Ordinary Generating Functions

0 Introduction

Generating functions are traditionally used to encode sequences as the coefficients of a polynomial or power series.

Definition 1. For a finite sequence $\{a_0, \ldots, a_n\}$, the corresponding **ordinary generating function** is the polynomial

$$f(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

Example 1. What is the ordinary generating function of 1, 5, 10, 10, 5, 1?

Solution. Let us index our sequence at $i = 0$. Our corresponding generating function is

$$f(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 = \boxed{(1+x)^5.}$$

Note that the generating functions of rows of Pascal's Triangle are given by the Binomial Theorem. We will examine this interaction in a later section.

Definition 2. For an infinite sequence $\{a_0, a_1, a_2, \ldots\}$, the corresponding ordinary generating function is the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots$$

Example 2. What is the generating function of the Fibonacci sequence 1, 1, 2, 3, 5, 8, ...?

Solution. Let us index our sequence at $i = 1$. Our corresponding generating function is

$$\boxed{f(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots}$$

Does this function have a closed-form representation? We will examine this in a later section.

These functions are used in a wide-variety of mathematical fields, and are powerful problem-solving tools. Non-ordinary generating functions such as the Exponential, Poisson, Lambert series, Bell Series, and Dirichlet series generating functions also exist. They are presented for reference here, but are beyond the scope of this handout to discuss.

$$\mathrm{EG}(a_k; x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}, \quad \mathrm{PG}(a_k; x) = \sum_{k=0}^{\infty} a_k e^{-x} \frac{x^k}{k!}, \quad \mathrm{LG}(a_k; x) = \sum_{k=1}^{\infty} a_k \frac{x^k}{1 - x^k},$$

$$\mathrm{BG}_p(a_k; x) = \sum_{k=0}^{\infty} a_{p^k} x^k, \quad \mathrm{DG}(a_k; s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}.$$

1 Background

Our introduction will be from the perspective of counting problems. Some key combinatorial and algebraic results lie at the core of generating functions and will be presented here for review.

- [Binomial Coefficients] Choose k elements from a population containing n distinguishable elements without replacement and without distinguishing the order in which elements are chosen. We are only concerned about which elements are selected. We say that there are "n choose k" ways to do this, denoted $\binom{n}{k}$.

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

> **Example 3.** A convex polygon is a polygon in which every interior angle is less than 180°. A diagonal of a convex polygon is a line segment that connects two non-adjacent vertices. How many diagonals does a convex n-gon have? Here, $n > 2$ is an integer.
>
> *Solution.* To count the number of line segments between any two of the n vertices, we can simply use $\binom{n}{2} = \frac{n(n-1)}{2}$, as any line is uniquely determined by an unordered pair of the n points. However, not all of these line segments are diagonals, as this method also counts the edges of the polygon. We thus subtract the n edges of the polygon that are over-counted to obtain $\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \boxed{\frac{n(n-3)}{2}.}$

- [Pascal's Triangle] Consider an equilateral triangular grid of dots such that the first row has one dot, the second row has two dots, and so on. Starting at the topmost dot $(0,0)$, how many ways can one reach the k^{th} dot on row n by only moving down and to the left or down and to the right?

  One will find that these numbers correspond to the binomial coefficients $\binom{n}{k}$ for $n \geq k$ and $n, k \in \{0, 1, 2, \dots\}$. This is exactly how Pascal's Triangle comes into fruition.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array} \qquad \begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \end{array}$$

> **Example 4.** In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3 : 4 : 5$? (AIME 1992 #4)
>
> *Solution.* In Pascal's Triangle, we know that the binomial coefficients of the n^{th} row are $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$. Let our row be the n^{th} row such that the three consecutive entries are $\binom{n}{k}$, $\binom{n}{k+1}$, and $\binom{n}{k+2}$. After expanding and dividing one entry by another to clean up the factorials, we see that $\frac{3}{4} = \frac{k+1}{n-k}$ and $\frac{4}{5} = \frac{k+2}{n-k-1}$. Solving yields $n = \boxed{62.}$

- [Binomial Theorem] The coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is the k^{th} element of the n^{th} row of Pascal's Triangle.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

Proof. We can write $(x+y)^n = \underbrace{(x+y)(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$. Repeatedly using the distributive property, we see that for a term $x^{n-k}y^k$, we must choose k of the n terms to contribute a y to the term, and then each of the other $n-k$ terms of the product must contribute a x. Thus, the coefficient of $x^{n-k}y^k$ is the number of ways to choose k objects from a set of size n, or $\binom{n}{k}$. Extending this to all possible values of k from 0 to n, we see that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$. $\square$

Example 5. Prove the identity $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ algebraically.

