Further Applications of Expectation Solutions

First PSet

1. A fair coin is tossed repeatedly until 5 consecutive heads occur. What is the expected number of coin tosses?

   Solution. Let X_n be a random variable that represents the number of tosses until n consecutive heads occur. Denote $\mathrm{E}[X_n]$ by E_n for ease of notation. In order to obtain n consecutive heads, we need to first obtain $n-1$ consecutive heads, and get heads on the next coin flip. If we get tails on the next coin flip, we are back at the start. As with Example 3,

   $$E_n = \frac{1}{2}(E_{n-1}+1) + \frac{1}{2}(E_{n-1}+1+E_n) \implies E_n = 2E_{n-1}+2.$$

   We can quickly verify that $E_0 = 0$, since we are guaranteed to get 0 consecutive heads after 0 coin flips. This allows us to calculate $E_1 = 2$, $E_2 = 6$, $E_3 = 14$, $E_4 = 30$, $E_5 = \boxed{62}$. This recurrence can be solved explicitly as $E_n = 2^{n+1} - 2$.

2. The casino *Magnicifecto* was having difficulties attracting its hotel guests down to the casino floor. The empty casino prompted management to take drastic measures, and they decided to forgo the house cut. They decided to offer an "even value" game. Whatever bet size the player places (say $\$A$), there is a 50% chance that he will get $+\$A$ and a 50% chance he will get $-\$A$. They felt that since the expected value of every game is 0, they should not be making or losing money in the long run.

   Scrooge, who was on vacation, decided to exploit this even value game. He has an infinite bankroll (money) and decides to play the first round (entire series) in the following manner:

   - First make a bet of \$10.
   - If he wins, keep the earnings and leave.
   - Each time that he loses, he doubles the size of his previous bet and plays again.

   What is the expected value of Scrooge's (total) winnings?

   Solution. If Scrooge wins the first match, he gets \$10 and goes home. If he loses, he bets \$20 in a second match. Now, if he wins his second match, he makes a net total (his earnings minus the money he lost) of \$20\$10 = \$10. If he loses, he bets \$40 in a third match . If he wins the third match, he will make a net total of $\$40 - \$20 - \$10 = \$10$. If he loses, he bets \$80 in a fourth match, and so on. Note that no matter which match it is, if he wins that match, he will make \$10 overall. Eventually, as the number of matches played increases, Scrooge will win one match at some point, since he has a fifty percent chance of winning and fifty percent chance of losing each time. This means that Scrooge will eventually make a net total of $\boxed{\$10}$.

3. Call the vertices of a regular tetrahedron A, B, C, and D. A bug starts on vertex A. Every second it randomly walks along one edge to another vertex. What is the expected value of the number of seconds it will take for it to reach the vertex D?

   Solution. Each time the bug moves it has a $\frac{1}{3}$ probability of getting to vertex D from any vertex and $\frac{2}{3}$ probability of effectively ending up where he started by symmetry. Thus, $E = 1 + \frac{2}{3}E \implies E = \boxed{3}$.

4. Brooke has two boxes. The left box contains 1 red ball and 2 blue balls while the right box contains 3 red balls and 2 blue balls. In a game, Brooke has to randomly and simultaneously pick up one ball from each box using both hands. If Brooke gets a blue ball from the left box, she'll be rewarded \$4, but she'll lose \$5 if she gets a red. On the other hand, a blue ball drawn from the right box will reward \$8 but a red ball will lose \$7 dollars. What is the expected value of a single round of this game?

   Solution. Since the draws from both boxes are independent from each other, we can simply add the expected values from both scenarios. This yields

$$4 \cdot \frac{2}{3} - 5 \cdot \frac{1}{3} - 7 \cdot \frac{3}{5} + 8 \cdot \frac{2}{5} = \boxed{0}.$$

5. A bug starts on one vertex A of a regular dodecahedron. A second vertex adjacent to the one it starts on is called B. Every second, it randomly walks along one edge to another vertex. What is the expected number of seconds it will take for the bug to reach the vertex B?

