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Campos de Killing

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Autor: Gustavo Gancedo Crespo

Director: Diego Herranz Muñoz

Abstract

The purpose of this thesis is to make a rigorous and approachable introduction to Killing fields and their applications to any undergrad that is or has studied a general relativity course.

Contents

1 Motivation and introduction to symmetry in physics

The cornerstone of modern physics is, without a shadow of a doubt, symmetry. This is thanks to one of the most beautiful theorems of physics. **The Noether theorem**.

The Noether theorem establishes a one to one connection between symmetries and conservation laws, thus explaining the origin of conservation of energy, charge and all other conservation laws widely used in physics.

The first question that might arise is, What constitutes a symmetry? What is it? A symmetry is a transformation, such as translations, rotations or even of the internal degrees of freedom a theory might have, that maintains some aspect of the theory invariant.

In the case of classical mechanics the main object that encapsulates the behavior of the system allowing for the computation of the equations of motion. Therefore a symmetry in classical mechanics is any transformation to the lagrangian that doesn't change the equations of motion.

In our case we will find that the corresponding symmetries en general relativity are those that preserve the geometry of spacetime, meaning, the metric. The so called isometries. Killing fields are nothing more than the generators of isometries.

2 Formalism of general relativity

In order to understand symmetry and motivate the definition of Killing fields first it is required to understand is, in the mathematical sense, spacetime and define flows and Lie derivatives.

To do this we will introduce little by little mathematical structure based on the qualities that a spacetime should have

2.1. Spacetimes

A spacetime in the formalism of general relativity is defined as a pseudo-Riemannian manifold. We will start by understanding this ideas.

2.2. Continuity

First of all, a spacetime has a notion that it is continuous, further than that, it is path connected, meaning one can connect any point to any other point by a continuous path[1].

The notion of continuity is defined in the mathematical field of topology A topological space is a pair of sets (M, τ), the first of these is the set of all the points in the space, the second is called the topology of the space and represents all of the open sets. The core idea behind having a topology is introducing a notion of 'closeness' without introducing a metric, in our case there will be an additional notion of closeness defined because of the metric but this idea has to be introduced later. Any topology obeys the following relations of closure

$$\begin{gathered}\emptyset, M \in \tau \\ x_i \in \tau \Rightarrow \bigcup_{i=0}^{\infty} x_i \in \tau \\ x_i \in \tau \Rightarrow \bigcap_{i=0}^{n} x_i \in \tau\end{gathered} \tag{2.1}$$

This allows to define what a continuous function is, the idea of continuity is that any two 'close' points in the input of the function will be 'close' in the output. On topological spaces the definition is related to how open sets transform, here a function between topological spaces $f : (M, \tau_M) \to (N, \tau_N)$ is continuous if $\forall V \in \tau_N, f^{-1}(V) \in \tau_M$ meaning all open sets in the output are open sets in the input. This definition is inspired by the $\varepsilon - \delta$ definition usually defined for metric spaces[2], in fact if one uses the topology defined by the open balls (sets of points closer than some distance) the definitions are equivalent.

2.3. Coordinates

Whenever one talks about any kind of state in physics it is talked about in a coordinate system. It would be expected that in spacetimes one can do the same thing and label the points in spacetime. This is covered in the mathematical field of manifolds. A manifold is a topological space that additionally can

[1]This path is not required to be physical, it could be superluminic.

[2]Spaces with the notion of distance

be locally mapped to a cartesian coordinate system, meaning for any open set V there is a continuous bijection φ from V to $\mathbb{R}^n$ such that φ^{-1} is continuous too.

Additionally it is required that for any two mappings $\varphi_1 : V_1 \to \mathbb{R}^n$ and $\varphi_2 : V_2 \to \mathbb{R}^n$ such that $V_1 \cap V_2 \neq \emptyset$ there has to be a function from $\psi : \varphi_1(V_1 \cap V_2) \to \varphi_2(V_1 \cap V_2)$ that is a bijection, continuous and has a continuous inverse. This means that one can 'translate' one coordinate system to another if they map the same region.

In the case of physics it is additionally required that ψ is infinitely differentiable, this is the definition for smooth manifolds. This is necessary because otherwise a smooth function would be smooth on one coordinate system but it would not be smooth on a different coordinate system because of the chain rule.

The set of all coordinate systems with a smooth coordinate change is called an atlas or $\mathcal{A}$.

Another representation for a coordinate system is a collection of n functions $x^\mu : \mathbb{E} \longrightarrow \mathbb{R}$ such that x^0 gives the 0-th component of a coordinate system φ, x^1 the first component and so on. This representation is more common in physics and will be widely used in this thesis.

2.4. Fields on the spacetime

Now it is time to start talking about what can we 'place' on spacetime.

Scalar fields on spacetimes

A scalar field assigns a number to each point of our spacetime $\mathbb{E}$. So it will be any function of the form

$$\phi : \mathbb{E} \longrightarrow \mathbb{R} \tag{2.2}$$

This function can be 'placed' in a coordinate system by defining $\phi_\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by taking a coordinate system from the atlas $\varphi \in \mathcal{A}$ and applying the inverse to the input $\phi_\varphi = \phi \circ \varphi$. From now on ϕ_φ will be denoted just ϕ whenever the coordinate system is clear.

