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r}}dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \qquad \theta^{(\vartheta)} = rd\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.7}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.2.9}$$

Structure coefficients:

$$c^{(t)}_{(t)(r)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad c^{(\vartheta)}_{(r)(\vartheta)} = c^{(\varphi)}_{(r)(\varphi)} = -\frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \qquad c^{(\varphi)}_{(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.11a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.11b}$$

The covariant derivatives of the Riemann tensor read

$$R_{(t)(r)(t)(r);(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi);(r)} = \frac{3r_s}{r^5}\sqrt{r(r - r_s)}, \tag{2.2.12a}$$

$$R_{(t)(r)(r)(\vartheta);(\vartheta)} = R_{(t)(r)(t)(\varphi);(\varphi)} = R_{(t)(\vartheta)(t)(\vartheta);(r)} = R_{(t)(\varphi)(t)(\varphi);(r)} =$$

$$= R_{(r)(\varphi)(\vartheta)(\varphi);(\vartheta)} = -\frac{3r_s}{2r^5}\sqrt{r(r - r_s)}, \tag{2.2.12b}$$

$$R_{(r)(\vartheta)(r)(\vartheta);(r)} = R_{(r)(\vartheta)(\vartheta)(\varphi);(\varphi)} = R_{(r)(\varphi)(r)(\varphi);(r)} = \frac{3r_s}{2r^5}\sqrt{r(r - r_s)}. \tag{2.2.12c}$$

Newman-Penrose tetrad:

$$\mathbf{l} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_{(t)} + \mathbf{e}_{(r)}\right), \qquad \mathbf{n} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_{(t)} - \mathbf{e}_{(r)}\right), \qquad \mathbf{m} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_{(\vartheta)} + i\mathbf{e}_{(\varphi)}\right). \tag{2.2.13}$$

Non-vanishing spin coefficients:

$$\rho = \mu = -\frac{1}{\sqrt{2}r}\sqrt{1 - \frac{r_s}{r}}, \quad \gamma = \varepsilon = \frac{r_s}{4\sqrt{2}r^2\sqrt{1 - r_s/r}}, \quad \alpha = -\beta = -\frac{\cot\vartheta}{2\sqrt{2}r}. \tag{2.2.14}$$

Embedding:
The embedding function reads

$$z = 2\sqrt{r_s}\sqrt{r - r_s}. \tag{2.2.15}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \tag{2.2.16}$$

with the constants of motion $k = (1 - r_s/r)c^2\dot{t}$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.8.2). For timelike geodesics, the effective potential has the extremal points

$$r_\pm = \frac{h^2 \pm h\sqrt{h^2 - 3c^2 r_s^2}}{c^2 r_s}, \tag{2.2.17}$$

where r_+ is a maximum and r_- is a minimum. The innermost timelike circular geodesic follows from $h^2 = 3c^2 r_s^2$ and reads $r_{\text{itcg}} = 3r_s$. Null geodesics, however, have only a maximum at $r_{\text{po}} = \frac{3}{2}r_s$. The corresponding circular orbit is called photon orbit.

Further reading:
Schwarzschild[Sch16, Sch03], MTW[MTW73], Rindler[Rin01], Wald[Wal84], Chandrasekhar[Cha06], Müller[Mül08b, Mül09].

2.2.2 Schwarzschild in pseudo-Cartesian coordinates

The Schwarzschild spacetime in pseudo-Cartesian coordinates (t,x,y,z) reads

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(\frac{x^2}{1-r_s/r}+y^2+z^2\right)\frac{dx^2}{r^2} + \left(x^2+\frac{y^2}{1-r_s/r}+z^2\right)\frac{dy^2}{r^2}$$
$$+ \left(x^2+y^2+\frac{z^2}{1-r_s/r}\right)\frac{dz^2}{r^2} + \frac{2r_s}{r^2(r-r_s)}\left(xy\,dxdy+xz\,dxdz+yz\,dydz\right),$$

(2.2.18)

where $r^2 = x^2+y^2+z^2$. For a natural local tetrad that is adapted to the x-axis, we make the following ansatz:

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_t, \qquad \mathbf{e}_{(1)} = A\partial_x, \qquad \mathbf{e}_{(2)} = B\partial_x+C\partial_y, \qquad \mathbf{e}_{(3)} = D\partial_x+E\partial_y+F\partial_z.$$

(2.2.19)

$$A = \frac{1}{\sqrt{g_{xx}}}, \qquad B = \frac{-g_{xy}}{g_{xx}\sqrt{-g_{xy}^2/g_{xx}+g_{yy}}}, \qquad C = \frac{1}{\sqrt{-g_{xy}^2/g_{xx}+g_{yy}}},$$

(2.2.20a)

$$D = \frac{g_{xy}g_{yz}-g_{xz}g_{yy}}{\sqrt{NW}}, \qquad E = \frac{g_{xz}g_{xy}-g_{xx}g_{yz}}{\sqrt{NW}}, \qquad F = \frac{\sqrt{N}}{\sqrt{W}},$$

(2.2.20b)

with

$$N = g_{xx}g_{yy}-g_{xy}^2,$$

(2.2.21a)

$$W = g_{xx}g_{yy}g_{zz}-g_{xz}^2g_{yy}+2g_{xz}g_{xy}g_{yz}-g_{xy}^2g_{zz}-g_{xx}g_{yz}^2.$$

(2.2.21b)

2.2.3 Isotropic coordinates

Spherical isotropic coordinates

The Schwarzschild metric (2.2.1) in spherical isotropic coordinates $(t,\rho,\vartheta,\varphi)$ reads

$$ds^2 = -\left(\frac{1-\rho_s/\rho}{1+\rho_s/\rho}\right)^2 c^2dt^2 + \left(1+\frac{\rho_s}{\rho}\right)^4\left[d\rho^2+\rho^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\right],$$

(2.2.22)

where

$$r = \rho\left(1+\frac{\rho_s}{\rho}\right)^2 \qquad \text{or} \qquad \rho = \frac{1}{4}\left(2r-r_s\pm 2\sqrt{r(r-r_s)}\right)$$

(2.2.23)

is the coordinate transformation between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ, see e.g. MTW[MTW73] page 840. The event horizon is given by $\rho_s = r_s/4$. The photon orbit and the innermost timelike circular geodesic read

$$\rho_{\text{po}} = \left(2+\sqrt{3}\right)\rho_s \qquad \text{and} \qquad \rho_{\text{itcg}} = \left(5+2\sqrt{6}\right)\rho_s.$$

(2.2.24)

Christoffel symbols:

$$\Gamma^\rho_{tt} = \frac{2(\rho-\rho_s)\rho^4\rho_s c^2}{(\rho+\rho_s)^7}, \quad \Gamma^t_{t\rho} = \frac{2\rho_s}{\rho^2-\rho_s^2}, \qquad \Gamma^\rho_{\rho\rho} = -\frac{2\rho_s}{(\rho+\rho_s)\rho},$$

(2.2.25a)

$$\Gamma^\vartheta_{\rho\vartheta} = \frac{\rho-\rho_s}{(\rho+\rho_s)\rho}, \qquad \Gamma^\varphi_{\rho\varphi} = \frac{\rho-\rho_s}{(\rho+\rho_s)\rho}, \qquad \Gamma^\rho_{\vartheta\vartheta} = -\rho\frac{\rho-\rho_s}{\rho+\rho_s},$$

(2.2.25b)

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\rho_{\varphi\varphi} = -\frac{(\rho-\rho_s)\rho\sin^2\vartheta}{\rho+\rho_s}, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta.$$

(2.2.25c)

Riemann-Tensor:

$$R_{t\rho t\rho} = -4\frac{(\rho-\rho_s)^2\rho_s c^2}{(\rho+\rho_s)^4\rho}, \qquad R_{t\vartheta t\vartheta} = 2\frac{(\rho-\rho_s)^2\rho\rho_s c^2}{(\rho+\rho_s)^4}, \tag{2.2.26a}$$

$$R_{t\varphi t\varphi} = 2\frac{(\rho-\rho_s)^2\rho c^2\rho_s\sin^2\vartheta}{(\rho+\rho_s)^4}, \qquad R_{\rho\vartheta\rho\vartheta} = -2\frac{(\rho+\rho_s)^2\rho_s}{\rho^3}, \tag{2.2.26b}$$

$$R_{\rho\varphi\rho\varphi} = -2\frac{(\rho+\rho_s)^2\rho_s\sin^2\vartheta}{\rho^3}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{4(\rho+\rho_s)^2\rho_s\sin^2\vartheta}{\rho}. \tag{2.2.26c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathscr{K} = 192\frac{r_s^2}{\rho^6\left(1+\rho_s/\rho\right)^{12}} = 12\frac{r_s^2}{r(\rho)^6}. \tag{2.2.27}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1+\rho_s/\rho}{1-\rho_s/\rho}\frac{\partial_t}{c}, \qquad \mathbf{e}_{(r)} = \frac{1}{[1+\rho_s/\rho]^2}\partial_\rho, \tag{2.2.28a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\rho\,[1+\rho_s/\rho]^2}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\rho\,[1+\rho_s/\rho]^2\sin^2\vartheta}\partial_\varphi. \tag{2.2.28b}$$

Ricci rotation coefficients:

$$\gamma_{(\rho)(t)(t)} = \frac{2\rho_s\rho^2}{(\rho+\rho_s)^3(\rho-\rho_s)}, \quad \gamma_{(\vartheta)(\rho)(\vartheta)} = \gamma_{(\varphi)(\rho)(\varphi)} = \frac{\rho(\rho-\rho_s)}{(\rho+\rho_s)^3}, \tag{2.2.29a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\rho\cot\vartheta}{(\rho+\rho_s)^2}. \tag{2.2.29b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\rho)} = \frac{2\rho(\rho^2-\rho\rho_s+\rho_s^2)}{(\rho+\rho_s)^3(\rho-\rho_s)}, \qquad \gamma_{(\vartheta)} = \frac{\rho\cot\vartheta}{(\rho+\rho_s)^2}. \tag{2.2.30}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \tag{2.2.31a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \tag{2.2.31b}$$

Further reading:
Buchdahl[Buc85].

Cartesian isotropic coordinates

The Schwarzschild metric (2.2.1) in Cartesian isotropic coordinates (t,x,y,z) reads,

$$\boxed{ds^2 = -\left(\frac{1-\rho_s/\rho}{1+\rho_s/\rho}\right)^2 c^2dt^2 + \left(1+\frac{\rho_s}{\rho}\right)^4\left[dx^2+dy^2+dz^2\right],} \tag{2.2.32}$$

where $\rho^2 = x^2+y^2+z^2$ and, as before,

$$r = \rho\left(1+\frac{\rho_s}{\rho}\right)^2. \tag{2.2.33}$$

Christoffel symbols:

$$\Gamma^x_{tt} = \frac{2c^2\rho^3\rho_s\,(\rho-\rho_s)\,x}{(\rho+\rho_s)^7}, \qquad \Gamma^y_{tt} = \frac{2c^2\rho^3\rho_s\,(\rho-\rho_s)\,y}{(\rho+\rho_s)^7}, \qquad \Gamma^z_{tt} = \frac{2c^2\rho^3\rho_s\,(\rho-\rho_s)\,z}{(\rho+\rho_s)^7}, \tag{2.2.34a}$$

$$\Gamma^t_{tx} = \frac{2\rho_s x}{\rho^3\left[1-\rho_s^2/\rho^2\right]}, \qquad \Gamma^t_{ty} = \frac{2\rho_s y}{\rho^3\left[1-\rho_s^2/\rho^2\right]}, \qquad \Gamma^t_{tz} = \frac{2\rho_s z}{\rho^3\left[1-\rho_s^2/\rho^2\right]}, \tag{2.2.34b}$$

$$\Gamma^x_{xx} = \Gamma^y_{xy} = \Gamma^z_{xz} = -\Gamma^x_{yy} = -\Gamma^x_{zz} = -\frac{2\rho_s}{\rho^3}\frac{x}{1+\rho_s/\rho}, \tag{2.2.34c}$$

$$\Gamma^y_{xx} = -\Gamma^x_{xy} = -\Gamma^y_{yy} = -\Gamma^z_{yz} = \Gamma^y_{zz} = \frac{2\rho_s}{\rho^3}\frac{y}{1+\rho_s/\rho}, \tag{2.2.34d}$$

$$\Gamma^z_{xx} = -\Gamma^x_{xz} = \Gamma^z_{yy} = -\Gamma^y_{yz} = -\Gamma^z_{zz} = \frac{2\rho_s}{\rho^3}\frac{z}{1+\rho_s/\rho}. \tag{2.2.34e}$$

2.2.4 Eddington-Finkelstein

The transformation of the Schwarzschild metric (2.2.1) from the usual Schwarzschild time coordinate t to the advanced null coordinate v with

$$cv = ct + r + r_s \ln(r-r_s) \tag{2.2.35}$$

leads to the ingoing Eddington-Finkelstein[Edd24, Fin58] metric with coordinates (v,r,ϑ,φ),

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dv^2 + 2c\,dv\,dr + r^2\left(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\right).} \tag{2.2.36}$$

Metric-Tensor:

$$g_{vv} = -c^2\left(1-\frac{r_s}{r}\right), \qquad g_{vr} = c, \qquad g_{\vartheta\vartheta} = r^2, \qquad g_{\varphi\varphi} = r^2\sin^2\vartheta. \tag{2.2.37}$$

Christoffel symbols:

$$\Gamma^v_{vv} = \frac{cr_s}{2r^2}, \qquad \Gamma^r_{vv} = \frac{c^2 r_s(r-r_s)}{2r^3}, \qquad \Gamma^r_{vr} = -\frac{cr_s}{2r^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.2.38a}$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^v_{\vartheta\vartheta} = -\frac{r}{c}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r-r_s), \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.2.38b}$$

$$\Gamma^v_{\varphi\varphi} = -\frac{r\sin^2\vartheta}{c}, \quad \Gamma^r_{\varphi\varphi} = -(r-r_s)\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.38c}$$

Partial derivatives

$$\Gamma^v_{vv,r} = -\frac{cr_s}{r^3}, \qquad \Gamma^r_{vv,r} = -\frac{(2r-3r_s)c^2 r_s}{2r^4}, \qquad \Gamma^r_{vr,r} = \frac{cr_s}{r^3}, \tag{2.2.39a}$$

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^v_{\vartheta\vartheta,r} = -\frac{1}{c}, \tag{2.2.39b}$$

$$\Gamma^r_{\vartheta\vartheta,r} = -1, \qquad \Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^v_{\varphi\varphi,r} = -\frac{\sin^2\vartheta}{c}, \tag{2.2.39c}$$

$$\Gamma^v_{\varphi\varphi,\vartheta} = -\frac{r\sin(2\vartheta)}{c}, \qquad \Gamma^r_{\varphi\varphi,r} = -\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.2.39d}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -(r-r_s)\sin(2\vartheta). \tag{2.2.39e}$$

Riemann-Tensor:

$$R_{vrvr} = -\frac{c^2 r_s}{r^3}, \qquad R_{v\vartheta v\vartheta} = \frac{c^2 r_s(r-r_s)}{2r^2}, \qquad R_{v\vartheta r\vartheta} = -\frac{cr_s}{2r}, \tag{2.2.40a}$$

$$R_{v\varphi v\varphi} = \frac{c^2 r_s(r-r_s)\sin^2\vartheta}{2r^2}, \qquad R_{v\varphi r\varphi} = -\frac{cr_s\sin^2\vartheta}{2r}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.40b}$$

While the Ricci tensor and the Ricci scalar vanish identically, the Kretschmann scalar is $\mathscr{K} = 12r_s^2/r^6$.

Static local tetrad:

$$\mathbf{e}_{(v)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v, \quad \mathbf{e}_{(r)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v + \sqrt{1-\frac{r_s}{r}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.41}$$

Dual tetrad:

$$\theta^{(v)} = c\sqrt{1-\frac{r_s}{r}}dv - \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\varphi)} = r\sin\vartheta d\varphi. \tag{2.2.42}$$

Ricci rotation coefficients:

$$\gamma_{(r)(v)(v)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.43}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r-3r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.2.44}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(v)(r)(v)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.45a}$$

$$R_{(v)(\vartheta)(v)(\vartheta)} = R_{(v)(\varphi)(v)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.45b}$$

2.2.5 Kruskal-Szekeres

The Schwarzschild metric in Kruskal-Szekeres[Kru60, Wal84] coordinates (T,X,ϑ,φ) reads

$$\boxed{ds^2 = \frac{4r_s^3}{r}e^{-r/r_s}\left(-dT^2+dX^2\right)+r^2d\Omega^2,} \tag{2.2.46}$$

where $r \in \mathbb{R}_+ \setminus \{0\}$ is given by means of the LambertW-function $\mathscr{W}$,

$$\left(\frac{r}{r_s}-1\right)e^{r/r_s} = X^2-T^2 \quad \text{or} \quad r = r_s\left[\mathscr{W}\left(\frac{X^2-T^2}{e}\right)+1\right]. \tag{2.2.47}$$

The Schwarzschild coordinate time t in terms of the Kruskal coordinates T and X reads

$$t = 2r_s\text{arctanh}\frac{T}{X}, \quad r > r_s, \tag{2.2.48a}$$

$$t = 2r_s\text{arctanh}\frac{X}{T}, \quad r < r_s, \tag{2.2.48b}$$

$$t = \infty, \quad r = r_s. \tag{2.2.48c}$$

The transformations between Kruskal- and Schwarzschild coordinates read

$$X = \sqrt{1-\frac{r}{r_s}}e^{r/(2r_s)}\sinh\frac{ct}{2r_s}, \quad T = \sqrt{1-\frac{r}{r_s}}e^{r/(2r_s)}\cosh\frac{ct}{2r_s}, \quad 0 < r < r_2, \tag{2.2.49a}$$

$$X = \sqrt{\frac{r}{r_s}-1}e^{r/(2r_s)}\cosh\frac{ct}{2r_s}, \quad T = \sqrt{\frac{r}{r_s}-1}e^{r/(2r_s)}\sinh\frac{ct}{2r_s}, \quad r \geq r_s. \tag{2.2.49b}$$

Christoffel symbols:

$$\Gamma^T_{TT} = \Gamma^X_{TX} = \Gamma^T_{XX} = \frac{Tr_s(r+r_s)}{r^2}e^{-r/r_s},$$ (2.2.50a)

$$\Gamma^X_{TT} = \Gamma^T_{TX} = \Gamma^X_{XX} = -\frac{Xr_s(r+r_s)}{r^2}e^{-r/r_s},$$ (2.2.50b)

$$\Gamma^\vartheta_{T\vartheta} = -\frac{2r_s^2 T}{r^2}e^{-r/r_s}, \qquad\qquad \Gamma^\vartheta_{X\vartheta} = \frac{2r_s^2 X}{r^2}e^{-r/r_s},$$ (2.2.50c)

$$\Gamma^T_{\vartheta\vartheta} = -\frac{r}{2r_s}T, \qquad\qquad \Gamma^X_{\vartheta\vartheta} = \frac{r}{2r_s}X,$$ (2.2.50d)

$$\Gamma^T_{\vartheta\vartheta} = -\frac{r}{2r_s}T\sin^2\vartheta, \qquad\qquad \Gamma^X_{\vartheta\vartheta} = \frac{r}{2r_s}X\sin^2\vartheta,$$ (2.2.50e)

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad\qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta.$$ (2.2.50f)

Riemann-Tensor:

$$R_{TXTX} = -16\frac{r_s^7}{r^5}e^{-2r/r_s}, \qquad\qquad R_{T\vartheta T\vartheta} = \frac{2r_s^4}{r^2}e^{-r/r_s},$$ (2.2.51a)

$$R_{T\varphi T\varphi} = \frac{2r_s^4}{r^2}e^{-r/r_s}\sin^2\vartheta, \qquad R_{X\vartheta X\vartheta} = -\frac{2r_s^4}{r^2}e^{-r/r_s},$$ (2.2.51b)

$$R_{X\varphi X\varphi} = -\frac{2r_s^4}{r^2}e^{-r/r_s}\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta.$$ (2.2.51c)

The *Ricci-Tensor* as well as the *Ricci-scalar* vanish identically.

