Finite-Agent Stochastic Differential Games on Large Graphs: I. The Linear-Quadratic Case

Ruimeng Hu* Jihao Long † Haosheng Zhou ‡ June 17, 2024

Abstract

In this paper, we study finite-agent linear-quadratic games on graphs. Specifically, we propose a comprehensive framework that extends the existing literature by incorporating heterogeneous and interpretable player interactions. Compared to previous works, our model offers a more realistic depiction of strategic decision-making processes. For general graphs, we establish the convergence of fictitious play, a widely-used iterative solution method for determining the Nash equilibrium of our proposed game model. Notably, under appropriate conditions, this convergence holds true irrespective of the number of players involved. For vertex-transitive graphs, we develop a semi-explicit characterization of the Nash equilibrium. Through rigorous analysis, we demonstrate the well-posedness of this characterization under certain conditions. We present numerical experiments that validate our theoretical results and provide insights into the intricate relationship between various game dynamics and the underlying graph structure.

Key words: Stochastic differential game on graphs, Nash equilibrium, convergence, fictitious play, vertex-transitive graph

1 Introduction

Game theory, established in the 1940s [VNM07], has become a premier method for modeling human interactions. Similarly, graph theory, which addresses pairwise relationships between objects, has gained significant interest since Euler's solution to the "seven bridges of Königsberg" problem. Graphs, representing various types of information, have been extensively used in modeling biological, informational, and social systems [MV07, RA11, MR20]. Thus, exploring the interdisciplinary synthesis of game theory and graph theory is crucial. Existing literature shows the efficacy of combining games and graphs in modeling significant phenomena, such as network security [MPPS05], vehicle-to-vehicle communication [HCH+20], and advertising dynamics [HKSE18]. We focus on stochastic differential games with graph structures, governed by systems of stochastic differential equations [Fri72, Car16], which model continuous-time games with continuous state and action spaces. Incorporating graph structures captures direct and indirect player interactions, introducing heterogeneity. These games have garnered substantial attention for understanding behaviors within economic and financial frameworks, including interbank lending [CSY20], systemic risk [CFS15], default contagion [FS19], and credit networks [NS20].

^{*}Department of Mathematics, and Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106-3080, rhu@ucsb.edu.

[†]Institute for Advanced Algorithms Research, Shanghai, China, longih1998@qmail.com.

[‡]Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106-3110, hzhou593@ucsb.edu.

In the literature, there are two major types of stochastic differential games: cooperative games, where players work towards a collective goal, and non-cooperative games, where players aim to maximize their own advantages. In the context of non-cooperative game theory, which we focus on in this paper, one key solution concept is the Nash equilibrium (NE) [Nas50, Nas51], where no player can benefit by unilaterally changing their own strategy. Due to its significance, the analytic and numerical solutions of the NE have been core topics in non-cooperative game theory.

Regarding the analytic solution for the NE, linear-quadratic games represent the majority of game models that have a closed-form solution for the NE. The linear-quadratic flocking model [CD18], the systemic risk model [CFS15], and other models presented in [BSYY16, LS22, GCH23] are notable examples. Despite their simple structures, linear-quadratic games are powerful enough to explain real-world phenomena and have been adopted in various fields [Yon02, BSYY16, ZSZ20, DIP23]. For a comprehensive discussion on linear-quadratic games, we refer readers to [SY20]. For a family of important examples of linear-quadratic games with mean-field interactions, see [CD18].

For a linear-quadratic game, numerically solving for the NE can always be transformed into numerically solving the associated coupled Riccati equations. This task is relatively straightforward due to the existence of numerous efficient numerical schemes for solving ODEs. However, solving for the NE of a general stochastic differential game is challenging, especially when the game is fully nonlinear (with controlled diffusion terms), involves a relatively large number of players, and/or has high-dimensional states. This complexity gives rise to fictitious play (FP) as an efficient method for numerically solving for the NE of an non-cooperative game in an iterative way, initially introduced by Brown [Bro49, Bro51]. The main idea behind FP is to recast the problem into a sequence of stochastic control problems faced by each player to be solved repeatedly. However, the convergence of FP to the NE is not guaranteed, even for tabular games [Sha63], which motivates research on the convergence of FP. In the context of continuous-time differential games, affirmative results include [CH17, BC18] for mean-field games and [Hu21, HHL22] for finite-player games.

Moreover, we emphasize that this paper focuses on directly tackling finite-agent games rather than using mean-field game approximations. Mean-field games (MFGs), first proposed by Lasry and Lions [LL07], provide asymptotic approximations to finite-agent games with mean-field interactions. MFGs have been extensively used to model human behaviors in various fields, including Aiyagari's growth model [CD18] in economics, the systemic risk model [CFS15] in finance, and the cryptocurrency mining model [LRS24] in blockchain technology, among others. Our choice of directly addressing the finite population problem is due to several reasons. Despite the usefulness of mean-field games, their approximations to the finite population counterpart rely on the assumption of a large number of identical players, which does not always hold. The recently developed graphon MFGs [CH19, CH21, GCH23] address this issue by modeling the heterogeneous interactions through graphons, which are the limits of dense graphs. However, sparse social networks are also of significant research interest. In contrast, finite-player games can model heterogeneous interactions, sparse or dense, directly and can accommodate a relatively smaller number of players, which MFGs cannot. For further research on learning graphon mean-field games, we refer readers to [CK21, FCK23].

Main contribution. Our main contributions in this paper are as follows: (i) We propose a new finite-agent linear-quadratic game on graphs, aiming to provide a more realistic modeling of players with heterogeneous and interpretable interactions. This substantially extends the scope of existing research, such as [CFS15], which primarily focuses games on equally-weighted complete graphs, and [LS22] by incorporating information flow from neighboring players. (ii) With arbitrary graphs, we establish the convergence of fictitious play for solving the Markovian NE of our model under a smallness condition, regardless of the number of players. This improves the previous analysis

[Hu21], which addresses the open-loop Nash equilibrium of games featuring homogeneous players on complete graphs, and also provides additional theoretical backing for the feasibility of FP. (iii) For vertex-transitive graphs, we construct a semi-explicit characterization of the NE. The semi-explicit characterization under our model is intricately linked to solving a matrix-valued ODE, which degenerates to a scalar ODE in a special case [LS22]. We investigate the existence and uniqueness of the solution to the matrix-valued ODE, which requires substantial effort to establish. (iv) The numerical experiments not only validate our theoretical results but also offer crucial insights into the relationship between various aspects of the game and the underlying graph structure.

Organization of the paper. In the rest of the paper, we present multiple results from different perspectives, uncovering the connection between graph structures and the stochastic differential games on them. In Section 2, we introduce the game on graphs to be investigated, derive the Hamilton-Jacobi-Bellman (HJB) system via the dynamic programming principle, and reduce it into coupled Riccati equations as a preparation for finding the Markovian Nash equilibrium. Section 3 analyzes the iterative scheme – fictitious play – for solving the Ricatti equations and proves its convergence. We restrict ourselves to vertex-transitive graphs in Section 4. This class of graphs exhibits symmetry, from which we derive a semi-explicit expression of the Markovian Nash equilibrium, along with the development of the global existence and uniqueness result. Section 5 presents numerical experiments solving linear-quadratic games on different graphs, providing verification for the theorems proved in the previous sections and crucial insights into the connection between various aspects of the game and the graph structure. The paper concludes in Section 6.

2 A Linear-Quadratic Game on Graphs

In this section, we consider an N-player linear-quadratic game associated with a finite graph G = (V, E) that is connected, simple, and has undirected edges. When associated with games, each vertex in the graph G can be interpreted as a player, while each edge in the graph is identified as the interaction between two players.

We start by fixing some notations and definitions. Denote [N] as the collection of integers $\{1, 2, \ldots, N\}$ and $\mathbb{S}^{N \times N}$ as the collection of symmetric $N \times N$ matrices. The graph-related concepts are defined as follows.

Definition 2.1. Let G = (V, E) be a finite connected graph with simple undirected edges, where $V = \{v_1, v_2, \ldots, v_N\}$ denotes the collection of vertices, and each edge $e \in E$ is an unordered pair e = (u, v) for some $u, v \in V$. Two vertices $u, v \in V$ are said to be adjacent $u \sim v$ if and only if $(u, v) \in E$. The degree of the vertex $v \in V$, denoted by d_v , is the number of edges that are connected to v. The neighborhood of a vertex $v \in V$ is defined as the collection of vertices that are one-step away from v, i.e., $N_G(v) = \{u \in V : u \sim v\}$.

In this paper, the connection between a stochastic differential game and a graph is established through the graph Laplacian L, which, validated by the spectral graph theory [Chu97], reveals crucial graph properties, e.g., graph connectivity, graph isoperimetric property, etc.

Definition 2.2 (Normalized graph Laplacian). For a connected simple undirected graph G with N vertices, the (normalized) graph Laplacian $L \in \mathbb{S}^{N \times N}$ is defined as

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } i \neq j \text{ and } v_i \sim v_j , \ \forall i, j \in [N]. \\ 0 & \text{else} \end{cases}$$

2.1 The game setup

Consider a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, supporting independent Brownian motions $\{W_t^1\}, \ldots, \{W_t^N\}$ and $\mathscr{F}_t = \sigma(W_t^i, \forall i \in [N])$. The state process of player i, denoted by $\{X_t^i\}$, is controlled through her strategy process $\{\alpha_t^i\}$ and subject to idiosyncratic noises modeled by $\{W_t^i\}$:

$$dX_t^i = \left[a \left(\frac{1}{\sqrt{d_{v_i}}} \sum_{j: v_j \sim v_i} \frac{1}{\sqrt{d_{v_j}}} X_t^j - X_t^i \right) + \alpha_t^i \right] dt + \sigma dW_t^i, \quad \forall i \in [N].$$
 (2.1)

Here, the parameter $a \ge 0$ models the speed of mean reversion, and the parameter $\sigma > 0$ indicates the volatility of the state dynamics. For further interpretation and motivation of the model, see Remark 2.3. Without loss of generality, the state and strategy processes are both assumed to take values in \mathbb{R} .

The generic model of our concern operates on the finite time horizon [0, T]. Player i chooses her strategy $\{\alpha_t^i\}_{t\in[0,T]}$ within the admissible set \mathscr{A} :

$$\mathscr{A} = \left\{ \alpha : \alpha \text{ is progressively measurable w.r.t. } \{\mathscr{F}_t\}, \ \mathbb{E} \int_0^T |\alpha_t|^2 dt < \infty \right\}, \tag{2.2}$$

to minimize her expected cost of the form:

$$J^{i}(\alpha) := \mathbb{E}\left[\int_{0}^{T} f^{i}(t, X_{t}, \alpha_{t}^{i}) dt + g^{i}(X_{T})\right]. \tag{2.3}$$

The notations $X_t := [X_t^1, \dots, X_t^N]^T$ and $\alpha_t := (\alpha_t^1, \dots, \alpha_t^N)$ represent the state processes and the strategies of all players. Under the notation of $x = [x^1, \dots, x^N]^T \in \mathbb{R}^N$ as the state variable and $\alpha \in \mathbb{R}$ as the control variable, the running and terminal costs of player $i \in [N]$ are set as

$$f^{i}(t, x, \alpha) = \frac{1}{2}(\alpha)^{2} - q\alpha \left[\frac{1}{\sqrt{d_{v_{i}}}} \sum_{j: v_{i} \sim v_{i}} \frac{1}{\sqrt{d_{v_{j}}}} x^{j} - x^{i} \right] + \frac{\varepsilon}{2} \left[\frac{1}{\sqrt{d_{v_{i}}}} \sum_{j: v_{i} \sim v_{i}} \frac{1}{\sqrt{d_{v_{j}}}} x^{j} - x^{i} \right]^{2}, (2.4)$$

$$g^{i}(x) = \frac{c}{2} \left[\frac{1}{\sqrt{d_{v_{i}}}} \sum_{j:v_{j} \sim v_{i}} \frac{1}{\sqrt{d_{v_{j}}}} x^{j} - x^{i} \right]^{2}.$$
 (2.5)

The parameters are in the range q > 0, c > 0, $\varepsilon > 0$, and the condition $q^2 \le \varepsilon$ is required to ensure the joint convexity of f^i and the well-posedness of the problem.

The model considered encompasses several models from the literature as special cases. For instance, when $G = K_N$, a complete graph with N vertices, our model aligns with the one in [CFS15]; and when $a = q = \varepsilon = 0$ and G is a vertex-transitive graph, our model aligns with the one in [LS22].

Remark 2.3 (Model interpretation). Using Definition 2.2, the mean reversion level in (2.1) can be rewritten as

$$\frac{1}{\sqrt{d_{v_i}}} \sum_{j: v_j \sim v_i} \frac{1}{\sqrt{d_{v_j}}} X_t^j = \frac{1}{d_{v_i}} \sum_{j: v_j \sim v_i} \sqrt{\frac{d_{v_i}}{d_{v_j}}} X_t^j, \tag{2.6}$$

which is a weighted state average among all the players in the neighborhood of v_i . We interpret the square root weight in equation (2.6) from two perspectives.

The first perspective comes from the connection with graph Laplacian L. Let $\{e_1, \ldots, e_N\}$ be the standard basis of \mathbb{R}^N . From Definition 2.2, it is clear that

$$\frac{1}{\sqrt{d_{v_i}}} \sum_{j: v_i \sim v_i} \frac{1}{\sqrt{d_{v_j}}} X_t^j - X_t^i = -e_i^T L X_t.$$
 (2.7)

This concise representation facilitates the connection between the graph structure and various aspects of stochastic differential games on graphs in later discussions.

The second perspective comes from considering the random walk on graphs as a process for transmitting substances or diverging information across a network. A standard result for the random walk on graph, as outlined in [Chu97], is the stationary distribution $\pi_i = d_{v_i}/\text{vol}(G)$, where the volume of the graph G is defined as $\text{vol}(G) = \sum_{i=1}^N d_{v_i}$, serving as a normalization factor. Let p_{ij} denote the one-step transition probability from v_i to v_j . The detailed balance equation of this random walk on graphs is given by $d_{v_i}/d_{v_j} = p_{ji}/p_{ij}$. Now, consider the symmetry constraint that the transmission/divergence of substance/information along each undirected edge is symmetric in both directions, i.e., if $p_{ji} = h(j,i)$ for some function h, then $p_{ij} = h(i,j)$ must also hold. Under the detailed balance equation and the symmetry constraint along undirected edges, $p_{ji} = \sqrt{d_{v_i}/d_{v_j}}$ provides a natural way to construct the weights for the mean reversion level of the model.

Note that if G is a regular graph, meaning all vertices have the same degree, the mean reversion level becomes the arithmetic average of the states of all players in the neighborhood of v_i .

Remark 2.4. Although G is required to be a connected graph in Definition 2.1, this model also works for disconnected graphs since one can always put up the same model on each connected component of a disconnected graph.

In a stochastic differential game, the information set on which each player makes its decision must be specified. The difference in the information set results in different notions of strategy, e.g., open-loop strategy, closed-loop strategy, Markovian strategy, etc. In this paper, our discussion is restricted to the Markovian strategy, i.e., $\alpha_t^i = \phi^i(t, X_t)$ for some deterministic feedback function $\phi^i: [0,T] \times \mathbb{R}^N \to \mathbb{R}$ of player i. To match the restrictions in the definition of \mathscr{A} (cf. (2.2)), the feedback function ϕ^i shall be Borel measurable such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^N} \frac{|\phi^i(t,x)|}{1+||x||} < \infty$. In the setting of a competitive game, the Nash equilibrium defined below is of general interest.

Definition 2.5 (Nash equilibrium). The collection of all players' strategies $\hat{\alpha} := (\hat{\alpha}^1, \dots, \hat{\alpha}^N)$ is defined as an Nash equilibrium (NE) if

$$J^i((\alpha,\hat{\alpha}^{-i})) \geq J^i(\hat{\alpha}), \ \forall i \in [N], \ \forall \alpha \ \in \mathscr{A}.$$

The notation $(\alpha, \hat{\alpha}^{-i}) := (\hat{\alpha}^1, \dots, \hat{\alpha}^{i-1}, \alpha, \hat{\alpha}^{i+1}, \dots, \hat{\alpha}^N)$ is the collection of strategies replacing player *i*'s strategy with α while maintaining all other players' strategies.

In simple words, an Nash equilibrium is a collection of strategies from which any player does not have the motivation to deviate, given that all other players stick to their Nash equilibrium strategies.