The set of all infinitely differentiable scalar fields on a manifold will be denoted $\mathcal{C}^\infty(M)$

Parametric curves

If one wishes to keep track of the path of a particle on a spacetime one would naturally use this kind of object. A parametric curve may be defined as a function

$$\gamma : \mathbb{R} \longrightarrow \mathbb{E} \tag{2.3}$$

Again this path can be represented in a coordinate system by composing it with a map $\varphi \in \mathcal{A}$, $\gamma_\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n, \gamma_\varphi = \varphi \circ \gamma$.

Vector fields on the spacetime

Motivated from the 'classical' version of a vector field defined as $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ it might be tempting to define $X : \mathbb{E} \longrightarrow \mathbb{R}^n$ as a vector field on an n dimensional smooth manifold. This definition has one big problem, it is not coordinate independent.

Imagine one has a 3-dimensional manifold with a coordinate system $\{x^\mu\}$ and a primed coordinate system $\{x'^\mu\}$ such that $x'^0 = x^1, x'^1 = x^0, x'^2 = x^2$. Now lets define a constant vector field on the 'x' direction $X(p) = (1, 0, 0)$. In the x^μ coordinate system this field points in the x^0 direction while on the x'^μ coordinate system points in the x'^0 direction, this would correspond to the x^0 direction by the coordinate transformations defined. Therefore this definition of a vector field is not independent of coordinate choice.

There are two equivalent definitions for vectors on a manifold at a point p that are coordinate independent.

The first of these is in terms of tangent vectors of curves, since a curve on the manifold is defined independently of the coordinate system it would be expected that the tangent vector is coordinate independent too. In this way the set of all vectors at a point $p \in \mathbb{E}$ is defined as the set of curves γ such that $\gamma(0) = p$. Here we will have to add an equivalence relation, similarly to how rational numbers are not all of the fractions but the fractions with the fact that two fractions are equal when they follow the relation $\frac{a}{b} \sim \frac{c}{d} \Leftrightarrow ad = bc$ here two of our vectors will be 'equal' if for any $\varphi \in \mathcal{A}$

$$\gamma \sim \hat{\gamma} \Leftrightarrow (\varphi \circ \gamma)' = (\varphi \circ \hat{\gamma})' \tag{2.4}$$

where $'$ is the usual derivative.

The second definitions is via derivations. A derivation at a point p is defined as a linear functional

$$D : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R} \tag{2.5}$$

that also obeys the product rule

$$\begin{gathered} f, g \in \mathcal{C}^\infty \\ D(f \cdot g) = f(p)D(g) + D(f)g(p) \end{gathered} \tag{2.6}$$

Any curve can be assigned a derivation via the following definition

$$D_\gamma f = (f \circ \gamma)'(0) \tag{2.7}$$

The equivalence of definitions may be proben by first proving both spaces have the same dimension. After that Eq. (2.7) gives a one to one correspondence on both spaces. When given a coordinate system the space of derivations has a basis defined by

$$\partial_\mu(p) = \frac{\partial}{\partial x^\mu}(p) \tag{2.8}$$

Where (p) denotes evaluation of the partial derivative at p

With any of the two definitions the vector space of all vectors at a point p of a manifold M is denoted T_pM.

By defining the set of all vectors tangent to the manifold $TM = \bigcup_{p \in M} T_pM$ a vector field may be defined as

$$\begin{aligned} X : M &\longrightarrow TM \\ p &\longrightarrow X(p) \in T_pM \end{aligned} \tag{2.9}$$

When given a coordinate system a vector field may be written as

$$X(p) = X^\mu(p) \frac{\partial}{\partial x^\mu}(p) \tag{2.10}$$

So a **smooth vector field** is defined as a vector field whose component functions, X^μ, are smooth. The set of all smooth vector fields is denoted as $\mathfrak{X}(M)$

For some proofs the notation $X(p, f) = X(f)(p) = X^\mu(p) \frac{\partial f}{\partial x^\mu}(p)$ will be useful

Covectors

It is easy now to define covectors. A covector at a point p is defined as a linear function

$$\omega : T_pM \longrightarrow \mathbb{R} \tag{2.11}$$

so the cotangent space T_p^*M is the space of all covectors at a point p and the set of all covectors $T^*M = \bigcup_{p\in M} T_p^*M$ a covector field is

$$\begin{aligned} \omega : M &\longrightarrow T^*M \\ p &\longrightarrow \omega(p) \in T_p^*M \end{aligned} \tag{2.12}$$

for any basis ∂_μ the canonical basis for the covector space can be defined as a covector collection such that $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$ where δ^μ_ν is the Kronecker delta.