Proof. By the Binomial Theorem, $2^n = (1+1)^n = \sum_{i=0}^{n}\binom{n}{i}1^{n-i}1^i = \sum_{i=0}^{n}\binom{n}{i}$. $\square$

- [Stars and Bars] The number of ways to distribute n indistinguishable objects into k distinguishable boxes if each box must contain at least one item is

$$\binom{n-1}{k-1}.$$

Proof. Imagine arranging the n objects in a line. In order to divide them into k groups, we need to insert $k-1$ dividers. There are $n-1$ slots between objects into which the dividers can be inserted and we need to choose $k-1$ of these slots. $\square$

The number of ways to distribute n indistinguishable objects into k distinct boxes if some boxes may remain empty is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Proof. Now one may place multiple dividers between two objects as well as place dividers before the first object or after the last object to represent empty boxes. We are now essentially ordering $n+k-1$ (n objects and $k-1$ dividers) in a line. Since any two objects are indistinguishable and so are any two dividers, we must divide by the internal orderings of objects and dividers. The number of ways to do so is $\frac{(n+k-1)!}{n!(k-1)!} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$. $\square$

Example 6. How many ways can 11 identical lollipops be given to 6 children if each child must receive at least one lollipop? What if children can receive 0 lollipops? How does this relate to ordered integer solutions (a,b,c,d,e,f) to the equation $a+b+c+d+e+f = 11$ where a,b,c,d,e,f are first strictly positive and then non-negative?

Solution. For the first case we have $\binom{11-1}{6-1} = \boxed{252}$ ways. For the second case we have $\binom{11+6-1}{6-1} = \boxed{4368}$ ways. These answers directly correspond to the ordered 6-tuple integer solutions of $a+b+c+d+e+f = 11$.

2 Generating Functions and Counting

Example 7. There are three urns. One has 2 red balls, one has 2 green balls, and one has 3 blue balls. Balls of the same color are indistinguishable. In how many ways can I choose 4 balls from the urns?

Solution. From the red urn, we can choose 0, 1, or 2 red balls. Since balls of the same color are indistinguishable, there is only 1 way per choice of 0, 1, or 2. The generating function for the number of ways to choose balls from the red urn is $1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 = 1 + x + x^2$. The sequence of coefficients represent the number of ways to choose 0 red balls, 1 red ball, and 2 red balls, respectively. Similarly, we see that the generating function for the green urn is also $1 + x + x^2$ and that the generating function for the blue urn is $1 + x + x^2 + x^3$. To find the generating function of multiple urns, we simply need to multiply their individual generating functions together by the rule of product. Thus,

$$(1 + x + x^2)(1 + x + x^2)(1 + x + x^2 + x^3) = 1 + 3x + 6x^2 + 8x^3 + 8x^4 + 6x^5 + 3x^6 + x^7.$$

The coefficient of x^4 is 8, so there are $\boxed{8}$ ways to choose 4 balls.

Example 8. What if balls of the same color were distinguishable in Example 7?

Solution. Our generating function for the red urn would become $1 + 2x + x^2$, as there are now two ways to choose a single red ball. Our generating function for the green urn would also be $1 + 2x + x^2$ and for the blue urn it is $1 + 3x + 3x^2 + x^3$. Multiplying these three yields $1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$. The coefficient of x^4 is $\boxed{35.}$

Example 9. Let there be two abnormal dice. One die has a 2 on three faces, a 3 on two faces, and a 5 on the sixth face. The other die has a 1 on one face, a 4 on four faces, and a 6 on the sixth face. If both dice are rolled, what is the probability that the sum is greater than 6 but less than 10?

Solution. The generating function for the first die is $3x^2 + 2x^3 + x^5$ and the generating function for the second die is $x + 4x^4 + x^6$. The sum of the dice is represented by the product of the generating functions since exponents are additive upon multiplication. Thus,

$$(3x^2 + 2x^3 + x^5)(x + 4x^4 + x^6) = 3x^3 + 2x^4 + 13x^6 + 8x^7 + 3x^8 + 6x^9 + x^11.$$

The coefficients sum to 36, so this generating function encompasses all 36 possible ordered rolls. There are $8 + 3 + 6 = 17$ ways to roll a sum of 7, 8, or 9, so the probability is $\boxed{\frac{17}{36}.}$

Example 10. Joe is ordering 3 hot dogs. He can choose among 5 toppings for each hot dog. In how many different ways can he choose 6 toppings for the 3 hot dogs?

