SC42025 FILTERING AND IDENTIFICATION

TURBULENCE MODELING FOR ADAPTIVE OPTICS

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Contents

INTRODUCTION

This assignment deals with modeling an *Adaptive Optics* (AO) system in which three different data-driven turbulence modeling methods are used to achieve optimal control performances, viz.

- a random-walk process
- a Vector-Auto-Regressive model
- a stochastic state-space model

Each model has some questions associated with it, and we solve them in chronological sequence taking one model at a time.

1. RANDOM WALK MODEL

Question 1

We know from the assignment's equation (2) that:

$$s_o(k) = G\phi(k) + e(k)$$

We have the values of the wavefront sensor data in open-loop, $s_o(k)$, and also the value of the matrix G. To compute the value of $\hat{\phi}(k)$, given no prior information on it, we follow the linear least-squares approach:

First, we determine whether the matrix G is full-rank or not. We load the *systemMatrices.mat* file which contains the matrix G, and then run the *rank* command in MATLAB to get a rank value of **47**, which is less than *min(number of rows, number of columns)* of G. Thus, we need to employ a linear least-squares method that doesn't assume the matrix G to be full-rank.

We know that there are multiple solutions to this problem, and for uniqueness we go with one such that the optimal solution, $\hat{\phi}(k)$, has a minimal 2-norm, thus leading to the original linear least-squares problem being reformulated as:

$\min_{\phi(k)\in\Gamma} \|\phi(k)\|_2^2$ with $\Gamma = \left\{\phi(k) : \phi(k) = \arg\min_z \|Gz - s_o(k)\|_2^2\right\}$

By performing an SVD operation on the matrix G, we obtain:

$$G = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T$$

Here, $\Sigma \in \Re^{47x47}$ is non-singular, by the definition of SVD.

Now, let us define a partitioned vector,

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} z$$

Thus, our problem becomes:

$\underset{\xi_1}{min} \| U_1 \Sigma \xi_1 z - s_o(k) \|_2^2$

We get $\hat{\xi}_1 = \Sigma^{-1} U_1^T s_o(k)$, since Σ is a non-singular matrix. ξ_2 has no effect on the above expression and can be chosen arbitrarily. Thus, we get the optimal solution,

$$\hat{z} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} = V_1 \Sigma^{-1} U_1^T s_o(k) + V_2 \hat{\xi}_2$$

Since we're choosing a vector $\phi(k)$ with the smallest 2-norm, and $V_1^T V_2 = 0$ we get:

$\|\phi(k)\|_2^2 = \|V_1 \Sigma^{-1} U_1^T s_o(k)\|_2^2 + \|V_2 \hat{\xi}_2\|_2^2$

As we're minimizing the norm, we take $\hat{\xi}_2 = 0$ and we finally get the value of $\hat{\phi}(k)$ to be:

$$\hat{\phi}(\mathbf{k}) = \mathbf{V_1} \Sigma^{-\mathbf{1}} \mathbf{U_1^T} \mathbf{s_o}(\mathbf{k})$$

Question 2

We are provided with some prior information about the wavefront, viz.:

- $\mathrm{E}[\phi(\mathrm{k})] = 0$
- $\mathrm{E}[\phi(k)\phi(k)^T] = C_\phi(0)$
- noise variance $= \sigma_e^2$

Based on equation (8) from the assignment, we approximate the value of $C_\phi(0)$ as:

$$C_\phi(0) = \frac{1}{N} \sum_{i=1}^{N} \phi(i)\phi(i)^T \tag{1}$$

We have the data necessary to formulate our problem of determining $\phi(k)$ as a stochastic linear least-squares problem, and hence we define our linear estimator $\tilde{\phi}(k)$ accordingly:

$$\tilde{\phi}(k) = \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} s_o(k) \\ E[\phi(k)] \end{bmatrix}$$

such that $\mathrm{E}\left[(\tilde{\phi}(k)-\phi(k))(\tilde{\phi}(k)-\phi(k))^T\right]$ is minimized and $\mathrm{E}[\tilde{\phi}(k)] = \mathrm{E}[\phi(k)] = 0$