A covector will be smooth if for a coordinate system the covector has components ω_μ defined by

$$\omega = \omega_\mu(p)\,\mathrm{d}x^\mu \tag{2.13}$$

are $\mathcal{C}^\infty(M)$ functions

Again the notation

$$\omega(p, X) = \omega_\mu(p)\,\mathrm{d}x^\mu(X)(p) \tag{2.14}$$

will be useful

Tensors

A tensor represents a multilinear map, meaning that for any input slot

$$T(a, b, ..., \alpha c + \beta d, ..., z) = \alpha T(a, b, ..., c, ..., z) + \beta T(a, b, ..., d, ..., z) \tag{2.15}$$

The most basic definition of a tensor one can come up with is

$$T : V_1 \times V_2 \times ... \times V_n \to \mathbb{R} \tag{2.16}$$

This is a tensor that takes n vectors as input and as output gives a number

It could be also output more vectors defining

$$\hat{T} : V_1 \times V_2 \times ... \times V_n \longrightarrow V_{n+1} \tag{2.17}$$

however by evaluating the output of $\hat{T}$ with a covector the result is a number representing some component. So it is common to represent this kind of tensors by

$$\hat{T} : V_1 \times V_2 \times ... \times V_n \times V^*_{n+1} \longrightarrow \mathbb{R} \tag{2.18}$$

Therefore the definition of a tensor over a vector space V of kind (q, p) or q times contravariant, p times covariant is defined as

$$T : \underbrace{V^* \times ... \times V^*}_{q \text{ copies}} \times \underbrace{V \times ... \times V}_{p \text{ copies}} \longrightarrow \mathbb{R} \tag{2.19}$$

In our case the corresponding vector spaces are the T_pM and a tensor field will be a map

$$T : \underbrace{T^*M \times ... \times T^*M}_{q \text{ copies}} \times \underbrace{TM \times ... \times TM}_{p \text{ copies}} \longrightarrow \mathcal{C}^\infty(M) \tag{2.20}$$

The components of a tensor can be obtained by feeding it some vectors and applying Eq. (2.10) and Eq. (2.13)

$$T(\omega, ..., X, ...) = T\big(\omega_\mu\,\mathrm{d}x^\mu, ..., X^\nu \partial_\nu, ...\big) = \omega_\mu X^\nu ... T(\mathrm{d}x^\mu, ..., \partial_\nu, ...) =: \omega_\mu X^\nu ... T^{\mu ...}_{\nu ...} \tag{2.21}$$

So a tensor field is called smooth if the component functions $T^{\mu\cdots}_{\nu\cdots}$ are $\mathcal{C}^{\infty}(M)$

Another notation that will be useful is

$$T(p, \omega, ..., X, ...) = \omega_{\mu_1}(p)...X^{\nu_1}(p)...T^{\mu_1\cdots}_{\nu_1\cdots}(p) \tag{2.22}$$

2.5. Metrics

The last piece for constructing a spacetime is adding a notion of magnitude to our vectors and distance. This is constructed by adding a tensor field to the spacetime Manifold which we will call the metic.

The metric defines a dot product between vectors

$$X \cdot Y = g(X, Y) = X^{\mu}Y^{\nu}g_{\mu\nu} \tag{2.23}$$

also allowing to lower the indices of vectors and tensors by contracting with the metric

$$X_{\mu} = g_{\mu\nu}X^{\nu} \tag{2.24}$$

since we would like to be able to invert the relation it is defined $g^{\mu\nu}$ such that

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu} \tag{2.25}$$

so $X^{\mu} = g^{\mu\nu}X_{\nu}$

A Manifold equipped with a metric is called Riemannian if a metric can be diagonalized with all positive eigenvalues and pseudo-Riemannian if it can have both positive and negative.

In general relativity the equivalence principle can be stated in terms of the metric so that for any point there is a coordinate system such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2) \tag{2.26}$$

where η is the Minkowski metric.

This allows to define a distance[3] function between two points of the manifold by denoting $\Gamma(p, q)$ the set of all curves starting at p and ending at q

$$d(p, q) = \min_{\gamma \in \Gamma(p,q)} \int_{\gamma} (g(\gamma'(\tau), \gamma'(\tau)))\, \mathrm{d}\tau \tag{2.27}$$

where γ' is the tangent vector to γ

[3]This will only be a distance function whenever the manifold is Riemannian, for pseudo-Riemannian it is not since it can be negative, in general relativity the sign will be a way to encode when a path moves in the 'time direction', in the 'space direction' or like light.

3 Transformations of manifolds

In order to study the symmetries of our spacetime one has to first understand how to make a transformation.

First will start by defining a smooth transformation between manifolds M, N as a function $F : M \to N$ such that for any coordinate system of M, φ, that contains p, and any coordinate system of N, φ', that contains $F(p)$, the function $\varphi' \circ F \circ \varphi^{-1}$ is smooth.

A **diffeomorphism** is a smooth map that is also bijective and with a smooth inverse. Any pair of manifolds that have a diffeomorphism relating them will be called diffeomorphic manifolds.

Diffeomorphic manifolds are equivalent in the sense that any field, may it be scalar, vectorial or tensorial defined on one of the manifolds. Has an equivalent definition on the other. The operations that map a field on one of the manifolds to the other are called pullback and pushforward.

3.1. Pullback and pushforward of scalar fields

Given a function $F : M \to N$. A pullback will map fields defined on N to fields defined on M. The simplest case is for scalar fields. The pullback of a scalar field $f \in \mathcal{C}^\infty(N)$ is defined as

$$F^* f = f \circ F \tag{3.1}$$

so that F maps points of M to N and then f maps it to $\mathbb{R}$ so the complete map is $M \to \mathbb{R}$.