   Solution. If we place the vertex opposite B at the origin and the vertex B on the positive x-axis then we can consider the following five groups of vertices that share a common x-coordinate:

   - Group 0: 1 point, the vertex opposite B.
   - Group 1: 3 points, all points neighboring the initial vertex.
   - Group 2: 6 points.
   - Group 3: 6 points.
   - Group 4: 3 points (A is in this group).
   - Group 5: 1 point, the vertex B.

   Now denote by E_n the expected number of moves to get from any vertex in Group n to vertex B. This yields the system of linear equations

$$\begin{aligned}
E_0 &= 1 + E_1, \\
E_1 &= \frac{1}{3}(E_0 + 1) + \frac{2}{3}(E_2 + 1), \\
E_2 &= \frac{1}{3}(E_1 + 1) + \frac{1}{3}(E_2 + 1) + \frac{1}{3}(E_3 + 1), \\
E_3 &= \frac{1}{3}(E_2 + 1) + \frac{1}{3}(E_3 + 1) + \frac{1}{3}(E_4 + 1), \\
E_4 &= \frac{1}{3}(1) + \frac{2}{3}(E_3 + 1).
\end{aligned}$$

   A is in Group 4, and solving yields $E_4 = \boxed{19}$.

6. A fair coin is flipped several times. Find the expected number of flips needed to make in order to see the pattern TXT, where T is tails, and X is either heads or tails.

   Solution. Let E_0 represent the expected number of flips needed at the very beginning. Let E_1 be the expected number of flips when the last flip is a tails (not preceded by another tails). Let E_2 be the expected number of flips when the last two coins are TT (but one hasn't seen TXT yet). Let E_3 be the expected number of flips when the last two flips are TH (but one haven't seen TXT yet). To illustrate our equation, if we are in "state" E_0, we flip once and can end in state E_1 with probability $\frac{1}{2}$ or back at state E with probability $\frac{1}{2}$. The other states follow similar logic, and we derive the system

$$\begin{aligned}
E_0 &= \frac{1}{2}(E_0+1)+\frac{1}{2}(E_1+1),\\
E_1 &= \frac{1}{2}(E_2+1)+\frac{1}{2}(E_3+1),\\
E_2 &= \frac{1}{2}(1)+\frac{1}{2}(E_3+1),\\
E_3 &= \frac{1}{2}(1)+\frac{1}{2}(E_0+1).
\end{aligned}$$

   Solving obtains $E_0 = \boxed{39}$.

7. A bug starts on one vertex of a regular icosahedron. Every second it randomly walks along one edge to another vertex. What is the expected value of the number of seconds it will take for it to reach the vertex opposite to the original vertex he was on?

   Solution. There are four stages to the bug's journey. There is the starting vertex A; the adjacent set of 5 vertices, which we will call B; the next set of 5 vertices C; and the finishing vertex D. This splits the 12 vertices of the icosahedron into four groups/sets. The expected time to reach a particular vertex is the sum of the expected times for all adjacent vertices, plus (in each case) 1 second for the time to get there, weighted by the probability of the bug choosing to walk to that vertex. Denote by E_A the expected time to reach the opposite vertex from any vertex in set A and define E_B, E_C, and E_D similarly. We have

$$\begin{aligned}
E_A &= 1+E_B,\\
E_B &= \frac{1}{5}(E_A+1)+\frac{2}{5}(E_B+1)+\frac{2}{5}(E_C+1),\\
E_C &= \frac{2}{5}(E_B+1)+\frac{2}{5}(E_C+1)+\frac{1}{5}(E_D+1),\\
E_D &= 0.
\end{aligned}$$

   Solving yields $E_A = \boxed{15}$.

Competition Problems

1. [AMC 12A 2002] Tina randomly selects two distinct numbers from the set $\{1, 2, 3, 4, 5\}$, and Sergio randomly selects a number from the set $\{1, 2, \ldots, 10\}$. What is the probability that Sergio's number is larger than the sum of the two numbers chosen by Tina?