Kretschmann scalar:

$$\mathscr{K} = \frac{12r_s^2}{r^6}.$$ (2.2.52)

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}}e^{r/(2r_s)}\partial_T, \quad \mathbf{e}_{(X)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}}e^{r/(2r_s)}\partial_X, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi$$ (2.2.53)

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(X)(T)(X)} = R_{(X)(\vartheta)(X)(\vartheta)} = R_{(X)(\varphi)(X)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3},$$ (2.2.54a)

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = \frac{r_s}{2r^3}.$$ (2.2.54b)

2.2.6 Tortoise coordinates

The Schwarzschild metric represented by tortoise coordinates $(t,\rho,\vartheta,\varphi)$ reads

$$ds^2 = -\left(1 - \frac{r_s}{r(\rho)}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r(\rho)}\right)d\rho^2 + r(\rho)^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right),$$ (2.2.55)

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The tortoise radial coordinate ρ and the Schwarzschild radial coordinate r are related by

$$\rho = r + r_s\ln\left(\frac{r}{r_s} - 1\right) \qquad \text{or} \qquad r = r_s\left\{1 + \mathscr{W}\left[\exp\left(\frac{\rho}{r_s} - 1\right)\right]\right\}.$$ (2.2.56)

Christoffel symbols:

$$\Gamma^{\rho}_{tt} = \frac{c^2 r_s}{2r(\rho)^2}, \qquad \Gamma^{t}_{t\rho} = \frac{r_s}{2r(\rho)^2}, \qquad \Gamma^{\rho}_{\rho\rho} = \frac{r_s}{2r(\rho)^2}, \tag{2.2.57a}$$

$$\Gamma^{\vartheta}_{\rho\vartheta} = \frac{1}{r(\rho)} - \frac{1}{r_s}, \qquad \Gamma^{\varphi}_{\rho\varphi} = \frac{1}{r(\rho)} - \frac{1}{r_s}, \qquad \Gamma^{\rho}_{\vartheta\vartheta} = -r(\rho), \tag{2.2.57b}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\rho}_{\varphi\varphi} = -r(\rho)\sin^2\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.57c}$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -\frac{c^2 r_s}{r(\rho)^3}\left(1 - \frac{r_s}{r(\rho)}\right)^2, \qquad R_{t\vartheta t\vartheta} = \frac{c^2}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)}, \tag{2.2.58a}$$

$$R_{t\varphi t\varphi} = \frac{c^2\sin^2\vartheta}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)}, \qquad R_{\rho\vartheta\rho\vartheta} = -\frac{1}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)} \tag{2.2.58b}$$

$$R_{\rho\varphi\rho\varphi} = -\frac{\sin^2\vartheta}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)}, \qquad R_{\vartheta\varphi\vartheta\varphi} = r(\rho)r_s\sin^2\vartheta. \tag{2.2.58c}$$

The Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathscr{K} = 12\frac{r_s^2}{r(\rho)^6}. \tag{2.2.59}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r(\rho)}}\partial_t, \quad \mathbf{e}_{(\rho)} = \frac{1}{\sqrt{1 - r_s/r(\rho)}}\partial_\rho, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r(\rho)}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r(\rho)\sin\vartheta}\partial_\varphi. \tag{2.2.60}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r(\rho)}}\,dt, \quad \theta^{(\rho)} = \sqrt{1 - \frac{r_s}{r(\rho)}}\,d\rho, \quad \theta^{(\vartheta)} = r(\rho)\,d\vartheta, \quad \theta^{(\varphi)} = r(\rho)\sin\vartheta\,d\varphi. \tag{2.2.61}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \tag{2.2.62a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \tag{2.2.62b}$$

Further reading:
MTW[MTW73]

2.2.7 Painlevé-Gullstrand

The Schwarzschild metric expressed in Painlevé-Gullstrand coordinates[MP01] reads

$$\boxed{ds^2 = -c^2 dT^2 + \left(dr + \sqrt{\frac{r_s}{r}}c\,dT\right)^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right),} \tag{2.2.63}$$

where the new time coordinate T follows from the Schwarzschild time t in the following way:

$$cT = ct + 2r_s\left(\sqrt{\frac{r}{r_s}} + \frac{1}{2}\ln\left|\frac{\sqrt{r/r_s} - 1}{\sqrt{r/r_s} + 1}\right|\right). \tag{2.2.64}$$

Metric-Tensor:

$$g_{TT} = -c^2\left(1 - \frac{r_s}{r}\right), \qquad g_{Tr} = c\sqrt{\frac{r_s}{r}}, \qquad g_{rr} = 1, \qquad g_{\vartheta\vartheta} = r^2, \qquad g_{\varphi\varphi} = r^2\sin^2\vartheta. \tag{2.2.65}$$

Christoffel symbols:

$$\Gamma^T_{TT} = \frac{cr_s}{2r^2}\sqrt{\frac{r_s}{r}}, \qquad\qquad \Gamma^r_{TT} = \frac{c^2 r_s(r - r_s)}{2r^3}, \qquad \Gamma^T_{Tr} = \frac{r_s}{2r^2}, \tag{2.2.66a}$$

$$\Gamma^r_{Tr} = -\frac{cr_s}{2r^2}\sqrt{\frac{r_s}{r}}, \qquad\qquad \Gamma^T_{rr} = \frac{r_s}{2cr^2}\sqrt{\frac{r}{r_s}}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r^2}, \tag{2.2.66b}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad\qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad\qquad \Gamma^T_{\vartheta\vartheta} = -\frac{r}{c}\sqrt{\frac{r_s}{r}}, \tag{2.2.66c}$$

$$\Gamma^r_{\vartheta\vartheta} = -(r - r_s), \qquad\qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^T_{\varphi\varphi} = -\frac{r}{c}\sqrt{\frac{r_s}{r}}\sin^2\vartheta, \tag{2.2.66d}$$

$$\Gamma^r_{\varphi\varphi} = -(r - r_s)\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.66e}$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{c^2 r_s}{r^3}, \qquad\qquad R_{T\vartheta T\vartheta} = \frac{c^2 r_s(r - r_s)}{2r^2}, \qquad R_{T\vartheta r\vartheta} = -\frac{cr_s}{2r}\sqrt{\frac{r_s}{r}}, \tag{2.2.67a}$$

$$R_{T\varphi T\varphi} = \frac{c^2 r_s(r - r_s)\sin^2\vartheta}{2r^2}, \quad R_{T\varphi r\varphi} = -\frac{cr_s}{2r}\sqrt{\frac{r_s}{r}}\sin^2\vartheta, \quad R_{r\vartheta r\vartheta} = -\frac{r_s}{2r}, \tag{2.2.67b}$$

$$R_{r\varphi r\varphi} = -\frac{r_s\sin^2\vartheta}{2r}, \qquad\qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.67c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathscr{K} = 12r_s^2/r^6. \tag{2.2.68}$$

For the Painlevé-Gullstrand coordinates, we can define two natural local tetrads.

Static local tetrad:

$$\hat{\mathbf{e}}_{(T)} = \frac{1}{c\sqrt{1 - r_s/r}}\partial_T, \quad \hat{\mathbf{e}}_{(r)} = \frac{\sqrt{r_s}}{c\sqrt{r - r_s}}\partial_T + \sqrt{1 - \frac{r_s}{r}}\partial_r, \quad \hat{\mathbf{e}}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \hat{\mathbf{e}}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi, \tag{2.2.69}$$

Dual tetrad:

$$\hat{\theta}^{(T)} = c\sqrt{1 - \frac{r_s}{r}}dT - \frac{dr}{\sqrt{r/r_s - 1}}, \quad \hat{\theta}^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \quad \hat{\theta}^{(\vartheta)} = r\,d\vartheta, \quad \hat{\theta}^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.70}$$

Freely falling local tetrad:

$$\mathbf{e}_{(T)} = \frac{1}{c}\partial_T - \sqrt{\frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.71}$$

Dual tetrad:

$$\theta^{(T)} = c\,dT, \qquad \theta^{(r)} = c\sqrt{\frac{r_s}{r}}dT + dr, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.72}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(r)(T)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.73a}$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.73b}$$

2.2.8 Israel coordinates

The Schwarzschild metric in Israel coordinates $(x, y, \vartheta, \varphi)$ reads[SKM$^+$03]

$$ds^2 = r_s^2 \left[4dx \left(dy + \frac{y^2 dx}{1+xy} \right) + (1+xy)^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right], \tag{2.2.74}$$

where the coordinates x and y follow from the Schwarzschild coordinates via

$$t = r_s \left(1 + xy + \ln \frac{y}{x} \right) \qquad \text{and} \qquad r = r_s(1+xy). \tag{2.2.75}$$

Christoffel symbols:

$$\Gamma^x_{xx} = -\frac{y(2+xy)}{(1+xy)^2}, \qquad \Gamma^y_{xx} = \frac{y^3(3+xy)}{(1+xy)^3}, \qquad \Gamma^y_{xy} = \frac{y(2+xy)}{(1+xy)^2}, \tag{2.2.76a}$$

$$\Gamma^\vartheta_{x\vartheta} = \frac{y}{1+xy}, \qquad \Gamma^\varphi_{x\varphi} = \frac{y}{1+xy}, \qquad \Gamma^\vartheta_{y\vartheta} = \frac{x}{1+xy}, \tag{2.2.76b}$$

$$\Gamma^\varphi_{x\varphi} = \frac{x}{1+xy}, \qquad \Gamma^x_{\vartheta\vartheta} = -\frac{x}{2}(1+xy), \qquad \Gamma^y_{\vartheta\vartheta} = -\frac{y}{2}(1-xy), \tag{2.2.76c}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^x_{\varphi\varphi} = -\frac{x}{2}(1+xy)\sin^2\vartheta, \ \ \Gamma^y_{\varphi\varphi} = -\frac{y}{2}(1-xy)\sin^2\vartheta, \tag{2.2.76d}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.76e}$$

Riemann-Tensor:

$$R_{xyxy} = -4\frac{r_s^2}{(1+xy)^3}, \qquad R_{x\vartheta x\vartheta} = -2\frac{y^2 r_s^2}{(1+xy)^2}, \quad R_{x\vartheta y\vartheta} = -\frac{r_s^2}{1+xy}, \tag{2.2.77a}$$

$$R_{x\varphi x\varphi} = -2\frac{r_s^2 y^2 \sin^2\vartheta}{(1+xy)^2}, \quad R_{x\varphi y\varphi} = -\frac{r_s^2 \sin^2\vartheta}{1+xy}, \quad R_{\vartheta\varphi\vartheta\varphi} = (1+xy)r_s^2 \sin^2\vartheta. \tag{2.2.77b}$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathscr{K} = \frac{12}{r_s^4(1+xy)^6}. \tag{2.2.78}$$

Local tetrad:

$$\mathbf{e}_{(0)} = -\frac{\sqrt{1+xy}}{2r_s y}\partial_x + \frac{y}{r_s\sqrt{1+xy}}\partial_y, \qquad \mathbf{e}_{(1)} = \frac{\sqrt{1+xy}}{2r_s y}\partial_x, \tag{2.2.79a}$$

$$\mathbf{e}_{(2)} = \frac{1}{r_s(1+xy)}\partial_\vartheta, \qquad \qquad \mathbf{e}_{(3)} = \frac{1}{r_s(1+xy)\sin\vartheta}\partial_\varphi. \tag{2.2.79b}$$

Dual tetrad:

$$\theta^{(0)} = \frac{r_s\sqrt{1+xy}}{y}dy, \qquad \theta^{(1)} = \frac{2r_s y}{\sqrt{1+xy}}dx + \frac{r_s\sqrt{1+xy}}{y}dy, \tag{2.2.80a}$$

$$\theta^{(2)} = r_s(1+xy)d\vartheta, \qquad \theta^{(3)} = r_s(1+xy)\sin\vartheta\,d\varphi. \tag{2.2.80b}$$

2.3   Alcubierre Warp

The Warp metric given by Miguel Alcubierre[Alc94] reads

$$ds^2 = -c^2 dt^2 + (dx - v_s f(r_s) dt)^2 + dy^2 + dz^2 \qquad (2.3.1)$$

where

$$v_s = \frac{dx_s(t)}{dt}, \qquad (2.3.2a)$$

$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}, \qquad (2.3.2b)$$

$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)}. \qquad (2.3.2c)$$

The parameter $R > 0$ defines the radius of the warp bubble and the parameter $\sigma > 0$ its thickness.

Metric-Tensor:

$$g_{tt} = -c^2 + v_s^2 f(r_s)^2, \qquad g_{tx} = -v_s f(r_s), \qquad g_{xx} = g_{yy} = g_{zz} = 1. \qquad (2.3.3)$$

Christoffel symbols:

$$\Gamma_{tt}^t = \frac{f^2 f_x v_s^3}{c^2}, \qquad\qquad \Gamma_{tt}^z = -f f_z v_s^2, \qquad \Gamma_{tt}^y = -f f_y v_s^2, \qquad (2.3.4a)$$

$$\Gamma_{tt}^x = \frac{f^3 f_x v_s^4 - c^2 f f_x v_s^2 - c^2 f_t v_s}{c^2}, \qquad \Gamma_{tx}^t = -\frac{f f_x v_s^2}{c^2}, \qquad \Gamma_{tx}^x = -\frac{f^2 f_x v_s^3}{c^2}, \qquad (2.3.4b)$$

$$\Gamma_{tx}^y = \frac{f_y v_s}{2}, \qquad\qquad \Gamma_{tx}^z = \frac{f_z v_s}{2}, \qquad \Gamma_{ty}^t = -\frac{f f_y v_s^2}{2c^2}, \qquad (2.3.4c)$$

$$\Gamma_{ty}^x = -\frac{f^2 f_y v_s^3 + c^2 f_y v_s}{2c^2}, \qquad \Gamma_{tz}^t = -\frac{f f_z v_s^2}{2c^2}, \qquad \Gamma_{tz}^x = -\frac{f^2 f_z v_s^3 + c^2 f_z v_s}{2c^2}, \qquad (2.3.4d)$$

$$\Gamma_{xx}^t = \frac{f_x v_s}{c^2}, \qquad\qquad \Gamma_{xx}^x = \frac{f f_x v_s^2}{c^2}, \qquad \Gamma_{xy}^t = \frac{f_y v_s}{2c^2}, \qquad (2.3.4e)$$

$$\Gamma_{xy}^x = \frac{f f_y v_s^2}{2c^2}, \qquad\qquad \Gamma_{xz}^t = \frac{f_z v_s}{2c^2}, \qquad \Gamma_{xz}^x = \frac{f f_z v_s^2}{2c^2}, \qquad (2.3.4f)$$

with derivatives

$$f_t = \frac{df(r_s)}{dt} = \frac{-v_s \sigma (x - x_s(t))}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2 (\sigma(r_s + R)) - \operatorname{sech}^2 (\sigma(r_s - R)) \right] \qquad (2.3.5a)$$

$$f_x = \frac{df(r_s)}{dx} = \frac{\sigma (x - x_s(t))}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2 (\sigma(r_s + R)) - \operatorname{sech}^2 (\sigma(r_s - R)) \right] \qquad (2.3.5b)$$

$$f_y = \frac{df(r_s)}{dy} = \frac{\sigma y}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2 (\sigma(r_s + R)) - \operatorname{sech}^2 (\sigma(r_s - R)) \right] \qquad (2.3.5c)$$

$$f_z = \frac{df(r_s)}{dz} = \frac{\sigma z}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2 (\sigma(r_s + R)) - \operatorname{sech}^2 (\sigma(r_s - R)) \right] \qquad (2.3.5d)$$

Riemann- and Ricci-tensor as well as Ricci- and Kretschman-scalar are shown only in the Maple worksheet.

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c} \left(\partial_t + v_s f \partial_x \right), \quad \mathbf{e}_{(1)} = \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \qquad (2.3.6)$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{c^2 - v_s^2 f^2}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{v_s f}{c\sqrt{c^2 - v_s^2 f^2}} \partial_t + \frac{\sqrt{c^2 - v_s^2 f^2}}{c} \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \qquad (2.3.7)$$

Further reading:
Pfenning[PF97], Clark[CHL99], Van Den Broeck[Bro99]

2.4 Barriola-Vilenkin monopol

The Barriola-Vilenkin metric describes the gravitational field of a global monopole[BV89]. In spherical coordinates $(t, r, \vartheta, \varphi)$, the metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + k^2 r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \right), \tag{2.4.1}$$

where k is the scaling factor responsible for the deficit/surplus angle.

Christoffel symbols:

$$\Gamma^r_{\vartheta\vartheta} = -k^2 r, \qquad \Gamma^r_{\varphi\varphi} = -k^2 r \sin^2\vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.4.2a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{2.4.2b}$$

Partial derivatives

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -k^2, \tag{2.4.3a}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -k^2 \sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.4.3b}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -k^2 r \sin(2\vartheta). \tag{2.4.3c}$$

Riemann-Tensor:

$$R_{\vartheta\varphi\vartheta\varphi} = (1-k^2)k^2 r^2 \sin^2\vartheta. \tag{2.4.4}$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{\vartheta\vartheta} = (1-k^2), \qquad R_{\varphi\varphi} = (1-k^2)\sin^2\vartheta, \qquad \mathscr{R} = 2\frac{1-k^2}{k^2 r^2}, \qquad \mathscr{K} = 4\frac{(1-k^2)^2}{k^4 r^4}. \tag{2.4.5}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(1-k^2)}{3k^2 r^2}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{6}(1-k^2), \qquad C_{t\varphi t\varphi} = \frac{c^2}{6}(1-k^2)\sin^2\vartheta, \tag{2.4.6a}$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{6}(1-k^2), \quad C_{r\varphi r\varphi} = -\frac{1}{6}(1-k^2)\sin^2\vartheta, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{k^2 r^2}{3}(1-k^2)\sin^2\vartheta. \tag{2.4.6b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{kr}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{kr\sin\vartheta}\partial_\varphi. \tag{2.4.7}$$

Dual tetrad:

$$\theta^{(t)} = c\,dt, \qquad \theta^{(r)} = dr, \qquad \theta^{(\vartheta)} = kr\,d\vartheta, \qquad \theta^{(\varphi)} = kr\sin\vartheta\,d\varphi. \tag{2.4.8}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{kr}. \tag{2.4.9}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{kr}. \tag{2.4.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{1-k^2}{k^2 r^2}.$$

(2.4.11)

Ricci-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{1-k^2}{k^2 r^2}.$$

(2.4.12)

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{1-k^2}{3k^2 r^2},$$

(2.4.13a)

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{1-k^2}{6k^2 r^2}.$$

(2.4.13b)

Embedding:
The embedding function, see Sec. 1.7, for $k < 1$ reads

$$z = \sqrt{1-k^2}\, r.$$

(2.4.14)

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{h_1^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(\frac{h_2^2}{k^2 r^2} - \kappa c^2\right),$$

(2.4.15)

with the constants of motion $h_1 = c^2 \dot{t}$ and $h_2 = k^2 r^2 \dot{\varphi}$.

The point of closest approach r_{pca} for a null geodesic that starts at $r = r_i$ with $\mathbf{y} = \pm\mathbf{e}_{(t)} + \cos\xi\,\mathbf{e}_{(r)} + \sin\xi\,\mathbf{e}_{(\varphi)}$ is given by $r = r_i \sin\xi$. Hence, the r_{pca} is independent of k. The same is also true for timelike geodesics.

Further reading:
Barriola and Vilenkin[BV89], Perlick[Per04].

2.5 Bertotti-Kasner

The Bertotti-Kasner spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ reads[Rin98]

$$ds^2 = -c^2 dt^2 + e^{2\sqrt{\Lambda}ct} dr^2 + \frac{1}{\Lambda} \left(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right), \tag{2.5.1}$$

where the cosmological constant Λ must be positive.

Christoffel symbols:

$$\Gamma^r_{tr} = c\sqrt{\Lambda}, \qquad \Gamma^t_{rr} = \frac{\sqrt{\Lambda}}{c} e^{2\sqrt{\Lambda}ct}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.5.2}$$

Partial derivatives

$$\Gamma^t_{rr,t} = 2\Lambda e^{2\sqrt{\Lambda}ct}, \qquad \Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta). \tag{2.5.3}$$

Riemann-Tensor:

$$R_{trtr} = -\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{\sin^2\vartheta}{\Lambda}. \tag{2.5.4}$$

Ricci-Tensor:

$$R_{tt} = -\Lambda c^2, \qquad R_{rr} = \Lambda e^{2\sqrt{\Lambda}ct}, \qquad R_{\vartheta\vartheta} = 1, \qquad R_{\varphi\varphi} = \sin^2\vartheta. \tag{2.5.5}$$

The Ricci and Kretschmann scalars read

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = 8\Lambda^2. \tag{2.5.6}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{2}{3}\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \qquad C_{t\vartheta t\vartheta} = \frac{c^2}{3}, \qquad\qquad C_{t\varphi t\varphi} = -\frac{1}{3} e^{2\sqrt{\Lambda}ct}, \tag{2.5.7a}$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{3} e^{2\sqrt{\Lambda}ct}, \qquad C_{r\varphi r\varphi} = -\frac{1}{3} e^{2\sqrt{\Lambda}ct} \sin^2\vartheta, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3}\frac{\sin^2\vartheta}{\Lambda}. \tag{2.5.7b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = e^{-\sqrt{\Lambda}ct}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \sqrt{\Lambda}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{\sqrt{\Lambda}}{\sin\vartheta}\partial_\varphi. \tag{2.5.8}$$

Dual tetrad:

$$\theta^{(t)} = c\,dt, \qquad \theta^{(r)} = e^{\sqrt{\Lambda}ct}dr, \qquad \theta^{(\vartheta)} = \frac{1}{\sqrt{\Lambda}}d\vartheta, \qquad \theta^{(\varphi)} = \frac{\sin\vartheta}{\sqrt{\Lambda}}d\varphi. \tag{2.5.9}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \sqrt{\Lambda}, \qquad \gamma_{(\vartheta)(\varphi)(\varphi)} = -\sqrt{\Lambda}\cot\vartheta. \tag{2.5.10}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\sqrt{\Lambda}, \qquad \gamma_{(\vartheta)} = \sqrt{\Lambda}\cot\vartheta. \tag{2.5.11}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\Lambda. \tag{2.5.12}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = -R_{(\vartheta)(\vartheta)} = -R_{(\varphi)(\varphi)} = -\Lambda. \tag{2.5.13}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2\Lambda}{3}, \tag{2.5.14a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{\Lambda}{3}. \tag{2.5.14b}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$c^2 \dot{t}^2 = h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa \tag{2.5.15}$$

with the constants of motion $h_1 = \dot{r}e^{2\sqrt{\Lambda}ct}$ and $h_2 = \dot{\varphi}/\Lambda$. Thus,

$$\lambda = \frac{1}{c\sqrt{\Lambda}\sqrt{\Lambda h_2^2 - \kappa}} \ln\left(\frac{1+q(t)}{1-q(t)}\frac{1-q(t_i)}{1+q(t_i)}\right), \qquad q(t) = \frac{h_1^2 e^{-2\sqrt{\Lambda}ct}}{\Lambda h_2^2 - \kappa} + 1, \tag{2.5.16}$$

where t_i is the initial time. We can also solve the orbital equation:

$$r(t) = w(t) - w(t_i) + r_i, \qquad w(t) = -\frac{\sqrt{h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa}}{h_1\sqrt{\Lambda}}, \tag{2.5.17}$$

where r_i is the initial radial position.

Further reading:
Rindler[Rin98]: *"Every spherically symmetric solution of the generalized vacuum field equations $R_{ij} = \Lambda g_{ij}$ is either equivalent to Kottler's generalization of Schwarzschild space or to the [...] Bertotti-Kasner space (for which Λ must be necessarily be positive)."*

2.6 Bessel gravitational wave

D. Kramer introduced in [Kra99] an exact gravitational wave solution of Einstein's vacuum field equations. According to [Ste03] we execute the substitution $x \to t$ and $y \to z$.

2.6.1 Cylindrical coordinates

The metric of the Bessel wave in cylindrical coordinates reads

$$ds^2 = e^{-2U} \left[e^{2K} \left(d\rho^2 - dt^2 \right) + \rho^2 d\varphi^2 \right] + e^{2U} dz^2. \tag{2.6.1}$$

The functions U and K are given by

$$U := CJ_0(\rho)\cos(t), \tag{2.6.2}$$

$$K := \frac{1}{2}C^2\rho \left\{ \rho \left[J_0(\rho)^2 + J_1(\rho)^2 \right] - 2J_0(\rho)J_1(\rho)\cos^2(t) \right\}, \tag{2.6.3}$$

where $J_n(\rho)$ are the Bessel functions of the first kind.