The pushforward is the opposite transformation to the pullback, mapping fields from M to N. In the case of diffeomorphisms it can be defined as the pullback by the inverse function. So if one has a function $f \in \mathcal{C}^\infty(M)$ the pushforward by F is defined as

$$F_* f = \left(F^{-1}\right)^* f = f \circ F^{-1} \tag{3.2}$$

The motivation behind this definition is that, if one pushes forward a function and then pulls it back, it would be reasonable for the function to remain unchanged therefore $F^* F_* f = f$

It is easy to see that the pullback and pushforward are linear since composition is linear so that

$$\begin{aligned} F_*(\alpha f + \beta g) &= \alpha F_* f + \beta F_* g \\ F^*(\alpha f + \beta g) &= \alpha F^* f + \beta F^* g \end{aligned} \tag{3.3}$$

Under this definitions diffeomorphisms may be thought rather than as mappings between manifolds, as coordinate changes, since for any coordinate system on N, x'^μ, it can be thought of as a coordinate system on M defined as $x^\mu = F^* x'^\mu$. This will come up later in the chapter in the notion of passive vs active transformations.

3.2. Pullback and pushforward of vector fields

Since a vector field was defined as a collection of derivations, it may be thought of as a function $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, that maps a function to the directional derivative of the function at that point.

The pushforward of vector fields may be thought of as first pulling back the vector field the corresponding function then pushing forward the result, for clarity $X(p, f)$ denotes evaluating $X(f)$ at p, $p \in M$, $\hat{P} \in N$ $g \in \mathcal{C}^\infty(N)$

$$(F_* X)(g) = X(F^{-1}(\hat{p}), F^* g) \tag{3.4}$$

The pullback is defined as the pushforward by the inverse function

$$F^* X = (F^{-1})_* X \tag{3.5}$$

Again pushforward and pullback of vector fields is linear

$$\begin{aligned} F_*(\alpha X + \beta Y)(f) &= (\alpha X + \beta Y)(f \circ F) = \alpha X(f \circ F) + \beta Y(f \circ F) \\ &= \alpha F_*(X)(f) + \beta F_*(Y)(f) \end{aligned} \tag{3.6}$$

Also by defining multiplication of vector fields and scalar fields fX as

$$(fX)(p, g) = f(p) X(p, g) \tag{3.7}$$

the pushforward of this composition is linear in the following sense

$$\begin{aligned} F_*(fX)(\hat{p}, g) &= (f, X)(F^{-1}(\hat{p}), F^* g) = f(F^{-1}(\hat{p})) X(F^{-1}(\hat{p}), F^* g) \\ &= (F_* f)(\hat{p}) \cdot (F_* X)(\hat{p}, g) \end{aligned} \tag{3.8}$$

where $\cdot$ denotes the product of real numbers

equivalently for the pullback

$$F^*(fX) = (F^* f)(F^* X) \tag{3.9}$$

This equations are coordinate independent, however for computations it is easier to obtain the transformations by coordinate systems, in order to obtain the coordinate transformation we will write the coordinate system of M as x^μ and the coordinate system of N obtained as the pushforward of x^μ, x'^μ

Now our vector field $X \in \mathfrak{X}(M)$ can be written

$$X = X^\mu(x^\mu) \frac{\partial}{\partial x^\mu} \tag{3.10}$$

by defining a vector field on N, X' as the pushforward of X

$$X' = F_* X = X'(x'^\mu) \frac{\partial}{\partial x'^\mu} \tag{3.11}$$

Since $x'^\mu = F(x^\mu)$ for a function in $f \in \mathcal{C}^\infty(N)$ and defining $p \in M, \hat{p} \in N$ so that $\hat{p} = F(p)$

$$\begin{aligned} F_* X(\hat{p}, f) &= F_* \left(X^\mu \frac{\partial}{\partial x^\mu} \right)(\hat{p}, f) = X^\mu(F^{-1}(\hat{p})) \frac{\partial f \circ F}{\partial x^\mu}(F^{-1}(\hat{p})) \\ &= X^\mu(p) \frac{\partial f \circ F}{\partial x^\mu}(p) = X^\mu(p) \frac{\partial f}{\partial x'^\nu}(\hat{p}) \frac{\partial x'^\nu}{\partial x^\mu}(p) \end{aligned} \tag{3.12}$$

Therefore by comparing the Eq. (3.11) and Eq. (3.12) the resulting transformation on a coordinate system is

$$X'^\mu(\hat{p}) = \frac{\partial x'^\mu}{\partial x^\nu}(F^{-1}(\hat{p})) X^\nu(F^{-1}(\hat{p})) \tag{3.13}$$

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This equation might seem purely mathematical but it explains the physical transformations that we will find. These have a translation component, encoded on the term of $X^\nu(F^{-1}(\hat{p}))$, that because of the F^{-1} term shifts the position of the $X^{\mu(p)}$ vector. The other component are rotations, or expansions, encoded on the $\frac{\partial x'^\mu}{\partial x^\nu}(F^{-1}(\hat{p}))$, this is because this term mixes the components and allows for changing the direction of the vector or length of the vector.