   Solution. The expected value of a number randomly selected form the set $\{1, 2, 3, 4, 5\}$ is 3. Therefore, Tina's expected sum is $3 + 3 = 6$. The probability that Sergio selects a number larger than 6 from his set is $\boxed{\frac{2}{5}}$ by symmetry.

2. [HMMT 2010] Indecisive Andy starts out at the midpoint of the 1-unit-long segment HT. He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H, and if the coin is tails, he moves halfway towards endpoint T. After his 2010 moves, what is the expected distance between Andy and the midpoint of HT?

   Solution. Let Andy's position be x units from the H end after 2009 flips. If Any moves towards the H end, he ends up at $\frac{x}{2}$, a distance of $\frac{1-x}{2}$ from the midpoint. If Andy moves towards the T end, he ends up at $\frac{1+x}{2}$, a distance of $\frac{x}{2}$ from the midpoint. His expected distance from the midpoint is then $\frac{\frac{1-x}{2}+\frac{x}{2}}{2} = \boxed{\frac{1}{4}}$ since this does not depend on x.

3. [HMMT 2010] Let $\triangle ABC$ be a triangle with $AB = 8$, $BC = 15$, and $AC = 17$. Point X is chosen at random on line segment AB. Point Y is chosen at random on line segment BC. Point Z is chosen at random on line segment CA. What is the expected area of triangle $\triangle XYZ$?

   Solution. Let $\mathrm{E}(X)$ denote the expected value of X, and let $[S]$ denote the area of S. Then,

$$\begin{aligned}\mathrm{E}([\triangle XYZ]) &= \mathrm{E}([\triangle ABC] - [\triangle XYB] - [\triangle ZYC] - [\triangle XBZ]) \\ &= [\triangle ABC] - \mathrm{E}([\triangle XYB]) - E([\triangle ZYC]) - [\triangle XBZ])\end{aligned}$$

   where the last step follows from linearity of expectation. But $[\triangle XYB] = \frac{1}{2} \cdot BX \cdot BY \cdot \sin(B)$. The $\frac{1}{2}\sin(B)$ term is constant, and BX and BY are both independent with expected values $\frac{AB}{2}$ and $\frac{BC}{2}$, respectively. Thus $\mathrm{E}([\triangle XYB]) = \frac{1}{8} AB \cdot BC \cdot \sin(B) = \frac{1}{4}[\triangle ABC]$. Similarly, $E([\triangle ZYC]) = \mathrm{E}([\triangle ZBX]) = \frac{1}{4}[\triangle ABC]$. Then we have

$$\mathrm{E}([\triangle XYZ]) = \left(1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right)[\triangle ABC] = \frac{1}{4}[\triangle ABC] = \boxed{15}.$$

4. [HMMT 2015] Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability. It cannot stay in the same square. What is the expected number of minutes until the rook reaches the upper right corner?

   Solution. Let the expected number of minutes it will take the rook to reach the upper right corner from the top or right edges be E_e, and let the expected number of minutes it will take the rook to reach the upper right corner from any other square be E_c. The expected time from any square on the top or right edges is the same, as is the expected time from any other square (swapping any two rows or columns doesn't change the movement of the rook).

$$E_c = \frac{2}{14}(E_e + 1) + \frac{12}{14}(E_c + 1), \quad E_e = \frac{1}{14}(1) + \frac{6}{14}(E_e + 1) + \frac{7}{14}(E_c + 1)$$

   which gives the solution $E_e = 63$, $E_c = \boxed{70}$.

5. [HMMT 2015] Consider a 10×10 grid of squares. One day, Daniel drops a burrito in the top left square, where a wingless pigeon happens to be looking for food. Every minute, if the pigeon and the burrito are in the same square, the pigeon will eat 10% of the burrito's original size and accidentally throw it into a random square (possibly the one it is already in). Otherwise, the pigeon will move to an adjacent square, decreasing the distance between it and the burrito. What is the expected number of minutes before the pigeon has eaten the entire burrito?