Christoffel symbols:

$$\Gamma^t_{tt} = \Gamma^\rho_{t\rho} = \Gamma^t_{\rho\rho} = -\frac{\partial U}{\partial t} + \frac{\partial K}{\partial t}, \qquad \Gamma^\varphi_{t\varphi} = \Gamma^z_{tz} = -\frac{\partial U}{\partial t}, \qquad \Gamma^t_{\varphi\varphi} = -e^{-2K}\rho^2\frac{\partial U}{\partial t}, \tag{2.6.4a}$$

$$\Gamma^\rho_{tt} = \Gamma^t_{t\rho} = \Gamma^\rho_{\rho\rho} = -\frac{\partial U}{\partial \rho} + \frac{\partial K}{\partial \rho}, \qquad \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho} - \frac{\partial U}{\partial \rho}, \qquad \Gamma^\rho_{zz} = -e^{4U-2K}\frac{\partial U}{\partial \rho}, \tag{2.6.4b}$$

$$\Gamma^\rho_{\varphi\varphi} = \rho e^{-2K} \left(\rho\frac{\partial U}{\partial \rho} - 1 \right), \qquad \Gamma^z_{\rho z} = \frac{\partial U}{\partial \rho}, \qquad \Gamma^t_{zz} = e^{4U-2K}\frac{\partial U}{\partial t}. \tag{2.6.4c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = e^{U-K}\partial_t, \quad \mathbf{e}_{(\rho)} = e^{U-K}\partial_\rho, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho}e^U\partial_\varphi, \quad \mathbf{e}_{(z)} = e^{-U}\partial_z. \tag{2.6.5}$$

Dual tetrad:

$$\theta^{(t)} = e^{K-U}dt, \quad \theta^{(\rho)} = e^{K-U}d\rho, \quad \theta^{(\varphi)} = \rho e^{-U}d\varphi, \quad \theta^{(z)} = e^U dz. \tag{2.6.6}$$

2.6.2 Cartesian coordinates

In Cartesian coordinates with $\rho = \sqrt{x^2+y^2}$ the metric (2.6.1) reads

$$ds^2 = -e^{2(K-U)}dt^2 + \frac{e^{-2U}}{x^2+y^2}\left[\left(e^{2K}x^2 + y^2\right)dx^2 + 2xy\left(e^{2K} - 1\right)dxdy \right.$$
$$\left. + \left(x^2 + e^{2K}y^2\right)dy^2 \right] + e^{2U}dz^2. \tag{2.6.7}$$

Local tetrad:

$$\mathbf{e}_{(t)} = e^{U-K}\partial_t, \qquad \mathbf{e}_{(x)} = e^U\sqrt{\frac{x^2+y^2}{e^{2K}x^2+y^2}}\partial_x,$$

$$\mathbf{e}_{(y)} = e^{U-K}\sqrt{\frac{e^{2K}x^2+y^2}{x^2+y^2}}\partial_y + xy\frac{e^{U-K}\left(e^{2K}-1\right)}{\sqrt{(x^2+y^2)(e^{2K}x^2+y^2)}}\partial_x, \qquad \mathbf{e}_{(z)} = e^{-U}\partial_z \tag{2.6.8}$$

2.7 Cosmic string in Schwarzschild spacetime

A cosmic string in the Schwarzschild spacetime represented by Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 \left(d\vartheta^2 + \beta^2 \sin^2 \vartheta d\varphi^2\right), \tag{2.7.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and β is the string parameter, compare Aryal et al[AFV86].

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s(r - r_s)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s}{2r(r - r_s)}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r(r - r_s)}, \tag{2.7.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r - r_s), \tag{2.7.2b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -(r - r_s)\beta^2 \sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\beta^2 \sin\vartheta\cos\vartheta. \tag{2.7.2c}$$

Partial derivatives

$$\Gamma^r_{tt,r} = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \qquad \Gamma^t_{tr,r} = -\frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad \Gamma^r_{rr,r} = \frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \tag{2.7.3a}$$

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -1, \tag{2.7.3b}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -\beta^2 \sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\beta^2 \cos(2\vartheta), \tag{2.7.3c}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -(r - r_s)\beta^2 \sin(2\vartheta). \tag{2.7.3d}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}\frac{c^2 (r - r_s) r_s}{r^2}, \qquad R_{t\varphi t\varphi} = \frac{1}{2}\frac{c^2 (r - r_s) r_s \beta^2 \sin^2\vartheta}{r^2}, \tag{2.7.4a}$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2}\frac{r_s}{r - r_s}, \quad R_{r\varphi r\varphi} = -\frac{1}{2}\frac{r_s \beta^2 \sin^2\vartheta}{r - r_s}, \quad R_{\vartheta\varphi\vartheta\varphi} = r r_s \beta^2 \sin^2\vartheta. \tag{2.7.4b}$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12\frac{r_s^2}{r^6}. \tag{2.7.5}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\beta\sin\vartheta}\partial_\varphi. \tag{2.7.6}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r}}dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \qquad \theta^{(\vartheta)} = r d\vartheta, \qquad \theta^{(\varphi)} = r\beta\sin\vartheta\, d\varphi. \tag{2.7.7}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.7.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.7.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.7.10a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.7.10b}$$

Embedding:
The embedding function for $\beta^2 < 1$ reads

$$z = (r - r_s)\sqrt{\frac{r}{r - r_s} - \beta^2} - \frac{r_s}{2\sqrt{1 - \beta^2}}\ln\frac{\sqrt{r/(r - r_s) - \beta^2} - \sqrt{1 - \beta^2}}{\sqrt{r/(r - r_s) - \beta^2} + \sqrt{1 - \beta^2}}. \tag{2.7.11}$$

If $\beta^2 = 1$, we have the embedding function of the standard Schwarzschild metric, compare Eq.(2.2.15).

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(\frac{h^2}{r^2\beta^2} - \kappa c^2\right) \tag{2.7.12}$$

with the constants of motion $k = (1 - r_s/r)c^2\dot{t}$ and $h = r^2\beta^2\dot{\varphi}$. The maxima of the effective potential V_{eff} lead to the same critical orbits $r_{\text{po}} = \frac{3}{2}r_s$ and $r_{\text{itcg}} = 3r_s$ as in the standard Schwarzschild metric.

2.8   Ernst spacetime

"The Ernst metric is a static, axially symmetric, electro-vacuum solution of the Einstein-Maxwell equations with a black hole immersed in a magnetic field."[KV92]

In spherical coordinates $(t, r, \vartheta, \varphi)$, the Ernst metric reads[Ern76] $(G = c = 1)$

$$ds^2 = \Lambda^2 \left[-\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\vartheta^2 \right] + \frac{r^2 \sin^2 \vartheta}{\Lambda^2} d\varphi^2, \tag{2.8.1}$$

where $\Lambda = 1 + B^2 r^2 \sin^2 \vartheta$. Here, M is the mass of the black hole and B the magnetic field strength.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{\left(2B^2 r^3 \sin^2 \vartheta - 3MB^2 r^2 \sin^2 \vartheta + M\right)(r - 2M)}{r^3 \Lambda}, \quad \Gamma^\vartheta_{tt} = \frac{2(r - 2M) B^2 \sin \vartheta \cos \vartheta}{r \Lambda}, \tag{2.8.2a}$$

$$\Gamma^t_{tr} = \frac{2B^2 r^3 \sin^2 \vartheta - 3MB^2 r^2 \sin^2 \vartheta + M}{r(r - 2M)\Lambda}, \qquad \Gamma^t_{t\vartheta} = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \tag{2.8.2b}$$

$$\Gamma^r_{rr} = \frac{2B^2 r^3 \sin^2 \vartheta - 5MB^2 r^2 \sin^2 \vartheta - M}{r(r - 2M)\Lambda}, \qquad \Gamma^\vartheta_{rr} = -\frac{2B^2 r \sin \vartheta \cos \vartheta}{(r - 2M)\Lambda}, \tag{2.8.2c}$$

$$\Gamma^r_{r\vartheta} = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{3B^2 r^2 \sin^2 \vartheta + 1}{r \Lambda}, \tag{2.8.2d}$$

$$\Gamma^\varphi_{r\varphi} = \frac{1 - B^2 r^2 \sin^2 \vartheta}{r \Lambda}, \qquad \Gamma^r_{\vartheta\vartheta} = \frac{\left(3B^2 r^2 \sin^2 \vartheta + 1\right)(r - 2M)}{\Lambda}, \tag{2.8.2e}$$

$$\Gamma^\vartheta_{\vartheta\vartheta} = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \frac{\Xi \cos \vartheta}{\Lambda}, \tag{2.8.2f}$$

$$\Gamma^r_{\varphi\varphi} = \frac{(r - 2M) \Xi \sin^2 \vartheta}{\Lambda^5}, \tag{2.8.2g}$$

$$\Gamma^\vartheta_{\varphi\varphi} = \frac{\Xi \sin \vartheta \cos \vartheta}{\Lambda^5}. \tag{2.8.2h}$$

with $\Xi = 1 - B^2 r^2 \sin^2 \vartheta$.

Riemann-Tensor:

$$R_{trtr} = \frac{2}{r^3} \left[B^4 r^4 \sin^4 \vartheta (3M - r) - M + 2r^5 B^4 \sin^2 \vartheta \cos^2 \vartheta + B^2 r^2 \sin^2 \vartheta (r - 2M) \right], \tag{2.8.3a}$$

$$R_{trt\vartheta} = 2B^2 \sin \vartheta \cos \vartheta \left[(3B^2 r^2 \sin^2 \vartheta (2M - 3r) + r - 2M \right], \tag{2.8.3b}$$

$$R_{t\vartheta t\vartheta} = \frac{1}{r^2} \left[B^4 r^4 (r - 2M)(4r - 9M) \sin^4 \vartheta + 2\Xi B^2 r^3 (r - 2M) \cos^2 \vartheta + M(r - 2M) \right], \tag{2.8.3c}$$

$$R_{t\varphi t\varphi} = \frac{1}{\Lambda^4 r^2} \left[(2B^2 r^3 - 3B^2 M r^2 \sin^2 \vartheta + M) \Xi (r - 2M) \sin^2 \vartheta \right], \tag{2.8.3d}$$

$$R_{r\vartheta r\vartheta} = -\frac{(2B^2 r^3 - 3B^2 M r^2 \sin^2 \vartheta + M) \Xi}{r - 2M}, \tag{2.8.3e}$$

$$R_{r\varphi r\varphi} = -\frac{\sin^2 \vartheta}{\Lambda^4 (r - 2M)} \left[B^4 r^4 (4r - 9M) \sin^4 \vartheta + 2B^2 r^2 (8M - 4r\vartheta) \sin^2 \vartheta + 2\Xi B^2 r^3 \cos^2 \vartheta + M \right], \tag{2.8.3f}$$

$$R_{r\varphi\vartheta\varphi} = -\frac{2B^2 r^3 \sin^3 \vartheta \cos \vartheta (3B^2 r^2 \sin^2 \vartheta - 5)}{\Lambda^4}, \tag{2.8.3g}$$

$$R_{\vartheta\varphi\vartheta\varphi} = \frac{r \sin^2 \vartheta}{\Lambda^4} \left[2B^4 r^4 (r - 3M) \sin^4 \vartheta + 4B^2 r^3 \cos^2 \vartheta (1 + \Xi) + 2B^2 r^2 \sin^2 \vartheta (2M - r) + 2M \right]. \tag{2.8.3h}$$

Ricci-Tensor:

$$R_{tt} = \frac{4B^2(r-2M)(r+2M\sin^2\vartheta)}{r^2\Lambda^2}, \quad R_{rr} = -\frac{4B^2[r\cos^2\vartheta - (r-2M)\sin^2\vartheta]}{(r-2M)\Lambda^2},$$ (2.8.4a)

$$R_{r\vartheta} = \frac{8B^2 r\sin\vartheta\cos\vartheta}{\Lambda^2}, \qquad R_{\vartheta\vartheta} = \frac{4B^2 r\left[r\cos^2\vartheta + (r-2M)\sin^2\vartheta\right]}{\Lambda^2},$$ (2.8.4b)

$$R_{\varphi\varphi} = \frac{4B^2 r\sin^2\vartheta\left(r+2M\sin^2\vartheta\right)}{\Lambda^6}.$$ (2.8.4c)

Ricci and Kretschmann scalars:

$$R = 0,$$ (2.8.5a)

$$\begin{aligned}
\mathscr{K} = \frac{16}{r^6\Lambda^8}\Bigg[& 3B^8 r^8\left(4r^2 - 18Mr + 21M^2\right)\sin^8\vartheta \\
& + 2B^4 r^4\left(31M^2 - 37Mr - 24B^2 r^4\cos^2\vartheta + 42B^2 Mr^3\cos^2\vartheta + 10r^2 + 6B^4 r^6\cos^4\vartheta\right)\sin^6\vartheta \\
& + 2B^2 r^2\left(-3Mr + 20B^2 r^4\cos^2\vartheta + 6M^2 - 46B^2 Mr^3\cos^2\vartheta - 12B^4 r^6\cos^4\vartheta\right)\sin^4\vartheta \\
& - 6B^6 r^6\left(6B^2 Mr^3\cos^2\vartheta + 4r^2 - 4B^2 r^4\cos^2\vartheta + 18M^2 - 17Mr\right) \\
& + 20B^4 r^6\cos^4\vartheta + 12B^2 Mr^3\cos^2\vartheta + 3M^2\Bigg].
\end{aligned}$$ (2.8.5b)

Static local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{\Lambda\sqrt{1-2m/r}}\partial_t, \quad \mathbf{e}_{(r)} = \frac{\sqrt{1-2m/r}}{\Lambda}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\Lambda r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\Lambda}{r\sin\vartheta}\partial_\varphi.$$ (2.8.6)

Dual tetrad:

$$\theta^{(t)} = \Lambda\sqrt{1-\frac{2m}{r}}\,dt, \quad \theta^{(r)} = \frac{\Lambda}{\sqrt{1-2m/r}}dr, \quad \theta^{(\vartheta)} = \Lambda r\,d\vartheta, \quad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\Lambda}\,d\varphi.$$ (2.8.7)

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{r}^2 + \frac{h^2(1-r_s/r)}{r^2} - \frac{k^2}{\Lambda^4} + \kappa\frac{1-r_s/r}{\Lambda^2} = 0$$ (2.8.8)

with constants of motion $k = \Lambda^2(1-r_s/r)\dot{t}$ and $h = (r^2/\Lambda^2)\dot{\varphi}$.

Further reading:
Ernst[Ern76], Dhurandhar and Sharma[DS83], Karas and Vokrouhlicky[KV92], Stuchlík and Hledík[SH99].

2.9 Friedman-Robertson-Walker

The Friedman-Robertson-Walker metric describes a general homogeneous and isotropic universe. In a general form it reads:

$$ds^2 = -c^2 dt^2 + R^2 d\sigma^2 \tag{2.9.1}$$

with $R = R(t)$ being an arbitrary function of time only and $d\sigma^2$ being a metric of a 3-space of constant curvature for which three explicit forms will be described here.

In all formulas in this section a dot denotes differentiation with respect to t, e.g. $\dot{R} = dR(t)/dt$.

2.9.1 Form 1

$$ds^2 = -c^2 dt^2 + R^2 \left\{ \frac{d\eta^2}{1 - k\eta^2} + \eta^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right\} \tag{2.9.2}$$

Christoffel symbols:

$$\Gamma^\eta_{t\eta} = \frac{\dot{R}}{R}, \qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.3a}$$

$$\Gamma^t_{\eta\eta} = \frac{R\dot{R}}{c^2(1 - k\eta^2)}, \quad \Gamma^\eta_{\eta\eta} = \frac{k\eta}{1 - k\eta^2}, \qquad \Gamma^\vartheta_{\eta\vartheta} = \frac{1}{\eta}, \tag{2.9.3b}$$

$$\Gamma^\varphi_{\eta\varphi} = \frac{1}{\eta}, \qquad \Gamma^t_{\vartheta\vartheta} = \frac{R\eta^2\dot{R}}{c^2}, \qquad \Gamma^\eta_{\vartheta\vartheta} = (k\eta^2 - 1)\eta, \tag{2.9.3c}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^t_{\varphi\varphi} = \frac{R\eta^2 \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma^\eta_{\varphi\varphi} = (k\eta^2 - 1)\eta \sin^2 \vartheta, \tag{2.9.3d}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.9.3e}$$

Riemann-Tensor:

$$R_{t\eta t\eta} = \frac{R\ddot{R}}{k\eta^2 - 1}, \qquad\qquad R_{t\vartheta t\vartheta} = -R\eta^2\ddot{R}, \tag{2.9.4a}$$

$$R_{t\varphi t\varphi} = -R\eta^2 \sin^2 \vartheta \ddot{R}, \qquad R_{\eta\vartheta\eta\vartheta} = -\frac{R^2\eta^2 \left(\dot{R}^2 + kc^2\right)}{c^2(k\eta^2 - 1)}, \tag{2.9.4b}$$

$$R_{\eta\varphi\eta\varphi} = -\frac{R^2\eta^2 \sin^2 \vartheta \left(\dot{R}^2 + kc^2\right)}{c^2(k\eta^2 - 1)}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\eta^4 \sin^2 \vartheta \left(\dot{R}^2 + kc^2\right)}{c^2}. \tag{2.9.4c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad\qquad R_{\eta\eta} = \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(1 - k\eta^2)}, \tag{2.9.5a}$$

$$R_{\vartheta\vartheta} = \eta^2 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2}, \qquad R_{\varphi\varphi} = \eta^2 \sin^2 \vartheta \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2}. \tag{2.9.5b}$$

The *Ricci scalar and Kretschmann scalar read:*

$$\mathscr{R} = 6\frac{R\ddot{R} + \dot{R}^2 + kc^2}{R^2 c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2 R^2 + \dot{R}^4 + 2\dot{R}^2 kc^2 + k^2 c^4}{R^4 c^4}. \tag{2.9.6}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\eta)} = \frac{\sqrt{1 - k\eta^2}}{R}\partial_\eta, \qquad e_\vartheta = \frac{1}{R\eta}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\eta \sin\vartheta}\partial_\varphi. \tag{2.9.7}$$

Ricci rotation coefficients:

$$\gamma_{(\eta)(t)(\eta)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\eta)(\vartheta)} = \gamma_{(\varphi)(\eta)(\varphi)} = \frac{\sqrt{1 - k\eta^2}}{R\eta},$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{R\eta}.$$

(2.9.8)

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = \frac{2\sqrt{1 - k\eta^2}}{R\eta}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\eta}.$$

(2.9.9)

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}$$

(2.9.10a)

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + kc^2}{R^2 c^2}.$$

(2.9.10b)

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2 + 2kc^2}{R^2 c^2}.$$

(2.9.11)

2.9.2 Form 2

$$\boxed{ds^2 = -c^2 dt^2 + \frac{R^2}{(1 + \frac{k}{4}r^2)^2}\left\{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right\}}$$

(2.9.12)

Christoffel symbols:

$$\Gamma^r_{tr} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R},$$

(2.9.13a)

$$\Gamma^t_{rr} = 16\frac{R\dot{R}}{c^2(4 + kr^2)^2}, \quad \Gamma^r_{rr} = -\frac{2kr}{4 + kr^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{4 - kr^2}{(4 + kr^2)r},$$

(2.9.13b)

$$\Gamma^\varphi_{r\varphi} = \frac{4 - kr^2}{(4 + kr^2)r}, \qquad \Gamma^t_{\vartheta\vartheta} = 16\frac{Rr^2\dot{R}}{c^2(4 + kr^2)^2}, \quad \Gamma^r_{\vartheta\vartheta} = \frac{r(kr^2 - 4)}{4 + kr^2},$$

(2.9.13c)

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad\qquad \Gamma^t_{\varphi\varphi} = 16\frac{Rr^2\sin^2\vartheta\dot{R}}{c^2(4 + kr^2)^2}, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta,$$

(2.9.13d)

$$\Gamma^r_{\varphi\varphi} = \frac{r\sin^2\vartheta(kr^2 - 4)}{4 + kr^2}.$$

(2.9.13e)

Riemann-Tensor:

$$R_{trtr} = -16\frac{R\ddot{R}}{(4 + kr^2)^2}, \qquad\qquad R_{t\vartheta t\vartheta} = -16\frac{Rr^2\ddot{R}}{(4 + kr^2)^2},$$

(2.9.14a)

$$R_{t\varphi t\varphi} = -16\frac{Rr^2\sin^2\vartheta\ddot{R}}{(4 + kr^2)^2}, \qquad\qquad R_{r\vartheta r\vartheta} = 256\frac{R^2 r^2\left(\dot{R}^2 + kc^2\right)}{c^2(4 + kr^2)^4},$$

(2.9.14b)

$$R_{r\varphi r\varphi} = 256\frac{R^2 r^2\sin^2\vartheta\left(\dot{R}^2 + kc^2\right)}{c^2(4 + kr^2)^4}, \quad R_{\vartheta\varphi\vartheta\varphi} = 256\frac{R^2 r^4\sin^2\vartheta\left(\dot{R}^2 + kc^2\right)}{c^2(4 + kr^2)^4}.$$

(2.9.14c)

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad\qquad R_{rr} = 16\frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(4 + kr^2)^2},$$

(2.9.15a)

$$R_{\vartheta\vartheta} = 16r^2\frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(4 + kr^2)^2}, \qquad R_{\varphi\varphi} = 16r^2\sin^2\vartheta\frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(4 + kr^2)^2}.$$

(2.9.15b)

The *Ricci scalar and Kretschmann scalar read:*

$$\mathscr{R} = 6\frac{R\ddot{R} + \dot{R}^2 + kc^2}{R^2 c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2 R^2 + \dot{R}^4 + 2\dot{R}^2 kc^2 + k^2 c^4}{R^4 c^4}. \tag{2.9.16}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(r)} = \frac{1 + \frac{k}{4}r^2}{R}\partial_r, \qquad e_\vartheta = \frac{1 + \frac{k}{4}r^2}{Rr}\partial_\vartheta, \qquad e_\varphi = \frac{1 + k/4r^2}{Rr\sin\vartheta}\partial_\varphi. \tag{2.9.17}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = -\frac{\frac{k}{4}r^2 - 1}{Rr}, \tag{2.9.18a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{(\frac{k}{4}r^2 + 1)\cot\vartheta}{Rr}. \tag{2.9.18b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{1 - \frac{k}{4}r^2}{Rr}, \qquad \gamma_{(\vartheta)} = \frac{(\frac{k}{4}r^2 + 1)\cot\vartheta}{Rr}. \tag{2.9.19}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \tag{2.9.20a}$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + kc^2}{R^2 c^2}. \tag{2.9.20b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2 + 2kc^2}{R^2 c^2}. \tag{2.9.21}$$

2.9.3   Form 3

The following forms of the metric are obtained from 2.9.2 by setting $\eta = \sin\psi, \psi, \sinh\psi$ for $k = 1, 0, -1$ respectively.