For the pullback the result is equivalent by changing $x'^\mu \to x^\mu$, $x^\mu \to x'^\mu$ $\hat{p} \to p$ and $F \to F^{-1}$

so

$$X'^\mu(p) = \frac{\partial x^\mu}{\partial x'^\nu}(F(p))X^\nu(F(p)) \tag{3.14}$$

3.3. Pullback and pushforward of covector fields

Just as we did with vector fields, covector fields map vector fields to scalar fields the definitions and results are equivalent so for a covector ω

$$F_*\omega(\hat{p}, X) = \omega(F^{-1}(\hat{p}), F^*X) \tag{3.15}$$

and for the pullback

$$F^*\omega(p, X) = \omega(F(p), F_*X) \tag{3.16}$$

Again these are linear over addition of covectors and products by real numbers, and by defining the product of covectors by

$$(f\omega)(p, X) = f(p)\omega(p, X) \tag{3.17}$$

the pushforward is 'linear' over these in the sense that

$$\begin{aligned} F_*(f\omega)(\hat{p}, X) &= (f\omega)(F^{-1}(\hat{p}), F^*X) = f(F^{-1}(\hat{p}))\omega(F^{-1}(\hat{p}), F^*X) \\ &= (F_*f)(\hat{p})(F_*\omega)(\hat{p}, X) \end{aligned} \tag{3.18}$$

and equivalently for the pullback

$$F^*(f\omega) = (F^*f)(F^*\omega) \tag{3.19}$$

When given a coordinate system for M and N, x^μ and x'^μ respectively, then the covector on M may be written as

$$\omega = \omega_\mu(p)\,\mathrm{d}x^\mu \tag{3.20}$$

and the pushforward

$$\omega' = F_*\omega = \omega_\mu(\hat{p})\,\mathrm{d}x'^\mu \tag{3.21}$$

so by applying the definition of pushforward of a covector field, Eq. (3.15), one finds by setting $X' \in \mathfrak{X}(N)$

$$\begin{aligned} F_*\omega(\hat{p}, X') &= F_*(\omega_\mu\,\mathrm{d}x^\mu)(\hat{p}, X') = F_*(\omega_\mu)(\hat{p})F_*(\mathrm{d}x^\mu)(\hat{p}, X') = \omega_\mu(p)\,\mathrm{d}x^\mu(p, F^*X') \\ &= \omega_\mu(p)\,\mathrm{d}x^\mu\left(p, X'^\nu(\hat{p})\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\right) = \omega_\mu(p)X'^\nu(\hat{p})\frac{\partial x^\alpha}{\partial x'^\nu}\underbrace{\mathrm{d}x^\mu\left(p, \frac{\partial}{\partial x^\alpha}\right)}_{\delta^\mu_\alpha} \\ &= \omega_\mu(p)X'^\nu(\hat{p})\frac{\partial x^\alpha}{\partial x'^\nu}(\hat{p})\delta^\mu_\alpha = \frac{\partial x^\mu}{\partial x'^\nu}(\hat{p})\omega_\mu(p)X'^\nu(\hat{p}) \end{aligned} \tag{3.22}$$

now, by feeding the same input to the Eq. (3.21) one obtains

$$\begin{aligned} \omega'(\hat{p}, X') = \omega'_\mu(\hat{p})\, \mathrm{d}x'^\mu(X') = \omega'_\mu(\hat{p}) X'^\nu(\hat{p})\, \mathrm{d}x'^\mu\left(\frac{\partial}{\partial x'^\nu}\right) \\ = \omega'_\mu(\hat{p}) X'^\nu(\hat{p}) \delta^\mu_\nu = \omega'_\mu(\hat{p}) X'^\nu(\hat{p}) \end{aligned} \tag{3.23}$$

by comparing Eq. (3.22) and Eq. (3.23)

one obtains

$$\omega'_\mu(\hat{p}) = \frac{\partial x^\nu}{\partial x'^\mu}(\hat{p}) \omega_\nu(F^{-1}(\hat{p})) \tag{3.24}$$

The equivalent reasoning for the pullback gives

$$\omega_\mu(p) = \frac{\partial x'^\nu}{\partial x^\mu}(p) \omega'_\nu(F(p)) \tag{3.25}$$

Again here one can identify a translation and a rotation or expansion term, however here the rotation is inverted.

3.4. Tensor pullbacks and pushforwards

The pushforward of a tensor field, just as we did before with vectors and covectors is defined by pulling back the vector and covector fields and then pushing forward the results

$$F_* T(p, \omega, ..., X, ...) = T(F^{-1}(p), F^* \omega, ..., F^* X, ...) \tag{3.26}$$

and equivalently for the pullback

$$F^* T(p, \omega, ..., X, ...) = T(F(p), F_* \omega, ..., F_* X, ...) \tag{3.27}$$

Just as proven with the method in the Eq. (3.22) it can be proben that if T is a tensor in M and T' is the pushforward on N, and by choosing a coordinate system x^μ on M and the pushforward of this coordinate system to N, x'^μ, one obtains the relationship between the coordinate systems of both as

$$T'^{\nu_1 \cdots}_{\mu_1 \cdots}(\hat{p}) = \frac{\partial x'^{\nu_1}}{\partial x^{\alpha_1}}(F^{-1}(\hat{p}))...\frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}}(\hat{p}) T^{\alpha_1 \cdots}_{\beta_1 \cdots}(F^{-1}(\hat{p})) \tag{3.28}$$

3.5. Isometries

An isometry, is a diffeomorphism between Riemannian or pseudo-Riemannian manifolds, $F : M \to N$, where g_M is the metric on M and g_N is the metric on N then F is an isometry if

$$g_N = F_* g_M \tag{3.29}$$

thus preserving the metric.