Solution. On an individual hot dog, there are $\binom{5}{k}$ ways for Joe to have k toppings. Thus, our generating function for an individual hot dog is $(1 + x)^5$. Since there are 3 hot dogs, the generating function becomes $(1+x)^{15}$. The coefficient of x^6 in the expansion is $\binom{15}{6} = \boxed{5005}$ by the Binomial Theorem.

What if we had to work with power series instead of polynomials? In those sorts of situations, we will often obtain a generating function of the form $(1+x+x^2+x^3+\ldots)^n$. Recall that the sum of an infinite geometric series starting at 1 with common ratio x is

$$\sum_{k=0}^{\infty} x^k = 1+x+x^2+x^3+\ldots = \frac{1}{1-x}$$

when $|x|<1$. In our cases, as long as the algebraic manipulation works for some nonzero values of x, this substitution is valid, so we do not need to worry about whether $|x|<1$ from a generating function standpoint. We thus obtain

$$(1+x+x^2+\ldots)^n = \frac{1}{(1-x)^n} = \left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} x^k = \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k.$$

This is similar to the stars and bars configuration that we have already encountered! The proof of this result utilizes the hockey stick identity, and with it, generating functions can be used to simplify complicated stars and bars problems.

Example 11. Find the number of ways to distribute \$15 to five people if each person gets a whole number of dollars between \$0 and \$4 inclusive.

Solution. The generating function for the amount that each person receives is $1+x+x^2+x^3+x^4$, so the total generating function is $(1+x+x^2+x^3+x^4)^5$. Recall the sum of the finite geometric series:

$$\sum_{k=0}^{n} x^k = 1+x+\ldots+x^{n-1}+x^n = \frac{1-x^{n+1}}{1-x}.$$

This yields $(1+x+x^2+x^3+x^4)^5 = \left(\frac{1-x^5}{1-x}\right)^5$. Expand the numerator with the Binomial Theorem and the denominator with our above result to obtain

$$\left(\frac{1-x^5}{1-x}\right)^5 = \left(1-5x^5+10x^{10}-10x^{15}+5x^{20}-x^{25}\right)\left(\sum_{k=0}^{\infty}\binom{4+k}{4}x^k\right).$$

To obtain that the coefficient of the x^{15} term, we only have to consider the products

$$\begin{aligned}
1 \cdot \binom{4+15}{4} x^{15} &= 3876x^{15}, \\
-5x^5 \cdot \binom{4+10}{4} x^{10} &= -5005x^{15}, \\
10x^{10} \cdot \binom{4+5}{4} x^{5} &= 1260x^{15}, \\
-10x^{15} \cdot \binom{4+0}{4} x^{0} &= -10x^{15}.
\end{aligned}$$

Summing these yields $121x^{15}$, so our desired coefficient is $\boxed{121.}$

Example 12. 100 pieces of candy are distributed to 5 children. The two youngest children want at most 1 piece. The middle child will take any number of pieces. The two oldest children demand an odd number of pieces. How many ways can this be done?

Solution. Note that the generating function for each of the youngest children is $1 + x$. The generating function for the middle child is $1 + x + x^2 + \dots$ The generating function for each of the oldest children is $x + x^3 + x^5 + \dots$ The combined generating function is thus

$$\begin{aligned}
&(1+x)^2(1+x+x^2+\dots)(x+x^3+x^5+\dots)^2 \\
&= x^2(1+x)^2(1+x+x^2+\dots)(1+x^2+x^4+\dots)^2 \\
&= x^2(1+x)^2\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)^2 = \frac{x^2(1+x)^2}{(1-x)(1-x^2)^2}.
\end{aligned}$$

Factoring simplifies our generating function to $\frac{x^2}{(1-x)^3}$. We want the coefficient of x^{100} in this expression, and that will be the same as the coefficient of x^{98} in $\frac{1}{(1-x)^3}$. By our above result, we know that this is $\binom{100}{2} = \boxed{4950.}$

3 Generating Functions and Partitions

Definition 3. A partition of a positive integer n is a decomposition of n into a sum of positive integers, not all necessarily distinct, without regard to the order in which we list the integers in the sum.

Example 13. List the partitions of 4 and 5.

Solution. The five partitions of 4 are $\boxed{4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1}$. The seven partitions of 5 are $\boxed{5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1}$.

Example 14. Let p_n be the number of partitions of n. Determine the generating function

$$f(x) = p_0 + p_1 x + p_2 x^2 + \dots$$

for counting the number of partitions of any positive integer. By convention, $p_0 = 1$.

