Richard A. Epstein

SECOND EDITION

The Theory of GAMBLING AND STATISTICAL LOGIC

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The Theory of Gambling and Statistical Logic

Second Edition

Richard A. Epstein

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It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge. . . . The most important questions of life are, for the most part, really only problems of probability.

Pierre Simon, Marquis de Laplace
Théorie Analytique des Probabilitiés, 1812

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Preface

After publication of the revised edition of this work (1977), I smugly presumed that the fundamentals of Gambling Theory had been fully accounted for—an attitude comparable to that attributed to the U.S. Patent Office at the end of the 19th century: "Everything worth inventing has already been invented."

In the more than 30 years since, progress in the field of gambling has not faltered. Improved computational facilities have led to the solution of games hitherto deemed unsolvable—most notably, Checkers. Such solutions have a regenerative effect, creating more problems that demand yet further solutions. Computer simulations have provided numerical answers to problems that remain resistant to closed-form solutions. Abstract branches of mathematics have been awakened and applied to gambling issues. And, of paramount significance, Parrondo's Paradox has advanced the startling notion that two losing ventures can be combined to form a winning prospect.

Reader and author are equally admonished: *Never bet against the future*.

Gambling remains a near universal pastime. Perhaps more people with less knowledge succumb to its lure than to any other sedentary avocation. As Balzac averred, "the gambling passion lurks at the bottom of every heart." It finds outlets in business, war, politics; in the formal overtures of the gambling casinos; and in the less ceremonious exchanges among individuals of differing opinions.

To some, the nature of gambling appears illusive, only dimly perceivable through a curtain of numbers. To others, it inspires a quasi-religious emotion: true believers are governed by mystical forces such as "luck," "fate," and "chance." To yet others, gambling is the algorithmic Circe in whose embrace lies the roadmap to El Dorado: the "foolproof" system. Even mathematicians have fallen prey to the clever casuistry of gambling fallacies. Special wards in lunatic asylums could well be populated with mathematicians who have attempted to predict random events from finite data samples.

It is, then, the intent of this book to dissipate the mystery, myths, and misconceptions that abound in the realm of gambling and statistical phenomena.

The mathematical theory of gambling enjoys a distinguished pedigree. For several centuries, gamblers' pastimes provided both the impetus and the only concrete basis for the development of the concepts and methods of probability theory. Today, games of chance are used to isolate, in pure form, the logical structures underlying real-life systems, while games of skill provide testing grounds for the study of multistage decision processes in practical contexts. We can readily confirm A.M. Turing's conviction that games constitute an ideal model system leading toward the development of machine intelligence.

It is also intended that a unified and complete theory be advanced. Thus, it is necessary to establish formally the fundamental principles underlying the phenomena of gambling before citing examples illustrating their behavior. A majority of the requisite mathematical exposition for this goal has been elaborated and is available in the technical literature. Where deficiencies remain, we have attempted to forge the missing links.

The broad mathematical disciplines associated with the theory of gambling are Probability Theory and Statistics, which are usually applied to those contests involving a statistical opponent ("nature"), and Game Theory, which is pertinent to conflict among "intelligent" contestants. To comprehend the operation of these disciplines normally requires only an understanding of the elementary mathematical tools (e.g., basic calculus). In only a few isolated instances do rigorous proofs of certain fundamental principles dictate a descent into the pit of more abstract and esoteric mathematics (such as, Set Theory).

If this book is successful, readers previously susceptible to the extensive folklore of gambling will view the subject in a more rational light; readers previously acquainted with the essentials of Gambling Theory will possess a more secure footing. The profits to be reaped from this knowledge strongly depend on the individual. To any moderately intelligent person, it is self-evident that the interests controlling the operations of gambling casinos are not engaged in philanthropy. Furthermore, each of the principal games of chance or skill has been thoroughly analyzed by competent statisticians. Any inherent weakness, any obvious loophole, would have been uncovered long ago; and any of the multitude of miraculous "systems" that deserved their supernal reputation would have long since pauperized every gambling establishment in existence. The systems that promise something for nothing inevitably produce nothing for something.