Positive Curvature

$$\boxed{ds^2 = -c^2 dt^2 + R^2 \left\{ d\psi^2 + \sin^2\psi \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2 \right) \right\}} \tag{2.9.22}$$

Christoffel symbols:

$$\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.23a}$$

$$\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad\qquad \Gamma^\vartheta_{\psi\vartheta} = \cot\psi, \qquad\qquad \Gamma^\varphi_{\psi\varphi} = \cot\psi, \tag{2.9.23b}$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{R\sin^2\psi\dot{R}}{c^2}, \qquad \Gamma^\psi_{\vartheta\vartheta} = -\sin\psi\cos\psi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \tag{2.9.23c}$$

$$\Gamma^t_{\varphi\varphi} = \frac{R\sin^2\psi\sin^2\vartheta\dot{R}}{c^2}, \Gamma^\psi_{\varphi\varphi} = -\sin\psi\cos\psi\sin^2\vartheta, \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.9.23d}$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad\qquad R_{t\vartheta t\vartheta} = -R\sin^2\psi\ddot{R}, \qquad\qquad (2.9.24a)$$

$$R_{t\varphi t\varphi} = -R\sin^2\psi\sin^2\vartheta\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sin^2\psi\left(\dot{R}^2+c^2\right)}{c^2}, \qquad (2.9.24b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sin^2\psi\sin^2\vartheta\left(\dot{R}^2+c^2\right)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sin^4\psi\sin^2\vartheta\left(\dot{R}^2+c^2\right)}{c^2}. \qquad (2.9.24c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad\qquad R_{\psi\psi} = \frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}, \qquad (2.9.25a)$$

$$R_{\vartheta\vartheta} = \sin^2\psi\frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}, \qquad R_{\varphi\varphi} = \sin^2\vartheta\sin^2\psi\frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}. \qquad (2.9.25b)$$

The *Ricci scalar and Kretschmann* read

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+c^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2c^2+c^4}{R^4c^4}. \qquad (2.9.26)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\sin\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\sin\psi\sin\vartheta}\partial_\varphi. \qquad (2.9.27)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\cot\psi}{R}, \qquad (2.9.28a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sin\psi}. \qquad (2.9.28b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{\cot\psi}{R}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sin\psi}. \qquad (2.9.29)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \qquad (2.9.30a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+c^2}{R^2c^2}. \qquad (2.9.30b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2(\dot{R}^2+c^2)}{R^2c^2}. \qquad (2.9.31)$$

Vanishing Curvature

$$\boxed{ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \psi^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\right\}} \qquad (2.9.32)$$

Christoffel symbols:

$$\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \qquad (2.9.33a)$$

$$\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad\qquad \Gamma^\vartheta_{\psi\vartheta} = \frac{1}{\psi}, \qquad\qquad \Gamma^\varphi_{\psi\varphi} = \frac{1}{\psi}, \qquad (2.9.33b)$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{R\psi^2\dot{R}}{c^2}, \qquad\qquad \Gamma^\psi_{\vartheta\vartheta} = -\psi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \qquad (2.9.33c)$$

$$\Gamma^t_{\varphi\varphi} = \frac{R\psi^2\sin^2\vartheta\dot{R}}{c^2}, \qquad \Gamma^\psi_{\varphi\varphi} = -\psi\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \qquad (2.9.33d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad\qquad R_{t\vartheta t\vartheta} = -R\psi^2\ddot{R}, \qquad\qquad (2.9.34a)$$

$$R_{t\varphi t\varphi} = -R\psi^2\sin^2\vartheta\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\psi^2\dot{R}^2}{c^2}, \qquad\qquad (2.9.34b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\psi^2\sin^2\vartheta\dot{R}^2}{c^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\psi^4\sin^2\vartheta\dot{R}^2}{c^2}. \qquad (2.9.34c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad\qquad R_{\psi\psi} = \frac{R\ddot{R}+2\dot{R}^2}{c^2}, \qquad\qquad (2.9.35a)$$

$$R_{\vartheta\vartheta} = \psi^2\frac{R\ddot{R}+2\dot{R}^2}{c^2}, \qquad R_{\varphi\varphi} = \sin^2\vartheta\,\psi^2\frac{R\ddot{R}+2\dot{R}^2}{c^2}. \qquad (2.9.35b)$$

The *Ricci scalar and Kretschmann read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4}{R^4c^4}. \qquad\qquad (2.9.36)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\psi\sin\vartheta}\partial_\varphi. \qquad (2.9.37)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{1}{R\psi}, \qquad (2.9.38a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\vartheta)}{R\psi}. \qquad\qquad (2.9.38b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = \frac{2}{R\psi}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\psi}. \qquad\qquad (2.9.39)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \qquad\qquad (2.9.40a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2}{R^2c^2}. \qquad\qquad (2.9.40b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2}{R^2c^2}. \qquad (2.9.41)$$

Negative Curvature

$$\boxed{ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \sinh^2\psi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right\}} \qquad\qquad (2.9.42)$$

Christoffel symbols:

$$\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad\qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \qquad (2.9.43a)$$

$$\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad\qquad \Gamma^\vartheta_{\psi\vartheta} = \coth\psi, \qquad \Gamma^\varphi_{\psi\varphi} = \coth\psi, \qquad (2.9.43b)$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{R\sinh^2\psi\dot{R}}{c^2}, \qquad \Gamma^\psi_{\vartheta\vartheta} = -\sinh\psi\cosh\psi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad (2.9.43c)$$

$$\Gamma^t_{\varphi\varphi} = \frac{R\sinh^2\psi\sin^2\vartheta\dot{R}}{c^2}, \quad \Gamma^\psi_{\varphi\varphi} = -\sinh\psi\cosh\psi\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \qquad (2.9.43d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad\qquad R_{t\vartheta t\vartheta} = -R\sinh^2\psi\ddot{R}, \qquad\qquad (2.9.44\text{a})$$

$$R_{t\varphi t\varphi} = -R\sinh^2\psi\sin^2\vartheta\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sinh^2\psi\left(\dot{R}^2-c^2\right)}{c^2}, \qquad (2.9.44\text{b})$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sinh^2\psi\sin^2\vartheta\left(\dot{R}^2-c^2\right)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sinh\psi^4\sin^2\vartheta\left(\dot{R}^2-c^2\right)}{c^2}. \qquad (2.9.44\text{c})$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad\qquad R_{\psi\psi} = \frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}, \qquad (2.9.45\text{a})$$

$$R_{\vartheta\vartheta} = \sinh^2\psi\frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}, \quad R_{\varphi\varphi} = \sin^2\vartheta\sin^2\psi\frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}. \qquad (2.9.45\text{b})$$

The *Ricci scalar and Kretschmann read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2-c^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4-2\dot{R}^2c^2+c^4}{R^4c^4}. \qquad (2.9.46)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\sinh\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\sinh\psi\sin\vartheta}\partial_\varphi. \qquad (2.9.47)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\coth\psi}{R}, \qquad (2.9.48\text{a})$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sinh\psi}. \qquad (2.9.48\text{b})$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{\coth\psi}{R}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sinh\psi}. \qquad (2.9.49)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \qquad (2.9.50\text{a})$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2-c^2}{R^2c^2}. \qquad (2.9.50\text{b})$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2(\dot{R}^2-c^2)}{R^2c^2}. \qquad (2.9.51)$$

Further reading:
Rindler[Rin01]

2.10 Gödel Universe

Gödel introduced a homogeneous and rotating universe model in [Göd49]. We follow the notation of [KWSD04]

2.10.1 Cylindrical coordinates

The Gödel metric in cylindrical coordinates is

$$ds^2 = -c^2 dt^2 + \frac{dr^2}{1 + [r/(2a)]^2} + r^2 \left[1 - \left(\frac{r}{2a} \right)^2 \right] d\varphi^2 + dz^2 - 2r^2 \frac{c}{\sqrt{2}a} dt d\varphi, \tag{2.10.1}$$

where $2a$ is the Gödel radius.

Christoffel symbols:

$$\Gamma^t_{tr} = \frac{r}{2a^2} \frac{1}{1 + [r/(2a)]^2}, \qquad \Gamma^\varphi_{tr} = -\frac{c}{\sqrt{2}ar} \frac{1}{1 + [r/(2a)]^2}, \tag{2.10.2a}$$

$$\Gamma^r_{t\varphi} = \frac{cr}{\sqrt{2}a} \left[1 + \left(\frac{r}{2a} \right) \right]^2, \qquad \Gamma^r_{rr} = -\frac{r}{4a^2} \frac{1}{1 + [r/(2a)]^2}, \tag{2.10.2b}$$

$$\Gamma^t_{r\varphi} = \frac{r^3}{4\sqrt{2}ca^3} \frac{1}{1 + [r/(2a)]^2}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \frac{1}{1 + [r/(2a)]^2}, \tag{2.10.2c}$$

$$\Gamma^r_{\varphi\varphi} = r \left[1 + \left(\frac{r}{2a} \right)^2 \right] \left[1 - \frac{1}{2} \left(\frac{r}{a} \right)^2 \right]. \tag{2.10.2d}$$

Riemann-Tensor:

$$R_{trtr} = \frac{c^2}{2a^2} \frac{1}{1 + [r/(2a)]^2}, \quad R_{trr\varphi} = -\frac{cr^2}{2\sqrt{2}a^3} \frac{1}{1 + [r/(2a)]^2}, \tag{2.10.3a}$$

$$R_{t\varphi t\varphi} = \frac{c^2 r^2}{2a^2} \frac{1}{1 + [r/(2a)]^2}, \quad R_{r\varphi r\varphi} = \frac{r^2}{2a^2} \frac{1 + 3[r/(2a)]^2}{1 + [r/(2a)]^2}. \tag{2.10.3b}$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2}{a^2}, \quad R_{t\varphi} = \frac{r^2 c}{\sqrt{2}a^3}, \quad R_{\varphi\varphi} = \frac{r^4}{2a^4}. \tag{2.10.4}$$

Ricci and Kretschmann scalar

$$\mathscr{R} = -\frac{1}{a^2}, \qquad \mathscr{K} = \frac{3}{a^4}. \tag{2.10.5}$$

cosmological constant:

$$\Lambda = \frac{R}{2} \tag{2.10.6}$$

Killing vectors:
An infinitesimal isometric transformation $x'^\mu = x^\mu + \varepsilon \xi^\mu(x^\nu)$ leaves the metric unchanged, that is $g'_{\mu\nu}(x'^\sigma) = g_{\mu\nu}(x'^\sigma)$. A killing vector field ξ^μ is solution to the killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$. There exist five killing vector fields in Gödel's spacetime:

$$\xi^\mu_a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi^\mu_b = \frac{1}{\sqrt{1 + [r/(2a)]^2}} \begin{pmatrix} \frac{r}{\sqrt{2}c} \cos\varphi \\ a \left(1 + [r/(2a)]^2 \right) \sin\varphi \\ \frac{a}{r} \left(1 + 2[r/(2a)]^2 \right) \cos\varphi \\ 0 \end{pmatrix}, \quad \xi^\mu_c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{2.10.7a}$$

$$\xi^\mu_d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi^\mu_e = \frac{1}{\sqrt{1 + [r/(2a)]^2}} \begin{pmatrix} \frac{r}{\sqrt{2}c} \sin\varphi \\ -a \left(1 + [r/(2a)]^2 \right) \cos\varphi \\ \frac{a}{r} \left(1 + 2[r/(2a)]^2 \right) \sin\varphi \\ 0 \end{pmatrix}. \tag{2.10.7b}$$

An arbitrary linear combination of killing vector fields is again a killing vector field.

Local tetrad:

For the local tetrad in Gödel's spacetime an ansatz similar to the local tetrad of a rotating spacetime in spherical coordinates (Sec. 1.4.7) can be used. After substituting $\vartheta \to z$ and swapping base vectors $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$ an orthonormalized and right-handed local tetrad is obtained.

$$\mathbf{e}_{(0)} = \Gamma \left(\partial_t + \zeta \partial_\varphi \right), \quad \mathbf{e}_{(1)} = \sqrt{1 + [r/(2a)]^2} \partial_r, \quad \mathbf{e}_{(2)} = \Delta \Gamma \left(A \partial_t + B \partial_\varphi \right), \quad \mathbf{e}_{(3)} = \partial_z, \tag{2.10.8a}$$

where

$$A = -\frac{r^2 c}{\sqrt{2}a} + \zeta r^2 \left(1 - [r/(2a)]^2 \right), \qquad\qquad B = c^2 + \frac{\zeta r^2 c}{\sqrt{2}a}, \tag{2.10.9a}$$

$$\Gamma = \frac{1}{\sqrt{c^2 + \zeta r^2 c \sqrt{2}/a - \zeta^2 r^2 \left(1 - [r/(2a)]^2 \right)}}, \qquad \Delta = \frac{1}{rc\sqrt{1 + [r/(2a)]^2}}. \tag{2.10.9b}$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ:

$$y^0 = y^{(0)}\Gamma + y^{(2)}\Delta\Gamma A, \quad y^1 = y^{(1)}\sqrt{1 + [r/(2a)]^2}, \quad y^2 = y^{(0)}\Gamma\zeta + y^{(2)}\Delta\Gamma B, \quad y^3 = y^{(3)}. \tag{2.10.10}$$

with the above abbreviations.

2.10.2 Scaled cylindrical coordinates

If we apply the simple transformation

$$T = \frac{t}{r_G}, \qquad R = \frac{r}{r_G}, \qquad \phi = \varphi, \qquad Z = \frac{z}{r_G}, \tag{2.10.11}$$

with $r_G = 2a$, we find a formulation for the metric scaling with r_G, which is

$$\boxed{ds^2 = r_G^2 \left(-c^2 dT^2 + \frac{dR^2}{1 + R^2} + R^2(1 - R^2)D\phi^2 + dZ^2 - 2\sqrt{2}cR^2 \, dT \, d\phi \right).} \tag{2.10.12}$$

Christoffel symbols:

$$\Gamma_{TR}^T = \frac{2R}{1 + R^2}, \qquad\qquad \Gamma_{TR}^\phi = -\frac{\sqrt{2}c}{R(1 + R^2)}, \tag{2.10.13a}$$

$$\Gamma_{T\phi}^R = \sqrt{2}cR(1 + R^2), \qquad \Gamma_{RR}^R = -\frac{R}{1 + R^2}, \tag{2.10.13b}$$

$$\Gamma_{R\phi}^T = \frac{\sqrt{2}R^3}{c(1 + R^2)}, \qquad \Gamma_{R\phi}^\phi = \frac{1}{R(1 + R^2)}, \tag{2.10.13c}$$

$$\Gamma_{\phi\phi}^R = R(1 + R^2)(2R^2 - 1). \tag{2.10.13d}$$

Riemann-Tensor:

$$R_{TRTR} = \frac{2r_G^2 c^2}{1 + R^2}, \qquad\qquad R_{TRR\phi} = -\frac{2\sqrt{2}r_G^2 cR^2}{1 + R^2}, \tag{2.10.14a}$$

$$R_{T\phi T\phi} = 2c^2 r_G^2 R^2(1 + R^2), \quad R_{R\phi R\phi} = \frac{2r_G^2 R^2(1 + 3R^2)}{1 + R^2}. \tag{2.10.14b}$$

Ricci-Tensor:

$$R_{TT} = 4c^2, \quad R_{T\phi} = 4\sqrt{2}cR^2, \quad R_{\phi\phi} = 8R^4. \tag{2.10.15}$$

Ricci and Kretschmann scalar

$$\mathcal{R} = -\frac{4}{r_G^2}, \qquad \mathcal{K} = \frac{48}{r_G^4}. \tag{2.10.16}$$

cosmological constant:

$$\Lambda = \frac{R}{2} \tag{2.10.17}$$

Killing vectors:
The Killing vectors read

$$\underset{a}{\xi}^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{b}{\xi}^\mu = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2}c}\cos\varphi \\ \frac{1}{2}(1+R^2)\sin\varphi \\ \frac{1}{2R}(1+2R^2)\cos\varphi \\ 0 \end{pmatrix}, \quad \underset{c}{\xi}^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{2.10.18a}$$

$$\underset{d}{\xi}^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underset{e}{\xi}^\mu = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2}c}\sin\varphi \\ -\frac{1}{2}(1+R^2)\cos\varphi \\ \frac{1}{2R}(1+2R^2)\sin\varphi \\ 0 \end{pmatrix}. \tag{2.10.18b}$$

Local tetrad:
After the transformation to scaled cylindrical coordinates, the local tetrad reads

$$\mathbf{e}_{(0)} = \frac{\Gamma}{r_G}\left(\partial_T + \zeta\partial_\phi\right), \quad \mathbf{e}_{(1)} = \frac{1}{r_G}\sqrt{1+R^2}\,\partial_R, \quad \mathbf{e}_{(2)} = \frac{\Delta\Gamma}{r_G}\left(A\partial_T + B\partial_\phi\right), \quad \mathbf{e}_{(3)} = \frac{1}{r_G}\partial_Z, \tag{2.10.19a}$$

where

$$A = R^2\left[-\sqrt{2}c + (1-R^2)\zeta\right], \qquad\qquad B = c^2 + \sqrt{2}R^2c\zeta, \tag{2.10.20a}$$

$$\Gamma = \frac{1}{\sqrt{c^2 + 2\sqrt{2}R^2c\zeta - R^2(1-R^2)\zeta^2}}, \qquad \Delta = \frac{1}{Rc\sqrt{1+R^2}}. \tag{2.10.20b}$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ:

$$y^0 = \frac{\Gamma}{r_G}y^{(0)} + \frac{\Delta\Gamma A}{r_G}y^{(2)}, \qquad y^1 = \frac{1}{r_G}\sqrt{1+R^2}y^{(1)}, \qquad y^2 = \frac{\Gamma\zeta}{r_G}y^{(0)} + \frac{\Delta\Gamma B}{r_G}y^{(2)}, \qquad y^3 = \frac{1}{r_G}y^{(3)}, \tag{2.10.21}$$

and the back transformation is given by

$$y^{(0)} = \frac{r_G}{\Gamma}\frac{By^0 - Ay^2}{B - \zeta A}, \qquad y^{(1)} = \frac{r_G}{\sqrt{1+R^2}}y^1, \qquad y^{(2)} = \frac{r_G}{\Delta\Gamma}\frac{y^2 - \zeta y^0}{B - \zeta A}, \qquad y^{(3)} = r_G y^3. \tag{2.10.22a}$$

2.11 Halilsoy standing wave

The standing wave metric by Halilsoy[Hal88] reads

$$ds^2 = V \left[e^{2K} \left(d\rho^2 - dt^2 \right) + \rho^2 d\varphi^2 \right] + \frac{1}{V} \left(dz + A \, d\varphi \right)^2, \tag{2.11.1}$$

where

$$V = \cosh^2 \alpha \, e^{-2CJ_0(\rho)\cos(t)} + \sinh^2 \alpha \, e^{2CJ_0(\rho)\cos(t)}, \tag{2.11.2a}$$

$$K = \frac{C^2}{2} \left[\rho^2 \left(J_0(\rho)^2 + J_1(\rho)^2 \right) - 2\rho J_0(\rho) J_1(\rho) \cos^2 t \right], \tag{2.11.2b}$$

$$A = -2C \sinh(2\alpha) \rho J_1(\rho) \sin(t). \tag{2.11.2c}$$

with spherical Bessel functions $J_{1,2}$ and parameters α and C.

Local tetrad:

$$\mathbf{e}_{(0)} = \frac{e^{-K}}{\sqrt{V}} \partial_t, \qquad \mathbf{e}_{(1)} = \frac{e^{-K}}{\sqrt{V}} \partial_\rho, \qquad \mathbf{e}_{(2)} = \frac{1}{\rho\sqrt{V}} \partial_\varphi - \frac{A}{\rho\sqrt{V}} \partial_z, \qquad \mathbf{e}_{(3)} = \sqrt{V} \partial_z. \tag{2.11.3}$$

dual tetrad:

$$\theta^{(0)} = \sqrt{V} e^K \, dt, \qquad \theta^{(2)} = \sqrt{V} e^K \, d\rho, \qquad \theta^{(2)} = \sqrt{V} \rho \, d\varphi, \qquad \theta^{(3)} = \frac{1}{\sqrt{V}} \left(dz + A \, d\varphi \right). \tag{2.11.4}$$

2.12   Janis-Newman-Winicour

The Janis-Newman-Winicour[JNW68] spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ is represented by the line element

$$ds^2 = -\alpha^\gamma c^2 dt^2 + \alpha^{-\gamma} dr^2 + r^2 \alpha^{-\gamma+1} \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right), \tag{2.12.1}$$

where $\alpha = 1 - r_s/(\gamma r)$. The Schwarzschild radius $r_s = 2GM/c^2$ is defined by Newton's constant G, the speed of light c, and the mass parameter M. For $\gamma = 1$, we obtain the Schwarzschild metric (2.2.1).