2.2 The associated HJB system

A standard approach to solving the Markovian NE involves constructing a value function and the Hamilton-Jacobi-Bellman (HJB) system. Denote by $v^i(t,x):[0,T]\times\mathbb{R}^N\to\mathbb{R}$ the value function

of player i, where $t \in [0,T]$ is the time variable and $x \in \mathbb{R}^N$ is the state variable. By dynamic programming principle, the HJB system is given by

$$\partial_t v^i + \inf_{\alpha^i} \left\{ \sum_{k=1}^N (\alpha^k - ae_k^{\mathrm{T}} Lx) \partial_{x^k} v^i + \frac{\sigma^2}{2} \sum_{k=1}^N \partial_{x^k x^k} v^i + \frac{1}{2} (\alpha^i)^2 + q\alpha^i e_i^{\mathrm{T}} Lx + \frac{\varepsilon}{2} (e_i^{\mathrm{T}} Lx)^2 \right\} = 0, \quad (2.8)$$

with a terminal condition $v^i(T,x) = \frac{c}{2}(e_i^T L x)^2$. Solving for the infimum in the above equation gives

$$\hat{\alpha}^{i}(t,x) = -qe_{i}^{\mathrm{T}}Lx - \partial_{x^{i}}v^{i}(t,x). \tag{2.9}$$

Using a quadratic (in x) ansatz,

$$v^{i}(t,x) = \frac{1}{2}x^{\mathrm{T}}F_{t}^{i}x + h_{t}^{i}, \tag{2.10}$$

where $F^i:[0,T]\to\mathbb{S}^{N\times N}$ and $h^i:[0,T]\to\mathbb{R}$ are both deterministic functions in time. Plugging equations (2.9)–(2.10) back into equation (2.8), and collecting the coefficients of x yield a Riccati equation for F^i :

$$\dot{F}_{t}^{i} + \sum_{k=1}^{N} [-(a+q)L - F_{t}^{k}] e_{k} e_{k}^{\mathrm{T}} F_{t}^{i} + \sum_{k=1}^{N} F_{t}^{i} e_{k} e_{k}^{\mathrm{T}} [-(a+q)L - F_{t}^{k}] + (\varepsilon - q^{2}) L e_{i} e_{i}^{\mathrm{T}} L + F_{t}^{i} e_{i} e_{i}^{\mathrm{T}} F_{t}^{i} = 0, \quad F_{T}^{i} = c L e_{i} e_{i}^{\mathrm{T}} L. \quad (2.11)$$

By collecting terms of order 1, one can deduce an ODE for h^i :

$$\dot{h}_t^i + \frac{1}{2}\sigma^2 \text{Tr}(F_t^i) = 0, \quad h_T^i = 0,$$

which is in first order involving F^i . Therefore, h^i can be easily represented by F^i , while the main difficulty lies in solving for F^i . If a solution (F_t^1, \ldots, F_t^N) to the coupled Riccati system (2.11) exists, a corresponding Markovian NE $\hat{\alpha}$ exists, which is given by

$$\hat{\alpha}^i(t,x) = -qe_i^{\mathrm{T}}Lx - e_i^{\mathrm{T}}F_t^i x. \tag{2.12}$$

This equilibrium depends on F^i but not on h^i . Consequently, the rest of the paper will focus on analyzing the Riccati system (2.11).

3 The Equilibrium via Fictitious Play

Initially introduced by Brown [Bro49, Bro51], the key concept of FP involves transforming the calculation of the NE into a sequence of stochastic control problems faced by each player. Specifically, player i optimizes her objective while assuming all other players' strategies remain fixed and follow their historical strategies. Solving this stochastic control problem yields a new optimal strategy for player i, completing one round of fictitious play for player i. Each round of fictitious play consists of performing this procedure for each player once. The ultimate goal is to observe the convergence of all players' strategies converging to the NE after a sufficient number of rounds of fictitious play have been carried out.

The convergence of FP to the NE, however, is not guaranteed, even for tabular games [Sha63], which motivates research on the convergence of FP. In the context of continuous-time differential games, affirmative answers include [CH17, BC18] for mean-field games and [Hu21, HHL22] for

finite-player games, but none of them take into account the graph structure. In this section, we provide proof for the convergence of FP for the generic model described in Section 2.1. This result contributes in two significant ways. Firstly, it validates the effectiveness of FP for solving stochastic differential games on graphs. Secondly, it elucidates the connection between the graph structure and the convergence of FP.

3.1 The mathematical formulation

To prepare for stating the convergence result in later contexts, we describe the procedure of FP in mathematical terms. Firstly, rewrite the state dynamics and the cost functionals in terms of the graph Laplacian L. Using equation (2.7), the state dynamics (2.1) is rewritten as

$$dX_t^i = (\alpha_t^i - ae_i^T L X_t) dt + \sigma dW_t^i,$$

and the cost functionals (2.4)–(2.5) are rewritten as

$$f^{i}(t,x,\alpha) = \frac{1}{2}(\alpha)^{2} + q\alpha e_{i}^{\mathrm{T}}Lx + \frac{\varepsilon}{2} \left(e_{i}^{\mathrm{T}}Lx\right)^{2}, \quad g^{i}(x) = \frac{c}{2} \left(e_{i}^{\mathrm{T}}Lx\right)^{2}. \tag{3.1}$$

To provide a clear description of how FP works, we summarize the notations used as follows.

Notation 3.1. For any object A, $A^{i,k}$ refers to the object associated with player $i \in [N]$ at FP stage $k \in \mathbb{N}$. Denote by A^k the collection of objects across all the players at FP stage k. In the following context, i and j stand for the player indices, while k represents the FP stage index.

Without loss of generality, let us consider the procedure of FP for player i. Assume all players have their strategies at stage k determined through a feedback functions $\phi^{i,k}$ that $\alpha^{i,k}_t = \phi^{i,k}(t, X^k_t)$, $i \in [N]$. At stage k+1, player i optimizes her strategy to get a new feedback function $\phi^{i,k+1}$ based on the environment at stage k while assuming all the other players stick to their feedback functions at stage k, i.e., player $j \neq i$ is using feedback function $\phi^{j,k}$. As a result, the stochastic control problem faced by player i has state dynamics as

$$\begin{cases} dX_t^{i,k+1} = (\alpha_t^i - ae_i^T L X_t^{k+1}) dt + \sigma dW_t^i, \\ dX_t^{j,k+1} = [\phi^{j,k}(t, X_t^{k+1}) - ae_i^T L X_t^{k+1}] dt + \sigma dW_t^j, \ \forall j \neq i. \end{cases}$$

Player i faces a running cost of $f^i(t, X_t^{k+1}, \alpha_t^i)$ and terminal cost $g^i(X_T^{k+1})$ where f^i and g^i are provided in equation (3.1).

Let $v^{i,k+1}:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ be the value function of player i at stage k+1. It satisfies

$$\partial_{t}v^{i,k+1} + \inf_{\alpha} \left\{ (\alpha - ae_{i}^{T}Lx)\partial_{x^{i}}v^{i,k+1} + \sum_{j \neq i} (\phi^{j,k}(t,x) - ae_{j}^{T}Lx)\partial_{x^{j}}v^{i,k+1} + \frac{\sigma^{2}}{2} \sum_{j=1}^{N} \partial_{x^{j}x^{j}}v^{i,k+1} + \frac{1}{2}\alpha^{2} + q\alpha e_{i}^{T}Lx + \frac{\varepsilon}{2}(e_{i}^{T}Lx)^{2} \right\} = 0, \quad v^{i,k+1}(T,x) = \frac{c}{2}(e_{i}^{T}Lx)^{2}. \quad (3.2)$$

The first-order condition in the above equation gives

$$\phi^{i,k+1}(t,x) = -\partial_{x^i} v^{i,k+1}(t,x) - q e_i^{\mathrm{T}} L x.$$
(3.3)

Plugging the above feedback function back into equation (3.2) yields

$$\partial_{t}v^{i,k+1} - (a+q)e_{i}^{T}Lx \ \partial_{x^{i}}v^{i,k+1} + \frac{\sigma^{2}}{2}\sum_{j=1}^{N}\partial_{x^{j}x^{j}}v^{i,k+1} - \frac{1}{2}(\partial_{x^{i}}v^{i,k+1})^{2} + \frac{\varepsilon - q^{2}}{2}(e_{i}^{T}Lx)^{2} + \sum_{j\neq i}\left[\phi^{j,k}(t,x) - ae_{j}^{T}Lx\right]\partial_{x^{j}}v^{i,k+1} = 0. \quad (3.4)$$

The term $\phi^{j,k}(t,x)$ in the above equation hinders us from reducing the HJB equation to a Riccati equation directly. To proceed, we make the following assumption on the initial strategies $\phi^{i,0}$, for all players $i \in [N]$.

Assumption 3.2. Assume that for every player $i, i \in [N]$, FP starts with $\phi^{i,0}$ that is of the form $\phi^{i,0}(t,x) = (\varphi_t^{i,0})^T x$, for some $\varphi^{i,0} : [0,T] \to \mathbb{R}^N$.

Lemma 3.3. Under Assumption 3.2, the optimal strategy $\phi^{i,k}$ for player i at FP stage k, can always be represented as $\phi^{i,k}(t,x) = (\varphi_t^{i,k})^T x$ for some $\varphi^{i,k} : [0,T] \to \mathbb{R}^N$, for $i \in [N]$ and $k \in \mathbb{N}$.

Proof. We present a proof by induction on k. Let us assume that the statement holds at stage k, and consider a quadratic (in x) ansatz for the value function $v^{i,k+1}(t,x) = \frac{1}{2}x^{\mathrm{T}}F_t^{i,k+1}x + h_t^{i,k+1}$, where $F^{i,k+1}:[0,T]\to\mathbb{S}^{N\times N}$ and $h^{i,k+1}:[0,T]\to\mathbb{R}$ are both deterministic functions in time. By plugging it into equation (3.4) and using the induction hypothesis, one can collect terms of x and obtain a Riccati equation for $F^{i,k+1}$:

$$\begin{split} \dot{F}_{t}^{i,k+1} - F_{t}^{i,k+1} e_{i} e_{i}^{\mathrm{T}} F_{t}^{i,k+1} + (\varepsilon - q^{2}) L e_{i} e_{i}^{\mathrm{T}} L - a L F_{t}^{i,k+1} - a F_{t}^{i,k+1} L + \sum_{j \neq i} F_{t}^{i,k+1} e_{j} (\varphi_{t}^{j,k})^{\mathrm{T}} \\ + \sum_{j \neq i} \varphi_{t}^{j,k} e_{j}^{\mathrm{T}} F_{t}^{i,k+1} - q L e_{i} e_{i}^{\mathrm{T}} F_{t}^{i,k+1} - q F_{t}^{i,k+1} e_{i} e_{i}^{\mathrm{T}} L = 0, \quad F_{T}^{i,k+1} = c L e_{i} e_{i}^{\mathrm{T}} L. \end{split}$$

Plugging the ansatz into equation (3.3) yields $\phi^{i,k+1}(t,x) = -e_i^{\mathrm{T}}(F_t^{i,k+1} + qL)x$. Therefore, $\forall i \in [N]$ and $\forall k \in \mathbb{N}$,

$$\varphi_t^{i,k+1} = -(F_t^{i,k+1} + qL)e_i. \tag{3.5}$$

This completes the proof.

This lemma indicates that if the FP procedure begins (at stage 0) with linear strategies (in x) for all players, it will result in every strategy being linear in x for all subsequent stages.

Note that $\{F^{i,k+1}\}_{i\in[N]}$ and $\{F^{i,k}\}_{i\in[N]}$ are coupled through the term $\varphi_t^{j,k}$, represented as $\varphi_t^{j,k} = -(F_t^{j,k} + qL)e_j$ (cf. (3.5)). This yields the following recursive Riccati equation, which describes the update from $\{F^{i,k}\}_{i\in[N]}$ to $\{F^{i,k+1}\}_{i\in[N]}$:

$$\dot{F}_{t}^{i,k+1} - F_{t}^{i,k+1} e_{i} e_{i}^{\mathrm{T}} F_{t}^{i,k+1} + (\varepsilon - q^{2}) L e_{i} e_{i}^{\mathrm{T}} L - (a+q) L F_{t}^{i,k+1} - (a+q) F_{t}^{i,k+1} L - F_{t}^{i,k+1} \sum_{j \neq i} e_{j} e_{j}^{\mathrm{T}} F_{t}^{j,k} - \sum_{j \neq i} F_{t}^{j,k} e_{j} e_{j}^{\mathrm{T}} F_{t}^{i,k+1} = 0, \quad F_{T}^{i,k+1} = c L e_{i} e_{i}^{\mathrm{T}} L. \quad (3.6)$$

Therefore, the FP procedure for the game introduced in Section 2.1 has been completely described by system (3.6) following from Assumption 3.2.

3.2 Convergence results

In this section, we prove that the iterative scheme described by (3.6) converges to a limit, which is the Markovian NE. We first define a notion of convergence when discussing the convergence of strategies. Since the players' strategies are represented by feedback functions of state processes, we examine the pointwise convergence of the feedback function $\phi^{i,k}(t,x)$ as $k \to \infty$ for any fixed $i \in [N]$. As implied by Lemma 3.3 and equation (3.5), the pointwise convergence of $\phi^{i,k}$ is equivalent to the convergence of $F^{i,k}$. Therefore, it suffices to prove that the solution $F^{i,k}$ to system (3.6) admits a limit as $k \to \infty$ for any fixed $i \in [N]$ (Theorem 3.6), and the limit satisfies equation (2.11) (Theorem 3.7). The proof relies on the following two technical lemmas. The first lemma provides a way to prove the uniform convergence of a function sequence.

Lemma 3.4. Let $f_k : [0,T] \to \mathbb{R}_+$ be a sequence of continuous non-negative functions. If there exists a constant C > 0 such that

$$f_k(t) \le C \int_t^T [f_k(s) + f_{k-1}(s)] ds, \ \forall t \in [0, T], \ \forall k \in \mathbb{N},$$

then f_k converges to $f \equiv 0$, uniformly in $t \in [0,T]$, as $k \to \infty$.

Proof. Consider $g_k(t) := e^{\beta t} f_k(t)$ for any $\beta > 0$. It is clear that

$$g_k(t) \le Ce^{\beta t} \int_t^T e^{-\beta s} [g_k(s) + g_{k-1}(s)] ds.$$

Since $g_k(t)$ is a continuous function on a compact domain, $\sup_{s \in [0,T]} g_k(s) < \infty$, which provides an upper bound for the integral above:

$$g_k(t) \le \frac{C}{\beta} (1 - e^{-\beta(T-t)}) \left(\sup_{s \in [0,T]} g_k(s) + \sup_{s \in [0,T]} g_{k-1}(s) \right).$$

Specifying the value of β to be large enough such that $\frac{C}{\beta} < \frac{1}{3}$, and taking the supremum on both sides with respect to $t \in [0, T]$ yields

$$\sup_{t \in [0,T]} g_k(t) \le \frac{1}{2} \sup_{t \in [0,T]} g_{k-1}(t),$$

which implies that

$$\sup_{t \in [0,T]} g_k(t) \le \frac{1}{2^k} \sup_{t \in [0,T]} g_0(t) \to 0 \ (k \to \infty).$$

Note that $\sup_{t \in [0,T]} f_k(t) \leq \sup_{t \in [0,T]} g_k(t)$, setting $k \to \infty$ concludes the proof.

The second lemma proves a matrix inequality. Unless otherwise specified, all matrix norms are understood as the matrix 2-norm, and the inequality between symmetric matrices is defined in terms of the semi-positive-definite sense. That is, $A \geq B$ if and only if A - B is semi-positive-definite, for $A, B \in \mathbb{S}^{N \times N}$.