Solution. Label the squares using coordinates, letting the top left corner be $(0,0)$. The burrito will end up in 10 (not necessarily different) squares. Call them $p_1 = (x_1, y_1) = (0,0)$, $p_2 = (x_2, y_2)$, $\ldots$, $p_{10} = (x_{10}, y_{10})$. See that p_2 through p_{10} are uniformly distributed throughout the square. Let $d_i = |x_{i+1} - x_i| + |y_{i+1} - y_i|$ be the taxicab distance between p_i and p_{i+1}. After 1 minute, the pigeon will eat 10% of the burrito. Note that if, after eating the burrito, the pigeon throws it to a square taxicab distance d from the square its currently in, it will take exactly d minutes for it to reach that square, regardless of the path it takes, and another minute for it to eat 10% of the burrito. Hence, the expected number of minutes it takes for the pigeon to eat the whole burrito is

$$\begin{aligned}
1 + \mathrm{E}\left[\sum_{i=1}^{9}(d_i + 1)\right] &= 1 + \mathrm{E}\left[\sum_{i=1}^{9}(|x_{i+1} - x_i| + |y_{i+1} - y_i| + 1)\right] \\
&= 10 + 2\,\mathrm{E}\left[\sum_{i=1}^{9}|x_{i+1} - x_i|\right] \\
&= 10 + 2\left(\mathrm{E}[|x_2|] + \mathrm{E}\left[\sum_{i=2}^{9}|x_{i+1} - x_i|\right]\right) \\
&= 10 + 2(\mathrm{E}[|x_2|] + 8\,\mathrm{E}[|x_{i+1} - x_i|]) \\
&= 10 + 2\left(4.5 + \frac{8}{100}\sum_{k=1}^{9}k(20 - 2k)\right) \\
&= 10 + 2(4.5 + 8 \cdot 3.3) = \boxed{71.8}.
\end{aligned}$$

6. [HMMT 2018] A permutation of $\{1, 2, ..., 7\}$ is chosen uniformly at random. A partition of the permutation into contiguous blocks is correct if, when each block is sorted independently, the entire permutation becomes sorted. For example, the permutation $(3, 4, 2, 1, 6, 5, 7)$ can be partitioned correctly into the blocks $[3, 4, 2, 1]$ and $[6, 5, 7]$, since when these blocks are sorted, the permutation becomes $(1, 2, 3, 4, 5, 6, 7)$. Find the expected value of the maximum number of blocks into which the permutation can be partitioned correctly.

Solution. Let σ be a permutation on $\{1, \ldots, n\}$. Call $m \in \{1, \ldots, n\}$ a *breakpoint* of σ if $\{\sigma(1), \ldots, \sigma(m)\} = \{1, \ldots, m\}$. Notice that the maximum partition is into k blocks, where k is the number of breakpoints: if our breakpoints are $m_1, \ldots, m_k$, then we take $\{1, \ldots, m_1\}$, $\{m_1 + 1, \ldots, m_2\}$, $\ldots$, $\{m_{k-1}, \ldots, m_k\}$ as our contiguous blocks. We wish to find $\mathrm{E}[k] = \mathrm{E}[X_1 + \ldots + X_n]$, where $X_i = 1$ if i is a breakpoint and $X_i = 0$ otherwise. By linearity of expectation, this yields $\mathrm{E}[X_i] = \frac{i!(n-i)!}{n!}$ since this is the probability that the first i numbers are just $1, \ldots, i$ in some order. Thus,

$$\mathrm{E}[k] = \sum_{i=1}^{n}\frac{i!(n-i)!}{n!} = \sum_{i=1}^{n}\frac{1}{\binom{n}{i}}, \qquad \sum_{i=1}^{7}\frac{1}{\binom{n}{i}} = \boxed{\frac{151}{105}}.$$

7. [SMT 2018] Morris plays a game using a fair coin. He starts with \$2 and proceeds using the following rules:

   - If Morris flips a heads, he gains \$2.
   - If Morris flips a tails, he loses half his money.
   - If Morris flips two tails in a row, the game ends (but he doesn't lose any more money)

   For instance, if Morris flips the sequence $THTT$, he will end up with \$1.50. What is the expected amount of money in dollars Morris will have after the game ends?