Thus, from the assignment's equation (2), based on $s_o(k)$'s expression, we can say,

$$\tilde{\phi}(k) = MG\phi(k) + Me(k) + NE\left[\phi(k)\right]$$

Since $\mathrm{E}[\tilde{\phi}(k)] = \mathrm{E}[\phi(k)] = 0$, we have:

$$\tilde{\phi}(k) = MG\phi(k) + Me(k)$$

Furthermore,

$$\phi(k) - \tilde{\phi}(k) = (I - MG)\phi(k) - Me(k)$$

Thus, the covariance of the above expression is computed as:

$$E\left[\left(\phi(k)-\tilde{\phi}(k)\right)\left(\phi(k)-\tilde{\phi}(k)\right)^T\right] = E\left[\left((I-MG)\phi(k)-Me(k)\right)\left((I-MG)\phi(k)-Me(k)\right)^T\right]$$

On expanding the right side of the equation and taking the error e(k) to be uncorrelated with the wavefront vector $\phi(k)$, and based on the statistical data provided to us, we get the following expression:

$$E\left[\left(\phi(k)-\tilde{\phi}(k)\right)\left(\phi(k)-\tilde{\phi}(k)\right)^T\right] = (I-MG)C_\phi(0)(I-MG)^T + M\sigma_e^2 IM^T$$

On further factorization, using the Schur complement of the factorized version, and the application of the "completion of squares" argument to the resulting equation, we get the optimum value of the matrix M that minimizes the covariance expression as:

$$M = C_\phi(0)G^T(GC_\phi(0)G^T + \sigma^2 I)^{-1}$$

(A point to note here: the term in brackets is invertible since $C_\phi(0)$ is a positive definite matrix and $\sigma^2 I$ is non-singular)

Accordingly, we also get the optimum estimate of the wavefront vector,

$$\hat{\phi}(k|k) = C_\phi(0)G^T(GC_\phi(0)G^T + \sigma_e^2 I)^{-1}s_o(k)$$

For questions 3 to 5, we assume $E[\epsilon(k)] = 0$ *and* $E[\epsilon(k)\epsilon(k)^T] = C_\phi(0)$

Question 3

Now, we consider the closed-loop system, and proceed to derive a UMVE of $\epsilon(k)$ using the given measurement set s(k). As in the previous question, we are provided with some prior information about the wavefront, viz.:

- $\mathrm{E}[\epsilon(k)] = 0$
- $\mathrm{E}\left[\epsilon(k)\epsilon(k)^T\right] = C_\phi(0)$
- noise variance = σ_e^2

We can clearly see that the equations (2) and (5) in the given assignment are similar, and we are told that the statistical data (the wavefront's and noise's mean and covariance values) are the same. Hence, as in the previous question, the optimum estimate of the wavefront vector is quite similar in the closed-loop system, and the only difference is in the value of the output vector, which in this case will be the closed-loop slope vector s(k):

$$\hat{\epsilon}(k|k) = C_\phi(0)G^T(GC_\phi(0)G^T + \sigma_e^2 I)^{-1}s(k)$$

Question 4

In this question, we make use of the random walk model represented by equation (9) in the assignment.

We know that:

$$\begin{aligned} &\epsilon(k) = \phi(k) - Hu(k-1) \\ \implies &\phi(k) = \epsilon(k) + Hu(k-1) \\ \implies &\phi(k+1) = \epsilon(k+1) + Hu(k) \end{aligned}$$

From the random walk model, we can relate $\phi(k)$ and $\phi(k+1)$, and substituting the above expressions respectively yields:

$$\epsilon(k+1) + Hu(k) = \epsilon(k) + Hu(k-1) + \eta(k)$$

If we consider $\hat{\epsilon}(k|k)$ to be the current optimal estimate, we know that the optimal one-step ahead prediction should not be stochastic in nature, and must be estimated based on the current optimal estimate. Based on the above equation, we can thus say,

$$\hat{\epsilon}(k+1|k) = \hat{\epsilon}(k|k) + Hu(k-1) - Hu(k) \tag{2}$$

Question 5

We denote $\delta u(k) := u(k) - u(k-1)$. We know from the previous question's equation 2 that:

$$\hat{\epsilon}(k+1|k) = \hat{\epsilon}(k|k) - H\delta u(k)$$

Thus, the minimization problem as described in the assignment's equation (6) can be reformulated as:

$$\underset{\delta u(k)}{min} \|\hat{\epsilon}(k|k) - H\delta u(k)\|_2^2$$

This is clearly a linear least-squares problem, and we know from running the *rank* command on the matrix H that it is full-rank. Hence, we can say that the optimal increment, $\hat{\delta} u(k)$ for minimizing the 2-norm is:

$$\hat{\delta} u(k) = (H^T H)^{-1} H^T \hat{\epsilon}(k|k)$$

We computed the expression for $\hat{\epsilon}(k|k)$ in question 3, and substituting the expression in the above equation yields the value of $\hat{\delta} u(k)$ as:

$$\hat{\delta} u(k) = (H^T H)^{-1} H^T C_\phi(0) G^T (G C_\phi(0) G^T + \sigma_e^2 I)^{-1} s(k)$$