Lemma 3.5. For symmetric semi-positive-definite matrices A_1, \ldots, A_N , the following inequality holds:

$$\|\sum_{i=1}^{N} e_i e_i^{\mathrm{T}} A_i\| \le \|\sum_{i=1}^{N} A_i\|.$$

Proof. To simplify the notation, let $B_i := \sum_{i=1}^N e_i e_i^T A_i$. Since $\sum_{i=1}^N A_i e_i e_i^T A_i \le \sum_{i=1}^N A_i^2$, it is clear

$$||B_i||^2 = ||B_i^{\mathrm{T}} B_i|| = ||\sum_{i=1}^N A_i e_i e_i^{\mathrm{T}} A_i|| \le ||\sum_{i=1}^N A_i^2||.$$

Due to the matrix inequality $\sum_{i=1}^{N} A_i^2 \leq \sum_{i=1}^{N} ||A_i|| A_i \leq \max_{i \in [N]} ||A_i|| \sum_{i=1}^{N} A_i$, the following inequality holds:

$$\|\sum_{i=1}^{N} A_i^2\| \le \max_{i \in [N]} \|A_i\| \|\sum_{i=1}^{N} A_i\| \le \|\sum_{i=1}^{N} A_i\|^2,$$

which concludes the proof.

We next state the first main result of this section, concerning the convergence of $F^{i,k}$ as $k \to \infty$.

Theorem 3.6. For any function $A^{i,k}:[0,T]\to\mathbb{S}^{N\times N}$, denote $\Delta A^{i,k}=A^{i,k+1}-A^{i,k}$ as the difference from stage k to k+1.

Under Assumption 3.2, if the fictitious play starts from $F^{i,0} \equiv 0$ for any $i \in [N]$, and the following condition

$$\max \left\{ [c + T(\varepsilon - q^2)] \|L\|^2, \frac{2}{\log 2} (a + q) T \|L\| \right\} \le 1, \quad T \le \frac{1}{100}, \tag{3.7}$$

holds, then $\sup_{i\in[N]} \|\Delta F_t^{i,k}\|$ converges to 0, uniformly in $t\in[0,T]$, as $k\to\infty$, where $F_t^{i,k}$ is the solution to system (3.6).

Consequently, $F_t^{i,k}$ converges for any fixed $i \in [N]$, uniformly in $t \in [0,T]$, as $k \to \infty$. We denote the uniform limit as $F_t^{i,\infty} := \lim_{k \to \infty} F_t^{i,k}$.

Proof. Step 1: The a priori bound for $||F_t^{i,k}||$. A change of variable in time $W_t^{i,k} := F_{T-t}^{i,k}$ yields

$$\begin{split} \dot{W}_t^{i,k+1} &= -(W_t^{i,k+1} - W_t^{i,k})e_ie_i^{\mathrm{T}}(W_t^{i,k+1} - W_t^{i,k}) + (\varepsilon - q^2)Le_ie_i^{\mathrm{T}}L - (a+q)LW_t^{i,k+1} - (a+q)W_t^{i,k+1}L \\ &- W_t^{i,k+1}\sum_{j=1}^N e_je_j^{\mathrm{T}}W_t^{j,k} - \sum_{j=1}^N W_t^{j,k}e_je_j^{\mathrm{T}}W_t^{i,k+1} + W_t^{i,k}e_ie_i^{\mathrm{T}}W_t^{i,k}, \quad W_0^{i,k+1} = cLe_ie_i^{\mathrm{T}}L. \end{split}$$

Using $W_t^{i,k+1} = W_0^{i,k+1} + \int_0^t \dot{W}_s^{i,k+1} \, \mathrm{d}s$ and $(W_t^{i,k+1} - W_t^{i,k}) e_i e_i^{\mathrm{T}} (W_t^{i,k+1} - W_t^{i,k}) \geq 0$, followed by summing both sides with respect to $i \in [N]$, gives the following matrix inequality:

$$\begin{split} & \sum_{i=1}^{N} W_t^{i,k+1} \leq [c+t(\varepsilon-q^2)]L^2 - (a+q)L \int_0^t \sum_{i=1}^{N} W_s^{i,k+1} \, \mathrm{d}s - (a+q) \int_0^t \sum_{i=1}^{N} W_s^{i,k+1} \, \mathrm{d}s \, L \\ & - \int_0^t \sum_{i=1}^{N} W_s^{i,k+1} \sum_{i=1}^{N} e_j e_j^\mathrm{T} W_s^{j,k} \, \mathrm{d}s - \int_0^t \sum_{i=1}^{N} W_s^{j,k} e_j e_j^\mathrm{T} \sum_{i=1}^{N} W_s^{i,k+1} \, \mathrm{d}s + \int_0^t \sum_{i=1}^{N} W_s^{i,k} e_i e_i^\mathrm{T} W_s^{i,k} \, \mathrm{d}s. \end{split}$$

Let $A_t^k := \sum_{i=1}^N W_t^{i,k}$ and $B_t^k := \sum_{i=1}^N e_i e_i^T W_t^{i,k}$. The matrix inequality above is rewritten as

$$\begin{split} A_t^{k+1} &\leq [c+t(\varepsilon-q^2)]L^2 - (a+q)L\int_0^t A_s^{k+1}\,\mathrm{d}s - (a+q)\int_0^t A_s^{k+1}\,\mathrm{d}s\,L \\ &-\int_0^t A_s^{k+1}B_s^k\,\mathrm{d}s - \int_0^t (B_s^k)^\mathrm{T}A_s^{k+1}\,\mathrm{d}s + \int_0^t (B_s^k)^\mathrm{T}B_s^k\,\mathrm{d}s. \end{split} \tag{3.8}$$

For any $i \in [N]$, $k \in \mathbb{N}$, the cost functionals f^i and g^i stay non-negative, so does the value function $v^{i,k}(t,x) = \frac{1}{2}x^{\mathrm{T}}F_t^{i,k}x + h_t^{i,k}, \ \forall x \in \mathbb{R}^N$. Consequently, one has $F_t^{i,k} \geq 0, \ W_t^{i,k} \geq 0, \ \forall t \in [0,T]$. Then Lemma 3.5 tells $\|B_t^k\| \leq \|A_t^k\|$. Combining with inequality (3.8), we have

$$||A_t^{k+1}|| \le [c + t(\varepsilon - q^2)]||L||^2 + 2\int_0^t \left[(a+q)||L|| + ||A_s^k|| \right] ||A_s^{k+1}|| \, \mathrm{d}s + \int_0^t ||A_s^k||^2 \, \mathrm{d}s.$$

Then one can apply Grönwall's inequality to get:

$$||A_t^{k+1}|| \le \left([c + t(\varepsilon - q^2)] ||L||^2 + \int_0^t ||A_s^k||^2 \, \mathrm{d}s \right) e^{2\int_0^t (a+q)||L|| + ||A_s^k|| \, \mathrm{d}s}.$$

Taking the supremum on both sides with respect to $t \in [0, T]$ yields

$$\sup_{t \in [0,T]} \|A_t^{k+1}\| \le \left(\left[c + T(\varepsilon - q^2) \right] \|L\|^2 + T \sup_{t \in [0,T]} \|A_t^k\|^2 \right) e^{2T(a+q)\|L\| + 2T \sup_{t \in [0,T]} \|A_t^k\|}.$$

Define $h: \mathbb{R}_+ \to \mathbb{R}_+$ as a continuous increasing function:

$$h(x) := ([c + T(\varepsilon - q^2)] \|L\|^2 + Tx^2) e^{2T(a+q)\|L\|} e^{2Tx}.$$
(3.9)

Under condition (3.7), $[c + T(\varepsilon - q^2)] ||L||^2 \le 1$ and $e^{2T(a+q)||L||} \le 2$, so

$$h(x) \le 2(1 + 0.01x^2)e^{0.02x} := g(x).$$

Since g(2) > 2, g(3) < 3, by the intermediate value theorem, g has a fixed point $x^* \in (2,3)$. Since FP starts from $F^{i,0} \equiv 0$, $\sup_{t \in [0,T]} \|A_t^0\| = 0$. A simple induction argument implies that

$$\sup_{t \in [0,T]} \|A_t^{k+1}\| \le h \left(\sup_{t \in [0,T]} \|A_t^k\| \right) \le g \left(\sup_{t \in [0,T]} \|A_t^k\| \right) \le g(x^*) = x^*, \ \forall k \in \mathbb{N}.$$

This inequality is equivalent to $\|\sum_{i=1}^N F_t^{i,k}\| \le x^*$, $\forall t \in [0,T]$, $\forall k \in \mathbb{N}$. Combining with the fact that $F_t^{i,k}$ takes values as semi-positive-definite matrices proves the a priori bound $\|F_t^{i,k}\| \le x^*$, $\forall k \in \mathbb{N}, \forall t \in [0,T], \forall i \in [N]$.

Step 2: The uniform convergence. Using $F_t^{i,k+1} = F_T^{i,k+1} - \int_t^T \dot{F}_s^{i,k+1} ds$ and replacing $\dot{F}_s^{i,k+1}$ by equation (3.6) yields the following upper bound:

$$\begin{split} \|\Delta F_t^{i,k}\| & \leq 2(a+q)\|L\| \int_t^T \|\Delta F_s^{i,k}\| \, \mathrm{d}s + \int_t^T \left(\|F_s^{i,k}\| \cdot \|\Delta F_s^{i,k}\| + \|F_s^{i,k+1}\| \cdot \|\Delta F_s^{i,k}\| \right) \, \mathrm{d}s \\ & + 2 \int_t^T \left(\|\Delta F_s^{i,k}\| \cdot \sum_{j \neq i} \|F_s^{j,k-1}\| + \|F_s^{i,k+1}\| \cdot \sum_{j \neq i} \|\Delta F_s^{j,k-1}\| \right) \, \mathrm{d}s. \end{split}$$

Taking the supremum on both sides with respect to $i \in [N]$, and using the a priori bound $||F_t^{i,k}|| \le x^*$ established in the last step, the following inequality is proved:

$$\sup_{i \in [N]} \|\Delta F_t^{i,k}\| \leq \left[2(a+q)\|L\| + 2x^* + 2Nx^*\right] \int_t^T \sup_{i \in [N]} \|\Delta F_s^{i,k}\| \, \mathrm{d}s + 2Nx^* \int_t^T \sup_{i \in [N]} \|\Delta F_s^{i,k-1}\| \, \mathrm{d}s.$$

By Lemma 3.4, $\sup_{i \in [N]} \|\Delta F_t^{i,k}\|$ converges to 0, uniformly in $t \in [0,T]$, as $k \to \infty$. This concludes the proof.

In addition, we argue that the limit $F_t^{i,\infty}$ identified in Theorem 3.6 corresponds to the Markovian NE, formulated below as Theorem 3.7.

Theorem 3.7. Under Assumption 3.2, if the fictitious play starts from $F^{i,0} \equiv 0$ for any $i \in [N]$ and condition (3.7) holds, then the scheme (3.6) produced by fictitious play converges to the Markovian NE for the linear-quadratic games on graphs (2.1)–(2.3).

Proof. It suffices to prove that $(F_t^{1,\infty},\cdots,F_t^{N,\infty})$, the limit of $(F_t^{1,k},\cdots,F_t^{N,k})$ as $k\to\infty$, is a solution to the coupled Riccati system (2.11). Set $k\to\infty$ in equation (3.6), and it remains to justify the interchange of the differentiation with respect to t and the limit with respect to t. To this end, we need to show $\lim_{k\to\infty} \dot{F}_t^{i,k} = \dot{F}_t^{i,\infty}$.

this end, we need to show $\lim_{k\to\infty} \dot{F}_t^{i,k} = \dot{F}_t^{i,\infty}$. Rewrite equation (3.6) as $\dot{F}_t^{i,k+1} = H(F_t^{i,k+1}, F_t^{1,k}, \cdots, F_t^{i-1,k}, F_t^{i+1,k}, \cdots, F_t^{N,k})$ for some continuous function $H: (\mathbb{S}^{N\times N})^N \to \mathbb{S}^{N\times N}$. From the proof of Theorem 3.6, one has $||F_t^{i,k}|| \leq x^*$ for every $k \in \mathbb{N}$, $i \in [N]$, $t \in [0,T]$. Hence, the domain of H is compact, and H is uniformly continuous. From Theorem 3.6, the convergence of $F_t^{i,k}$ is uniform w.r.t. $t \in [0,T]$. It's well-known that uniformly continuous transformations preserve uniform convergence. As a result, $\dot{F}_t^{i,k+1}$ converges uniformly w.r.t. $t \in [0,T]$ as $k \to \infty$. The interchange of the limit and the differentiation is justified, which concludes the proof.

Lastly, we rescale the time horizon to relax condition (3.7), as shown in Corollary 3.8. This concludes the investigation of the convergence of FP for our linear-quadratic games on graphs.

Corollary 3.8. Under Assumption 3.2, if the fictitious play starts from $F^{i,0} \equiv 0$ for any $i \in [N]$ and the following condition

$$C_L := \max \left\{ 100T[c + T(\varepsilon - q^2)] \|L\|^2, \frac{2}{\log 2}(a+q)T\|L\| \right\} \le 1, \tag{3.10}$$

holds, then the scheme (3.6) produced by fictitious play converges to the Markovian NE for the linear-quadratic games on graphs (2.1)–(2.3).

Proof. Define $G_t^{i,k} := \frac{T}{T'} F_{\frac{T}{T'}t}^{i,k}$ for some positive T', equation (3.6) becomes:

$$\dot{G}_{t}^{i,k+1} - G_{t}^{i,k+1}e_{i}e_{i}^{\mathrm{T}}G_{t}^{i,k+1} + \left(\frac{T}{T'}\right)^{2}(\varepsilon - q^{2})Le_{i}e_{i}^{\mathrm{T}}L - \frac{T}{T'}(a+q)LG_{t}^{i,k+1} - \frac{T}{T'}(a+q)G_{t}^{i,k+1}L$$
$$-G_{t}^{i,k+1}\sum_{j\neq i}e_{j}e_{j}^{\mathrm{T}}G_{t}^{j,k} - \sum_{j\neq i}G_{t}^{j,k}e_{j}e_{j}^{\mathrm{T}}G_{t}^{i,k+1} = 0, \quad G_{T'}^{i,k+1} = \frac{T}{T'}cLe_{i}e_{i}^{\mathrm{T}}L.$$

This implies that the FP procedure for the linear-quadratic games on graphs (2.1)–(2.3) with model parameters $(T, a, q, \varepsilon, c)$ is equivalent to the one with model parameters $(T', a', q', \varepsilon', c')$, where

$$a' = \frac{T}{T'}a, \quad q' = \frac{T}{T'}q, \quad \varepsilon' = \left(\frac{T}{T'}\right)^2 \varepsilon, \quad c' = \frac{T}{T'}c.$$

Condition (3.7) imposed on the new set of model parameters $(T', a', q', \varepsilon', c')$ requires:

$$\max \left\{ [c' + T'(\varepsilon' - (q')^2)] \|L\|^2, \frac{2}{\log 2} (a' + q')T'\|L\| \right\} \le 1, \quad T' = \frac{1}{100},$$

which concludes the proof.

Remark 3.9 (Discussion on Condition (3.10)). The condition (3.10) imposes a smallness constraint on the model parameters, a requirement that is commonly found in the literature. For instance, the convergence of FP to the open-loop NE for linear-quadratic games on complete graphs requires that T or $(c, q, \epsilon - q^2)$ are sufficiently small, or a is sufficiently large [Hu21]. The convergence of the Deep BSDE method [EHJ17] for stochastic control problems with Lipschitz models requires a small time duration or small Lipschitz coefficients [HHL22], as well as the well-posedness of forward-backward stochastic differential equations (FBSDEs) [MWZZ15]. Condition (3.10) aligns with these requirements, and it is certainly not the strictest constraint and can be easily extended. As long as reasonable bounds are imposed on model parameters such that the function h in equation (3.9) has a fixed point in \mathbb{R}_+ , Theorems 3.6–3.7 and Corollary 3.8 remain valid and provide a standard way to find a condition, under which FP is guaranteed to converge. Lastly, we want to emphasize two points: (i) FP does not necessarily have to start from $F^{i,0} \equiv 0$. From the proof of Theorem 3.6, the convergence results still hold as long as $\sup_{t \in [0,T]} \|\sum_{i=1}^N F_t^{i,0}\| \le x^*$. (ii) Condition (3.10) does not impose any restrictions on the number of players N. The convergence result still holds even when there is a large number of players, i.e., $N \to \infty$.