Solution. Consider the seven partitions of 5:

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

We can write these as the product of monomials that multiply to x^5:

$$x^5, \quad (x^4)(x), \quad (x^3)(x^2), \quad (x^3)(x)^2, \quad (x^2)^2(x), \quad (x^2)(x)^3, \quad (x)^5.$$

This motivates us to think of a partition as a choice of exponents $a_1, a_2, \dots$ such that

$$(x)^{a_1}(x^2)^{a_2}(x^3)^{a_3}\cdots = x^n.$$

This choice corresponds to the partition of n consisting of a_1 1's, a_2 2's, a_3 3's, and so on. This is because

$$a_1 + 2a_2 + 3a_3 + \dots = n.$$

Now consider the product

$$(1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots)\cdots$$

Every x^n term in this product will come from choosing an $(x)^{a_1}$ term from the first power series, an $(x^2)^{a_2}$ term from the second power series, and so on such that $(x)^{a_1}(x^2)^{a_2}(x^3)^{a_3}\cdots = x^n$. The number of ways to do this is exactly the number of ways that n can be broken up into a sum of a_1 1's, a_2 2's, and so on, and is thus the number of partitions of n. Thus,

$$f(x) = (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots)\cdots$$

We can simplify $f(x)$ by using the fact that each term of the product is a geometric series to obtain

$$\boxed{f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.}$$

4 Generating Functions and the Fibonacci Numbers

Definition 4. The sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, ... is known as the Fibonacci sequence. Named after Italian mathematician Fibonacci of Pisa, the sequence has the property that every number is the sum of the previous two (except for the leading 1's). Let $F_1 = 1$ and $F_2 = 1$. The n^{th} Fibonacci number for $n > 2$ is given by

$$F_n = F_{n-1} + F_{n-2}.$$

Example 15. Brooke is climbing a flight of 10 stairs. With each step, she will climb either 1 or 2 stairs at a time. How many different ways can Brooke climb the stairs?

Solution. Note that when Brooke reaches the tenth stair, she would have either climbed 2 stairs from the eighth or 1 stair from the ninth. We can formulate this recursion as $f(10) = f(9) + f(8)$, where $f(x)$ is the number of ways to climb x stairs. Note that $f(9)$ and $f(8)$ can further be broken down recursively, and this exactly represents the n^{th} Fibonacci number, but shifted by one index. The shift is because here $f(1) = 1$ and $f(2) = 2$. Thus, the number of ways to climb 10 stairs this way is F_{11}, or $\boxed{89.}$

- [Binet's Formula] If F_n is the nth Fibonacci number, then

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Proof. Note that the quotient of consecutive terms of the sequence approach around 1.618. $\frac{34}{21} \approx 1.61904761905$, $\frac{55}{34} \approx 1.61764705882$, ... Thus we have a sequence resembling that of a geometric sequence. We let such a geometric sequence have terms $G_n = a \cdot r^n$. Then, $F_{n+1} = F_n + F_{n-1} \implies a \cdot r^{n+1} = a \cdot r^n + a \cdot r^{n-1} \implies r^2 = r + 1$. Using the quadratic formula, we find $r = \frac{1 \pm \sqrt{5}}{2}$. We now have two sequences $G_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $H_n = a\left(\frac{1-\sqrt{5}}{2}\right)^n$, but neither match up with the Fibonacci sequence. In particular, $F_0 = 0$, but for G_0 and H_0 to be zero, we need $a = 0$, but then the sequence just generates a constant 0. After a bit of experimenting with these two sequences to find a sequence where the zeroth term is zero, notice that $G_{n+1} - H_{n+1} = G_n - H_n + G_{n-1} - H_{n-1}$, so $G_n - H_n$ also satisfies this recurrence. If we match up the numbers of F_n and $G_n - H_n = a\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$, we note that $F_0 = G_0 - H_0 = 0$. However, $F_1 = 1 = G_1 - H_1$, which implies that $a = \frac{1}{\sqrt{5}}$. Now, $G_n - H_n$ satisfies the same recurrence as F_n and has the same initial terms, so we are done. □

- The generating function of the Fibonacci sequence is given by

$$f(x) = \frac{x}{1 - x - x^2}.$$

Proof. Define $g(x) = (1 - x - x^2)(x + x^2 + 2x^3 + 3x^4 + \ldots)$. We wish to show that $g(x) = x$. $g(x)$ clearly has no constant term. The coefficient of x in the expansion will be 1, and the coefficient of x^2 in the expansion will be 0. For higher degree terms, note that

$$1(F_i x^i) - x(F_{i-1}x^{i-1}) - x^2(F_{i-2}x^{i-2}) = (F_i - F_{i-1} - F_{i-2})x^i.$$

However, $F_i = F_{i-1} + F_{i-2}$ by definition, so the above coefficient is 0. Thus, $g(x) = x$. □

Example 16. Use the generating function of the Fibonacci numbers to derive Binet's formula for the n^{th} Fibonacci number.