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{r_s c^2}{2r^2} \alpha^{2\gamma-1}, \qquad \Gamma^t_{tr} = \frac{r_s}{2\gamma r^2 \alpha}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2\gamma r^2 \alpha}, \tag{2.12.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \qquad \Gamma^\varphi_{r\varphi} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \qquad \Gamma^r_{\vartheta\vartheta} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma}, \tag{2.12.2b}$$

$$\Gamma^r_{\varphi\varphi} = \Gamma^r_{\vartheta\vartheta} \sin^2 \vartheta, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot \vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin \vartheta \cos \vartheta. \tag{2.12.2c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{r_s c^2 \left[2\gamma r - r_s(\gamma+1) \right] \alpha^{\gamma-2}}{2\gamma r^4}, \qquad R_{t\vartheta t\vartheta} = \frac{r_s c^2 \left[2\gamma r - r_s(\gamma+1) \right] \alpha^{\gamma-1}}{4\gamma r^2}, \tag{2.12.3a}$$

$$R_{t\varphi t\varphi} = \frac{r_s c^2 \left[2\gamma r - r_s(\gamma+1) \right] \alpha^{\gamma-1} \sin^2 \vartheta}{4\gamma r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s \left[2\gamma^2 r - r_s(\gamma+1) \right]}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \tag{2.12.3b}$$

$$R_{r\varphi r\varphi} = -\frac{r_s \left[2\gamma^2 r - r_s(\gamma+1) \right] \sin^2 \vartheta}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s \left[4\gamma^2 r - r_s(\gamma+1)^2 \right] \sin^2 \vartheta}{4\gamma^2 \alpha^\gamma}. \tag{2.12.3c}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{r_s c^2 \alpha^{\gamma-2} \beta}{6\gamma^2 r^4}, \qquad C_{t\vartheta t\vartheta} = \frac{r_s c^2 \alpha^{\gamma-1} \beta}{12\gamma^2 r^2}, \tag{2.12.4a}$$

$$C_{t\varphi t\varphi} = \frac{r_s c^2 \alpha^{\gamma-1} \beta \sin^2 \vartheta}{12\gamma^2 r^2}, \qquad C_{r\vartheta r\vartheta} = -\frac{r_s \beta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \tag{2.12.4b}$$

$$C_{r\varphi r\varphi} = -\frac{r_s \beta \sin^2 \vartheta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \qquad C_{\vartheta\varphi\vartheta\varphi} = \frac{r_s \beta \sin^2 \vartheta}{6\gamma^2 \alpha^\gamma}, \tag{2.12.4c}$$

where $\beta = 6\gamma^2 r - r_s(\gamma+1)(2\gamma+1)$.

Ricci-Tensor:

$$R_{rr} = \frac{r_s^2 (1 - \gamma^2)}{2\gamma^2 r^4 \alpha^2}. \tag{2.12.5}$$

The *Ricci scalar* reads

$$\mathscr{R} = \frac{r_s^2 (1 - \gamma^2) \alpha^{\gamma-2}}{2\gamma^2 r^4}, \tag{2.12.6}$$

whereas the *Kretschmann scalar* is given by

$$\mathscr{K} = \frac{r_s^2 \alpha^{2\gamma-4}}{4\gamma^4 r^8} \left[7\gamma^2 r_s^2 (2 + \gamma^2) + 48\gamma^4 r^2 \alpha + 8\gamma r_s (2\gamma^2 + 1)(r_s - 2\gamma r) + 3r_s^2 \right]. \tag{2.12.7}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\alpha^{\gamma/2}} \partial_t, \qquad \mathbf{e}_{(r)} = \alpha^{\gamma/2} \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{\alpha^{(\gamma-1)/2}}{r} \partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{\alpha^{(\gamma-1)/2}}{r\sin \vartheta} \partial_\varphi. \tag{2.12.8}$$

Dual tetrad:

$$\theta^{(t)} = c\alpha^{\gamma/2}dt, \qquad \theta^{(r)} = \frac{dr}{\alpha^{\gamma/2}}, \qquad \theta^{(\vartheta)} = \frac{r}{\alpha^{(\gamma-1)/2}}d\vartheta, \qquad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\alpha^{(\gamma-1)/2}}d\varphi. \tag{2.12.9}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \tag{2.12.10a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.12.10b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4\gamma r - r_s(2+\gamma)}{2\gamma r^2}\alpha^{(\gamma-1)/2}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.12.11}$$

Structure coefficients:

$$c^{(t)}_{(t)(r)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \qquad c^{(\vartheta)}_{(r)(\vartheta)} = c^{(\varphi)}_{(r)(\varphi)} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \tag{2.12.12a}$$

$$c^{(\varphi)}_{(\vartheta)(\varphi)} = -\frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.12.12b}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\alpha^{\gamma}\left(\frac{h^2\alpha^{\gamma-1}}{r^2} - \kappa c^2\right) \tag{2.12.13}$$

with the constants of motion $h = r^2\alpha^{-\gamma+1}\dot{\varphi}$ and $k = \alpha^{\gamma}c^2\dot{t}$. For null geodesics ($\kappa = 0$) and $\gamma > \frac{1}{2}$, there is an extremum at

$$r = r_s\frac{1+2\gamma}{2\gamma}. \tag{2.12.14}$$

Embedding:
The embedding function $z = z(r)$ for $r \in [r_s(\gamma+1)^2/(4\gamma^2), \infty)$ follows from

$$\frac{dz}{dr} = \sqrt{\frac{r_s[4r\gamma^2 - r_s(1+\gamma)^2]}{4r^2\gamma^2\alpha^{\gamma+1}}}. \tag{2.12.15}$$

However, the analytic solution

$$z(r) = 2\sqrt{r_s r}F_1\left(-\frac{1}{2};\frac{\gamma+1}{2},-\frac{1}{2};\frac{1}{2};\frac{r_s}{r\gamma},\frac{r_s(1+\gamma)^2}{4r\gamma^2}\right) - \frac{2\pi\gamma}{\gamma+1}{}_2F_1\left(-\frac{1}{2},\frac{\gamma+1}{2};1;\frac{4\gamma}{(\gamma+1)^2}\right), \tag{2.12.16}$$

depends on the Appell-F_1- and the Hypergeometric-${}_2F_1$-function.

2.13   Kasner

The Kasner spacetime in Cartesian coordinates (t,x,y,z) is represented by the line element[MTW73, Kas21] $(c=1)$

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}tz^2,$$
(2.13.1)

where p_1, p_2, p_3 have to fulfill the two conditions

$$p_1 + p_2 + p_3 = 1 \quad \text{and} \quad p_1^2 + p_2^2 + p_3^2 = 1.$$
(2.13.2)

These two conditions can also be represented by the Khalatnikov-Lifshitz parameter u with

$$p_1 = -\frac{u}{1+u+u^2}, \qquad p_2 = \frac{1+u}{1+u+u^2}, \qquad p_3 = \frac{u(1+u)}{1+u+u^2}.$$
(2.13.3)

Christoffel symbols:

$$\Gamma_{tx}^x = \frac{p_1}{t}, \qquad \Gamma_{ty}^y = \frac{p_2}{t}, \qquad \Gamma_{tz}^z = \frac{p_3}{t},$$
(2.13.4a)

$$\Gamma_{xx}^t = \frac{p_1 t^{2p_1}}{t}, \qquad \Gamma_{yy}^t = \frac{p_2 t^{2p_2}}{t}, \qquad \Gamma_{zz}^t = \frac{p_3 t^{2p_3}}{t}.$$
(2.13.4b)

Partial derivatives

$$\Gamma_{tx,t}^x = -\frac{p_1}{t^2}, \qquad \Gamma_{ty,t}^t = -\frac{p_2}{t^2}, \qquad \Gamma_{tz,t}^z = -\frac{p_3}{t^2},$$
(2.13.5a)

$$\Gamma_{xx,t}^t = p_1(2p_1-1)t^{2p_1-2}, \qquad \Gamma_{yy,t}^t = p_2(2p_2-1)t^{2p_2-2}, \qquad \Gamma_{zz,t}^t = p_3(2p_3-1)t^{2p_3-2}.$$
(2.13.5b)

Riemann-Tensor:

$$R_{txtx} = \frac{p_1(1-p_1)t^{2p_1}}{t^2}, \qquad R_{tyty} = \frac{p_2(1-p_2)t^{2p_2}}{t^2}, \qquad R_{tztz} = \frac{p_3(1-p_3)t^{2p_3}}{t^2},$$
(2.13.6a)

$$R_{xyxy} = \frac{p_1 p_2 t^{2p_1}t^{2p_2}}{t^2}, \qquad R_{xzxz} = \frac{p_1 p_3 t^{2p_1}t^{2p_3}}{t^2}, \qquad .R_{yzyz} = \frac{p_2 p_3 t^{2p_2}t^{2p_3}}{t^2}.$$
(2.13.6b)

The Ricci tensor as well as the Ricci scalar vanish identically. The Kretschmann scalar reads

$$\mathcal{K} = \frac{4}{t^4}\left(p_1^2 - 2p_1^3 + p_1^4 + p_2^2 - 2p_2^3 + p_2^4 + p_1^2 p_3^2 + p_3^2 - 2p_3^3 + p_3^4 + p_1^2 p_2^2 + p_2^2 p_3^2\right)$$
(2.13.7a)

$$= \frac{16u^2(1+u)^2}{t^4(1+u+u^2)^3}.$$
(2.13.7b)

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \qquad \mathbf{e}_{(x)} = t^{-p_1}\partial_x, \qquad \mathbf{e}_{(y)} = t^{-p_2}\partial_y, \qquad \mathbf{e}_{(z)} = t^{-p_3}\partial_z.$$
(2.13.8)

Dual tetrad:

$$\theta^{(t)} = dt, \qquad \theta^{(x)} = t^{p_1}dx, \qquad \theta^{(y)} = t^{p_2}dy, \qquad \theta^{(z)} = t^{p_3}dz.$$
(2.13.9)

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \frac{p_1}{t}, \qquad \gamma_{(t)(\vartheta)(\vartheta)} = \frac{p_2}{t}, \qquad \gamma_{(t)(\varphi)(\varphi)} = \frac{p_3}{t}.$$
(2.13.10)

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\frac{1}{t}.$$
(2.13.11)

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(y)(x)} = \frac{p_1(1-p_1)}{t^2}, \qquad R_{(t)(y)(t)(y)} = \frac{p_2(1-p_2)}{t^2}, \qquad R_{(t)(z)(t)(z)} = \frac{p_3(1-p_3)}{t^2},$$
(2.13.12a)

$$R_{(x)(y)(x)(y)} = \frac{p_1 p_2}{t^2}, \qquad R_{(x)(z)(x)(z)} = \frac{p_1 p_3}{t^2}, \qquad R_{(y)(z)(y)(z)} = \frac{p_2 p_3}{t^2}.$$
(2.13.12b)

2.14 Kerr

The Kerr spacetime, found by Roy Kerr in 1963[Ker63], describes a rotating black hole.

2.14.1 Boyer-Lindquist coordinates

The Kerr metric in Boyer-Lindquist coordinates

$$
\begin{aligned}
ds^2 \quad &= -\left(1 - \frac{r_s r}{\Sigma}\right) c^2 dt^2 - \frac{2 r_s a r \sin^2 \vartheta}{\Sigma} c\, dt\, d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\vartheta^2 \\
&\quad + \left(r^2 + a^2 + \frac{r_s a^2 r \sin^2 \vartheta}{\Sigma}\right) \sin^2 \vartheta\, d\varphi^2,
\end{aligned}
\tag{2.14.1}
$$

with $\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta = r^2 - r_s r + a^2$, and $r_s = 2GM/c^2$, is taken from Bardeen[BPT72]. M is the mass and a is the angular momentum per unit mass of the black hole. The contravariant form of the metric reads

$$
\partial_s^2 = -\frac{A}{c^2 \Sigma \Delta} \partial_t^2 - \frac{2 r_s a r}{c \Sigma \Delta} \partial_t \partial_\varphi + \frac{\Delta}{\Sigma} \partial_r^2 + \frac{1}{\Sigma} \partial_\vartheta^2 + \frac{\Delta - a^2 \sin^2 \vartheta}{\Sigma \Delta \sin^2 \vartheta} \partial_\varphi^2,
\tag{2.14.2}
$$

where $A = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2 \vartheta = \left(r^2 + a^2\right) \Sigma + r_s a^2 r \sin^2 \vartheta$.

The event horizon r_+ is defined by the outer root of Δ,

$$
r_+ = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2},
\tag{2.14.3}
$$

whereas the outer boundary r_0 of the ergosphere follows from the outer root of $\Sigma - r_s r$,

$$
r_0 = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2 \cos^2 \vartheta},
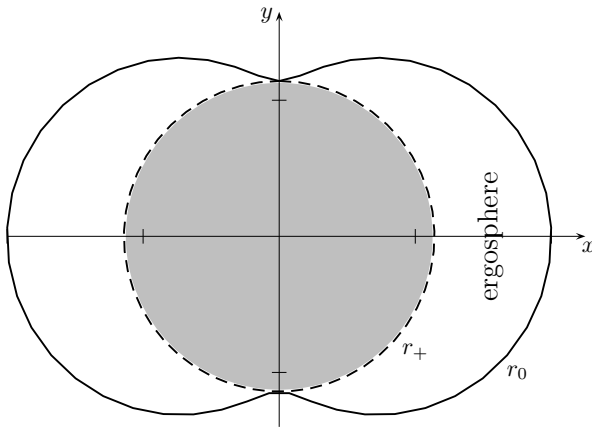\tag{2.14.4}
$$

Figure 2.1: Ergosphere and horizon (dashed circle) for $a = 0.99 \frac{r_s}{2}$.

Christoffel symbols:

$$\Gamma_{tt}^{r} = \frac{c^2 r_s \Delta (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^3}, \qquad \Gamma_{tt}^{\vartheta} = -\frac{c^2 r_s a^2 r \sin \vartheta \cos \vartheta}{\Sigma^3}, \tag{2.14.5a}$$

$$\Gamma_{tr}^{t} = \frac{r_s (r^2 + a^2)(r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^2 \Delta}, \qquad \Gamma_{tr}^{\varphi} = \frac{c r_s a (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^2 \Delta}, \tag{2.14.5b}$$

$$\Gamma_{t\vartheta}^{t} = -\frac{r_s a^2 r \sin \vartheta \cos \vartheta}{\Sigma^2}, \qquad \Gamma_{t\vartheta}^{\varphi} = -\frac{c r_s a r \cot \vartheta}{\Sigma^2}, \tag{2.14.5c}$$

$$\Gamma_{t\varphi}^{r} = -\frac{c \Delta r_s a \sin^2 \vartheta (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^3}, \qquad \Gamma_{t\varphi}^{\vartheta} = \frac{c r_s a r (r^2 + a^2) \sin \vartheta \cos \vartheta}{\Sigma^3}, \tag{2.14.5d}$$

$$\Gamma_{rr}^{r} = \frac{2r a^2 \sin^2 \vartheta - r_s (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma \Delta}, \qquad \Gamma_{rr}^{\vartheta} = \frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma \Delta}, \tag{2.14.5e}$$

$$\Gamma_{r\vartheta}^{r} = -\frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma}, \qquad \Gamma_{r\vartheta}^{\vartheta} = \frac{r}{\Sigma}, \tag{2.14.5f}$$

$$\Gamma_{\vartheta\vartheta}^{r} = -\frac{r\Delta}{\Sigma}, \qquad \Gamma_{\vartheta\vartheta}^{\vartheta} = -\frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma}, \tag{2.14.5g}$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \frac{\cot \vartheta}{\Sigma^2} \left[\Sigma^2 + r_s a^2 r \sin^2 \vartheta \right], \qquad \Gamma_{\vartheta\varphi}^{t} = \frac{r_s a^3 r \sin^3 \vartheta \cos \vartheta}{c\Sigma^2}, \tag{2.14.5h}$$

$$\Gamma_{r\varphi}^{t} = \frac{r_s a \sin^2 \vartheta \left[a^2 \cos^2 \vartheta (a^2 - r^2) - r^2 (a^2 + 3r^2) \right]}{2c\Sigma^2 \Delta}, \tag{2.14.5i}$$

$$\Gamma_{r\varphi}^{\varphi} = \frac{2r\Sigma^2 + r_s \left[a^4 \sin^2 \vartheta \cos^2 \vartheta - r^2 (\Sigma + r^2 + a^2) \right]}{2\Sigma^2 \Delta}, \tag{2.14.5j}$$

$$\Gamma_{\varphi\varphi}^{r} = \frac{\Delta \sin^2 \vartheta}{2\Sigma^3} \left[-2r\Sigma^2 + r_s a^2 \sin^2 \vartheta (r^2 - a^2 \cos^2 \vartheta) \right], \tag{2.14.5k}$$

$$\Gamma_{\varphi\varphi}^{\vartheta} = -\frac{\sin \vartheta \cos \vartheta}{\Sigma^3} \left[A\Sigma + (r^2 + a^2) r_s a^2 r \sin^2 \vartheta \right], \tag{2.14.5l}$$

General local tetrad:

$$\mathbf{e}_{(0)} = \Gamma \left(\partial_t + \zeta \partial_\varphi \right), \qquad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \tag{2.14.6a}$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \qquad \mathbf{e}_{(3)} = \frac{\Gamma}{c} \left(\mp \frac{g_{t\varphi} + \zeta g_{\varphi\varphi}}{\sqrt{\Delta} \sin \vartheta} \partial_t \pm \frac{g_{tt} + \zeta g_{t\varphi}}{\sqrt{\Delta} \sin \vartheta} \partial_\varphi \right), \tag{2.14.6b}$$

where $-\Gamma^{-2} = g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}$,

$$\Gamma^{-2} = \left(1 - \frac{r_s r}{\Sigma} \right) + \frac{2r_s a r \sin^2 \vartheta}{\Sigma} \frac{\zeta}{c} - \left(r^2 + a^2 + \frac{r_s a^2 r \sin^2 \vartheta}{\Sigma} \right) \frac{\zeta^2}{c^2} \sin^2 \vartheta \tag{2.14.7}$$

Non-rotating local tetrad ($\zeta = \omega$)**:**

$$\mathbf{e}_{(0)} = \sqrt{\frac{A}{\Sigma \Delta}} \left(\frac{1}{c} \partial_t + \omega \partial_\varphi \right), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \sqrt{\frac{\Sigma}{A}} \frac{1}{\sin \vartheta} \partial_\varphi, \tag{2.14.8}$$

where $\omega = -g_{t\varphi}/g_{\varphi\varphi} = r_s a r / A$.

Dual tetrad:

$$\theta^{(2)} = \sqrt{\frac{\Sigma \Delta}{A}} c\, dt, \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}} dr, \quad \theta^{(2)} = \sqrt{\Sigma} d\vartheta, \quad \theta^{(3)} = \sqrt{\frac{A}{\Sigma}} \sin \vartheta \left(d\varphi - \omega\, d\varphi \right). \tag{2.14.9}$$

The relation between the constants of motion E, L, Q, and μ (defined in Bardeen[BPT72]) and the initial direction υ, compare Sec. (1.4.5), with respect to the LNRF reads $(c = 1)$

$$\upsilon^{(0)} = \sqrt{\frac{A}{\Sigma\Delta}}E - \frac{r_s ra}{\sqrt{A\Sigma\Delta}}L, \qquad\qquad \upsilon^{(1)} = \sqrt{\frac{\Delta}{\Sigma}}p_r, \tag{2.14.10a}$$

$$\upsilon^{(2)} = \frac{1}{\sqrt{\Sigma}}\sqrt{Q - \cos^2\vartheta\left[a^2(\mu^2 - E^2) + \frac{L^2}{\sin^2\vartheta}\right]}, \qquad \upsilon^{(3)} = \sqrt{\frac{\Sigma}{A}}\frac{L}{\sin\vartheta}. \tag{2.14.10b}$$

Static local tetrad $(\zeta = 0)$**:**

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1 - r_s r/\Sigma}}\partial_t, \qquad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \qquad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \tag{2.14.11a}$$

$$\mathbf{e}_{(3)} = \pm\frac{r_s ar\sin\vartheta}{c\sqrt{1 - r_s r/\Sigma}\sqrt{\Delta\Sigma}}\partial_t \mp \frac{\sqrt{1 - r_s r/\Sigma}}{\sqrt{\Delta}\sin\vartheta}\partial_\varphi. \tag{2.14.11b}$$

Photon orbits:
The direct(-) and retrograd(+) photon orbits have radius

$$r_{\text{po}} = r_s\left[1 + \cos\left(\frac{2}{3}\arccos\frac{\mp 2a}{r_s}\right)\right]. \tag{2.14.12}$$

Marginally stable timelike circular orbits
are defined via

$$r_{\text{ms}} = \frac{r_s}{2}\left(3 + Z_2 \mp \sqrt{(3 - Z_1)(2 + Z_1 + 2Z_2)}\right), \tag{2.14.13}$$

where

$$Z_1 = 1 + \left(1 - \frac{4a^2}{r_s^2}\right)^{1/3}\left[\left(1 + \frac{2a}{r_s}\right)^{1/3} + \left(1 - \frac{2a}{r_s}\right)^{1/3}\right], \tag{2.14.14a}$$

$$Z_2 = \sqrt{\frac{12a^2}{r_s^2} + Z_1^2}. \tag{2.14.14b}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = 0 \tag{2.14.15}$$

with the effective potential

$$V_{\text{eff}} = \frac{1}{2r^3}\left\{h^2(r - r_s) + 2\frac{ahk}{c}r_s - \frac{k^2}{c^2}\left[r^3 + a^2(r + r_s)\right]\right\} - \frac{\kappa c^2\Delta}{r^2} \tag{2.14.16}$$

and the constants of motion

$$k = \left(1 - \frac{r_s}{r}\right)c^2\dot{t} + \frac{cr_s a}{r}\dot{\varphi}, \qquad h = \left(r^2 + a^2 + \frac{r_s a^2}{r}\right)\dot{\varphi} - \frac{cr_s a}{r}\dot{t}. \tag{2.14.17}$$

Further reading:
Boyer and Lindquist[BL67], Wilkins[Wil72], Brill[BC66].