Remark 3.10 (The graph structure and the convergence of fictitious play). We are particularly interested in the connection between the graph structure and the convergence of fictitious play (FP). This relationship can be analyzed through C_L defined in condition (3.10). Fixing the other model parameters, an increase in ||L|| leads to an increase in C_L , making it more likely to violate condition (3.10). Taking a closer look, we note that any graph Laplacian L must satisfy $0 \le ||L|| \le 2$. If a graph has no isolated vertices, its graph Laplacian satisfies $||L|| \ge \frac{N}{N-1}$ [Chu97]. A connected bipartite graph has the norm of the graph Laplacian achieve the upper bound ||L|| = 2, while a complete graph achieves the lower bound $||L|| = \frac{N}{N-1}$. Although the convergence rate of FP has not been accurately quantified in Corollary 3.8, numerical experiments show a connection between the graph structure and the convergence rate of FP. We refer the readers to Section 5.2 for a more detailed discussion.

4 Semi-Explicit Equilibrium under Vertex-Transitive Graph

In this section, we focus on the game on a vertex-transitive graph G, and aim to provide more explicit characterizations for the value functions v^i and equilibrium strategies $\hat{\alpha}^i$, compared to the general case characterized by the coupled matrix-valued Riccati equations (2.11)–(2.12). Vertex-transitive graphs, which are defined in Definition 4.1, have nice symmetry properties, and play a key role in our analysis.

Definition 4.1. A graph G is vertex-transitive if $\forall v_i, v_j \in V$, there exists $\varphi \in \operatorname{Aut}(G)$ such that $\varphi(v_i) = v_j$. Here $\operatorname{Aut}(G)$ denotes the set of automorphisms of the graph G:

$$\operatorname{Aut}(G) := \{ \text{bijection } \phi : V \to V | (u, v) \in E \text{ if and only if } (\phi(u), \phi(v)) \in E, \ \forall u, v \in V \}.$$

Since a vertex-transitive graph must be a regular graph, Remark 2.3 implies that for player i, the mean reversion level in the model dynamics is the arithmetic average of the states of all players in the neighborhood of v_i . As seen below, the symmetry of the graph allows for the possibility of deriving a semi-explicit form of the Markovian NE. The main result, Theorem 4.2, is presented in Section 4.1, followed by discussions and generalizations. Its constructions and proofs are provided in later sections. To proceed, we first need the following definitions.

Let $\mathscr{X} \subset \mathbb{S}^{N \times N}$ be a subset of all symmetric $N \times N$ matrices, such that

$$\mathscr{X} := \{X : X \ge 0 \text{ is a polynomial of } L\}, \tag{4.1}$$

where the inequality between symmetric matrices is defined in terms of the semi-positive-definite sense. That is, $A \geq B$ if and only if A - B is semi-positive-definite, for $A, B \in \mathbb{S}^{N \times N}$.

4.1 Main results

Theorem 4.2. Let G be a simple connected vertex-transitive graph with undirected edges. Assume $q^2 = \varepsilon$ in problem (2.1)–(2.3). Then for any T > 0, the solution to the following ODE, denoted as $R: [0,T] \to \mathcal{X}$, exists and is unique.

$$R'(t) = \frac{1}{c} \text{Tr} \left[Q'(R(t)) e^{-t(a+q)L} \right] e^{-t(a+q)L}, \quad R(0) = 0, \tag{4.2}$$

where the function $Q: \mathscr{X} \to \mathbb{R}$ is defined as

$$Q(X) := [\det(I + cXL)]^{\frac{1}{N}}. \tag{4.3}$$

The solution F^i to the Riccati system (2.11) is given by:

$$F_t^i = \frac{1}{\frac{\text{Tr}(P_t) - (a+q)\text{Tr}(L)}{N}} [P_t - (a+q)L]e_i e_i^{\text{T}} [P_t - (a+q)L], \tag{4.4}$$

where P is constructed in terms of R that

$$P_t = (a+q)L + R'(T-t)cL[I + R(T-t)cL]^{-1}.$$
(4.5)

The Markovian NE for player i is given by

$$\hat{\alpha}^i(t,x) = -qe_i^{\mathrm{T}}Lx - e_i^{\mathrm{T}}F_t^i x. \tag{4.6}$$

Proof. We first show that R takes values in \mathscr{X} . From equation (4.2), it is clear that R can be written as

$$R(t) = \int_0^t C(s)e^{-s(a+q)L} \,\mathrm{d}s,$$

where $C:[0,t]\to\mathbb{R}_+$ is a function containing model parameters. Since $L\in\mathbb{S}^{N\times N}$, it has a characteristic polynomial Λ with $\deg\Lambda=N$. By the Cayley–Hamilton theorem [Lax07], $\Lambda(L)=0$. The matrix exponential $e^{-s(a+q)L}:=f_s(L)$ is an analytic function of L, where $f_s(x)=\sum_{k=0}^{\infty}\frac{[-s(a+q)]^k}{k!}x^k$. There exists polynomials M_s and N_s , with $\deg N_s\leq N-1$, such that $f_s(x)=\Lambda(x)M_s(x)+N_s(x)$. Let $N_s(x)$ be $N_s(x)=\sum_{k=0}^{N-1}a_k(s)x^k$, this results in $e^{-s(a+q)L}=f_s(L)=N_s(L)=\sum_{k=0}^{N-1}a_k(s)L^k$, $\forall s\in[0,t]$, which leads to

$$R(t) = \sum_{k=0}^{N-1} \left(\int_0^t C(s) a_k(s) \, \mathrm{d}s \right) L^k.$$

Hence it is proved that R must take values in \mathscr{X} . The Markovian NE in equation (4.6) follows directly from equation (2.12).

Then, it suffices to establish the following: (i) The existence and uniqueness of solutions to (4.2), which will be discussed in Section 4.3. (ii) The function F^i as constructed in equation (4.4) is a solution to the Riccati system (2.11). This will be addressed in Section 4.4.

Remark 4.3 (The law of the equilibrium state process). Plugging the equilibrium strategy (4.6) into the state dynamics (2.1) yields the dynamics of the equilibrium state process $\hat{X}_t := [\hat{X}_t^1, \dots, \hat{X}_t^N]^T$:

$$\mathrm{d}\hat{X}_t = -P_t \hat{X}_t \, \mathrm{d}t + \sigma \, \mathrm{d}W_t,$$

where P_t is defined in equation (4.9) and has the representation (4.5) given by Theorem 4.2. Given the initial condition $\hat{X}_0 = X_0$, this Ornstein-Uhlenbeck dynamics has a closed-form solution:

$$\hat{X}_t = e^{-\int_0^t P_s \, ds} X_0 + \sigma \int_0^t e^{-\int_s^t P_u \, du} \, dW_s,$$

where we have used the fact that P_s and P_t commute for any $t, s \in [0, T]$. Therefore, \hat{X}_t is Gaussian distributed with mean $e^{-\int_0^t P_s \, \mathrm{d}s} X_0$ and variance $\sigma^2 \int_0^t e^{-2\int_s^t P_u \, \mathrm{d}u} \, \mathrm{d}s$. Using equations (4.18) and (4.31) in Section 4.2 below, which can be verified to be correct, the mean and variance of \hat{X}_t can be represented in terms of R, the solution to equation (4.2).

We note the following, (i) Theorem 4.2 applies specifically to the case where $q^2 = \varepsilon$. As will discussed in Remark 4.8, this condition results in an uncoupled nature of the underlying linear system, greatly simplifying the calculations. (ii) A numerical verification of Theorem 4.2 will be presented in Section 5.1. (iii) Theorem 4.2 can be generalized to provide a semi-explicit Markovian NE for games where the mean reversion level is the average of states within a more broadly defined neighborhood of a player, such as the l-neighborhood (see Definition 4.4 and Remark 4.5). This extends the concept of a graph neighborhood as introduced in Definition 2.1. The generalized result is briefly outlined below.

Definition 4.4. The *l*-neighborhood of a vertex v in a graph G, denoted $N_G^l(v)$, is the collection of all vertices that are exactly l steps away from v, i.e.,

 $N_G^l(v) = \{u \in V : \text{there exists a path of length } l \text{ between } u \text{ and } v\}$.

Remark 4.5 (Generalizations of Theorem 4.2). To set the mean reversion level as the average within the l-neighborhood, we consider the following generalized state dynamics on a vertex-transitive graph G:

$$dX_{t}^{i} = \left[a \left(\sum_{j: v_{j} \in N_{G}^{l}(v_{i})} \frac{n(v_{j}, v_{i}; l)}{|N_{G}^{l}(v_{i})|} X_{t}^{j} - X_{t}^{i} \right) + \alpha_{t}^{i} \right] dt + \sigma dW_{t}^{i}, \quad \forall i \in [N],$$
(4.7)

where $n(v_j, v_i; l)$ stands for the number of paths of length l between v_i and v_j . Similarly, the running and terminal costs of player $i \in [N]$ are set as

$$f^{i}(t, x, \alpha) = \frac{1}{2}(\alpha)^{2} - q\alpha \left(\sum_{j:v_{j} \in N_{G}^{l}(v_{i})} \frac{n(v_{j}, v_{i}; l)}{|N_{G}^{l}(v_{i})|} x^{j} - x^{i} \right) + \frac{\varepsilon}{2} \left(\sum_{j:v_{j} \in N_{G}^{l}(v_{i})} \frac{n(v_{j}, v_{i}; l)}{|N_{G}^{l}(v_{i})|} x^{j} - x^{i} \right)^{2},$$

$$g^{i}(x) = \frac{c}{2} \left(\sum_{j:v_{j} \in N_{G}^{l}(v_{i})} \frac{n(v_{j}, v_{i}; l)}{|N_{G}^{l}(v_{i})|} x^{j} - x^{i} \right)^{2}.$$

In the generalized state dynamics (4.7), the mean reversion level is interpreted as a weighted average of all the players' states within $N_G^l(v_i)$. Each vertex $v_j \in N_G^l(v_i)$ is assigned a weight of $n(v_j, v_i; l)/|N_G^l(v_i)|$, which is proportional to the number of paths of length l between v_i and v_j .

To express the mean reversion level in a matrix form, as done in equation (2.7), we define A as the adjacency matrix of G, and δ as the common degree shared by all vertices in G. Since the graph is vertex-transitive, the cardinality of the l-neighborhood can be calculated as follows:

$$|N_G^l(v_i)| = \sum_{j=1}^N (A^l)_{ij} = \sum_{k=1}^N (A^{l-1})_{ik} \sum_{j=1}^N A_{kj} = \delta \sum_{k=1}^N (A^{l-1})_{ik} = \dots = \delta^l,$$

for any $v_i \in V$. Due to the properties of the adjacency matrix, $n(v_j, v_i; l) = e_i^{\mathrm{T}} A^l e_j$, allowing us to rewrite the mean reversion level as $\sum_{j:v_j \in N_G^l(v_i)} \frac{n(v_j, v_i; l)}{|N_G^l(v_i)|} X_t^j = e_i^{\mathrm{T}} (A^l/\delta^l) X_t$. According to Definition 2.2, $L = I - \frac{1}{\delta} A$. Thus,

$$\sum_{j:v_j \in N_G^l(v_i)} \frac{n(v_j, v_i; l)}{|N_G^l(v_i)|} X_t^j - X_t^i = -e_i^{\mathrm{T}} M(l, L) X_t, \quad \text{where} \quad M(l, L) = I - (I - L)^l.$$

For the game described above, under the assumption that $q^2 = \varepsilon$, the Markovian NE can also be constructed semi-explicitly, similar to Theorem 4.2. The only modification needed is replacing the graph Laplacian L in Theorem 4.2 with M(l, L).

Another game with favorable results occurs when the mean reversion level is a weighted average of players' states that are at most l steps away, i.e., averaging among $v_j \in \bigcup_{k=1}^l N_G^k(v_i)$. In this case, the mean reversion term in the state dynamics and cost functions is:

$$\sum_{j:v_j \in \bigcup_{k=1}^l N_G^k(v_i)} \frac{\sum_{k=1}^l w^k n(v_j, v_i; k)}{\sum_{k=1}^l w^k |N_G^k(v_i)|} X_t^j - X_t^i, \tag{4.8}$$

where each vertex $v_j \in \bigcup_{k=1}^l N_G^k(v_i)$ is assigned a weight $\sum_{k=1}^l w^k n(v_j, v_i; k) / \sum_{k=1}^l w^k |N_G^k(v_i)|$, proportional to a discounted sum of the number of paths of length $k \in \{1, \ldots, l\}$ between v_i and v_i . Each path of length k is discounted by a factor of w^k , where w denotes the discount rate.

The expression in equation (4.8) can be written in matrix form:

$$-e_i^{\mathrm{T}} S(l,L) X_t$$
, where $S(l,L) = I - \frac{1 - w\delta}{1 - (w\delta)^l} (I - L) [I - (w\delta)^l (I - L)^l] [I - w\delta (I - L)]^{-1}$.

Then, under the assumption that $q^2 = \varepsilon$ and $w \in (0, 1/\delta)$, a semi-explicit Makovian NE can be constructed similarly to Theorem 4.2, with the graph Laplacian L replaced by S(l, L).

4.2 Heuristic derivations

To construct the solution in equation (4.4), we draw inspiration from a previous work [LS22] and employ two key techniques: a fixed point scheme that aids in representing the solution to the Riccati system and the regular representation of the graph automorphism group that leverages symmetry. We will discuss them below one by one.

During the heuristic derivations below, several assumptions must be made. For a smoother presentation flow, these assumptions (A.1)–(A.4) are summarized later in Remark 4.11, and are referred to during the heuristic derivation. However, we emphasize that verifying these assumptions is not necessary due to the existence of the verification procedure presented in Section 4.4.

Let $P:[0,T]\to\mathbb{R}^{N\times N}$ be a matrix-valued function defined in terms of F_t^1,\cdots,F_t^N :

$$P_t := \sum_{k=1}^{N} \left[(a+q)L + F_t^k \right] e_k e_k^{\mathrm{T}} = (a+q)L + \sum_{k=1}^{N} F_t^k e_k e_k^{\mathrm{T}}.$$
 (4.9)

Under assumption (A.1), the Riccati system (2.11) can be rewritten in terms of P_t :

$$\dot{F}_{t}^{i} - P_{t}F_{t}^{i} - F_{t}^{i}P_{t} + (\varepsilon - q^{2})Le_{i}e_{i}^{T}L + F_{t}^{i}e_{i}e_{i}^{T}F_{t}^{i} = 0, \quad F_{T}^{i} = cLe_{i}e_{i}^{T}L. \tag{4.10}$$

Treating P as given, the solution to equation (4.10) can be represented in terms of P_t , denoted as $F_t^1(P), \dots, F_t^N(P)$. Substituting $F_t^1(P), \dots, F_t^N(P)$ into equation (4.9) provides an equation solely for P:

$$P_t = (a+q)L + \sum_{k=1}^{N} F_t^k(P)e_k e_k^{\mathrm{T}}.$$

Once a fixed point P^* is obtained from the above equation, $F_t^k(P^*)$ naturally provides the solution. On the other hand, vertex-transitive graphs naturally exhibit rich symmetry, which can eliminate many quantities' dependence on the player index i. To facilitate the derivations, we use the regular representation of the graph automorphism group, which is defined below.

Definition 4.6. Recall from Definition 4.1 that $\operatorname{Aut}(G)$ denotes the set of automorphisms of G. For any $\varphi \in \operatorname{Aut}(G)$, the associated regular representation $R_{\varphi} \in \mathbb{R}^{N \times N}$ is defined as an invertible matrix such that $\forall i \in [N]$,

$$R_{\varphi}e_i = e_{\varphi(i)}.$$

Immediately from Definition 4.6, one has $R_{\varphi}R_{\psi} = R_{\varphi \circ \psi}$, $R_{\varphi}^{-1} = R_{\varphi^{-1}} = R_{\varphi}^{\mathrm{T}}$, $\forall \varphi, \psi \in \mathrm{Aut}(G)$. The following lemma outlines key properties for leveraging the symmetry of vertex-transitive graphs.

Lemma 4.7 ([LS22, Lemma 4.1]). For a vertex-transitive graph G with the vertex set V = [N], the following statements are true:

- (i) L commutes with R_{φ} for every $\varphi \in \text{Aut}(G)$.
- (ii) If $Y \in \mathbb{R}^{N \times N}$ commutes with R_{φ} for every $\varphi \in \operatorname{Aut}(G)$, then $Y_{ii} = \frac{1}{N}\operatorname{Tr}(Y)$, $\forall i \in [N]$.
- (iii) If $Y^1, \dots, Y^N \in \mathbb{R}^{N \times N}$ are such that $R_{\varphi}Y^i = Y^{\varphi(i)}R_{\varphi}$ holds for every $\varphi \in \operatorname{Aut}(G)$ and $i \in [N]$, then $Y^i_{ii} = Y^j_{ij}$, $\forall i, j \in [N]$.