Any object that only depends on the metric is called **intrinsic** and is preserved under isometries in the same sense that the metric is preserved.

A few examples are:
- The Levi-Civita connection (∇_μ)
- The Riemann tensor (${R^\mu}_{\nu\gamma\sigma}$)
- The length of a curve ($\int_\gamma \sqrt{g_{\mu\nu} \gamma'^\mu \gamma'^\nu}\, \mathrm{d}\tau$)

4 Flows

A flow, intuitively, is described as the movement of a liquid or a gas that at each point moves in one particular direction.

Mathematically this can be described by a velocity field, that describes the movement of the fluid.

This might not seem relevant to the study of transformations in general relativity, however this concept is the definition we will use to build all of the transformations.

First we will start by defining a flow as a curve that solves the following differential equation

$$\begin{cases} \frac{\partial \phi}{\partial \tau}(\tau, x_0) = V(\phi(\tau, x_0)) \\ \phi(0, x_0) = x_0 \end{cases} \tag{4.1}$$

where V is the velocity field and ϕ is a curve on the manifold. There are a few interesting properties of flows that will be important later.

First of all, since V is a smooth vector field, the solutions to $\phi(\tau, x_0)$ are unique, this also means that for any fixed τ the transformation $\phi_\tau : M \to M$ defined as $\phi_\tau(p) = \phi(\tau, p)$ is a diffeomorphism since the solutions are unique and since the function is differentiable with respect to τ it has to be smooth.

Another property that flow has is that these are defined except by a constant translation on the parameter τ. Meaning if $\phi(\tau, x_0)$ is a flow of a field V then $\phi(\tau + s)$ is also a flow of the field V. Unless stated otherwise the convention we will take is such that

$$\phi(0, x_0) = x_0 \tag{4.2}$$

By uniqueness of the solutions by defining $\phi(\tau', x_0) = \phi(\tau, \phi(s, x_0))$ and imposing that

TODO: prove this

$$\phi(\tau + s, x_0) = \phi(\tau, \phi(s, x_0)) \tag{4.3}$$

and thus

$$\phi(-\tau, \phi(\tau, x_0)) = x_0 \Rightarrow \phi_\tau^{-1}(p) = \phi_{-\tau}(p) \tag{4.4}$$

5 Lie derivatives

Finally after all of the mathematical conundrum we are finally ready to define the Lie derivative. The Lie derivative is an object that takes in a vector field V and some geometrical object, such as scalar fields, vector fields or tensor fields. And what the Lie derivative represents is, if someone following the flow of V, made a function of how they see these objects change as a function of τ and then took a derivative at $\tau = 0$, what would be the value of that derivative.

This intuitive image of what a Lie derivative wants to answer can be stated in two equivalent ways, and active way and a passive way. The active way models the path of the observer by keeping coordinates stationary and transforming the fields so that $\varphi \to \varphi'$ and the Lie derivative would be something like $(\varphi' - \varphi)/\varepsilon$. The passive approach transforms the coordinates so that the fields stay still and the observer is the one moving from this coordinate system. In this case $x^\mu \to x'^\mu$ and the Lie derivative would take the form of $(\varphi(x'^\mu) - \varphi(x^\mu))/\varepsilon$.

Here we will take the active approach and model the derivative by transforming the fields.

First we will motivate the equation with an example and then will start computing the Lie derivative in component form of multiple kinds of objects.

To obtain a Lie derivative of some field φ with respect to some vector field X, first of all the flow of X is computed obtaining $\phi_\tau(p)$. Now the manifold M is mapped by ϕ_τ to M' which is nothing else than the same manifold but with a different coordinate system. Here the interpretation is not that the coordinate system changed, when we interpret is that the coordinate systems of M and M' are the same but the fields changed. Now φ' is a field on M' so what we can do to compare it with φ is to pull it back to M and by taking the limit as $\tau \to 0$ one obtains the Lie derivative

$$\mathscr{L}_X \varphi = \lim_{\tau\to 0} \frac{\phi_\tau^* \varphi - \varphi}{\tau} = \lim_{\tau\to 0} \frac{\phi_{-\tau *} \varphi - \varphi}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} \phi_\tau^* \varphi \tag{5.1}$$

Now it is possible to compute Lie derivatives of several objects in a coordinate system.

By defining x^μ as a coordinate system on M and $x'^\mu = \phi_\tau(x^\mu)$ then it follows that $x'^\mu = x^\mu + V^\mu \tau + \mathcal{O}(\tau^2)$.