It will also be self-evident that the laws of chance cannot be suspended despite all earnest supplications to the whim of Tyche or genuflections before the deities of the Craps table. Such noumena cast small shadows on the real axis.

In the real world there is no "easy way" to ensure a financial profit at the recognized games of chance or skill; if there were, the rules of play would soon be changed. An effort to understand the mathematics validating each game, however, can produce a highly gratifying result. At least, it is gratifying to rationalize that we would rather lose intelligently than win ignorantly.

Richard A. Epstein
Winnetka, CA

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Norman Wattenberger, proprietor, Casino Vérité (*www.qfit.com*). His CD of CV Blackjack continues as a classic in its field. If there is a more proficient "sim" master in the U.S., that person has yet to surface. Norman programmed extensive tables for this book and has contributed numerical solutions to several problems posed herein—problems that would otherwise have remained wholly intractable.

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Chapter | one

Kubeiagenesis

Shortly after *pithecanthropus erectus* gained the ascendency, he turned his attention to the higher-order abstractions. He invented a concept that has since been variously viewed as a vice, a crime, a business, a pleasure, a type of magic, a disease, a folly, a weakness, a form of sexual substitution, an expression of the human instinct. He invented gambling.

Archaeologists rooting in prehistoric sites have uncovered large numbers of cube-shaped bones, called *astragalia*, that were apparently used in games some thousands of years ago. Whether our Stone Age ancestors cast these objects for prophecy or amusement or simply to win their neighbor's stone axe, they began a custom that has survived evolution and revolution.

Although virtually every culture has engaged in some form of dice play, centuries elapsed before thought was directed to the "fairness" of throwing dice or to the equal probability with which each face falls or should fall. The link between mathematics and gambling long remained unsuspected.

Most early civilizations were entrapped by the deep-rooted emotional appeal of absolute truth; they demanded Olympian certitude and could neither envision nor accept the inductive reliability sought by modern physics. "Arguments from probabilities are impostors," was the doctrine expressed in Plato's *Phaedo*. Carneades, in the second century B.C., was the first to shift from the traditional Greek rationalist position by developing an embryonic probability theory that distinguished three types of probability, or degrees of certainty. However, this considerable accomplishment (against the native grain) advanced the position of empiricist philosophy more than the understanding of events of chance.

Throughout the entire history of man preceding the Renaissance, all efforts aimed at explaining the phenomena of chance were characterized by comprehensive ignorance of the nature of probability. Yet gambling has flourished in various forms almost continuously from the time Paleolithic hominids cast polished knucklebones and painted pebbles. Lack of knowledge has rarely inhibited anyone from taking a chance.

Reasoned considerations relating games of chance to a rudimentary theory of probability first emerged in the 16th century. Gerolamo Cardano (1501–1576), physician, philosopher, scientist, astrologer, religionist, gambler, murderer, was responsible for the initial attempt to organize the concept of chance events into a cohesive discipline. In *Liber de Ludo Aleae* (The Book on Games of Chance), published posthumously in 1663, he expressed a rough concept of mathematical expectation, derived power laws for the repetition of events, and conceived the definition of probability as a frequency ratio. Cardano designated the "circuit" as the totality of permissible outcomes of an event. The circuit was 6 for one die, 36 for two dice, and 216 for three dice. He then defined the probability of a particular outcome as the sum of possible ways of achieving that outcome divided by the circuit.