4.2.1 Solving F^i in terms of P

Our first step is to solve the Riccati equation (4.10) for F_t^i , treating P as given, and represent the solution in terms of P. To this end, we consider the linear system for (Y_t^i, Λ_t^i) :

$$\begin{bmatrix} \dot{Y}_t^i \\ \dot{\Lambda}_t^i \end{bmatrix} = \begin{bmatrix} P_t & -(\varepsilon - q^2) L e_i e_i^{\mathrm{T}} L \\ e_i e_i^{\mathrm{T}} & -P_t \end{bmatrix} \begin{bmatrix} Y_t^i \\ \Lambda_t^i \end{bmatrix}, \tag{4.11}$$

with terminal conditions $Y_T^i = cLe_ie_i^{\mathrm{T}}L$, $\Lambda_T^i = I$. If the solution to equation (4.11) is found and Λ_t^i is invertible for any $t \in [0,T]$, then $F_t^i = Y_t^i(\Lambda_t^i)^{-1}$ is the solution to equation (4.10) [BJP00]. Under the condition $q^2 = \varepsilon$, the equation for Y_t^i becomes decoupled, resulting in $Y_t^i = \tilde{P}_t Y_T^i$, where

$$\tilde{P}_t = e^{-\int_t^T P_s \, ds}.\tag{4.12}$$

By substituting $Y_t^i = \tilde{P}_t Y_T^i$ into equation (4.11), we obtain an ODE for Λ_t^i : $\dot{\Lambda}_t^i = e_i e_i^{\mathrm{T}} \tilde{P}_t Y_T^i - P_t \Lambda_t^i$. The solution to this equation: $\Lambda_t^i = \tilde{P}_t^{-1} \left(\Lambda_T^i - \int_t^T \tilde{P}_s e_i e_i^{\mathrm{T}} \tilde{P}_s Y_T^i \, \mathrm{d}s \right)$. Therefore, incorporating the terminal conditions, the underlying linear system yields the following solution:

$$\begin{cases} Y_t^i = c\tilde{P}_t L e_i e_i^{\mathrm{T}} L \\ \Lambda_t^i = \tilde{P}_t^{-1} \left(I - c \int_t^T \tilde{P}_s e_i e_i^{\mathrm{T}} \tilde{P}_s L e_i e_i^{\mathrm{T}} L \, \mathrm{d}s \right) \end{cases} .$$

Assuming (A.2) and applying Lemma 4.7(ii) gives,

$$e_i^{\mathrm{T}}\tilde{P}_sLe_i = \frac{1}{N}\mathrm{Tr}(\tilde{P}_sL).$$
 (4.13)

This simplifies Λ_t^i to:

$$\Lambda_t^i = \tilde{P}_t^{-1} \left(I - c \int_t^T \frac{\text{Tr}(\tilde{P}_s L)}{N} \tilde{P}_s e_i e_i^{\text{T}} L \, ds \right).$$

Under assumption (A.3), the solution to equation (4.10) is provided as

$$F_t^i = c\tilde{P}_t L e_i e_i^{\mathrm{T}} L \left(I - c \int_t^T \frac{\mathrm{Tr}(\tilde{P}_s L)}{N} \tilde{P}_s e_i e_i^{\mathrm{T}} L \, \mathrm{d}s \right)^{-1} \tilde{P}_t. \tag{4.14}$$

Up until now, we've derived a solution F_t^i expressed in terms of P.

Remark 4.8. The condition $q^2 = \varepsilon$ ensures that the underlying linear system for (Y_t^i, Λ_t^i) is decoupled. Without this condition, we need to solve the full system (4.11), for which the existence of a closed-form solution remains unclear since:

$$\begin{bmatrix} P_t & -(\varepsilon-q^2)Le_ie_i^{\mathrm{T}}L \\ e_ie_i^{\mathrm{T}} & -P_t \end{bmatrix} \text{ does not commute with } \begin{bmatrix} P_s & -(\varepsilon-q^2)Le_ie_i^{\mathrm{T}}L \\ e_ie_i^{\mathrm{T}} & -P_s \end{bmatrix} \text{ when } t \neq s.$$

4.2.2 Expressing P in terms of η

To further simplify equation (4.14), we define

$$\Sigma_t^i := \int_t^T \frac{\text{Tr}(\tilde{P}_s L)}{N} \tilde{P}_s e_i e_i^{\text{T}} L \, \mathrm{d}s, \tag{4.15}$$

and

$$\eta_t^i := e_i^{\mathrm{T}} L \left(I - c \Sigma_t^i \right)^{-1} \tilde{P}_t e_i. \tag{4.16}$$

Thus F_t^i admits the representation $F_t^i = c\tilde{P}_t L e_i e_i^{\mathrm{T}} L \left(I - c\Sigma_t^i\right)^{-1} \tilde{P}_t$ and satisfies $F_t^i e_i = c\eta_t^i \tilde{P}_t L e_i$. Under assumption (A.2), using Lemma 4.7(i) and equation (4.15) leads to $R_{\varphi} \Sigma_t^i = \Sigma_t^{\varphi(i)} R_{\varphi}$, $\forall \varphi \in \mathrm{Aut}(G)$. With the additional assumption (A.4), we find:

$$R_{\varphi}L\left(I-c\Sigma_{t}^{i}\right)^{-1}=L\left(I-c\Sigma_{t}^{\varphi(i)}\right)^{-1}R_{\varphi},\ \forall \varphi\in \mathrm{Aut}(G).$$

By Lemma 4.7(iii), η_t^i becomes independent of the player index *i*. From this point, we will use the notation η_t instead of η_t^i , as this quantity is shared by all players.

We now proceed to derive an expression for P_t in terms of η_t . Substituting the relation $F_t^i e_i = c\eta_t \tilde{P}_t L e_i$ into equation (4.9) yields

$$P_t = (a+q)L + c\eta_t \tilde{P}_t L, \tag{4.17}$$

where \tilde{P} is defined as in equation (4.12). It remains to solve the above equation for P_t .

Under assumption (A.1), any two symmetric matrices in the collection $\{L\} \cup \{P_t, P_t : t \in [0, T]\}$ commute, which implies that this collection of matrices can be diagonalized simultaneously. Therefore, one can work with the spectra of the matrices rather than directly with the matrices themselves, providing greater convenience for calculations. Let $\rho_t^1, \ldots, \rho_t^N$ be the eigenvalues of P_t and $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of L, equation (4.17) becomes

$$\rho_t^k = (a+q)\lambda_k + c\eta_t \lambda_k e^{-\int_t^T \rho_s^k ds}, \ \forall k \in [N].$$

Multiplying both sides by $e^{-(T-t)(a+q)\lambda_k}$ and integrating with respect to t over [0,T] gives

$$e^{\int_{t}^{T} \rho_{s}^{k} - (a+q)\lambda_{k} ds} = 1 + c\lambda_{k} \int_{t}^{T} \eta_{s} e^{-(T-s)(a+q)\lambda_{k}} ds.$$
 (4.18)

Taking the logarithm on both sides and differentiating with respect to t yields

$$\rho_t^k = (a+q)\lambda_k + \frac{c\lambda_k \eta_t e^{-(T-t)(a+q)\lambda_k}}{1 + c\lambda_k \int_t^T \eta_s e^{-(T-s)(a+q)\lambda_k} \, \mathrm{d}s}.$$
(4.19)

Rewriting the above equation in matrix form yields the representation

$$P_t = (a+q)L + c\eta_t L e^{-(T-t)(a+q)L} \left(I + \int_t^T c\eta_s e^{-(T-s)(a+q)L} \, \mathrm{d}s \, L \right)^{-1}. \tag{4.20}$$

The above expression for P_t in terms of η_t is not entirely satisfactory because η , as defined in (4.16), depends on \tilde{P}_t , and therefore on P_t . Consequently, equation (4.20) does not provide an explicit construction for P_t . The next step (in Section 4.2.3) is to find a way to characterize and construct η based solely on model parameters. Once this is achieved, equation (4.20) can be used to construct P_t , as detailed in Section 4.2.4.

4.2.3 The governing equation for η

According to the definition of η in (4.16), the primary challenge in characterizing η_t arises from the inverse term $(I - c\Sigma_t^i)^{-1}$. Combining equations (4.13) and (4.15) gives

$$(\Sigma_t^i)^2 = \left[\int_t^T \left(\frac{\operatorname{Tr}(\tilde{P}_s L)}{N} \right)^2 ds \right] \Sigma_t^i.$$

Using assumption (A.4) and repeatedly applying the above equation produces

$$(I - c\Sigma_t^i)^{-1} = I + c \left[1 - c \int_t^T \left(\frac{\operatorname{Tr}(\tilde{P}_s L)}{N} \right)^2 ds \right]^{-1} \Sigma_t^i.$$
 (4.21)

Plugging it back into equation (4.16) and combining it with equation (4.13) yields

$$\eta_t = \frac{g_t}{1 - c \int_t^T g_s^2 \, \mathrm{d}s},\tag{4.22}$$

where g_t is a scalar function defined as:

$$g_t := \frac{\operatorname{Tr}(\tilde{P}_t L)}{N}.\tag{4.23}$$

A property of g_t will be repeatedly referred to later, so we derive it here. Direct calculations based on equation (4.17) gives:

$$\operatorname{Tr}(\tilde{P}_t L) = \frac{1}{c\eta_t} \left[\operatorname{Tr}(P_t) - (a+q)\operatorname{Tr}(L) \right]. \tag{4.24}$$

Given $\text{Tr}(P_t) = \sum_{k=1}^{N} \rho_t^k$, combining it with equation (4.19) yields:

$$Ng_t = \text{Tr}(\tilde{P}_t L) = \sum_{k=1}^N \frac{\lambda_k e^{-(T-t)(a+q)\lambda_k}}{1 + c\lambda_k \int_t^T \eta_s e^{-(T-s)(a+q)\lambda_k} \, \mathrm{d}s}.$$
 (4.25)

Back to equation (4.22), by multiplying both sides by cg_t , integrating in t, and taking the exponential, we obtain:

$$e^{c \int_t^T g_s \eta_s \, \mathrm{d}s} = e^{\int_t^T \frac{c g_s^2}{1 - c \int_s^T g_u^2 \, \mathrm{d}u} \, \mathrm{d}s} = \frac{1}{1 - c \int_t^T g_s^2 \, \mathrm{d}s}.$$
 (4.26)

Combining equations (4.23) and (4.25), one has

$$c \int_{t}^{T} g_{s} \eta_{s} \, ds = \frac{1}{N} \sum_{k=1}^{N} \int_{t}^{T} \frac{c \lambda_{k} \eta_{s} e^{-(T-s)(a+q)\lambda_{k}}}{1 + c \lambda_{k} \int_{s}^{T} \eta_{u} e^{-(T-u)(a+q)\lambda_{k}} \, du} \, ds.$$
 (4.27)

Combining equations (4.26)–(4.27) and using similar arguments as done for equation (4.18) implies

$$\frac{1}{1 - c \int_t^T g_s^2 \, \mathrm{d}s} = \left[\prod_{k=1}^N \left(1 + c\lambda_k \int_t^T \eta_s e^{-(T-s)(a+q)\lambda_k} \, \mathrm{d}s \right) \right]^{\frac{1}{N}}.$$
 (4.28)

Plugging equations (4.23), (4.25) and (4.28) into equation (4.22) yields

$$\eta_t = \frac{1}{N} \left(\sum_{k=1}^N \frac{\lambda_k e^{-(T-t)(a+q)\lambda_k}}{1 + c\lambda_k \int_t^T \eta_s e^{-(T-s)(a+q)\lambda_k} \, \mathrm{d}s} \right) \left[\prod_{k=1}^N \left(1 + c\lambda_k \int_t^T \eta_s e^{-(T-s)(a+q)\lambda_k} \, \mathrm{d}s \right) \right]^{\frac{1}{N}}.$$
(4.29)

Finally, rewriting equation (4.29) in the matrix form we obtain:

$$\eta_t = \frac{1}{N} \operatorname{Tr} \left[Le^{-(T-t)(a+q)L} \left(I + cL \int_t^T \eta_s e^{-(T-s)(a+q)L} \, \mathrm{d}s \right)^{-1} \right]$$

$$\left[\det \left(I + cL \int_t^T \eta_s e^{-(T-s)(a+q)L} \, \mathrm{d}s \right) \right]^{\frac{1}{N}}. \quad (4.30)$$

Observe that equation (4.30) involves only the model parameters and the graph Laplacian L without relying on P_t , which aligns with our goal of characterizing η_t . Nevertheless, equation (4.30) remains complex and does not easily allow for proving existence and uniqueness. To address this, we apply matrix calculus techniques. The lemma below plays an important role in formulating the governing equation for η_t .

Lemma 4.9 ([PP08, Equation (41)]). For a matrix-valued function $A:[0,T]\to\mathbb{R}^{N\times N}$, if A(t)takes values as nonsingular matrices for any $t \in [0,T]$, then

$$\frac{\mathrm{d}\det A(t)}{\mathrm{d}t} = \mathrm{Tr}\left(\mathrm{adj}[A(t)]\frac{\mathrm{d}A(t)}{\mathrm{d}t}\right),$$

where $\operatorname{adj} B = B^{-1} \operatorname{det} B$ is the adjugate matrix, well-defined for any nonsingular matrix B. For a matrix-valued function $A: \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$, if A(X) takes values as nonsingular matrices for any $X \in \mathbb{R}^{N \times N}$, then

$$\frac{\mathrm{d}\det A(X)}{\mathrm{d}X} = \frac{\mathrm{Tr}\left(\mathrm{adj}[A(X)]\mathrm{d}A(X)\right)}{\mathrm{d}X}.$$

The next corollary follows easily from Lemma 4.9 and the chain rule of matrix calculus.

Corollary 4.10. For a matrix-valued function $R:[0,T]\to\mathbb{S}^{N\times N}$, if R(t) takes values as nonsingular matrices for any $t \in [0, T]$, then

$$\frac{\mathrm{d}[\det(I+cLR(t))]^{\frac{1}{N}}}{\mathrm{d}t} = \frac{1}{N}\mathrm{Tr}\left([I+cLR(t)]^{-1}cLR'(t)\right)\left[\det(I+cLR(t))\right]^{\frac{1}{N}}.$$

For a matrix-valued function $\mathscr{X} \ni X \mapsto I + cXL$, I + cXL takes values as nonsingular symmetric matrices for any $X \in \mathcal{X}$. This leads to

$$\frac{d[\det(I + cXL)]^{\frac{1}{N}}}{dX} = \frac{1}{N} [\det(I + cXL)]^{\frac{1}{N}} (I + cXL)^{-1} cL.$$

Recall the definition of Q from equation (4.3): $Q(X) := [\det(I + cXL)]^{\frac{1}{N}}$. If we define the function $R:[0,T]\to\mathbb{S}^{N\times N}$ as follows:

$$R(t) := \int_{T-t}^{T} \eta_s e^{-(T-s)(a+q)L} \, \mathrm{d}s, \tag{4.31}$$

then, based on equation (4.30) and Corollary 4.10 and the definition of Q, R(t) solves the following matrix-valued ODE:

$$R'(t) = \frac{1}{c} \text{Tr} \left[Q'(R(t)) e^{-t(a+q)L} \right] e^{-t(a+q)L}, \quad R(0) = 0,$$

which is exactly equation (4.2). Therefore, once a solution R is found,

$$R'(T-t) = \eta_t e^{-(T-t)(a+q)L}$$
(4.32)

provides the corresponding value of η_t .

We conclude this section by summarizing all the assumptions used so far.

Remark 4.11 (Assumptions for heuristic derivations). The following assumptions are made in the heuristic construction:

- (A.1) P_t takes values as symmetric matrices and commutes with L for any $t \in [0, T]$. P_t commutes with P_s for any $t, s \in [0, T]$.
- (A.2) \tilde{P}_t commutes with R_{φ} , $\forall t \in [0, T]$, $\forall \varphi \in \text{Aut}(G)$.
- (A.3) Λ_t^i is invertible, $\forall t \in [0, T], \forall i \in [N]$.
- (A.4) The power series $(I c\Sigma_t^i)^{-1} = \sum_{n=0}^{\infty} c^n (\Sigma_t^i)^n$ is valid.