   Solution. Let $f(x)$ be the expected amount of money Morris has at the end of the game if he starts with $\$x$. We split the original game into two slightly different games, where the winnings of the original game is the sum of the winnings of the two other games.
   Game 1: Morris starts the game with $\$x$.

   - If Morris flips a heads, nothing happens.
   - If Morris flips a tails, he loses half his money.
   - If Morris flips two tails in a row, the game ends (but he doesn't lose any more money)

   Game 2: Morris starts the game with \$0.

   - If Morris flips a heads, he gains \$2.
   - If Morris flips a tails, he loses half his money.
   - If Morris flips two tails in a row, the game ends (but he doesnt lose any more money)

   Intuitively, the first game represents the decay of our original investment of \$2 by flipping tails, while the second game represents our winnings when we start with \$0. We first compute the expected value of game 1. The probability that we halve exactly once is $\frac{1}{2}$, since that means that we flipped another tails immediately after flipping our first one. Likewise, the probability that we halve exactly twice is $\frac{1}{4}$, and so on, so the probability that we exactly halve k times is $\frac{1}{2^k}$. The expected value is therefore

   $$\sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{x}{2^k} = \sum_{k=1}^{\infty} \frac{x}{4^k} = \frac{x}{3}.$$

   By definition, the expected value of game 2 is $f(0)$. Thus, the expected value of original the game when we start with \$x is $f(x) = \frac{x}{3} + f(0)$. Plugging in $x = 2$ gives us the equation $f(2) = \frac{2}{3} + f(0)$. On the other hand, consider the case when we start with \$0. We have a $\frac{1}{2}$ chance of flipping heads and gaining \$2, which is the same as starting the game again with \$2. We have a $\frac{1}{4}$ chance of flipping tails and then heads, which is equivalent to the scenario above. Finally we have a $\frac{1}{4}$ chance of flipping tails twice in a row and ending the game. This gives us the equation $f(0) = \frac{1}{2}f(2) + \frac{1}{4}f(2) + \frac{1}{4} \cdot 0$. Plugging this into the other equation and solving, we get $f(2) = \boxed{\frac{8}{3}}$.

Real-World Interview Problems

1. [Citadel Investment Group] What is the expected number of draws from a standard deck of 52 cards until an ace is drawn?

   Solution. Consider any card as just it's suit A, C, D, and S for ace, club, diamond, and spade, respectively. Since we don't care about any particular ace, we can encode the order of the deck as a string of repeated letters. In fact, we only care about aces, so our deck can be considered a binary string of 4 A's and 48 O's, where O stands for "other." The number of such strings is $\binom{52}{4}$. We now consider cases:

   The number of strings with A in the first position is $\binom{51}{3}$. The number of strings with A in the second position is $\binom{50}{3}$. This continues until the number of strings with A in the 49^{th} position is $\binom{3}{3}$.

   Since there are 4 A's, the first A can not be in the 50^{th} position or beyond. Note that for the first case, we need to turn 0 cards before getting an ace. For the second case, we need to turn 1 card before getting an ace. In the 49th case, we need to turn 48 cards before getting an ace. Therefore,the average will be

   $$\frac{48\binom{3}{3} + 47\binom{4}{3} + \ldots + 0\binom{51}{3}}{\binom{52}{4}} = \frac{\binom{52}{5}}{\binom{52}{4}} = \frac{48}{5}.$$

   Here, the numerator is calculated with repeated applications of the **Hockey-stick identity**. On average, we draw $\frac{48}{5}$ non-aces before an ace, so the expected number of cards we need to draw to get an ace is $1 + \frac{48}{5} = \boxed{\frac{53}{5}}$.