2.15 Kottler spacetime

The Kottler spacetime is represented in spherical coordinates $(t, r, \vartheta, \varphi)$ by the line element[Per04]

$$ds^2 = -\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right)c^2 dt^2 + \frac{1}{1 - r_s/r - \Lambda r^2/3}dr^2 + r^2 d\Omega^2, \qquad (2.15.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and Λ is the cosmological constant. If $\Lambda > 0$ the metric is also known as Schwarzschild-deSitter metric, whereas if $\Lambda < 0$ it is called Schwarzschild-anti-deSitter.
For the following, we define the two abbreviations

$$\alpha = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3} \qquad \text{and} \qquad \beta = \frac{r_s}{r} - \frac{2\Lambda}{3}r^2. \qquad (2.15.2)$$

The critical points of the Kottler metric follow from the roots of the cubic equation $\alpha = 0$. These can be found by means of the parameters $p = -1/\Lambda$ and $q = 3r_s/(2\Lambda)$. If $\Lambda < 0$, we have only one real root

$$r_1 = \frac{2}{\sqrt{-\Lambda}}\sinh\left[\frac{1}{3}\text{arsinh}\left(\frac{3r_s}{2}\sqrt{-\Lambda}\right)\right]. \qquad (2.15.3)$$

If $\Lambda > 0$, we have to distinguish whether $D \equiv q^2 + p^3 = 9r_s^2/(4\Lambda^2) - \Lambda^{-3}$ is positive or negative. If $D > 0$, there is no real positive root. For $D < 0$, the two real positive roots read

$$r_\pm = \frac{2}{\sqrt{\Lambda}}\cos\left[\frac{\pi}{3} \pm \frac{1}{3}\arccos\left(\frac{3r_s}{2}\sqrt{\Lambda}\right)\right] \qquad (2.15.4)$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 \alpha \beta}{2r}, \qquad \Gamma^t_{tr} = \frac{\beta}{2r\alpha}, \qquad \Gamma^r_{rr} = -\frac{\beta}{2r\alpha}, \qquad (2.15.5a)$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -\alpha r, \qquad (2.15.5b)$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -\alpha r \sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \qquad (2.15.5c)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2\left(3r_s + \Lambda r^3\right)}{3r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}c^2\alpha\beta, \qquad (2.15.6a)$$

$$R_{t\varphi t\varphi} = \frac{1}{2}c^2\alpha\beta\sin^2\vartheta, \qquad R_{r\vartheta r\vartheta} = -\frac{\beta}{2\alpha}, \qquad (2.15.6b)$$

$$R_{r\varphi r\varphi} = -\frac{\beta}{2\alpha}\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = r\left(r_s + \frac{\Lambda r^3}{3}\right)\sin^2\vartheta. \qquad (2.15.6c)$$

Ricci-Tensor:

$$R_{tt} = -c^2\alpha\Lambda, \qquad R_{rr} = \frac{\Lambda}{\alpha}, \qquad R_{\vartheta\vartheta} = \Lambda r^2, \qquad R_{\varphi\varphi} = \Lambda r^2 \sin^2\vartheta. \qquad (2.15.7)$$

The *Ricci scalar* and the *Kretschmann scalar* read

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = 12\frac{r_s^2}{r^6} + \frac{8\Lambda^2}{3}. \qquad (2.15.8)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad C_{t\vartheta t\vartheta} = \frac{c^2\alpha r_s}{2r}, \qquad C_{t\varphi t\varphi} = \frac{c^2\alpha r_s \sin^2\vartheta}{2r}, \qquad (2.15.9a)$$

$$C_{r\vartheta r\vartheta} = -\frac{r_s}{2r\alpha}, \qquad C_{r\varphi r\varphi} = -\frac{r_s\sin^2\vartheta}{2r\alpha}, \qquad C_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \qquad (2.15.9b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{\alpha}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\alpha}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.15.10}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{\alpha}\,dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{\alpha}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.15.11}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s - \frac{2}{3}\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{\alpha}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.15.12}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s - 2\Lambda r^3}{2r^2\sqrt{\alpha}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.15.13}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{\Lambda r^3 + 3r_s}{3r^3}, \tag{2.15.14a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{3r_s - 2\Lambda r^3}{6r^3}. \tag{2.15.14b}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.15.15a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.15.15b}$$

Embedding:
The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{r_s/r + \Lambda r^2/3}{1 - r_s/r - \Lambda r^2/3}}. \tag{2.15.16}$$

Euler-Lagrange:
The Euler-Lagrangian formalism[Rin01] yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \tag{2.15.17}$$

with the constants of motion $k = (1 - r_s/r - \Lambda r^2/3)c^2\dot{t}$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.8.2).
As in the Schwarzschild metric, the effective potential has only one extremum for null geodesics, the so called photon orbit at $r = \frac{3}{2}r_s$. For timelike geodesics, however, we have

$$\frac{dV_{\text{eff}}}{dr} = \frac{h^2(-6r + 9r_s) + c^2r^2(3r_s - 2r^3\Lambda)}{3r^4} \overset{!}{=} 0. \tag{2.15.18}$$

This polynomial of fifth order might have up to five extrema.

Further reading:
Kottler[Kot18], Weyl[Wey19], Hackmann[HL08], Cruz[COV05].

2.16 Morris-Thorne

The most simple wormhole geometry is represented by the metric of Morris and Thorne[MT88],

$$ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.16.1}$$

where b_0 is the throat radius and l is the proper radial coordinate; and $\{t \in \mathbb{R}, l \in \mathbb{R}, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$.

Christoffel symbols:

$$\Gamma^\vartheta_{l\vartheta} = \frac{l}{b_0^2 + l^2}, \qquad \Gamma^\varphi_{l\varphi} = \frac{l}{b_0^2 + l^2}, \qquad \Gamma^l_{\vartheta\vartheta} = -l, \tag{2.16.2a}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^l_{\varphi\varphi} = -l\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.16.2b}$$

Partial derivatives

$$\Gamma^\vartheta_{l\vartheta,l} = -\frac{l^2 - b_0^2}{(b_0^2 + l^2)^2}, \qquad \Gamma^\varphi_{l\varphi,l} = -\frac{l^2 - b_0^2}{(b_0^2 + l^2)^2}, \qquad \Gamma^l_{\vartheta\vartheta,l} = -1, \tag{2.16.3a}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^l_{\varphi\varphi,l} = -\sin^2\vartheta, \qquad \Gamma^l_{\varphi\varphi,\vartheta} = -l\sin(2\vartheta), \tag{2.16.3b}$$

$$\Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta). \tag{2.16.3c}$$

Riemann-Tensor:

$$R_{l\vartheta l\vartheta} = -\frac{b_0^2}{b_0^2 + l^2}, \quad R_{l\varphi l\varphi} = -\frac{b_0^2 \sin^2\vartheta}{b_0^2 + l^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = b_0^2 \sin^2\vartheta. \tag{2.16.4}$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{ll} = -2\frac{b_0^2}{\left(b_0^2 + l^2\right)^2}, \qquad \mathscr{R} = -2\frac{b_0^2}{\left(b_0^2 + l^2\right)^2}, \qquad \mathscr{K} = \frac{12 b_0^4}{\left(b_0^2 + l^2\right)^4}. \tag{2.16.5}$$

Weyl-Tensor:

$$C_{tltl} = -\frac{2}{3}\frac{c^2 b_0^2}{\left(b_0^2 + l^2\right)^2}, \qquad C_{t\vartheta t\vartheta} = \frac{1}{3}\frac{c^2 b_0^2}{b_0^2 + l^2}, \qquad C_{t\varphi t\varphi} = \frac{1}{3}\frac{c^2 b_0^2 \sin^2\vartheta}{b_0^2 + l^2}, \tag{2.16.6a}$$

$$C_{l\vartheta l\vartheta} = -\frac{1}{3}\frac{b_0^2}{b_0^2 + l^2}, \qquad C_{l\varphi l\varphi} = -\frac{1}{3}\frac{b_0^2 \sin^2\vartheta}{b_0^2 + l^2}, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3}b_0^2 \sin^2\vartheta. \tag{2.16.6b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(l)} = \partial_l, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{\sqrt{b_0^2 + l^2}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\sqrt{b_0^2 + l^2}\sin\vartheta}\partial_\varphi. \tag{2.16.7}$$

Dual tetrad

$$\theta^{(t)} = c\,dt, \qquad \theta^{(l)} = dl, \qquad \theta^{(\vartheta)} = \sqrt{b_0^2 + l^2}\,d\vartheta, \qquad \theta^{(\varphi)} = \sqrt{b_0^2 + l^2}\sin\vartheta\,d\varphi. \tag{2.16.8}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{l}{b_0^2 + l^2}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sqrt{b_0^2 + l^2}}. \tag{2.16.9}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2l}{b_0^2 + l^2}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sqrt{b_0^2 + l^2}}. \tag{2.16.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(l)(\vartheta)(l)(\vartheta)} = R_{(l)(\varphi)(l)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{b_0^2}{\left(b_0^2 + l^2\right)^2}. \tag{2.16.11}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(l)(l)} = -\frac{2b_0^2}{\left(b_0^2 + l^2\right)^2}. \tag{2.16.12}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(l)(t)(l)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2b_0^2}{3\left(b_0^2 + l^2\right)^2}, \tag{2.16.13a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(l)(\vartheta)(l)(\vartheta)} = -C_{(l)(\varphi)(l)(\varphi)} = \frac{b_0^2}{3\left(b_0^2 + l^2\right)^2}. \tag{2.16.13b}$$

Embedding:
The embedding function reads

$$z(r) = \pm b_0 \ln\left[\frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0}\right)^2 - 1}\right] \tag{2.16.14}$$

with $r^2 = b_0^2 + l^2$.

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{l}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(\frac{h^2}{b_0^2 + l^2} - \kappa c^2\right), \tag{2.16.15}$$

with the constants of motion $k = c^2\dot{t}$ and $h = (b_0^2 + l^2)\dot{\varphi}$. The shape of the effective potential V_{eff} is independend of the geodesic type. The maximum of the effective potential is located at $l = 0$.

A geodesic that starts at $l = l_i$ with direction $\mathbf{y} = \pm\mathbf{e}_{(t)} + \cos\xi\,\mathbf{e}_{(l)} + \sin\xi\,\mathbf{e}_{(\varphi)}$ approaches the wormhole throat asymptotically for $\xi = \xi_{\text{crit}}$ with

$$\xi_{\text{crit}} = \arcsin\frac{b_0}{\sqrt{b_0^2 + l_i^2}}. \tag{2.16.16}$$

This critical angle is independent of the type of the geodesic.

Further reading:
Ellis[Ell73], Visser[Vis95], Müller[Mül04, Mül08a]

2.17 Oppenheimer-Snyder collapse

2.17.1 Outer metric

The metric of the outer spacetime, $R > R_b$, in comoving coordinates $(\tau, R, \vartheta, \varphi)$ with $(c = 1)$ is given by

$$ds^2 = -d\tau^2 + \frac{R}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}dR^2 + \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right). \tag{2.17.1}$$

Christoffel symbols:

$$\Gamma^R_{\tau R} = \frac{1}{2}\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\vartheta_{\tau\vartheta} = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \tag{2.17.2a}$$

$$\Gamma^\varphi_{\tau\varphi} = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\tau_{RR} = \frac{R\sqrt{r_s}}{2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{5/3}}, \tag{2.17.2b}$$

$$\Gamma^R_{RR} = -\frac{3\sqrt{r_s}\tau}{4\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)R}, \qquad \Gamma^\vartheta_{R\vartheta} = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \tag{2.17.2c}$$

$$\Gamma^\varphi_{R\varphi} = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\tau_{\vartheta\vartheta} = -\sqrt{r_s}\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}, \tag{2.17.2d}$$

$$\Gamma^R_{\vartheta\vartheta} = -\frac{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}{\sqrt{R}}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.17.2e}$$

$$\Gamma^\tau_{\varphi\varphi} = -\sqrt{r_s}\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{2.17.2f}$$

$$\Gamma^R_{\varphi\varphi} = -\frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)\sin^2\vartheta}{\sqrt{R}}. \tag{2.17.2g}$$

Riemann-Tensor:

$$R_{\tau R\tau R} = -\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{8/3}}, \qquad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}, \tag{2.17.3a}$$

$$R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_s\sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}, \qquad R_{R\vartheta R\vartheta} = -\frac{1}{2}\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3}}, \tag{2.17.3b}$$

$$R_{R\varphi R\varphi} = -\frac{1}{2}\frac{Rr_s\sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}r_s\sin^2\vartheta. \tag{2.17.3c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathscr{K} = 12\frac{r_s^2}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^4}. \tag{2.17.4}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \qquad \mathbf{e}_{(R)} = \frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}}{\sqrt{R}}\partial_R, \tag{2.17.5a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}\sin\vartheta}\partial_\varphi. \tag{2.17.5b}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = -\frac{\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{2\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \tag{2.17.6a}$$

$$\gamma_{(R)(\varphi)(\varphi)} = \gamma_{(R)(\vartheta)(\vartheta)} = -\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \tag{2.17.6b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad \gamma_{(R)} = 2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}, \quad \gamma_{(\vartheta)} = \cot\vartheta\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \tag{2.17.7}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{4r_s}{\left(2R^{3/2} - 3\sqrt{r_s}\tau\right)^2}, \tag{2.17.8a}$$

$$R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = -R_{(R)(\vartheta)(R)(\vartheta)} = -R_{(R)(\varphi)(R)(\varphi)} = \frac{2r_s}{\left(2R^{3/2} - 3\sqrt{r_s}\tau\right)^2}. \tag{2.17.8b}$$

The Ricci tensor with respect to the local tetrad vanishes identically.

2.17.2 Inner metric

The metric of the inside, $R \le R_b$, reads

$$ds^2 = -d\tau^2 + \left(1 - \frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{4/3}\left[dR^2 + R^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]. \tag{2.17.9}$$

For the following components, we define

$$A_{\text{Oin}} := 1 - \frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau. \tag{2.17.10}$$

Christoffel symbols:

$$\Gamma^R_{\tau R} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \qquad \Gamma^\vartheta_{\tau\vartheta} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma^\varphi_{\tau\varphi} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \tag{2.17.11a}$$

$$\Gamma^\tau_{RR} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}, \quad \Gamma^\vartheta_{R\vartheta} = \frac{1}{R}, \qquad\qquad \Gamma^\varphi_{R\varphi} = \frac{1}{R}, \tag{2.17.11b}$$

$$\Gamma^R_{\vartheta\vartheta} = -R, \qquad\qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\tau_{\vartheta\vartheta} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2, \tag{2.17.11c}$$

$$\Gamma^R_{\varphi\varphi} = -R\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \quad \Gamma^\tau_{\varphi\varphi} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2\sin^2\vartheta. \tag{2.17.11d}$$

Riemann-Tensor:

$$R_{\tau R\tau R} = \frac{1}{2}\frac{r_s}{R_b^3 A_{\text{Oin}}^{2/3}}, \qquad\qquad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_s R^2}{R_b^3 A_{\text{Oin}}^{2/3}}, \qquad R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_s R^2\sin^2\vartheta}{R_b^3 A_{\text{Oin}}^{2/3}}, \tag{2.17.12a}$$

$$R_{R\varphi R\varphi} = \frac{r_s R^2\sin^2\vartheta}{R_b^3}A_{\text{Oin}}^{2/3}, \qquad R_{R\vartheta R\vartheta} = \frac{r_s R^2}{R_b^3}A_{\text{Oin}}^{2/3}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s R^4\sin^2\vartheta}{R_b^3}A_{\text{Oin}}^{2/3}. \tag{2.17.12b}$$

Ricci-Tensor:

$$R_{\tau\tau} = \frac{3}{2}\frac{r_s}{R_b^3 A_{\text{Oin}}^2}, \qquad R_{RR} = \frac{3}{2}\frac{r_s}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\vartheta\vartheta} = \frac{3}{2}\frac{r_s R^2}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\varphi\varphi} = \frac{3}{2}\frac{r_s R^2\sin^2\vartheta}{R_b^3 A_{\text{Oin}}^{2/3}}. \tag{2.17.13}$$

The *Ricci* and *Kretschmann* scalars read:

$$\mathscr{R} = \frac{3r_s}{R_b^3 A_{\text{Oin}}^2}, \qquad \mathscr{K} = 15\frac{r_s^2}{R_b^6 A_{\text{Oin}}^4}. \tag{2.17.14}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \qquad \mathbf{e}_{(R)} = \frac{1}{A_{\text{Oin}}^{2/3}}\partial_R, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{RA_{\text{Oin}}^{2/3}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{A_{\text{Oin}}^{2/3}R\sin\vartheta}\partial_\varphi. \tag{2.17.15}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \tag{2.17.16a}$$

$$\gamma_{(R)(\vartheta)(\vartheta)} = \gamma_{(R)(\varphi)(\varphi)} = -\frac{1}{RA_{\text{Oin}}^{2/3}}, \tag{2.17.16b}$$

$$\gamma_{(\vartheta)(\varphi)(\varphi)} = -\frac{\cot\vartheta}{RA_{\text{Oin}}^{2/3}}. \tag{2.17.16c}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \qquad \gamma_{(R)} = \frac{2}{RA_{\text{Oin}}^{2/3}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{RA_{\text{Oin}}^{2/3}}. \tag{2.17.17}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = \frac{r_s R_b^{-3}}{2A_{\text{Oin}}^2}, \tag{2.17.18a}$$

$$R_{(R)(\vartheta)(R)(\vartheta)} = R_{(R)(\varphi)(R)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s R_b^{-3}}{A_{\text{Oin}}^2}. \tag{2.17.18b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\tau)(\tau)} = R_{(R)(R)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{3r_s R_b^{-3}}{2A_{\text{Oin}}^2}. \tag{2.17.19}$$

Further reading:
Oppenheimer and Snyder[OS39].

2.18 Petrov-Type D – Levi-Civita spacetimes

The Petrov type D static vacuum spacetimes AI-C are taken from Stephani et al.[SKM+03], Sec. 18.6, with the coordinate and parameter ranges given in "Exact solutions of the gravitational field equations" by Ehlers and Kundt [EK62].

2.18.1   Case AI

In spherical coordinates, $(t, r, \vartheta, \varphi)$, the metric is given by the line element

$$ds^2 = r^2 \left(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) + \frac{r}{r-b} dr^2 - \frac{r-b}{r} dt^2. \tag{2.18.1}$$

This is the well known Schwarzschild solution if $b = r_s$, cf. Eq. (2.2.1). Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < \vartheta < \pi, \qquad \varphi \in [0, 2\pi), \qquad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{r}{r-b}} \partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}} \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \tag{2.18.2}$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{r-b}{r}} dt, \qquad \theta^{(r)} = \sqrt{\frac{r}{r-b}} dr, \qquad \theta^{(\vartheta)} = r d\vartheta, \qquad \theta^{(\varphi)} = r \sin \vartheta \, d\varphi. \tag{2.18.3}$$

Effective potential:
With the Hamilton-Jacobi formalism it is possible to obtain an effective potential fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K \frac{r-b}{r^3} - \kappa \frac{r-b}{r} \tag{2.18.4}$$

and the constants of motion

$$C_0^2 = \dot{t}^2 \left(\frac{r-b}{r} \right)^2, \tag{2.18.5a}$$

$$K = \dot{\vartheta}^2 r^4 + \dot{\varphi}^2 r^4 \sin^2 \vartheta. \tag{2.18.5b}$$

2.18.2   Case AII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 \left(dr^2 + \sinh^2 r \, d\varphi^2 \right) + \frac{z}{b-z} dz^2 - \frac{b-z}{z} dt^2. \tag{2.18.6}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < r, \qquad \varphi \in [0, 2\pi), \qquad 0 < z < b.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{z}{b-z}} \partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{z \sinh r} \partial_\varphi, \qquad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}} \partial_z. \tag{2.18.7}$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{b-z}{z}} dt, \qquad \theta^{(r)} = z dr, \qquad \theta^{(\varphi)} = z \sinh r \, d\varphi, \qquad \theta^{(z)} = \sqrt{\frac{z}{b-z}} dz. \tag{2.18.8}$$

2.18.3   Case AIII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 \left(dr^2 + r^2 d\varphi^2 \right) + z dz^2 - \frac{1}{z} dt^2. \tag{2.18.9}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < r, \qquad \varphi \in [0, 2\pi), \qquad 0 < z.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{z}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{zr}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}}\partial_z. \tag{2.18.10}$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{\sqrt{z}} dt, \qquad \theta^{(r)} = z\, dr, \qquad \theta^{(\varphi)} = z r\, d\varphi, \qquad \theta^{(z)} = \sqrt{z}\, dz. \tag{2.18.11}$$

2.18.4   Case BI

In spherical coordinates, the metric is given by the line element

$$ds^2 = r^2 \left(d\vartheta^2 - \sin^2 \vartheta\, dt^2 \right) + \frac{r}{r-b} dr^2 + \frac{r-b}{r} d\varphi^2. \tag{2.18.12}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < \vartheta < \pi, \qquad \varphi \in [0, 2\pi), \qquad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{r\sin\vartheta}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \sqrt{\frac{r}{r-b}}\partial_\varphi. \tag{2.18.13}$$

Dual tetrad:

$$\theta^{(t)} = r\sin\vartheta\, dt, \qquad \theta^{(r)} = \sqrt{\frac{r}{r-b}}\, dr, \qquad \theta^{(\vartheta)} = r\, d\vartheta, \qquad \theta^{(\varphi)} = \sqrt{\frac{r-b}{r}}\, d\varphi. \tag{2.18.14}$$

Effective potential:
With the Hamilton-Jacobi formalism, an effective potential for the radial coordinate can be calculated
fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K\frac{r-b}{r^3} - \kappa\frac{r-b}{r} \tag{2.18.15}$$

and the constants of motion

$$C_0^2 \;=\; \dot{\varphi}^2 \left(\frac{r-b}{r} \right)^2, \tag{2.18.16a}$$

$$K \;=\; \dot{\vartheta}^2 r^4 - \dot{t}^2 r^4 \sin^2 \vartheta. \tag{2.18.16b}$$

Note that the metric is not spherically symmetric. Particles or light rays fall into one of the poles if they
are not moving in the $\vartheta = \frac{\pi}{2}$ plane.