We emphasize again that they do not need to be verified, as the fact that F_t^i given in (4.4) solves (2.11) is directly verified in Section 4.4.

4.2.4 The construction of the Markovian NE

Once η_t has been constructed based on equations (4.2)–(4.3), the Markovian NE can be determined as follows. From equations (4.20) and (4.32), P_t can be represented in terms of R as follows:

$$P_t = (a+q)L + R'(T-t) cL[I + R(T-t) cL]^{-1}.$$

which coincides with equation (4.5). By combining equations (4.14)–(4.15) and (4.21), we obtain

$$F_t^i = \frac{c}{1 - c \int_t^T \left(\frac{\text{Tr}(\tilde{P}(s)L)}{N}\right)^2 ds} \tilde{P}_t L e_i e_i^T L \tilde{P}_t. \tag{4.33}$$

The denominator of F_t^i can be simplified using equations (4.23)–(4.24):

$$1 - c \int_t^T \left(\frac{\operatorname{Tr}(\tilde{P}_s L)}{N} \right)^2 ds = \frac{1}{cN\eta_t^2} [\operatorname{Tr}(P_t) - (a+q)\operatorname{Tr}(L)]. \tag{4.34}$$

Plugging equations (4.17) and (4.34) into equation (4.33) leads to the result (4.4).

To this point, we've completed the heuristic derivation of the Markvoian NE for the game (2.1)–(2.3) on vertex-transitive graphs. Specifically, we provide expressions for F_t^i that depend on P_t , and P_t that depends on the solution R to equation (4.2). Next, we aim to justify the well-posedness of equation (4.2) as well as verifying that equation (4.4) does provide an exact solution to the coupled Riccati system (2.11).

4.3 The well-posedness of equation (4.2) and the existence of η

Due to the essential dependence of the semi-explicit Markovian NE on R(t), it is both natural and essential to establish the existence and uniqueness of solutions to (4.2). We recall readers of equations (4.1)–(4.3) as the definitions of \mathcal{X} , Q, and the matrix-valued ODE for R(t):

$$\mathscr{X} := \{X : X \ge 0 \text{ is a polynomial in } L\}, \tag{4.1}$$

$$R'(t) = \frac{1}{c} \operatorname{Tr} \left[Q'(R(t)) e^{-t(a+q)L} \right] e^{-t(a+q)L}, \quad R(0) = 0.$$
 (4.2)

$$Q(X) := [\det(I + cXL)]^{\frac{1}{N}}, \tag{4.3}$$

The definition of \mathscr{X} in (4.1) is proposed such that any two matrices in the set $\{L, X_1, X_2\}$ commute, allowing them to be simultaneously diagonalizable for any $X_1, X_2 \in \mathscr{X}$. Next, we establish the well-posedness of equation (4.2) within the set \mathscr{X} . We begin by proving a local result and then extend it using a concavity argument.

Theorem 4.12. There exists $t_0 \in \mathbb{R}_+$ and $x_0 \in (0, \frac{1}{4c})$ such that the solution R(t) to equation (4.2) exists and is unique in \mathcal{X} , when $t \in [0, t_0]$ and $0 \le R(t) \le x_0 I$.

Proof. Define the function $f:[0,T]\times\mathscr{X}\to\mathscr{X}$ such that

$$f(t,X) = \frac{1}{c} \text{Tr} \left[Q'(X) e^{-t(a+q)L} \right] e^{-t(a+q)L},$$
 (4.35)

with Q(X) defined in (4.3). Then equation (4.2) can be rewritten as R'(t) = f(t, R(t)) with the initial condition R(0) = 0. In the following, we use the Picard-Lindelöf theorem [Tes12] to prove the local existence and uniqueness of the solution.

We first check that \mathscr{X} is closed under the Picard iteration. Given the k^{th} function $R_k(t)$, the $(k+1)^{th}$ iteration is given by:

$$R_{k+1}(t) = \int_0^t \frac{1}{c} \text{Tr} \left[Q'(R_k(s)) e^{-s(a+q)L} \right] e^{-s(a+q)L} ds.$$

Following the argument in the first part of the proof of Theorem 4.2, if R_k takes values in \mathscr{X} , then R_{k+1} also takes values in \mathscr{X} .

We proceed to prove that f is continuous in t and Lipschitz continuous in X for all $t \in [0, t_0]$ and $X \in \mathscr{X}$ such that $X \leq x_0 I$. Since the continuity in t is straightforward, our focus will be on proving the Lipschitz continuity in X. Given that all eigenvalues of L lie in the interval [0, 2], it's clear that $||e^{-t(a+q)L}|| \leq 1$. Combining it with the Cauchy-Schwarz inequality under the matrix inner product $\langle A, B \rangle := \text{Tr}(A^T B)$ yields

$$||f(t, X_2) - f(t, X_1)|| \le \frac{1}{c} \sqrt{\text{Tr}\left(e^{-2t(a+q)L}\right)} \sqrt{\text{Tr}\left[\left(Q'(X_2) - Q'(X_1)\right)^2\right]}.$$
 (4.36)

The first term on the right-hand side of the above inequality has a trivial bound:

$$\sqrt{\operatorname{Tr}\left(e^{-2t(a+q)L}\right)} = \sqrt{\sum_{k=1}^{N} e^{-2t(a+q)\lambda_k}} \le \sqrt{N}.$$
(4.37)

Therefore, we focus on bounding the second term on the right-hand side of inequality (4.36).

Denote the eigenvalues of X as x_1, \dots, x_N , it's clear that $0 \le x_i \le x_0$, for any $i \in [N]$. Since L and $\forall X \in \mathscr{X}$ are simultaneously diagonalizable, an upper bound for Q(X) can be established:

$$Q(X) = \left[\prod_{k=1}^{N} (1 + cx_k \lambda_k) \right]^{\frac{1}{N}} \le 1 + 2cx_0.$$
 (4.38)

Using Corollary 4.10 for Q(X) and combining it with the triangle inequality gives

$$\operatorname{Tr}\left[\left(Q'(X_2) - Q'(X_1)\right)^2\right] \leq \frac{2c^2}{N^2} \operatorname{Tr}\left(L^2\left[Q(X_2)(I + cX_2L)^{-1} - Q(X_2)(I + cX_1L)^{-1}\right]^2\right) + \frac{2c^2}{N^2} \operatorname{Tr}\left(L^2\left[Q(X_2)(I + cX_1L)^{-1} - Q(X_1)(I + cX_1L)^{-1}\right]^2\right). \tag{4.39}$$

Now, it remains to bound the two trace terms on the right-hand side of inequality (4.39).

Denote by $x_{j,1}, \dots, x_{j,N}$ the eigenvalues of X_j where $j \in \{1, 2\}$. Notice that any two matrices in the set $\{X_1, X_2, L\}$ are simultaneously diagonalizable for any $X_1, X_2 \in \mathcal{X}$. This yields

$$\operatorname{Tr}\left[\left(Q'(X_{2}) - Q'(X_{1})\right)^{2}\right] \leq \frac{8c^{2}}{N^{2}} \sum_{k=1}^{N} \left[Q(X_{2}) \frac{1}{1 + cx_{2,k}\lambda_{k}} - Q(X_{2}) \frac{1}{1 + cx_{1,k}\lambda_{k}}\right]^{2} + \frac{8c^{2}}{N^{2}} \sum_{k=1}^{N} \left[Q(X_{2}) \frac{1}{1 + cx_{1,k}\lambda_{k}} - Q(X_{1}) \frac{1}{1 + cx_{1,k}\lambda_{k}}\right]^{2}. \quad (4.40)$$

We next bound the two summations on the right-hand side of inequality (4.40), respectively.

For the first summation in inequality (4.40), using equation (4.38) together with the facts that $1 + cx_{1,k}\lambda_k \ge \frac{1}{2}$ and $||X_2 - X_1|| = \max_{k \in [N]} |x_{2,k} - x_{1,k}|$ produces

$$\sum_{k=1}^{N} \left[Q(X_2) \frac{1}{1 + cx_{2,k}\lambda_k} - Q(X_2) \frac{1}{1 + cx_{1,k}\lambda_k} \right]^2 \leq \sum_{k=1}^{N} (1 + 2cx_0)^2 \left[\frac{c\lambda_k(x_{1,k} - x_{2,k})}{(1 + cx_{1,k}\lambda_k)(1 + cx_{2,k}\lambda_k)} \right]^2 \leq 64c^2 (1 + 2cx_0)^2 N \|X_2 - X_1\|^2, \tag{4.41}$$

For the second summation in inequality (4.40), using the bound for $1 + cx_{1,k}\lambda_k$ once again yields

$$\sum_{k=1}^{N} \left[Q(X_2) \frac{1}{1 + cx_{1,k} \lambda_k} - Q(X_1) \frac{1}{1 + cx_{1,k} \lambda_k} \right]^2 \le 4 \sum_{k=1}^{N} [Q(X_2) - Q(X_1)]^2.$$
 (4.42)

Applying the intermediate value theorem for each entry of Q'(X), one obtains the bound:

$$[Q(X_2) - Q(X_1)]^2 \le \|X_2 - X_1\|_F^2 \sup_{\substack{-\frac{1}{4c}I \le X \le x_0I\\X \in \mathscr{X}}} \|Q'(X)\|_F^2 \le N^2 \|X_2 - X_1\|_F^2 \sup_{\substack{-\frac{1}{4c}I \le X \le x_0I\\X \in \mathscr{X}}} \|Q'(X)\|^2, \tag{4.43}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. The supremum term in inequality (4.43) is bounded by:

$$\sup_{\substack{-\frac{1}{4c}I \le X \le x_0I\\X \in \mathcal{X}}} \|Q'(X)\|^2 \le \frac{c}{N} (1 + 2cx_0) \sup_{\substack{-\frac{1}{4c} \le x_1, \cdots, x_n \le x_0\\X \in [N]}} \frac{\lambda_k}{1 + cx_k \lambda_k} \le \frac{4c}{N} (1 + 2cx_0). \tag{4.44}$$

Plugging inequalities (4.43)–(4.44) into inequality (4.42) gives

$$\sum_{k=1}^{N} \left[Q(X_2) \frac{1}{1 + cx_{1,k} \lambda_k} - Q(X_1) \frac{1}{1 + cx_{1,k} \lambda_k} \right]^2 \le 64c^2 N(1 + 2cx_0)^2 ||X_2 - X_1||^2. \tag{4.45}$$

Combining inequalities (4.36)-(4.37), (4.40)-(4.41) and (4.45) yields

$$||f(t, X_2) - f(t, X_1)|| \le 32c(1 + 2cx_0)||X_2 - X_1||_2.$$

Therefore, f is Lipschitz continuous in X with Lipschitz constant $32c(1+2cx_0)$. This concludes the proof.

In general, without the local constraint $R(t) \leq x_0 I$, the function f defined in (4.35) fails to be globally Lipschitz. However, through a concavity argument, we prove that the solution R(t) has an a priori upper bound. This upper bound implies the global existence and uniqueness of the solution, summarized as follows.

Theorem 4.13. For any T > 0, the solution R(t) to equation (4.2) exists and is unique in \mathscr{X} on the finite time horizon $t \in [0,T]$.

Proof. Notice that the proof of Theorem 4.12 does not actually rely on t_0 . Therefore, it suffices to establish an a priori upper bound for R(t), i.e., finding $x_0 > 0$ such that $R(t) \leq x_0 I$, for all $t \in [0, T]$.

Let $r_1(t), \dots, r_N(t)$ denote the eigenvalues of R(t), such that $r'_1(t), \dots, r'_N(t)$ represent the eigenvalues of R'(t). It's clear that $r_1(t), \dots, r_N(t)$ are positive for $\forall t \in [0, T]$. Since any two

matrices in the set $\{L, R(t), R'(t)\}$ commute for any fixed $t \in [0, T]$, they can be simultaneously diagonalized. Rewritting equation (4.2) using the spectra of matrices, one has

$$r'_{k}(t) = \frac{1}{N} \left[\prod_{j=1}^{N} (1 + c\lambda_{j} r_{j}(t)) \right]^{\frac{1}{N}} \sum_{j=1}^{N} \frac{\lambda_{j} e^{-t(a+q)\lambda_{j}}}{1 + c\lambda_{j} r_{j}(t)} \cdot e^{-t(a+q)\lambda_{k}}, \quad r_{k}(0) = 0.$$
 (4.46)

Define $y_j(t) := \frac{\lambda_j e^{-t(a+q)\lambda_j}}{1+c\lambda_j r_j(t)}$, and $Q_R(t) := Q(R(t)) = \left[\prod_{j=1}^N (1+c\lambda_j r_j(t))\right]^{\frac{1}{N}}$ to capture the dependence of Q(R(t)) on time t. Taking the logarithm of $Q_R(t)$ and then differentiating with respect to t yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_R(t) = Q_R(t)\frac{1}{N}\sum_{i=1}^N \frac{c\lambda_j r_j'(t)}{1 + c\lambda_j r_j(t)}.$$

Plugging equation (4.46) into the equation above produces

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{R}(t) = \frac{Q_{R}^{2}(t)}{N^{2}} \sum_{k=1}^{N} \frac{\lambda_{k}e^{-t(a+q)\lambda_{k}}}{1 + c\lambda_{k}r_{k}(t)} \sum_{j=1}^{N} \frac{c\lambda_{j}e^{-t(a+q)\lambda_{j}}}{1 + c\lambda_{j}r_{j}(t)} = \frac{c}{N^{2}}Q_{R}^{2}(t) \left(\sum_{j=1}^{N} y_{j}(t)\right)^{2}.$$
(4.47)

In what follows, we establish the concavity of Q_R , which is crucial for achieving global well-posedness. Differentiating equation (4.47) with respect to t yields:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Q_R(t) = \frac{2c}{N^2} Q_R(t) \cdot \sum_{j=1}^N y_j(t) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t} Q_R(t) \cdot \sum_{j=1}^N y_j(t) + Q_R(t) \sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}t} y_j(t) \right]. \tag{4.48}$$

Using equation (4.46), the derivative of $y_i(t)$ is computed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}y_j(t) = -(a+q)\lambda_j y_j(t) - cQ_R(t)y_j^2(t)\frac{1}{N}\sum_{k=1}^N y_k(t). \tag{4.49}$$

Plugging equations (4.47) and (4.49) into equation (4.48), we compute the second-order derivative:

$$\frac{d^2}{dt^2}Q_R(t) = \frac{2c}{N^2}Q_R(t)\sum_{j=1}^N y_j(t) \left[\frac{c}{N^2}Q_R^2(t)\left(\sum_{j=1}^N y_j(t)\right)^3 - \frac{cQ_R^2(t)}{N}\sum_{j=1}^N y_j^2(t)\sum_{j=1}^N y_j(t) - (a+q)Q_R(t)\sum_{j=1}^N \lambda_j y_j(t)\right].$$

Using the trivial bound $(a+q)Q_R(t)\sum_{j=1}^N \lambda_j y_j(t) \ge 0$ and the Cauchy-Schwarz inequality yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Q_R(t) \le \frac{2c^2}{N} Q_R^3(t) \sum_{j=1}^N y_j(t) \left[\left(\frac{1}{N} \sum_{j=1}^N y_j(t) \right)^3 - \left(\frac{1}{N} \sum_{j=1}^N y_j(t) \right) \left(\frac{1}{N} \sum_{j=1}^N y_j^2(t) \right) \right] \le 0.$$

This proves the concavity of Q_R in t.

The concavity of Q_R yields the following estimate:

$$Q_R(t) \le Q_R(0) + Q_R'(0)t, \ \forall T > 0, \forall t \in [0, T].$$

By evaluating $Q_R'(0)$ using equation (4.47), one has $Q_R(t) \leq 1 + ct$. Since $y_j(t) \leq \lambda_j$, the inequality, combined with equation (4.46), implies $r_k'(t) \leq 1 + ct$. Integrating both sides w.r.t. t, we obtain an a priori bound $r_k(t) \leq \int_0^t (1+cs) \, \mathrm{d}s \leq T + \frac{c}{2}T^2$. Therefore, setting $x_0 = T + \frac{c}{2}T^2$ completes the proof.

By proving Theorem 4.13, we have established the first part of Theorem 4.2.

4.4 Verification of heuristic derivations

This section aims to verify that equation (4.4) is indeed the solution to the Riccati system (2.11). Although proving the existence and uniqueness of the solution R(t) to the governing equation (4.2) is convenient, using (4.2) for verification complicates the process.