2. [Tower Trading Group] Find the expected number of cycles of length greater than $\frac{n}{2}$ in a random permutation of $\{1, \ldots, n\}$.

   Solution. We proceed similarly to Example 5. The expected number of cycles of length ℓ is $\frac{1}{\ell}$. We are essentially looking for

   $$E = \sum_{i=k+1}^{2k} \frac{1}{i}.$$

   Where $k+1 \sim \left\lceil \frac{n}{2} \right\rceil$ and $2k \sim n$. One can consider these as **truncated sums** of the harmonic series. Rewriting as

   $$\sum_{i=k+1}^{2k} \frac{1}{i} = \sum_{i=k+1}^{2k} \frac{1}{k} \cdot \frac{1}{\frac{i}{k}}$$

   allows us to identify this as a **Riemann sum**. To see this, divide the interval $[1, 2]$ into k equal-length sub-intervals. Then evaluate the function $f(x) = \frac{1}{x}$ at the right end of each sub-interval. As $k \to \infty$, the Riemann sum will then tend to the value of the definite integral

   $$\int_1^2 \frac{1}{x} dx = \boxed{\ln(2)}.$$

3. [Jump Trading] Aaron makes an observation from the Uniform(0,1) distribution. Then Brooke repeatedly makes observations from the same distribution until she obtains a number higher than Aaron's. How many observations is she expected to make?

   Solution. Call Aaron's number a (with associated random variable A) and let B be a random variable that represents Brooke's observation at any given point. We know that $P(B < a) = a$, $P(B > a) = 1 - a$, and of course $P(B = a) = 0$, as it is a continuous distribution. We now find the conditional expectation from the geometric distribution. Since the probability of getting a number higher than Aaron's is $1 - a$, we expect $\boxed{\frac{1}{1-a}}$ observations. Note that this is the conditional expectation of the number of observations given the event $A = a$. In an interview, one would also be expected to calculate the unconditional expectation of the number of observations. To do this, let N be a random variable that represents the unconditional number of observations before exceeding A. We have calculated $\mathrm{E}[N|A = a] = \frac{1}{1-a}$. To find $\mathrm{E}[N]$ explicitly, we must invoke the **Law of Iterated Expectation**. This can be done by noting $\mathrm{E}[N|A] = \frac{1}{1-A}$. Thus,

$$\mathrm{E}[N] = \mathrm{E}[\mathrm{E}[N|A]] = \mathrm{E}\left[\frac{1}{1-A}\right].$$

   To calculate this expectation, we will utilize the **Law of the Unconscious Statistician**.

$$\mathrm{E}[N] = \mathrm{E}\left[\frac{1}{1-A}\right] = \int_{a=0}^{1} \frac{1}{1-a}\, da = -\ln(1-a)\Big|_0^1 = \boxed{\text{unbounded}}.$$

   To verify this, we can also directly find the distribution of N. Since the function $N = \frac{1}{1-A} = g(A)$ is monotonically increasing on $(0, 1)$, it has an inverse on the interval. We can use the **inverse function method** to find the distribution of $N = g(A)$.

$$f_N(n) = f_A(g^{-1}(n))\frac{d}{dn}g^{-1}(n) = 1 \cdot \frac{d}{dn}\left(\frac{n-1}{n}\right) = \frac{1}{n^2} \text{ for } 0 < n < \infty.$$

   Thus, we can conclude

$$\mathrm{E}[N] = \sum_{n=1}^{\infty} n\frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} H_n = \boxed{\text{unbounded}}.$$

   The last line arises from the **divergence of the Harmonic series**.

4. [Five Rings Capital] Brooke is betting on a fair coin flip. She can bet any percentage of the $100. If she wins, she gains 1.2 times her bet (and her bet back), but if she loses, she loses her bet. What is her optimal bet size (proportion of her wealth) to maximize long-run expected earnings?

   Solution. $\boxed{8.3\%}$ This is exactly the result obtained from the **Kelly criterion**.