5.1. Lie derivative of a scalar field

The Lie derivative of a scalar field is the simplest Lie derivative to compute. This is because by Eq. (3.1) the Lie derivative may be computed as

$$\begin{aligned}\mathscr{L}_X f &= \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_\tau^* f\bigg|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau} f(\phi(\tau, x_0))\bigg|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau} f(x'^\mu)\bigg|_{\tau=0} = \frac{\partial f}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial \tau}\bigg|_{\tau=0} \\ &= \frac{\partial f}{\partial x^\nu}\big(\delta^\nu_\mu + \mathcal{O}(\tau)\big)\big(V^\mu + \mathcal{O}(\tau)\big)\bigg|_{\tau=0} = V^\mu \frac{\partial f}{\partial x^\mu}\end{aligned} \tag{5.2}$$

Here one can see that the Lie derivative of a scalar field with respect to some vector field is nothing else than the directional derivative of f in the direction of V. In general this is not the case, the reason for this is that scalar fields have no sense of 'direction' so they are not affected by rotations, as we will see now this is not the case for vector fields

5.2. Lie derivatives of vector fields

The same procedure as done in Eq. (5.2) to find the Lie derivative of a vector field. However here the corresponding pullback equation is Eq. (3.14)

$$\begin{aligned}(\mathscr{L}_X Y)^\mu &= \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^*_\tau Y\bigg|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x^\mu}{\partial x'^\nu}Y^\nu(x')\right)\bigg|_{\tau=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)Y^\nu(x')\bigg|_{\tau=0} + \frac{\partial x^\mu}{\partial x'^\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}Y^{\nu(x')}\bigg|_{\tau=0} \\ &= \left(-\frac{\partial X^\mu}{\partial x'^\nu} + \mathcal{O}(\tau)\right)Y^\nu(x')\bigg|_{\tau=0} + (\delta^\mu_\nu + \mathcal{O}(\tau))\frac{\mathrm{d}}{\mathrm{d}\tau}Y^\nu(x')\bigg|_{\tau=0} \\ &= -Y^\nu(x)\frac{\partial X^\mu}{\partial x^\nu} + (\delta^\mu_\nu + \mathcal{O}(\tau))\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau}\frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial Y^\nu}{\partial x^\beta}\bigg|_{\tau=0} \\ &= -Y^\nu(x)\frac{\partial X^\mu}{\partial x^\nu} + (\delta^\mu_\nu + \mathcal{O}(\tau))X^\alpha\frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial Y^\nu}{\partial x^\beta}\bigg|_{\tau=0} \\ &= -Y^\nu(x)\frac{\partial X^\mu}{\partial x^\nu} + X^\alpha\frac{\partial Y^\mu}{\partial x^\alpha}\end{aligned} \tag{5.3}$$

Here there are two terms, the first one is associated with translations. This term is the $X^\alpha \frac{\partial Y^\mu}{\partial x^\alpha}$. This is essentially what was obtained in the scalar field case and represents a directional derivative. The second term, $-Y^\alpha \frac{\partial X^\mu}{\partial x^\alpha}$, represents rotations and dilations. This is because it 'mixes' components of the vector Y and allows for changing the norm of the vector.

This means that for any two vector fields X, Y their Lie derivative applied to a scalar field is

$$\mathscr{L}_X Y(f) = X(Y(f)) - Y(X(f)) = [X, Y](f) \tag{5.4}$$

5.3. Lie derivatives of tensor fields

Once again, by using these methods one finds the general equation

$$\begin{aligned}(\mathscr{L}_X T)^{\alpha_1\cdots}{}_{\beta_1\cdots} = X^\alpha &\frac{\partial T^{\alpha_1\cdots}{}_{\beta_1\cdots}}{\partial x^\alpha} \\ &+T^{\alpha_1\cdots\alpha_{i-1}\sigma\alpha_{i+1}\cdots}{}_{\beta_1\cdots}\frac{\partial X^\alpha_i}{\partial x^\sigma} + \ldots \\ &-T^{\alpha_1\cdots}{}_{\beta_1\ldots\beta_{i-1}\sigma\beta_{i+1}\cdots}\frac{\partial X^\sigma}{\partial x^{\beta_i}} - \ldots\end{aligned} \tag{5.5}$$

This equation also can be written in the following form

$$(\mathscr{L}_X T)(A, B, \ldots) = \mathscr{L}_X(T(A, B, \ldots)) + T(\mathscr{L}_X A, B, \ldots) + T(A, \mathscr{L}_X B, \ldots) + \ldots \tag{5.6}$$

6 Killing fields

Finally we can write the definition of Killing fields. A Killing field, K, is a vector field such that the flow it generates is an isometry. In physical terms, moving along the 'velocity field' K doesn't change the metric tensor

$$\phi_\tau^* g = g \tag{6.1}$$

that can be written as

$$(\mathscr{L}_K g)_{\mu\nu} = 0 \tag{6.2}$$

In component form the Killing equation may be written in contravariant form as

$$K^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\alpha\nu} \frac{\partial K^\alpha}{\partial x^\mu} + g_{\mu\alpha} \frac{\partial K^\alpha}{\partial x^\nu} = 0 \tag{6.3}$$

Additionally, whenever a symmetric connection is used, such as the Levi-Civita connection used in general relativity, the following covariant from is equivalent

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{6.4}$$

6.1. Properties of Killing fields

Killing fields form vector spaces, this is because the Lie derivative is linear on the vector field wrt which it differentiates

$$\mathscr{L}_{\alpha X + \beta Y} g = \alpha\, \mathscr{L}_X g + \beta\, \mathscr{L}_Y g = 0 \tag{6.5}$$

This can easily be seen on the component definitions of Lie derivatives.