To achieve this, we propose an alternative governing system for η , showing its connection with equation (4.2), which allows for an easier verification procedure. We once again emphasize that verifying the assumptions in Remark 4.11 is unnecessary since the fact that F_t^i given in (4.4) solves (2.11) is directly verified below.

4.4.1 The alternative governing equation for η

Consider the coupled ODE system:

$$\begin{cases}
Q_1(t) = \left[\det(I + cLR_1(t))\right]^{\frac{1}{N}} \\
R'_1(T - t) = \sqrt{\frac{Q'_1(T - t)}{c}}e^{-(T - t)(a + q)L}
\end{cases}, \quad R_1(0) = 0, \quad R'_1(0) = I,$$
(4.50)

where the solutions $Q_1:[0,T]\to\mathbb{R}$ and $R_1:[0,T]\to\mathscr{X}$ are defined on [0,T]. We want to relate the above system to equation (4.2). Specifically, we demostrate that any solution R to equation (4.2) must also provide a solution $(Q_1,R_1\equiv R)$ to equation (4.50), thus equation (4.50) offers an alternative characterization for η_t .

Let R(t) be a solution to equation (4.2), we first check the initial conditions in equation (4.50). By Corollary 4.10, $Q'(0) = \frac{c}{N}L$, thus $R'(0) = \frac{1}{c}\operatorname{Tr}\left[Q'(0)\right]I = I$. So it remains to show

$$\left(\operatorname{Tr}\left[Q'(R(t))e^{-t(a+q)L}\right]\right)^2 = cQ_1'(t),\tag{4.51}$$

with Q_1 defined in (4.50). By Corollary 4.10 and equation (4.3), one has

$$\operatorname{Tr}\left(Q'(R(t))e^{-t(a+q)L}\right) = \operatorname{Tr}\left(\frac{1}{N}\left[\det\left(I + cLR(t)\right)\right]^{\frac{1}{N}}\left(I + cLR(t)\right)^{-1}cLe^{-t(a+q)L}\right).$$

On the other hand, a calculation of $Q'_1(t)$ based on Corollary 4.10 provides

$$Q'_1(t) = \text{Tr}\left(\frac{1}{N}\left[\det\left(I + cLR(t)\right)\right]^{\frac{1}{N}}\left(I + cLR(t)\right)^{-1}cLR'(t)\right).$$

Replacing R'(t) in the above equation using equation (4.2) yields

$$Q_1'(t) = \frac{1}{c} \text{Tr} \left(Q'(R(t)) e^{-t(a+q)L} \right) \text{Tr} \left(\frac{1}{N} \left[\det \left(I + cLR(t) \right) \right]^{\frac{1}{N}} \left(I + cLR(t) \right)^{-1} cL e^{-t(a+q)L} \right),$$

which achieves equation (4.51).

4.4.2 Verification through the alternative governing equation for η

To conclude the proof of Theorem 4.2, it remains to prove that F_t^i provided by equation (4.4) is a solution to the Riccati system (2.11). We first check the terminal condition of the Riccati system (2.11). Since R(0) = 0 and R'(0) = I, equation (4.5) implies that $P_T = (a + q)L + cL$. Combining this with equation (4.4) yields $F_T^i = cLe_ie_i^TL$. To verify that equation (4.4) follows the Riccati system (2.11), we need to account for P_t , which depends on η_t . This is where we will use the alternative governing system (4.50). The verification process is divided into several steps.

Step 1: Rebuild the relationship between F_t^i and P_t . Let us define $\tau(t)$ as follows:

$$\tau(t) := \frac{\operatorname{Tr}(P_t) - (a+q)\operatorname{Tr}(L)}{N}.$$
(4.52)

Based on equation (4.5) and Lemma 4.7(i), $P_t - (a+q)L$ takes values as symmetric matrices and commutes with R_{φ} for all $\varphi \in \operatorname{Aut}(G)$ for any $t \in [0,T]$. Lemma 4.7(ii) implies that $e_k^{\mathrm{T}}[P_t - (a+q)L]e_k = \tau(t)$ for each $k \in [N]$. Combining this with equation (4.4) yields

$$\sum_{k=1}^{N} F_t^k e_k e_k^{\mathrm{T}} + (a+q)L = \sum_{k=1}^{N} [P_t - (a+q)L] e_k e_k^{\mathrm{T}} + (a+q)L = P_t.$$

This affirms the validity of equation (4.9) within the context of verification, which has been a key element throughout the heuristic construction of the equilibrium.

Step 2: The spectral form of the Riccati system (2.11). Using the definition of $\tau(t)$ from (4.52), equation (4.4) can be rewritten as $F_t^i = \frac{1}{\tau(t)}[P_t - (a+q)L]e_ie_i^{\mathrm{T}}[P_t - (a+q)L]$. With this expression regarding the Riccati system (2.11), it suffices to verify:

$$-\frac{\tau'(t)}{\tau(t)}[P_t - (a+q)L]e_ie_i^{\mathrm{T}}[P_t - (a+q)L] + \dot{P}_te_ie_i^{\mathrm{T}}[P_t - (a+q)L] + [P_t - (a+q)L]e_ie_i^{\mathrm{T}}\dot{P}_t$$

$$-P_t[P_t - (a+q)L]e_ie_i^{\mathrm{T}}[P_t - (a+q)L] - [P_t - (a+q)L]e_ie_i^{\mathrm{T}}[P_t - (a+q)L]P_t$$

$$+\tau(t)[P_t - (a+q)L]e_ie_i^{\mathrm{T}}[P_t - (a+q)L] = 0. \quad (4.53)$$

Since any two matrices in the set $\{P_t, L, R(T-t), R'(T-t)\}$ commute for any fixed $t \in [0, T]$, they are simultaneously diagonalizable. Recall that we denote $\lambda_1, \ldots, \lambda_N$ as the eigenvalues of L, and $\rho_t^1, \ldots, \rho_t^N$ as the eigenvalues of P_t . According to equation (4.50), the j-th eigenvalues of R(T-t) and R'(T-t) are $\int_t^T \sqrt{\frac{Q_1'(T-s)}{c}} e^{-(T-s)(a+q)\lambda_j} \, \mathrm{d}s$ and $\sqrt{\frac{Q_1'(T-t)}{c}} e^{-(T-t)(a+q)\lambda_j}$ respectively. As a result, equation (4.5) can be expressed in the spectral form as

$$\rho_t^j = (a+q)\lambda_j + \frac{c\lambda_j \sqrt{\frac{Q_1'(T-t)}{c}} e^{-(T-t)(a+q)\lambda_j}}{1 + c\lambda_j \int_t^T \sqrt{\frac{Q_1'(T-s)}{c}} e^{-(T-s)(a+q)\lambda_j} \, \mathrm{d}s}.$$
 (4.54)

To verify equation (4.53), it suffices to verify its spectral version:

$$-\frac{\tau'(t)}{\tau(t)}[\rho_t^j - (a+q)\lambda_j][\rho_t^k - (a+q)\lambda_k] + \dot{\rho}_t^j[\rho_t^k - (a+q)\lambda_k] + [\rho_t^j - (a+q)\lambda_j]\dot{\rho}_t^k \\ - \rho_t^j[\rho_t^j - (a+q)\lambda_j][\rho_t^k - (a+q)\lambda_k] - [\rho_t^j - (a+q)\lambda_j][\rho_t^k - (a+q)\lambda_k]\rho_t^k \\ + \tau(t)[\rho_t^j - (a+q)\lambda_j][\rho_t^k - (a+q)\lambda_k] = 0, \ \forall j, k \in [N],$$

whose left-hand side can be factored as the sum of two products:

$$\left(\dot{\rho}_{t}^{k} - \left[\rho_{t}^{k} - (a+q)\lambda_{k}\right]\rho_{t}^{k} - \frac{\rho_{t}^{k} - (a+q)\lambda_{k}}{2} \left[\frac{\tau'(t)}{\tau(t)} - \tau(t)\right]\right) \left[\rho_{t}^{j} - (a+q)\lambda_{j}\right] \\
+ \left(\dot{\rho}_{t}^{j} - \left[\rho_{t}^{j} - (a+q)\lambda_{j}\right]\rho_{t}^{j} - \frac{\rho_{t}^{j} - (a+q)\lambda_{j}}{2} \left[\frac{\tau'(t)}{\tau(t)} - \tau(t)\right]\right) \left[\rho_{t}^{k} - (a+q)\lambda_{k}\right].$$

Thus, it remains to check

$$\dot{\rho}_t^k - [\rho_t^k - (a+q)\lambda_k]\rho_t^k - \frac{\rho_t^k - (a+q)\lambda_k}{2} \left[\frac{\tau'(t)}{\tau(t)} - \tau(t) \right] = 0, \ \forall k \in [N],$$

which is equivalent to

$$\frac{\dot{\rho}_t^k}{\rho_t^k - (a+q)\lambda_k} - \rho_t^k = \frac{1}{2} \left[\frac{\tau'(t)}{\tau(t)} - \tau(t) \right], \ \forall k \in [N].$$

$$(4.55)$$

Step 3: Verifying equation (4.55). We compute both sides of equation (4.55) separately. For the left-hand side, define $g_k : [0, T] \to \mathbb{R}$ such that

$$g_k(s) := c\sqrt{\frac{Q_1'(T-s)}{c}}e^{-(T-s)(a+q)\lambda_k}.$$
 (4.56)

Straightforward calculations based on equation (4.54) give

$$\frac{\dot{\rho}_t^k}{\rho_t^k - (a+q)\lambda_k} - \rho_t^k = \frac{g_k'(t)}{g_k(t)} - (a+q)\lambda_k, \tag{4.57}$$

which corresponds to the k-th eigenvalue of the matrix $-R''(T-t)[R'(T-t)]^{-1} - (a+q)L$, using equation (4.50).

For the right-hand side, rewrite τ as a function of g_k using equations (4.52), (4.54) and (4.56):

$$\tau(t) = \frac{1}{N} \sum_{k=1}^{N} [\rho_t^k - (a+q)\lambda_k] = \frac{1}{N} \sum_{k=1}^{N} \frac{\lambda_k g_k(t)}{1 + \lambda_k \int_t^T g_k(s) \, \mathrm{d}s}.$$
 (4.58)

Next, rewriting equation (4.58) as the derivative of a logarithm yields

$$\tau(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \log \left[\prod_{k=1}^{N} \left(1 + \lambda_k \int_{t}^{T} g_k(s) \, \mathrm{d}s \right) \right]^{\frac{1}{N}} = -\frac{\mathrm{d}}{\mathrm{d}t} \log \left[\det \left(I + cL \int_{t}^{T} R'(T-s) \, \mathrm{d}s \right) \right]^{\frac{1}{N}}.$$

Using equation (4.50), it follows that

$$\tau(t) = -\frac{d}{dt} \log Q_1(T - t) = \frac{Q_1'(T - t)}{Q_1(T - t)}.$$

Taking the derivative with respect to t on both sides results in

$$\tau'(t) = \frac{[Q_1'(T-t)]^2 - Q_1(T-t)Q_1''(T-t)}{Q_1^2(T-t)}.$$

Thus, the right-hand side of equation (4.55) reads

$$\frac{1}{2} \left[\frac{\tau'(t)}{\tau(t)} - \tau(t) \right] = -\frac{Q_1''(T-t)}{2Q_1'(T-t)}.$$
(4.59)

Combining results from equations (4.57) and (4.59), proving the spectral equation (4.55) requires demonstrating the following matrix equation:

$$-R''(T-t)[R'(T-t)]^{-1} - (a+q)L = -\frac{Q_1''(T-t)}{2Q_1'(T-t)}I.$$
 (4.60)

Differentiating both sides of equation (4.50) with respect to t produces

$$R''(T-t) = \frac{Q_1''(T-t)}{2\sqrt{cQ_1'(T-t)}}e^{-(T-t)(a+q)L} - \sqrt{\frac{Q_1'(T-t)}{c}}e^{-(T-t)(a+q)L}(a+q)L. \tag{4.61}$$

Combining equations (4.61) and (4.50) yields equation (4.60), thus concluding the verification process and proving the second part of Theorem 4.2.

5 Numerical Experiments

In this section, we conduct several numerical experiments concerning the NE of the game introduced in Section 2.1. Section 5.1 computes the semi-explicit equilibrium derived in Theorem 4.2 under various vertex-transitive graphs, and numerically verifies its consistency with solving the Riccati system (2.11). Implied by Corollary 3.8, fictitious play is shown to be a generally applicable technique for any connected graphs with provable convergence. In Section 5.2, we visualize this convergence procedure and quantify its rate of convergence for different graphs. Lastly, in Section 5.3, a comparison of the time complexity and the running time among previously mentioned numerical methods is presented.

5.1 Numerical verification of Theorem 4.2

This section aims to numerically verify that the results derived in Theorem 4.2 for vertex-transitive graphs are consistent with the baseline equilibrium strategies. The baseline equilibrium strategy refers to the one obtained by directly solving the Riccati system (2.11) numerically using the explicit Runge-Kutta method of order 8 [WH96]. For cases where G is a complete graph, an additional baseline is constructed using the closed-form solution presented in [CFS15]. Regarding the semi-explicit equilibrium strategy, we first solve equation (4.2) numerically for R, and then build F^i from R using equations (4.4)–(4.5).

In the numerical experiments, we fix the following model parameters:

$$\sigma = 0.5, \quad a = 0.1, \quad c = 1, \quad X_0 = x_0 \in \mathbb{R}^N,$$

where the initial state x_0 is randomly sampled from a uniform distribution $x_0^i \overset{\text{i.i.d.}}{\sim} \mathcal{U}(-1,1), \ \forall i \in [N]$. The remaining model parameters (q,ε) , N, T and G vary and will be specified in Table 1. Throughout the experiments, the relationship $q^2 = \varepsilon$ is maintained and G is taken as one of the following vertex-transitive graphs:

 $KG_{5,2}$: the 10-vertex Petersen graph, K_N : the N-vertex complete graph,

 C_N : the N-vertex cycle graph, Q_k : the 2^k -vertex hypercube graph,

 $C_N^{s_1,\ldots,s_k}$: the N-vertex circulant graph with jumps s_1,\ldots,s_k .

These graphs are of great research interest in their own, and we refer readers to [Chu97, GR01] for their definitions, constructions, and properties.

For the time discretization, we partition the time horizon [0,T] into $N_T = 1000$ subintervals of equal lengths $h := T/N_T$, and denote the discretization scheme by $\Delta := \{kh : k \in \{0,1,\ldots,N_T-1\}\}$, which is the collection of all the subintervals' endpoints. The difference between the baseline and semi-explicit equilibrium is measured by maximum absolute error (MAE) and maximum relative error (MRE):

$$MAE(\tilde{F}; \check{F}) := \max_{t \in \Delta} \max_{i \in [N]} \|\check{F}_t^i - \tilde{F}_t^i\|, \quad MRE(\tilde{F}; \check{F}) := \max_{t \in \Delta} \max_{i \in [N]} \frac{\|\check{F}_t^i - \tilde{F}_t^i\|}{\|\check{F}_t^i\|}, \tag{5.1}$$

where \check{F}^i represents the numerical solution of the Ricatti system (2.11), and \tilde{F}^i denotes the semi-explicit counterpart. Small values of MAE and MRE indicate the alignment between the baseline and the semi-explicit construction, providing evidence for the correctness of Theorem 4.2. Table 1 provides the MAE and MRE values for various models.