Moreover, since H is invertible, we can further simplify the expression as:

$$\hat{\delta} u(k) = H^{-1} C_\phi(0) G^T (G C_\phi(0) G^T + \sigma_e^2 I)^{-1} s(k)$$

Question 6

2. VAR Model

We are provided with a VAR model represented by:

$$\phi(k+1) = A\phi(k) + w(k) \tag{3}$$

We have the following information provided to us to demonstrate that w(k) is uncorrelated with the measurement noise and the wavefront:

- $E\left[w(k)e(k)^T\right] = 0$
- $E\left[w(k)\phi(k)^T\right] = 0$

We are also provided with statistical information about w(k): w(k) $\sim \mathcal{N}(0, C_w)$

Question 1

We have the following data:

- $C_\phi(0) = E\left[\phi(k)\phi(k)^T\right]$
- $C_\phi(1) = E\left[\phi(k+1)\phi(k)^T\right]$

Multiplying equation 3 with $\phi(k)^T$ on both sides yields:

$$\phi(k+1)\phi(k)^T = A\phi(k)\phi(k)^T + w(k)\phi(k)^T$$

Further, on taking the expectation:

$$E\left[\phi(k+1)\phi(k)^T\right] = E\left[A\phi(k)\phi(k)^T\right] + E\left[w(k)\phi(k)^T\right]$$

Thus, we get the relation:

$$C_\phi(1) = AC_\phi(0)$$

and hence we calculate:

$$A = C_\phi(1)/C_\phi(0)$$

Question 2

We make an assumption here that $\phi(k)$ is a WSS signal.

Multiplying equation 3 with $w(k)^T$ on both sides, and then taking the resulting expectation yields:

$$E\left[\phi(k+1)\,w(k)^T\right] = E\left[A\phi(k)\,w(k)^T\right] + E\left[w(k)\,w(k)^T\right]$$

Based on the data we have, we get the following relation based on the above equation:

$$E\left[\phi(k+1)\,w(k)^T\right] = C_w$$

Taking the transpose of equation 3, and multiplying with $\phi(k+1)$ on both sides, we get:

$$E\left[\phi(k+1)\phi(k+1)^T\right] = E\left[A\phi(k+1)\phi(k)^T\right] + E\left[\phi(k+1)\,w(k)^T\right]$$

Since we have assumed $\phi(k)$ to be WSS, we can say that:
$E\left[\phi(k+1)\phi(k+1)^T\right] = E\left[\phi(k)\phi(k)^T\right] = C_\phi(0)$

And we know from the previous question that $C_\phi(1) = AC_\phi(0)$
Thus,

$$C_\phi(0) = AC_\phi(1) + C_w$$
$$\implies C_\phi(0) = A^2 C_\phi(0) + C_w$$

So we derive the following relationship:

$$C_w = (I - A^2)C_\phi(0)$$

Question 3

We need to formulate a state-space model with $\epsilon(k)$ being the state vector and s(k) being the output.

Based on equation (4) from the assignment, we can write:

$$\phi(k) = \epsilon(k) + Hu(k-1)$$
$$\implies \phi(k+1) = \epsilon(k+1) + Hu(k)$$

Substituting this expression for $\phi(k)$ in equation (10), we get:

$$\epsilon(k+1) + Hu(k) = A\epsilon(k) + AHu(k-1) + w(k)$$
$$\implies \epsilon(k+1) = \epsilon(k) + AHu(k-1) - Hu(k) + w(k)$$

Combining the above relation with equation (5) from the assignment, we have the following state-space model as required:

$$\epsilon(k+1) = \epsilon(k) + \Gamma(k) + w(k)$$
$$s(k) = G\epsilon(k) + e(k)$$
$$where$$
$$\Gamma(k) = AHu(k-1) - Hu(k)$$

Question 4

Use an expression similar to Kalman Gain to get ideal predictor.

Question 5

Find Optimal Increment.

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