Additionally these form a Lie algebra, this is because Lie derivatives have the following property

$$\mathscr{L}_{[X,Y]} T = \mathscr{L}_X\, \mathscr{L}_Y T - \mathscr{L}_Y\, \mathscr{L}_X T \tag{6.6}$$

So that if X and Y are Killing fields, then $[X, Y]$ form a Killing field too.

The norm of a Killing vector field is constant along its own flow, this is easily probed by

$$\mathscr{L}_K K^2 = K^\nu \nabla_\nu (K^\mu K_\mu) = 2 K^\nu K^\mu \nabla_\nu K_\mu = K^\nu K^\mu \underbrace{(\nabla_\nu K_\mu + \nabla_\mu K_\nu)}_{\text{Killing equation}} = 0 \tag{6.7}$$

These are also divergenceless tensors. Intuitively one can think of Killing fields as flows of an incompresible fluid. One can prove this by multiplying Eq. (6.4) by the metric so that

$$0 = g^{\mu\nu} \nabla_\mu K_\nu + g^{\mu\nu} \nabla_\mu K_\nu = \nabla_\mu K^\mu + \nabla_\nu K^\nu = 2 \nabla_\mu K^\mu \Rightarrow \nabla_\mu K^\mu = 0 \tag{6.8}$$

The main property of Killing fields is that it allows us to build conserved quantities on geodesics, meaning, if $x^\mu(\tau)$ is a geodesic and K^μ a Killing field, then

$$\frac{\mathrm{d}(\dot{x}^\mu K_\mu)}{\mathrm{d}\tau} = \underbrace{\frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\tau}}_{\text{zero by geodesic}} K_\mu + \dot{x}^\mu \frac{\mathrm{d}K_\mu}{\mathrm{d}\tau} = \dot{x}^\mu \dot{x}^\nu \nabla_\nu K_\mu = \frac{1}{2}\dot{x}^\mu \dot{x}^\nu \underbrace{(\nabla_\nu K_\mu + \nabla_\mu K_\nu)}_{\text{Killing equation}} = 0 \tag{6.9}$$

It also allows for defining conserved currents for any divergenceless rank two symmetric tensor because

$$\nabla_\mu (T^{\mu\nu} K_\nu) = \cancel{\nabla_\mu T^{\mu\nu} K_\nu} + T^{\mu\nu} \nabla_\mu K_\nu = \frac{1}{2} T^{\mu\nu} (\nabla_\nu K_\mu + \nabla_\mu K_\nu) = 0 \tag{6.10}$$

as an example of such a tensor the Stress-energy tensor, allowing to define energy and momentum densities in curved spacetimes.

Number of Killing fields

A good question now is to ask "How many Killing fields does our space have?", since this will lead to n conserved quantities, simplifying the resulting equations.

As it turns out it is not possible, in general, to know exactly to know how many Killing fields there are without solving the equations, however, it is possible to place an upper bound on the number of Killing fields, and giving an interpretation of these.

To prove this the starting point is the Riemann tensor

$$R^\delta{}_{\alpha\beta\gamma} K_\delta = \nabla_\alpha \nabla_\beta K_\gamma - \nabla_\beta \nabla_\alpha K_\gamma \tag{6.11}$$

and the Bianchi identities

$$R^\delta{}_{\alpha\beta\gamma} + R^\delta{}_{\gamma\alpha\beta} + R^\delta{}_{\beta\gamma\alpha} = 0 \tag{6.12}$$

By multiplying the Bianchi identities by K_δ and applying Eq. (6.11) the result is

$$\begin{gathered}
0 = \nabla_\alpha \nabla_\beta K_\gamma - \nabla_\beta \nabla_\alpha K_\gamma + \nabla_\gamma \nabla_\alpha K_\beta - \nabla_\alpha \nabla_\gamma K_\beta + \nabla_\beta \nabla_\gamma K_\alpha - \nabla_\gamma \nabla_\beta K_\alpha = \\
\nabla_\alpha \nabla_\beta K_\gamma - \nabla_\alpha \nabla_\gamma K_\beta + \nabla_\beta \nabla_\gamma K_\alpha - \nabla_\beta \nabla_\alpha K_\gamma + \nabla_\gamma \nabla_\alpha K_\beta - \nabla_\gamma \nabla_\beta K_\alpha = \\
\nabla_\alpha (\nabla_\beta K_\gamma - \nabla_\gamma K_\beta) + \nabla_\beta (\nabla_\gamma K_\alpha - \nabla_\alpha K_\gamma) + \nabla_\gamma (\nabla_\alpha K_\beta - \nabla_\beta K_\alpha) \underset{\text{Killing equation}}{=} \\
2(\nabla_\alpha \nabla_\beta K_\gamma - \nabla_\gamma \nabla_\beta K_\alpha + \nabla_\gamma \nabla_\alpha K_\beta) = 2\left(R^\delta{}_{\alpha\beta\gamma} K_\delta + \nabla_\gamma \nabla_\alpha K_\beta\right) \Rightarrow \\
\Rightarrow R^\delta{}_{\alpha\beta\gamma} K_\delta = -\nabla_\gamma \nabla_\alpha K_\beta
\end{gathered} \tag{6.13}$$

This allows, by substituting into the Taylor series to obtain an expression of the solution to the Killing field equation.