2.18.5 Case BII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 \left(dr^2 - \sinh^2 r \, dt^2\right) + \frac{z}{b-z}dz^2 + \frac{b-z}{z}d\varphi^2. \tag{2.18.17}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad \varphi \in [0, 2\pi), \qquad 0 < z < b, \qquad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{z \sinh r}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \sqrt{\frac{z}{b-z}}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}}\partial_z. \tag{2.18.18}$$

Dual tetrad:

$$\theta^{(t)} = z \sinh r \, dt, \qquad \theta^{(r)} = z \, dr, \qquad \theta^{(\varphi)} = \sqrt{\frac{b-z}{z}}d\varphi, \qquad \theta^{(z)} = \sqrt{\frac{z}{b-z}}dz. \tag{2.18.19}$$

2.18.6 Case BIII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 \left(dr^2 - r^2 dt^2\right) + z dz^2 + \frac{1}{z}d\varphi^2. \tag{2.18.20}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad \varphi \in [0, 2\pi), \qquad 0 < z, \qquad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{zr}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \sqrt{z}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}}\partial_z. \tag{2.18.21}$$

Dual tetrad:

$$\theta^{(t)} = zr \, dt, \qquad \theta^{(r)} = z \, dr, \qquad \theta^{(\varphi)} = \frac{1}{\sqrt{z}}d\varphi, \qquad \theta^{(z)} = \sqrt{z}dz. \tag{2.18.22}$$

2.18.7 Case C

The metric is given by the line element

$$ds^2 = \frac{1}{(x+y)^2}\left(\frac{1}{f(x)}dx^2 + f(x)d\varphi^2 - \frac{1}{f(-y)}dy^2 + f(-y)dt^2\right) \tag{2.18.23}$$

with $f(u) := \pm(u^3 + au + b)$. Coordinates and parameters are restricted to

$$0 < x+y, \qquad f(-y) > 0, \qquad 0 > f(x).$$

Local tetrad:

$$\mathbf{e}_{(t)} = (x+y)\frac{1}{\sqrt{-y^3 - ay + b}}\partial_t, \qquad \mathbf{e}_{(x)} = (x+y)\sqrt{x^3 + ax + b}\partial_x, \tag{2.18.24a}$$

$$\mathbf{e}_{(y)} = (x+y)\sqrt{-y^3 - ay + b}\partial_y, \qquad \mathbf{e}_{(\varphi)} = (x+y)\frac{1}{\sqrt{x^3 + ax + b}}\partial_\varphi, \tag{2.18.24b}$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{x+y}\sqrt{-y^3 - ay + b}\, dt, \qquad \theta^{(x)} = \frac{1}{x+y}\frac{1}{\sqrt{x^3 + ax + b}}\, dx, \qquad (2.18.25a)$$

$$\theta^{(y)} = \frac{1}{x+y}\frac{1}{\sqrt{-y^3 - ay + b}}\, dy, \qquad \theta^{(\varphi)} = \frac{1}{x+y}\sqrt{x^3 + ax + b}\, d\varphi, \qquad (2.18.25b)$$

A coordinate change can eliminate the linear term in the polynom f generating a quadratic term instead. This brings the line element to the form

$$ds^2 = \frac{1}{A(x+y)^2}\left[\frac{1}{f(x)}dx^2 + f(x)dp^2 - \frac{1}{f(-y)}dy^2 + f(-y)dq^2\right] \qquad (2.18.26)$$

with $f(u) := \pm(-2mAu^3 - u^2 + 1)$ given in [PP01].
Furthermore, coordinates can be adapted to the boost-rotation symmetry with the line element in [PP01] from in [Bon83]

$$ds^2 = \frac{1}{z^2 - t^2}\left[e^\rho r^2(z\,dt - t\,dz)^2 - e^\lambda(z\,dz - t\,dt)^2\right] - e^\lambda\, dr^2 - r^2 e^{-\rho}\, d\varphi^2 \qquad (2.18.27)$$

with

$$e^\rho = \frac{R_3 + R + Z_3 - r^2}{4\alpha^2\,(R_1 + R + Z_1 - r^2)},$$

$$e^\lambda = \frac{2\alpha^2\left[R(R + R_1 + Z_1) - Z_1 r^2\right]\left[R_1 R_3 + (R + Z_1)(R + Z_3) - (Z_1 + Z_3)r^2\right]}{R_i R_3\left[R(R + R_3 + Z_3) - Z_3 r^2\right]},$$

$$R = \frac{1}{2}\left(z^2 - t^2 + r^2\right),$$

$$R_i = \sqrt{(R + Z_i)^2 - 2Z_i r^2},$$

$$Z_i = z_i - z_2,$$

$$\alpha^2 = \frac{1}{4}\frac{m^2}{A^6(z_2 - z_1)^2(z3 - z1)^2},$$

$$q = \frac{1}{4\alpha^2},$$

and $z_3 < z_1 < z_2$ the roots of $2A^4 z^3 - A^2 z^2 + m^2$.

Local tetrad:

Case $z^2 - t^2 > 0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{z^2 - t^2}}\left(qze^{-\rho/2}\partial_t + te^{-\lambda/2}\partial_z,\right), \qquad \mathbf{e}_{(r)} = e^{-\lambda/2}\partial_r, \qquad (2.18.28a)$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{z^2 - t^2}}\left(qte^{-\rho/2}\partial_t + ze^{-\lambda/2}\partial_z,\right), \qquad \mathbf{e}_{(\varphi)} = re^{\rho/2}\partial_\varphi. \qquad (2.18.28b)$$

Case $z^2 - t^2 < 0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{t^2 - z^2}}\left(qte^{-\rho/2}\partial_t + ze^{-\lambda/2}\partial_z,\right), \qquad \mathbf{e}_{(r)} = e^{-\lambda/2}\partial_r, \qquad (2.18.29a)$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{t^2 - z^2}}\left(qze^{-\rho/2}\partial_t + te^{-\lambda/2}\partial_z,\right), \qquad \mathbf{e}_{(\varphi)} = re^{\rho/2}\partial_\varphi. \qquad (2.18.29b)$$

Dual tetrad:

Case $z^2 - t^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^{\rho}}{z^2 - t^2}} \frac{1}{q} (z\,dt + t\,dz), \qquad \theta^{(r)} = e^{\lambda}\,dr, \tag{2.18.30a}$$

$$\theta^{(z)} = \sqrt{\frac{e^{\lambda}}{z^2 - t^2}} (t\,dt + z\,dz), \qquad \theta^{(\varphi)} = \frac{1}{re^{\rho}}\,d\varphi. \tag{2.18.30b}$$

Case $z^2 - t^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^{\lambda}}{t^2 - z^2}} (t\,dt + z\,dz), \qquad \theta^{(r)} = e^{\lambda}\,dr, \tag{2.18.31a}$$

$$\theta^{(z)} = \sqrt{\frac{e^{\rho}}{t^2 - z^2}} \frac{1}{q} (z\,dt + t\,dz), \qquad \theta^{(\varphi)} = \frac{1}{re^{\rho}}\,d\varphi. \tag{2.18.31b}$$

2.19   Plane gravitational wave

W. Rindler described in [Rin01] an exact plane gravitational wave which is bounded between two planes. The metric of the so called 'sandwich wave' with $u := t - x$ reads

$$ds^2 = -dt^2 + dx^2 + p^2(u)\,dy^2 + q^2(u)\,dz^2. \qquad (2.19.1)$$

The functions $p(u)$ and $q(u)$ are given by

$$p(u) := \begin{cases} p_0 = \text{const.} & u < -a \\ 1 - u & 0 < u \\ L(u)\,e^{m(u)} & \text{else} \end{cases} \quad \text{and} \quad q(u) := \begin{cases} q_0 = \text{const.} & u < -a \\ 1 - u & 0 < u \\ L(u)\,e^{-m(u)} & \text{else} \end{cases}, \qquad (2.19.2)$$

where a is the longitudinal extension of the wave. The functions $L(u)$ and $m(u)$ are

$$L(u) = 1 - u + \frac{u^3}{a^2} + \frac{u^4}{2a^3}, \qquad m(u) = \pm 2\sqrt{3} \int \sqrt{\frac{u^2 + au}{2a^3 u - 2au^3 - u^4 - 2a^3}}\, du. \qquad (2.19.3)$$

Christoffel symbols:

$$\Gamma^y_{ty} = -\Gamma^y_{xy} = \frac{1}{p}\frac{\partial p}{\partial u}, \qquad \Gamma^t_{zz} = \Gamma^x_{zz} = q\frac{\partial q}{\partial u}, \qquad \Gamma^z_{tz} = -\Gamma^z_{xz} = \frac{1}{q}\frac{\partial q}{\partial u}, \qquad \Gamma^t_{yy} = \Gamma^x_{yy} = p\frac{\partial p}{\partial u}. \qquad (2.19.4)$$

Riemann-Tensor:

$$R_{tyty} = R_{xyxy} = -R_{tyxy} = -p\frac{\partial^2 p}{\partial u^2}, \qquad R_{tztz} = R_{xzxz} = -R_{tzxz} = -q\frac{\partial^2 q}{\partial u^2}. \qquad (2.19.5)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \quad \mathbf{e}_{(x)} = \partial_x, \quad \mathbf{e}_{(y)} = \frac{1}{p}\partial_y, \quad \mathbf{e}_{(z)} = \frac{1}{q}\partial_z. \qquad (2.19.6)$$

Dual tetrad:

$$\theta^{(t)} = dt, \quad \theta^{(x)} = dx, \quad \theta^{(y)} = p\,dy, \quad \theta^{(z)} = q\,dz. \qquad (2.19.7)$$

2.20 Reissner-Nordstrøm

The Reissner-Nordstrøm black hole in spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ is defined by the metric[MTW73]

$$ds^2 = -A_{\mathrm{RN}}c^2dt^2 + A_{\mathrm{RN}}^{-1}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.20.1}$$

where

$$A_{\mathrm{RN}} = 1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2} \tag{2.20.2}$$

with $r_s = 2GM/c^2$, the charge Q, and $\rho = G/(\varepsilon_0 c^4) \approx 9.33 \cdot 10^{-34}$. As in the Schwarzschild case, there is a true curvature singularity at $r = 0$. However, for $Q^2 < r_s^2/(4\rho)$ there are also two critical points at

$$r = \frac{r_s}{2} \pm \frac{r_s}{2}\sqrt{1 - \frac{4\rho Q^2}{r_s^2}}. \tag{2.20.3}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{A_{\mathrm{RN}}c^2(r_s r - 2\rho Q^2)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s r - 2\rho Q^2}{2r^3 A_{\mathrm{RN}}}, \qquad \Gamma^r_{rr} = -\frac{r_s r - 2\rho Q^2}{2r^3 A_{\mathrm{RN}}}, \tag{2.20.4a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -rA_{\mathrm{RN}}, \tag{2.20.4b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -rA_{\mathrm{RN}}\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.20.4c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2(r_s r - 3\rho Q^2)}{r^4}, \qquad R_{t\vartheta t\vartheta} = \frac{A_{\mathrm{RN}}c^2(rs_r - 2\rho Q^2)}{2r^2}, \tag{2.20.5a}$$

$$R_{t\varphi t\varphi} = \frac{A_{\mathrm{RN}}c^2(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2}, \qquad R_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\mathrm{RN}}}, \tag{2.20.5b}$$

$$R_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2 A_{\mathrm{RN}}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = (r_s r - \rho Q^2)\sin^2\vartheta. \tag{2.20.5c}$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2\rho Q^2 A_{\mathrm{RN}}}{r^4}, \qquad R_{rr} = -\frac{\rho Q^2}{r^4 A_{\mathrm{RN}}}, \qquad R_{\vartheta\vartheta} = \frac{\rho Q^2}{r^2}, \qquad R_{\varphi\varphi} = \frac{\rho Q^2 \sin^2\vartheta}{r^2}. \tag{2.20.6}$$

While the Ricci scalar vanishes identically, the Kretschmann scalar reads

$$\mathcal{K} = 4\frac{3r_s^2 r^2 - 12r_s r\rho Q^2 + 14\rho^2 Q^4}{r^8}. \tag{2.20.7}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(r_s r - 2\rho Q^2)}{r^4}, \qquad C_{t\vartheta t\vartheta} = -\frac{A_{\mathrm{RN}}c^2(rs_r - 2\rho Q^2)}{2r^2}, \tag{2.20.8a}$$

$$C_{t\varphi t\varphi} = \frac{A_{\mathrm{RN}}c^2(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2}, \qquad C_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\mathrm{RN}}}, \tag{2.20.8b}$$

$$C_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2 A_{\mathrm{RN}}}, \qquad C_{\vartheta\varphi\vartheta\varphi} = (r_s r - 2\rho Q^2)\sin^2\vartheta. \tag{2.20.8c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{A_{\mathrm{RN}}}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{A_{\mathrm{RN}}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.20.9}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{A_{\mathrm{RN}}}\,dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{A_{\mathrm{RN}}}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.20.10}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{rr_s - 2\rho Q^2}{2r^3\sqrt{A_{\mathrm{RN}}}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{A_{\mathrm{RN}}}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.20.11}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r^2 - 3rr_s + 2\rho Q^2}{2r^3\sqrt{A_{\mathrm{RN}}}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.20.12}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -\frac{r_s r - 3\rho Q^2}{r^4}, \qquad R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s r - \rho Q^2}{r^4}, \tag{2.20.13a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \tag{2.20.13b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{\rho Q^2}{r^4}. \tag{2.20.14}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s r - 2\rho Q^2}{r^4}, \tag{2.20.15a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \tag{2.20.15b}$$

Embedding:
The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{1}{1 - r_s/r + \rho Q^2/r^2} - 1}. \tag{2.20.16}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\mathrm{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\mathrm{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \tag{2.20.17}$$

with constants of motion $k = A_{\mathrm{RN}}c^2\dot{t}$ and $h = r^2\dot{\varphi}$. For null geodesics, $\kappa = 0$, there are two extremal points

$$r_\pm = \frac{3}{4}r_s\left(1 \pm \sqrt{1 - \frac{32\rho Q^2}{9r_s^2}}\right), \tag{2.20.18}$$

where r_+ is a maximum and r_- a minimum.
Further reading:
Eiroa[ERT02]

2.21 de Sitter spacetime

The de Sitter spacetime with $\Lambda > 0$ is a solution of the Einstein field equations with constant curvature. A detailed discussion can be found for example in Hawking and Ellis[HE99]. Here, we use the coordinate transformations given by Bičák[BK01].

2.21.1 Standard coordinates

The de Sitter metric in standard coordinates $\{\tau \in \mathbb{R}, \chi \in [-\pi, \pi], \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads

$$ds^2 = -d\tau^2 + \alpha^2 \cosh^2 \frac{\tau}{\alpha} \left[d\chi^2 + \sin^2 \chi \left(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right], \tag{2.21.1}$$

where $\alpha^2 = 3/\Lambda$.

Christoffel symbols:

$$\Gamma^\chi_{\tau\chi} = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \qquad\qquad \Gamma^\vartheta_{\tau\vartheta} = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \qquad\qquad \Gamma^\varphi_{\tau\varphi} = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \tag{2.21.2a}$$

$$\Gamma^\tau_{\chi\chi} = \alpha \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \qquad\qquad \Gamma^\vartheta_{\chi\vartheta} = \cot \chi, \qquad\qquad \Gamma^\varphi_{\chi\varphi} = \cot \chi, \tag{2.21.2b}$$

$$\Gamma^\tau_{\vartheta\vartheta} = \alpha \sin^2 \chi \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \qquad \Gamma^\chi_{\vartheta\vartheta} = -\sin \chi \cos \chi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot \vartheta, \tag{2.21.2c}$$

$$\Gamma^\tau_{\varphi\varphi} = \alpha \sin^2 \chi \sin^2 \vartheta \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \Gamma^\chi_{\varphi\varphi} = -\sin^2 \vartheta \sin \chi \cos \chi, \Gamma^\vartheta_{\varphi\varphi} = -\sin \vartheta \cos \vartheta. \tag{2.21.2d}$$

Riemann-Tensor:

$$R_{\tau\chi\tau\chi} = -\cosh^2 \frac{\tau}{\alpha}, \qquad\qquad\qquad R_{\tau\vartheta\tau\vartheta} = -\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi, \tag{2.21.3a}$$

$$R_{\tau\varphi\tau\varphi} = -\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi \sin^2 \vartheta, \qquad\qquad R_{\chi\vartheta\chi\vartheta} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha} \right)^2 \sin^2 \chi, \tag{2.21.3b}$$

$$R_{\chi\varphi\chi\varphi} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha} \right)^2 \sin^2 \chi \sin^2 \vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha} \right)^2 \sin^4 \chi \sin^2 \vartheta. \tag{2.21.3c}$$

Ricci-Tensor:

$$R_{\tau\tau} = -\frac{3}{\alpha^2}, \quad R_{\chi\chi} = 3\cosh^2 \frac{\tau}{\alpha}, \quad R_{\vartheta\vartheta} = 3\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi, \quad R_{\varphi\varphi} = 3\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi \sin^2 \vartheta. \tag{2.21.4}$$

Ricci and Kretschmann scalars:

$$\mathscr{R} = \frac{12}{\alpha^2}, \qquad \mathscr{K} = \frac{24}{\alpha^4}. \tag{2.21.5}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(\chi)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha}} \partial_\chi, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha} \sin \chi} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha} \sin \chi \sin \vartheta} \partial_\varphi. \tag{2.21.6}$$

Dual tetrad:

$$\theta^{(\tau)} = d\tau, \quad \theta^{(\chi)} = \alpha \cosh \frac{\tau}{\alpha} d\chi, \quad \theta^{(\vartheta)} = \alpha \cosh \frac{\tau}{\alpha} \sin \chi \, d\vartheta, \quad \theta^{(\varphi)} = \alpha \cosh \frac{\tau}{\alpha} \sin \chi \sin \vartheta \, d\varphi. \tag{2.21.7}$$

2.21.2 Conformally Einstein coordinates

In conformally Einstein coordinates $\{\eta \in [0, \pi], \chi \in [-\pi, \pi], \vartheta \in [0, \pi], \varphi \in [0, 2\pi)\}$, the de Sitter metric reads

$$ds^2 = \frac{\alpha^2}{\sin^2 \eta} \left[-d\eta^2 + d\chi^2 + \sin^2 \chi \left(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2 \right) \right]. \tag{2.21.8}$$

It follows from the standard form (2.21.1) by the transformation

$$\eta = 2\arctan\left(e^{\tau/\alpha}\right). \tag{2.21.9}$$

2.21.3 Conformally flat coordinates

Conformally flat coordinates $\{T \in \mathbb{R}, r \in \mathbb{R}, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ follow from conformally Einstein coordinates by means of the transformations

$$T = \frac{\alpha \sin\eta}{\cos\chi + \cos\eta}, \quad r = \frac{\alpha \sin\chi}{\cos\chi + \cos\eta}, \quad \text{or} \quad \eta = \arctan\frac{2T\alpha}{\alpha^2 - T^2 + r^2}, \quad \chi = \arctan\frac{2r\alpha}{\alpha^2 + T^2 - r^2}. \tag{2.21.10}$$

For the transformation $(T,R) \to (\eta,\chi)$, we have to take care of the coordinate domains. In that case, if $\kappa^2 - T^2 + r^2 < 0$, we have to map $\eta \to \eta + \pi$. On the other hand, if $\kappa^2 + T^2 - r^2 < 0$, we have to consider the sign of r. If $r > 0$, then $\chi \to \chi + \pi$, otherwise $\chi \to \chi - \pi$.
The resulting metric reads

$$ds^2 = \frac{\alpha^2}{T^2}\left[-dT^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right]. \tag{2.21.11}$$

Note that we identify points $(r < 0, \vartheta, \varphi)$ with $(r > 0, \pi - \vartheta, \varphi - \pi)$.

Christoffel symbols:

$$\Gamma^T_{TT} = \Gamma^r_{Tr} = \Gamma^\vartheta_{T\vartheta} = \Gamma^\varphi_{T\varphi} = \Gamma^T_{rr} = -\frac{1}{T}, \quad \Gamma^\vartheta_{r\vartheta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \quad \Gamma^T_{\vartheta\vartheta} = -\frac{r^2}{T}, \quad \Gamma^r_{\vartheta\vartheta} = -r, \tag{2.21.12a}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \quad \Gamma^T_{\varphi\varphi} = -\frac{r^2\sin^2\vartheta}{T}, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.21.12b}$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{\alpha^2}{T^4}, \qquad R_{T\vartheta T\vartheta} = -\frac{\alpha^2 r^2}{T^4}, \qquad R_{T\varphi T\varphi} = -\frac{\alpha^2 r^2\sin^2\vartheta}{T^4}, \tag{2.21.13a}$$

$$R_{r\vartheta r\vartheta} = \frac{\alpha^2 r^2}{T^4}, \qquad R_{r\varphi r\varphi} = \frac{\alpha^2 r^2\sin^2\vartheta}{T^4}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{\alpha^2 r^4\sin^2\vartheta}{T^4}. \tag{2.21.13b}$$

Ricci-Tensor:

$$R_{TT} = -\frac{3}{T^2}, \qquad R_{rr} = \frac{3}{T^2}, \qquad R_{\vartheta\vartheta} = \frac{3r^2}{T^2}, \qquad R_{\varphi\varphi} = \frac{3r^2\sin^2\vartheta}{T^2}. \tag{2.21.14}$$

The *Ricci and Kretschmann scalar read:*

$$\mathscr{R} = \frac{12}{\alpha^2}, \qquad \mathscr{K} = \frac{24}{\alpha^4}. \tag{2.21.15}$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{T}{\alpha}\partial_T, \qquad \mathbf{e}_{(r)} = \frac{T}{\alpha}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{T}{\alpha r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{T}{\alpha r\sin\vartheta}\partial_\varphi. \tag{2.21.16}$$

2.21.4 Static coordinates

The de Sitter metric in static spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ reads

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)c^2 dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right). \tag{2.21.17}$$

It follows from the conformally Einstein form (2.21.8) by the transformations

$$t = \frac{\alpha}{2} \ln \frac{\cos \chi - \cos \eta}{\cos \chi + \cos \eta}, \quad r = \alpha \frac{\sin \chi}{\sin \eta}. \tag{2.21.18}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{(\Lambda r^2 - 3)}{9} c^2 \Lambda r, \qquad \Gamma^t_{tr} = \frac{\Lambda r}{\Lambda r^2 - 3}, \qquad \Gamma^r_{rr} = \frac{\Lambda r}{3 - \Lambda r^2}, \tag{2.21.19a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = \frac{(\Lambda r^2 - 3)r}{3}, \tag{2.21.19b}$$

$$\Gamma^\phi_{\vartheta\phi} = \cot(\vartheta), \qquad \Gamma^r_{\phi\phi} = \frac{\Lambda r^2 - 3}{3} r \sin^2(\vartheta), \qquad \Gamma^\vartheta_{\phi\phi} = -\sin(\vartheta)\cos(\vartheta). \tag{2.21.19c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{\Lambda}{3} c^2, \qquad R_{t\vartheta t\vartheta} = -\frac{3 - \Lambda r^2}{9} c^2 \Lambda r^2, \qquad R_{t\varphi t\varphi} = -\frac{3 - \Lambda r^2}{9} c^2 \Lambda r^2 \sin(\vartheta)^2, \tag{2.21.20a}$$

$$R_{r\vartheta r\vartheta} = \frac{\Lambda r^2}{-\Lambda r^2 + 3}, \qquad R_{r\varphi r\varphi} = \frac{\Lambda r^2 \sin(\theta)^2}{-\Lambda r^2 + 3}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r^4 \sin^2(\theta) \Lambda}{3}. \tag{2.21.20b}$$