Solution. Let $\phi_1 = \frac{-1+\sqrt{5}}{2}$ and $\phi_2 = \frac{-1-\sqrt{5}}{2}$. Factoring by the quadratic formula yields

$$f(x) = \frac{x}{1-x-x^2} = \frac{-x}{(\phi_1 - x)(\phi_2 - x)}.$$

We then decompose this faction utilizing partial fraction decomposition. Let

$$\begin{aligned} \frac{-x}{(\phi_1 - x)(\phi_2 - x)} &= \frac{A}{\phi_1 - x} + \frac{B}{\phi_2 - x}, \\ A(\phi_2 - x) + B(\phi_1 - x) &= -1. \end{aligned}$$

Solving yields $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Thus,

$$f(x) = \frac{x}{\sqrt{5}} \left(\frac{1}{\phi_1 - x} - \frac{1}{\phi_2 - x} \right).$$

For any constant α, note that

$$\begin{aligned} \frac{1}{\alpha - x} = \frac{1/\alpha}{1 - x/\alpha} &= \frac{1}{\alpha} \left(1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \dots \right) \\ &= \frac{1}{\alpha} + \frac{1}{\alpha^2} x + \frac{1}{\alpha^3} x^2 + \dots \end{aligned}$$

Therefore, the coefficient of x^n in $\frac{1}{\alpha - x}$ is $\frac{1}{\alpha^{n+1}}$. We can this use this identity to obtain

$$\begin{aligned} f(x) &= \frac{x}{\sqrt{5}} \left(\frac{1}{\phi_1 - x} - \frac{1}{\phi_2 - x} \right) \\ &= \frac{x}{\sqrt{5}} \left(\frac{1}{\phi_1} + \frac{1}{\phi_1^2} x + \frac{1}{\phi_1^3} x^2 + \dots - \frac{1}{\phi_2} - \frac{1}{\phi_2^2} x - \frac{1}{\phi_2^3} x^2 - \dots \right). \end{aligned}$$

We now have the coefficient of x^n in our generating function, which is the n^{th} Fibonacci number

$$\frac{1}{\sqrt{5}} \left(\left(\frac{1}{\phi_1} \right)^n - \left(\frac{1}{\phi_2} \right)^n \right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

This is exactly Binet's formula.

4.1 Summary

- Generating functions allow the use of algebraic techniques to solve counting problems.
- In a generating function $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ we think of the coefficient a_k of x^k as counting something that depends on a parameter k.
- Generating functions are a convenient way to keep track of many cases at once. They can eliminate the need for lengthy casework.
- A key step in finding the coefficients in a generating function is to only multiply the needed terms. There is no need to determine the x^{100} term if one is looking for the coefficient of x^{10}. Utilizing a power series approach can also help.
- One of the most common generating functions is for distributing indistinguishable items in a stars and bars fashion. Expressions for these problems will usually take the form $\frac{1}{(1-x)^n}$ and the coefficient of x^k will be of the form $\binom{n-1+k}{k}$.
- The generating function for the number of partitions of a positive integer n is $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$.
- The generating function for the Fibonacci numbers is $\frac{x}{1-x-x^2}$.

4.2 Further Reading

- https://artofproblemsolving.com/wiki/index.php?title=Generating_function
- https://brilliant.org/wiki/generating-functions-solving-recurrence-relations/
- https://www.imomath.com/index.php?options=353&lmm=1
- http://yufeizhao.com/olympiad/comb3.pdf

4.3 Problems

1. [AIME II 2006 #10] Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded a point and the loser gets 0 points. The total points are accumulated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is m/n, where m and n are relatively prime positive integers. Find $m+n$.

2. [AIME I 2010 #4] Jackie and Phil have two fair coins and a third coin that comes up heads with probability 4/7. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m+n$.

3. [AIME II 2010 #11] Define a T-grid to be a 3×3 matrix which satisfies the following two properties: (1) Exactly five of the entries are 1's, and the remaining four entries are 0's. (2) Among the eight rows, columns, and long diagonals (the long diagonals are $\{a_{13}, a_{22}, a_{31}\}$ and $\{a_{11}, a_{22}, a_{33}\}$, no more than one of the eight has all three entries equal. Find the number of distinct T-grids.