# of Players	N = 10	N = 30	N = 40	N = 64	N = 128	N = 150
Graph G	$KG_{5,2}$	K_{30}	C_{40}	Q_6	Q_7	C_{150}^{11}
Time T	1	1	1.5	1.5	2	2.5
(q,arepsilon)	(0,0)	(1, 1)	(0,0)	(2, 4)	(0, 0)	(1, 1)
MAE	3.88×10^{-6}	2.09×10^{-5}	7.12×10^{-6}	4.67×10^{-6}	1.92×10^{-6}	3.16×10^{-5}
MRE	5.07×10^{-6}	8.96×10^{-5}	1.44×10^{-5}	8.21×10^{-4}	5.30×10^{-6}	1.16×10^{-3}

Table 1: Model parameters, MAE, and MRE for the examples in Section 5.1

Next, we visualize the equilibrium state and strategy processes. To this end, we adopt the Euler scheme to equation (2.1) and simulate it with the feedback strategy α_t^i plugged in using the previously computed \tilde{F}_t^i and \tilde{F}_t^i (cf. (2.12) and (4.6)):

$$\hat{X}_{t+h}^{i} = \hat{X}_{t}^{i} + \left[a \left(\frac{1}{\sqrt{d_{v_{i}}}} \sum_{j: v_{j} \sim v_{i}} \frac{1}{\sqrt{d_{v_{j}}}} \hat{X}_{t}^{j} - \hat{X}_{t}^{i} \right) + \hat{\alpha}_{t}^{i} \right] h + \sigma \sqrt{h} \xi_{t}^{i}, \; \xi_{t}^{i \text{ i.i.d.}} \sim \mathcal{N}(0, 1),$$

$$\hat{\alpha}_{t}^{i} = -q e_{i}^{T} L \hat{X}_{t} - e_{i}^{T} F_{t}^{i} \hat{X}_{t}, \quad \forall i \in [N], \; \forall t \in \Delta, \text{ where } F^{i} = \check{F}^{i} \text{ or } \tilde{F}^{i}.$$

In Figures 1–2, trajectories of the equilibrium state process \hat{X} and the equilibrium strategy process $\hat{\alpha}$ from four randomly selected players are presented for complete and cycle graphs, with the choice of time discretization $N_T = 50$, time horizon T = 1, and model parameters $(q, \varepsilon) = (1, 1)$.

5.2 Numerical convergence of fictitious play

Next, we examine the convergence of FP, visualize the convergence procedure, and compare the rate of convergence for different graphs. Our experiment identifies a contraction-mapping-like structure, linking the rate of convergence to the graph structure. Moreover, as will be observed below, condition (3.10) for Corollary 3.8 is sufficient but not necessary for the convergence of FP. This further validates the general applicability of FP as a state-of-the-art numerical technique solving for the Nash equilibrium.

In the numerical experiments, we fix the following model parameters:

$$N = 50, \quad T = 1, \quad q = 0.5, \quad \varepsilon = 1, \quad \sigma = 0.5, \quad a = 0.1, \quad c = 1, \quad X_0 = x_0 \in \mathbb{R}^N,$$

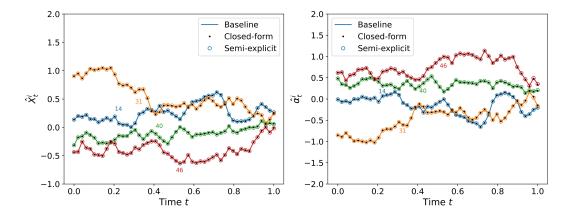


Figure 1: Comparisons of equilibrium state (left panel) and strategy (right panel) trajectories for N=50 players in the linear-quadratic game on the complete graph $G=K_{50}$. In both panels, the colored solid lines represent the baseline solution (by numerically solving the Riccati system (2.11)), the colored circles are obtained by numerically solving the semi-explicit solution in Theorem 4.2, and the black dots are computed by the closed-form solution given in [CFS15]. For the sake of clarity, only trajectories of four randomly selected players are plotted.

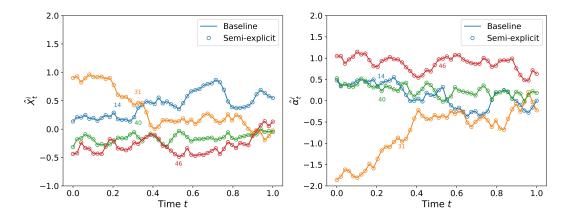


Figure 2: Comparisons of equilibrium state (left panel) and strategy (right panel) trajectories for N = 50 players in the linear-quadratic game on the cycle graph $G = C_{50}$. In both panels, the colored solid lines represent the baseline solution (by numerically solving the Riccati system (2.11)), and the colored circles are obtained by numerically solving the semi-explicit solution in Theorem 4.2. For the sake of clarity, only trajectories of four randomly selected players are plotted.

where the initial state x_0 is randomly sampled from a uniform distribution $x_0^i \stackrel{\text{i.i.d.}}{\sim} \mathcal{U}(-1,1), \ \forall i \in [N]$. Throughout the experiments, G is taken as one of the following graphs, with the specific choice given in Tables 2–3:

 K_N : the N-vertex complete graph, C_N : the N-vertex cycle graph,

 S_N : the N-vertex star graph,

 $K_{m,m}$: the 2m-vertex complete bipartite graph with partitions of size m and m,

 RSG_N^s : the N-vertex random sparse graph with edge density s,

 RMJ_N^d : the N-vertex d-regular random Ramanujan graph.

Among the graphs mentioned above, RSG_N^s and RMJ_N^d are random graphs, while the others are deterministic. For the construction of random sparse graphs, random edges are added to a randomly generated spanning tree until the sparsity measure of the graph satisfies certain conditions. Here, we select the edge density s as the sparsity measure, defined as $s := |E|/\binom{|V|}{2}$, which is the ratio between the number of edges present in the graph and the maximum possible number of edges the graph can contain. On the other hand, the d-regular Ramanujan graph [LPS88] is a randomly generated regular expander graph with common degree d, which essentially saturates the Alon–Boppana bound [HLW06]. Ramanujan graphs are important examples of highly connected sparse graphs [Chu97], demonstrating the possible coexistence of two seemingly contradictory graph properties. Therefore, we include them in the experiments.

Starting with initial strategies $\check{F}_t^{i,0} = 0$, $\forall i \in [N], t \in \Delta$, we compute $\{\check{F}_t^{i,k} : i \in [N], t \in \Delta\}$ at stage $k \in \mathbb{N}$ by numerically solving system (3.6) iteratively with a time discretization level $N_T = 1000$. We perform $N_{\text{round}} = 10$ rounds of FP for each numerical example and assess the convergence of FP by calculating $\text{MAE}(\check{F}^{\cdot,N_{\text{round}}};\check{F})$ and $\text{MRE}(\check{F}^{\cdot,N_{\text{round}}};\check{F})$, both of which have been previously defined in equation (5.1). The numerical results are organized in Tables 2–3: Table 2 for deterministic graphs and Table 3 for three different samples of random graphs RSG_N^s and RMJ_N^d .

Table 2: MAE and MRE for LQ games on deterministic graphs in Section 5.2

Problem	$G = K_{50}$	$G = C_{50}$	$G = S_{50}$	$G = K_{25,25}$
MAE MRE	$\begin{array}{ c c c c c c } 2.19 \times 10^{-3} \\ 3.21 \times 10^{-3} \end{array}$	8.58×10^{-2} 1.02×10^{-1}	1.50×10^{-1} 1.69×10^{-1}	1.76×10^{-2} 2.26×10^{-2}

Table 3: MAE and MRE for LQ games on random graphs in Section 5.2

Duablana	$G = RSG_{50}^{0.125}$			$G = \text{RMJ}_{50}^6$			
Problem	Sample 1	Sample 2	Sample 3	Sample 1	Sample 2	Sample 3	
	5.62×10^{-2}						
MRE	7.23×10^{-2}	6.03×10^{-2}	5.98×10^{-2}	4.82×10^{-2}	4.82×10^{-2}	4.83×10^{-2}	

The MAEs and MREs shown in Tables 2–3 are consistently small, implying the numerical convergence of FP, even when the model parameters are deliberately chosen to not satisfy condition (3.10) of Corollary 3.8. We observe that the magnitude of errors varies for different graph

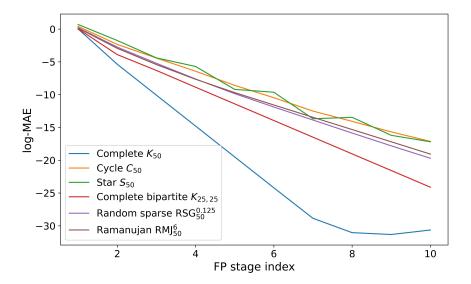


Figure 3: Comparisons of log-MAE curves for the linear-quadratic game with N=50 players on different graphs. The behavior at the tail of the log-MAE curves for the complete graphs is due to the numerical round-off error.

structures. To better compare the convergence rates, we plot $\log \text{MAE}(\check{F}^{\cdot,k-1};\check{F}^{\cdot,k})$ vs. the FP stage index k, as shown in Figure 3. The log-MAE curves are nearly straight lines with slopes depending on the graph structures. This implies a contraction-mapping-like structure

$$MAE(\check{F}^{\cdot,k}; \check{F}^{\cdot,k+1}) \approx C_G \cdot MAE(\check{F}^{\cdot,k-1}; \check{F}^{\cdot,k}), \ \forall k \in \mathbb{N},$$
(5.2)

where $C_G \in (0,1)$ is a constant that depends on the graph G. A simple linear regression gives

$$C_{K_{50}} = -4.55, \ C_{C_{50}} = -1.94, \ C_{S_{50}} = -2.01, \ C_{K_{25,25}} = -2.61, \ C_{\text{RSG}_{50}^{0.125}} = -2.06, \ C_{\text{RMJ}_{50}^6} = -2.07.$$

For the purpose of comparison, we provide below the norms of the graph Laplacians L_G used in the numerical experiments, denoted as $N_G := ||L_G|| \in [0, 2]$, that

$$N_{K_{50}} = 1.02, \ N_{C_{50}} = 2.00, \ N_{S_{50}} = 2.00, \ N_{K_{25,25}} = 2.00, \ N_{RSG_{50}^{0.125}} = 1.68, \ N_{RMJ_{50}^6} = 1.70.$$

A pairwise comparison between complete graphs and the other graphs shows that a large norm of the graph Laplacian possibly results in a slower rate of convergence of FP, which is consistent with the insights provided in Remark 3.10. However, the norm of the graph Laplacian is not the only factor that affects the rate of convergence of FP. For instance, the complete bipartite graph has $N_{K_{25,25}} = 2.00$, taking the largest value on [0, 2], yet FP converges faster on complete bipartite graphs compared to random sparse graphs and Ramanujan graphs. Consequently, accurately quantifying the rate of convergence of FP in terms of the graph structure requires extra work and is left for future studies.

5.3 The time complexity analysis and running time comparison

So far, we have implemented three numerical methods for solving the NE of linear-quadratic games on graphs: (i) the baseline construction by directly solving the Riccati system (2.11); (ii) the semi-explicit solution based on Theorem 4.2; and (iii) fictitious play by iteratively solving system (3.6).

In this section, we conduct a comparative study of the efficiency of these methods by investigating both the theoretical time complexity and the numerical running time, aiming to provide guidance on the trade-off between the efficiency and the generality of these numerical methods.

The time complexity analysis. We first discuss the time complexity of the three numerical methods, and the comparison intends to show the efficiency of constructing the semi-explicit NE using Theorem 4.2.

For illustration purpose, the analysis of time complexity uses the forward Euler method as the numerical ODE solver. In the baseline construction for solving the Riccati system (2.11), each Euler step computes $(\check{F}^1_{t+h}, \ldots, \check{F}^N_{t+h})$ based on $(\check{F}^1_t, \ldots, \check{F}^N_t)$. The calculation of the derivative \dot{F}^i_t requires $O(N^3)$ calculations, so a single Euler step costs $O(N^4)$, and constructing the entire baseline solution requires $O(N_T N^4)$. Meanwhile, only one matrix-valued ODE is numerically solved during the semi-explicit construction, so the whole procedure only costs $O(N_T N^3)$.

Lastly, we analyze the time complexity of FP. To conduct a fair comparison, the number of rounds of FP, denoted by N_{round} , should be such that $\check{F}^{\cdot,N_{\text{round}}}$ has approximately the same order of error as the baseline construction \check{F} . Since the Euler method has convergence order one [Kre12], i.e., $\text{MAE}(\check{F},F)=O(h)$ as $h\to 0$ with a time discretization scheme that has step size h, N_{round} should be such that $\text{MAE}(\check{F}^{\cdot,N_{\text{round}}},F)=O(h)$. Due to the contraction-mapping-like structure (5.2) empirically observed, $N_{\text{round}}=O(\log\frac{1}{h})$ and the entire procedure of FP costs $O(N_T\log N_T N^4)$. The above discussion is summarized in Table 4 below.

Table 4: Time complexity analysis of three numerical methods. All constructions have a maximum absolute error of the same order.

Construction Methods	Baseline	Semi-explicit	Fictitious play
Time Complexity	$O(N_T N^4)$	$O(N_T N^3)$	$O(N_T \log N_T N^4)$

Running time. Next, we compare the running time of the three methods under the following choice of model parameters:

$$G = K_N, \quad T = 1, \quad q = 1, \quad \varepsilon = 1, \quad \sigma = 0.5, \quad a = 0.1, \quad c = 1, \quad X_0 = x_0 \in \mathbb{R}^N,$$

where the initial state x_0 is randomly sampled from a uniform distribution $x_0^i \stackrel{\text{i.i.d.}}{\sim} \mathcal{U}(-1,1), \ \forall i \in [N]$. The remaining model parameter N varies and will be specified in Table 5. To ensure a fair comparison, we fix the error tolerance level at tol = 10^{-5} as indicated by Table 1, and stop the FP procedure at stage k once $\text{MAE}(\check{F}^{\cdot,k-1},\check{F}^{\cdot,k}) < \text{tol}$. The numerical outcomes are presented in Table 5 and align with the time complexity analysis shown in Table 4.

6 Conclusion and Future Studies

In this paper, we analyzed a class of linear-quadratic games on graphs (2.1)–(2.3), where the players' interactions are characterized by the graph Laplacian. For generic graphs, we employed fictitious play, one of the most important numerical techniques to compute Nash equilibria, and provided its convergence proof. For vertex-transitive graphs with nice symmetry properties, we developed a semi-explicit characterization of the Nash equilibrium and proved its well-posedness. Additionally, numerical experiments were conducted, providing numerical verifications for our previously proved theorems and a comparison among the numerical methods. The findings consist of: (i) The

Table 5: Running time comparison of three numerical methods: (i) Baseline from directly solving the Riccati system (2.11); (ii) Semi-explicit based on Theorem 4.2; and (iii) Fictitious play by iteratively solving system (3.6). For each value of N, the exact running time is measured in minutes, while the percentage is calculated relative to the baseline construction.

Running time (mins)	N = 32		N = 64		N = 128	
Baseline Semi-explicit	0.03	$100\% \\ 33\%$	$0.29 \\ 0.05$	100% 17% 579%	$2.50 \\ 0.25$	100% 10%
Fictitious play	0.18	600%	1.68	579%	10.01	400%

semi-explicit characterization (Theorem 4.2) is the most efficient method, but it only works for vertex-transitive graphs when the model parameters satisfy $\varepsilon = q^2$; (ii) The baseline method has intermediate efficiency and requires the linear-quadratic structure of the game, but it is applicable to any connected graphs with no restrictions on the model parameters; and (iii) Fictitious play is the least efficient method, but it can be parallelized and applied beyond linear-quadratic games.

As machine learning methods have become more powerful, there has been an increasing trend to solve control problems and games numerically using deep learning and reinforcement learning techniques. We refer readers to the survey [HL23] for a general overview of these methods. In particular, direct parameterization [HE16] and Deep BSDE [EHJ17] are two popular methods for control problems and have been generalized to various extents. Furthermore, [HH20, Hu21] use the deep version of fictitious play as a fundamental strategy for solving the NE. Along with this research direction, we are developing graph-based deep-learning algorithms for solving the Nash equilibrium of large sparse network games. This effort is mainly motivated by the theoretical results and numerical findings in this work. The algorithm accelerates training by reducing the number of parameters without significantly losing the accuracy. The details of the algorithm and the numerical outcomes will be presented in an upcoming work [HLZ].

Finally, we point out that several directions are worth further investigation. For instance, although the global convergence of fictitious play has been observed in numerical experiments even when all model parameters take relatively large values, we were only able to provide the proof of convergence (cf. Corollary 3.8) under a smallness condition. Further relaxation of the technical assumption would be valuable. Additionally, regarding the rate of convergence, fictitious play numerically exhibits a graph-dependent rate as seen in Section 5.2, which has not been accurately characterized in Corollary 3.8. These studies on the convergence of fictitious play are left to future investigation. Separately, for linear-quadratic games on vertex-transitive graphs, we encounter technical difficulties when developing the semi-explicit characterization for the NE when $\varepsilon \neq q^2$, and it remains unsolved. The game equilibrium and asymptotic correlations of the limiting regime $(N \to \infty)$, in the spirit of [LS22], are also interesting topics for future work.

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