Ricci-Tensor:

$$R_{tt} = \frac{\Lambda r^2 - 3}{3} c^2 \Lambda, \qquad R_{rr} = \frac{3\Lambda}{3 - \Lambda r^2}, \qquad R_{\vartheta\vartheta} = \Lambda r^2, \qquad R_{\varphi\varphi} = r^2 \sin^2(\vartheta) \Lambda. \tag{2.21.21}$$

The *Ricci scalar and Kretschmann scalar read:*

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = \frac{8}{3} \Lambda^2. \tag{2.21.22}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{3}{3 - \Lambda r^2}} \frac{\partial_t}{c}, \qquad \mathbf{e}_{(r)} = \sqrt{1 - \frac{\Lambda r^2}{3}} \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin(\vartheta)} \partial_\varphi. \tag{2.21.23}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(t)} = -\frac{\Lambda r}{\sqrt{9 - 3\Lambda r^2}}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{9 - 3\Lambda r^2}}{3r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \tag{2.21.24}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{\sqrt{9 - 3\Lambda r^2}(\Lambda r^2 - 2)}{(\Lambda r^2 - 3)r}, \qquad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \tag{2.21.25}$$

Riemann-Tensor with respect to local tetrad:

$$-R_{(t)(r)(t)(r)} = -R_{(t)(\vartheta)(t)(\vartheta)} = -R_{(t)(\varphi)(t)(\varphi)} = R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{1}{3}\Lambda. \tag{2.21.26}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \Lambda. \tag{2.21.27}$$

2.21.5   Lemaître-Robertson form

The de Sitter universe in the Lemaître-Robertson form reads

$$ds^2 = -c^2 dt^2 + e^{2Ht}\left[dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \right) \right], \qquad (2.21.28)$$

with Hubble's Parameter $H = \sqrt{\frac{\Lambda c^2}{3}} = \frac{c}{\alpha}$, which is assumed here to be time-independent.
This a special case of the first and second form of the Friedman-Robertson-Walker metric defined in Eqs. (2.9.2) and (2.9.12) with $R(t) = e^{Ht}$ and $k = 0$.
Christoffel symbols:

$$\Gamma^r_{tr} = H, \qquad\qquad \Gamma^\vartheta_{t\vartheta} = H, \qquad\qquad \Gamma^\varphi_{t\varphi} = H, \qquad (2.21.29a)$$

$$\Gamma^t_{rr} = \frac{e^{2Ht}H}{c^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad (2.21.29b)$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{e^{2Ht}r^2 H}{c^2}, \qquad \Gamma^r_{\vartheta\vartheta} = -r, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \qquad (2.21.29c)$$

$$\Gamma^t_{\varphi\varphi} = \frac{e^{2Ht}r^2 \sin^2(\theta)H}{c^2}, \qquad \Gamma^r_{\varphi\varphi} = -r\sin(\vartheta)^2, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta). \qquad (2.21.29d)$$

Riemann-Tensor:

$$R_{trtr} = -e^{2Ht}H^2, \qquad\qquad R_{t\vartheta t\vartheta} = -e^{2Ht}r^2 H^2, \qquad (2.21.30a)$$

$$R_{t\varphi t\varphi} = -e^{2Ht}r^2 \sin^2(\vartheta)H^2, \qquad R_{r\vartheta r\vartheta} = \frac{e^{4Ht}r^2 H^2}{c^2}, \qquad (2.21.30b)$$

$$R_{r\varphi r\varphi} = \frac{e^{4Ht}r^2 \sin^2(\vartheta)H^2}{c^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{e^{4Ht}r^4 \sin^2(\vartheta)H^2}{c^2}. \qquad (2.21.30c)$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \qquad R_{rr} = 3\frac{e^{2Ht}H^2}{c^2}, \qquad R_{\vartheta\vartheta} = 3\frac{e^{2Ht}r^2 H^2}{c^2}, \qquad R_{\varphi\varphi} = 3\frac{e^{2Ht}r^2 \sin^2(\vartheta)H^2}{c^2}. \qquad (2.21.31)$$

Ricci and Kretschmann scalars:

$$\mathscr{R} = \frac{12H^2}{c^2}, \qquad \mathscr{K} = \frac{24H^4}{c^4}. \qquad (2.21.32)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = e^{-Ht}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{e^{-Ht}}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{e^{-Ht}}{r\sin\vartheta}\partial_\varphi. \qquad (2.21.33)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{H}{c} \qquad (2.21.34a)$$

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{e^{Ht}r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\theta)}{e^{Ht}r}. \qquad (2.21.34b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}, \qquad \gamma_{(r)} = \frac{2}{e^{Ht}r}, \gamma_{(\vartheta)} = \frac{\cot(\theta)}{e^{Ht}r}. \qquad (2.21.35)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{H^2}{c^2} \qquad (2.21.36a)$$

$$R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{H^2}{c^2}. \qquad (2.21.36b)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 3\frac{H^2}{c^2}. \tag{2.21.37}$$

2.21.6 Cartesian coordinates

The de Sitter universe in Lemaître-Robertson form can also be expressed in Cartesian coordinates:

$$\boxed{ds^2 = -c^2 dt^2 + e^{2Ht}\left[dx^2 + dy^2 + dz^2\right].} \tag{2.21.38}$$

Christoffel symbols:

$$\Gamma^x_{tx} = H, \qquad \Gamma^y_{ty} = H, \qquad \Gamma^z_{tz} = H, \tag{2.21.39a}$$

$$\Gamma^t_{xx} = \frac{e^{2Ht}H}{c^2}, \qquad \Gamma^t_{yy} = \frac{e^{2Ht}H}{c^2}, \qquad \Gamma^t_{zz} = \frac{e^{2Ht}H}{c^2}. \tag{2.21.39b}$$

$$\tag{2.21.39c}$$

Partial derivatives

$$\Gamma^t_{xx,t} = \Gamma^t_{yy,t} = \Gamma^t_{zz,t} = \frac{2H^2 e^{2Ht}}{c^2}. \tag{2.21.40}$$

Riemann-Tensor:

$$R_{txtx} = R_{txtx} = R_{tztz} = -e^{2Ht}H^2, \qquad R_{xyxy} = R_{xzxz} = R_{yzyz} = \frac{e^{4Ht}H^2}{c^2}. \tag{2.21.41}$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \qquad R_{xx} = R_{yy} = R_{zz} = 3\frac{e^{2Ht}H^2}{c^2}. \tag{2.21.42}$$

The Ricci and Kretschmann scalar read:

$$\mathscr{R} = 12\frac{H^2}{c^2}, \qquad \mathscr{K} = 24\frac{H^4}{c^4}. \tag{2.21.43}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(x)} = e^{-Ht}\partial_x, \qquad \mathbf{e}_{(y)} = e^{-Ht}\partial_y, \qquad \mathbf{e}_{(z)} = e^{-Ht}\partial_z. \tag{2.21.44}$$

Ricci rotation coefficients:

$$\gamma_{(x)(t)(x)} = \gamma_{(y)(t)(y)} = \gamma_{(z)(t)(z)} = \frac{H}{c}. \tag{2.21.45}$$

The only non-vanishing contraction of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}. \tag{2.21.46}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(t)(x)} = R_{(t)(y)(t)(y)} = R_{(t)(z)(t)(z)} = -\frac{H^2}{c^2}, \tag{2.21.47a}$$

$$R_{(x)(y)(x)(y)} = R_{(x)(z)(x)(z)} = R_{(y)(z)(y)(z)} = \frac{H^2}{c^2}. \tag{2.21.47b}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(x)(x)} = R_{(y)(y)} = R_{(z)(z)} = 3\frac{H^2}{c^2}. \tag{2.21.48}$$

Further reading:
Tolman[Tol34, sec. 142], Bičák[BK01]

2.22   Straight spinning string

The metric of a straight spinning string in cylindrical coordinates (t, ρ, φ, z) reads

$$ds^2 = -(c\,dt - a\,d\varphi)^2 + d\rho^2 + k^2\rho^2 d\varphi^2 + dz^2, \tag{2.22.1}$$

where $a \in \mathbb{R}$ and $k > 0$ are two parameters, see Perlick[Per04].

Metric-Tensor:

$$g_{tt} = -c^2, \qquad g_{t\varphi} = ac, \qquad g_{\rho\rho} = g_{zz} = 1, \qquad g_{\varphi\varphi} = k^2\rho^2 - a^2. \tag{2.22.2}$$

Christoffel symbols:

$$\Gamma^t_{\rho\varphi} = \frac{a}{c\rho}, \qquad \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho}, \qquad \Gamma^\rho_{\varphi\varphi} = -k^2\rho. \tag{2.22.3}$$

Partial derivatives

$$\Gamma^t_{\rho\varphi,\rho} = -\frac{\alpha}{c\rho^2}, \qquad \Gamma^\varphi_{\rho\varphi,\rho} = -\frac{1}{\rho^2}, \qquad \Gamma^\rho_{\varphi\varphi,\rho} = -k^2. \tag{2.22.4}$$

The Riemann-, Ricci-, and Weyl-tensors as well as the Ricci- and Kretschmann-scalar vanish identically.

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(1)} = \partial_\rho, \qquad \mathbf{e}_{(2)} = \frac{1}{k\rho}\left(\frac{a}{c}\partial_t + \partial_\varphi\right), \qquad \mathbf{e}_{(3)} = \partial_z. \tag{2.22.5}$$

Dual tetrad:

$$\theta^{(0)} = c\,dt - a\,d\varphi, \qquad \theta^{(1)} = d\rho, \qquad \theta^{(2)} = k\rho\,d\varphi, \qquad \theta^{(3)} = dz. \tag{2.22.6}$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(2)(1)(2)} = \frac{1}{\rho}, \qquad \gamma_{(0)} = \gamma_{(2)} = \gamma_{(3)} = 0, \qquad \gamma_{(1)} = \frac{1}{\rho}. \tag{2.22.7}$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{k^2\rho^2 - a^2}}{k\rho}\left(\frac{1}{c}\partial_t - \frac{a}{k^2\rho^2 - a^2}\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \partial_\rho, \tag{2.22.8a}$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{k^2\rho^2 - a^2}}\partial_\varphi, \qquad\qquad\qquad \mathbf{e}_{(3)} = \partial_z. \tag{2.22.8b}$$

Dual tetrad:

$$\theta^{(0)} = \frac{k\rho}{\sqrt{k^2\rho^2 - a^2}}c\,dt, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = \frac{ac\,dt}{\sqrt{k^2\rho^2 - a^2}} + \sqrt{k^2\rho^2 - a^2}\,d\varphi, \quad \theta^{(3)} = dz. \tag{2.22.9}$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(0)(1)(0)} = \frac{a^2}{\rho\left(k^2\rho^2 - a^2\right)}, \quad \gamma_{(2)(1)(0)} = \gamma_{(0)(2)(1)} = \gamma_{(0)(1)(2)} = \frac{ak}{k^2\rho^2 - a^2}, \tag{2.22.10a}$$

$$\gamma_{(2)(1)(2)} = \frac{k^2\rho}{k^2\rho^2 - a^2}, \tag{2.22.10b}$$

$$\gamma_{(1)} = \frac{1}{\rho}. \tag{2.22.10c}$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{\rho}^2 + \frac{1}{k^2\rho^2}\left(h_2 - \frac{ah_1}{c}\right)^2 - \kappa c^2 = \frac{h_1^2}{c^2}, \tag{2.22.11}$$

with the constants of motion $h_1 = c(c\dot{t} - a\dot{\varphi})$ and $h_2 = a(c\dot{t} - a\dot{\varphi}) + k^2\rho^2\dot{\varphi}$.

The point of closest approach ρ_{pca} for a null geodesic that starts at $\rho = \rho_i$ with $\mathbf{y} = \pm\mathbf{e}_{(0)} + \cos\xi\,\mathbf{e}_{(1)} + \sin\xi\,\mathbf{e}_{(2)}$ with respect to the static tetrad is given by $\rho = \rho_i\sin\xi$. Hence, the ρ_{pca} is independent of a and k. The same is also true for timelike geodesics.

2.23 Sultana-Dyer spacetime

The Sultana-Dyer metric represents a black hole in the Einstein-de Sitter universe. In spherical coordinates $(t, r, \vartheta, \varphi)$, the metric reads[SD05] $(G = c = 1)$

$$ds^2 = t^4 \left[\left(1 - \frac{2M}{r} \right) dt^2 - \frac{4M}{r} dt\, dr - \left(1 + \frac{2M}{r} \right) dr^2 - r^2 d\Omega^2 \right], \tag{2.23.1}$$

where M is the mass of the black hole and $\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ is the spherical surface element. Note that here, the signature of the metric is $\mathrm{sign}(\mathbf{g}) = -2$.

Christoffel symbols:

$$\Gamma^t_{tt} = \frac{2\left(r^3 + 4M^2 r + M^2 t\right)}{tr^3}, \quad \Gamma^r_{tt} = \frac{M(r - 2M)(4r + t)}{tr^3}, \quad \Gamma^t_{tr} = \frac{M(r + 2M)(4r + t)}{tr^3}, \tag{2.23.2a}$$

$$\Gamma^r_{tr} = \frac{2\left(r^3 - 4M^2 r - M^2 t\right)}{tr^3}, \quad \Gamma^\vartheta_{t\vartheta} = \frac{2}{t}, \qquad\qquad \Gamma^\varphi_{t\varphi} = \frac{2}{t}, \tag{2.23.2b}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad\qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad\qquad \Gamma^t_{\vartheta\vartheta} = \frac{2\left(r^2 + 2Mr - Mt\right)}{t}, \tag{2.23.2c}$$

$$\Gamma^r_{\vartheta\vartheta} = -\frac{4Mr + tr - 2Mt}{t}, \quad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{2.23.2d}$$

$$\Gamma^t_{rr} = \frac{2\left(r^3 + 4Mr^2 + 4M^2 r + M^2 t + Mtr\right)}{tr^3}, \quad \Gamma^r_{rr} = -\frac{M\left(4r^2 + 8Mr + 2Mt + tr\right)}{tr^3}, \tag{2.23.2e}$$

$$\Gamma^t_{\varphi\varphi} = \frac{2\left(r^2 + 2Mr - Mt\right)\sin^2\vartheta}{t}, \qquad \Gamma^r_{\varphi\varphi} = -\frac{(4Mr + tr - 2Mt)\sin^2\vartheta}{t}. \tag{2.23.2f}$$

Riemann-Tensor:

$$R_{trtr} = \frac{2t^2\left(-2Mr^2 - r^3 + Mt^2 + 2Mtr\right)}{r^3}, \tag{2.23.3a}$$

$$R_{r\vartheta t\vartheta} = -\frac{t^2\left(2r^4 + 16M^2 r^2 + 4Mtr^2 - 4M^2 r^2 t + Mt^2 r - 2M^2 t^2\right)}{r^2}, \tag{2.23.3b}$$

$$R_{t\vartheta r\vartheta} = -\frac{2Mt^2(4r + t)(r^2 + 2Mr - Mt)}{r^2}, \tag{2.23.3c}$$

$$R_{r\varphi t\varphi} = -\frac{t^2\sin^2\vartheta\left(2r^4 + 16M^2 r^2 + 4Mtr^2 - 4M^2 r^2 t + Mt^2 r - 2M^2 t^2\right)}{r^2}, \tag{2.23.3d}$$

$$R_{t\varphi r\varphi} = -\frac{2Mt^2\sin^2\vartheta(4r + t)(r^2 + 2Mr - Mt)}{r^2}, \tag{2.23.3e}$$

$$R_{r\vartheta r\vartheta} = -\frac{t^2\left(4r^4 + 16Mr^4 - 4M^2 tr + 16M^2 r^2 - 2M^2 t^2 - Mt^2 r\right)}{r^2}, \tag{2.23.3f}$$

$$R_{r\varphi r\varphi} = -\frac{t^2\sin^2\vartheta\left(4r^4 + 16Mr^4 - 4M^2 tr + 16M^2 r^2 - 2M^2 t^2 - Mt^2 r\right)}{r^2}, \tag{2.23.3g}$$

$$R_{\vartheta\varphi\vartheta\varphi} = -2t^2 r\sin^2\vartheta\left(2r^3 + 4Mr^2 - 4Mtr + mt^2\right). \tag{2.23.3h}$$

Ricci-Tensor:

$$R_{tt} = \frac{2\left(3r^2 + 12M^2 + 2Mt\right)}{t^2 r^2}, \qquad R_{tr} = \frac{4M\left(3r + t + 6M\right)}{t^2 r^2}, \tag{2.23.4a}$$

$$R_{rr} = \frac{2\left(3r^2 + 12Mr + 2Mt + 12M^2\right)}{t^2 r^2}, \quad R_{\vartheta\vartheta} = \frac{6\left(r^2 + 2Mr - 2Mt\right)}{t^2}, \tag{2.23.4b}$$

$$R_{\varphi\varphi} = \frac{6\left(r^2 + 2Mr - 2Mt\right)\sin^2\vartheta}{t^2}. \tag{2.23.4c}$$

Ricci and Kretschmann scalars:

$$R = -\frac{12\left(r^2 + 2Mr - 2Mt\right)}{t^6 r^2},$$ (2.23.5a)

$$\mathcal{K} = \frac{48\left(M^2 t^4 + 20M^2 r^4 + 20Mr^5 + 8M^2 r^2 t^2 - 4Mr^4 t - 16M^2 r^3 t + 5r^6\right)}{t^{12} r^6}.$$ (2.23.5b)

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{1+2M/r}}{t^2}\partial_t - \frac{2M/r}{t^2\sqrt{1+2M/r}}\partial_r, \quad \mathbf{e}_{(1)} = \frac{1}{t^2\sqrt{1+2M/r}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r}\partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r \sin\vartheta}\partial_\varphi.$$ (2.23.6)

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{t^2\sqrt{1-2M/r}}\partial_t, \quad \mathbf{e}_{(1)} = \frac{2M/r}{t^2\sqrt{1-2M/r}}\partial_t + \frac{\sqrt{1-2M/r}}{t^2}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r}\partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r \sin\vartheta}\partial_\varphi.$$ (2.23.7)

Further reading:
Sultana and Dyer[SD05].

2.24   TaubNUT

The TaubNUT metric in Boyer-Lindquist like spherical coordinates $(t, r, \vartheta, \varphi)$ reads[BCJ02] $(G = c = 1)$

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt + 2\ell\cos\vartheta\,d\varphi\right)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\right), \tag{2.24.1}$$

where $\Sigma = r^2 + \ell^2$ and $\Delta = r^2 - 2Mr - \ell^2$. Here, M is the mass of the black hole and ℓ the magnetic monopol strength.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{\Delta\rho}{\Sigma^3}, \qquad \Gamma^t_{tr} = \frac{\rho}{\Delta\Sigma}, \qquad \Gamma^t_{t\vartheta} = -2\ell^2\cos\vartheta\,\frac{\Delta}{\Sigma^2}, \tag{2.24.2a}$$

$$\Gamma^\varphi_{t\vartheta} = \frac{\ell\Delta}{\Sigma^2\sin\vartheta}, \quad \Gamma^r_{t\varphi} = \frac{2\ell\rho\Delta\cos\vartheta}{\Sigma^3}, \quad \Gamma^\vartheta_{t\varphi} = -\frac{\ell\Delta\sin\vartheta}{\Sigma^2}, \tag{2.24.2b}$$

$$\Gamma^r_{rr} = -\frac{\rho}{\Sigma\Delta}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{r}{\Sigma}, \qquad \Gamma^\varphi_{r\varphi} = \frac{r}{\Sigma}, \quad \Gamma^r_{\vartheta\vartheta} = -\frac{r\Delta}{\Sigma}, \tag{2.24.2c}$$

$$\Gamma^t_{r\varphi} = \frac{-2\ell(r^3 - 3Mr^2 - 3r\ell^2 + M\ell^2)\cos\vartheta}{\Sigma\Delta}, \tag{2.24.2d}$$

$$\Gamma^t_{\vartheta\varphi} = -\frac{\ell\left[\cos^2\vartheta\left(6r^2\ell^2 - 8\ell^2 Mr - 3\ell^4 + r^4\right) + \Sigma^2\right]}{\Sigma^2\sin\vartheta}, \tag{2.24.2e}$$

$$\Gamma^r_{\varphi\varphi} = \frac{\Delta}{\Sigma^3}\left[\cos^2\vartheta\left(9r\ell^4 + 4\ell^2 Mr^2 - 4\ell^4 M + r^5 + 2r^3\ell^2\right) - r\Sigma^2\right], \tag{2.24.2f}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \frac{\left(4r^2\ell^2 - 4Mr\ell^2 - \ell^4 + r^4\right)\cot\vartheta}{\Sigma^2}, \tag{2.24.2g}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\frac{\left(6r^2\ell^2 - 8Mr\ell^2 - 3\ell^4 + r^4\right)\sin\vartheta\cos\vartheta}{\Sigma^2}, \tag{2.24.2h}$$

where $\rho = 2r\ell^2 + Mr^2 - M\ell^2$.

Static local tetrad:

$$\mathbf{e}_{(0)} = \sqrt{\frac{\Sigma}{\Delta}}\partial_t, \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \quad \mathbf{e}_{(3)} = -\frac{2\ell\cot\vartheta}{\sqrt{\Sigma}}\partial_t + \frac{1}{\sqrt{\Sigma}\sin\vartheta}\partial_\varphi. \tag{2.24.3}$$

Dual tetrad:

$$\theta^{(0)} = \sqrt{\frac{\Delta}{\Sigma}}\left(dt + 2\ell\cos\vartheta\,d\varphi\right), \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}}dr, \quad \theta^{(2)} = \sqrt{\Sigma}d\vartheta, \quad \theta^{(3)} = \sqrt{\Sigma}\sin\vartheta\,d\varphi. \tag{2.24.4}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\frac{\Delta}{\Sigma}\left(\frac{h^2}{\Sigma} - \kappa\right) \tag{2.24.5}$$

with the constants of motion $k = (\Delta/\Sigma)\dot{t}$ and $h = \Sigma\dot{\varphi}$. For null geodesics, we obtain a photon orbit at $r = r_{\text{po}}$ with

$$r_{\text{po}} = M + 2\sqrt{M^2 + \ell^2}\cos\left(\frac{1}{3}\arccos\frac{M}{\sqrt{M^2 + \ell^2}}\right) \tag{2.24.6}$$

Further reading:
Bini et al.[BCdMJ03].

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