Cardano investigated the probabilities of casting astragalia and undertook to explain the relative occurrence of card combinations, notably for the game of Primero (loosely similar to Poker). A century before Antoine de Méré, he posed the problem of the number of throws of two dice necessary to obtain at least one roll of two aces with an even chance (he answered 18 rather than the correct value of 24.6). Of this brilliant but erratic Milanese, the English historian Henry Morley wrote, "He was a genius, a fool, and a charlatan who embraced and amplified all the superstition of his age, and all its learning." Cardano was imbued with a sense of mysticism; his undoing came through a pathological belief in astrology. In a much-publicized event, he cast the horoscope of the frail 15-year-old Edward VI of England, including specific predictions for the 55th year, third month, and 17th day of the monarch's life. Edward inconsiderately expired the following year at the age of sixteen. Undismayed, Cardano then had the temerity to cast the horoscope of Jesus Christ, an act not viewed with levity by 16th-century theologians. Finally, when the self-predicted day of his own death arrived, with his health showing no signs of declining, he redeemed his reputation by committing suicide.

Following Cardano, several desultory assaults were launched on the incertitudes of gambling. Kepler issued a few words on the subject, and shortly after the turn of the 17th century, Galileo wrote a short treatise titled, *Considerazione sopra il Giuoco dei Dadi*.[1] A group of noblemen of the Florentine court had consulted Galileo in a bid to understand why the total 10 appears more often than 9 in throws of 3 dice. The famous physicist showed that 27 cases out of 216 possible total the number 10, while the number 9 occurs 25 times out of 216.

Then, in 1654, came the most significant event in the theory of gambling as the discipline of mathematical probability emerged from its chrysalis. A noted gambler and roué, Antoine Gombaud, Chevalier de Méré, posed to his friend, the Parisian mathematician Blaise Pascal, the following problem: "Why do the odds differ in throwing a 6 in four rolls of one die as opposed to throwing two 6s in 24 rolls of two dice?" In subsequent correspondence with Pierre de Fermat

[1]Originally, "Sopra le Scoperte dei Dadi."

(then a jurist in Toulouse) to answer this question, Pascal constructed the foundations on which the theory of probability rests today. In the discussion of various gambling problems, Pascal's conclusions and calculations were occasionally incorrect, while Fermat achieved greater accuracy by considering both dependent and independent probabilities.

Deriving a solution to the "Problem of points" (two players are lacking x and y points, respectively, to win a game; if the game is interrupted, how should the stakes be divided between them?), Pascal developed an approach similar to the calculus of finite differences. Pascal was an inexhaustible genius from childhood; much of his mathematical work was begun at age 16. At 19 he invented and constructed the first calculating machine in history.[2] He is also occasionally credited with the invention of the roulette wheel. Whoever of the two great mathematicians contributed more, Fermat and Pascal were first, based on considerations of games of chance, to place the theory of probability in a mathematical framework.

Curiously, the remaining half of the 17th century witnessed little interest in or extension of the work of Pascal and Fermat. In 1657, Christiaan Huygens published a treatise titled, *De Ratiociniis in Ludo Aleae* (Reasonings in Games of Chance), wherein he deals with the probability of certain dice combinations and originates the concept of "mathematical expectation." Leibnitz also produced work on probabilities, neither notable nor rigorous: he stated that the sums of 11 and 12, cast with two dice, have equal probabilities (*Dissertatio de Arte Combinatoria*, 1666). John Wallis contributed a brief work on combinations and permutations, as did the Jesuit John Caramuel. A shallow debut of the discipline of statistics was launched by John Graunt in his book on population growth, *Natural and Political Observations Made Upon the Bills of Mortality*. John de Witt analyzed the problem of annuities, and Edmund Halley published the first complete mortality tables.[3] By mathematical standards, however, none of these works can qualify as first-class achievements.

More important for the comprehension of probabilistic concepts was the pervasive skepticism that arose during the Renaissance and Reformation. The doctrine of certainty in science, philosophy, and theology was severely attacked. In England, William Chillingworth promoted the view that man is unable to find absolutely certain religious knowledge. Rather, he asserted, a limited certitude based on common sense should be accepted by all reasonable men. Chillingworth's theme was later applied to scientific theory and practice by Glanville, Boyle, and Newton, and given a philosophical exposition by Locke.

Turning into the 18th century, the "Age of Reason" set in, and the appeal of probability theory once again attracted competent mathematicians. In the *Ars Conjectandi* (*Art of Conjecturing*), Jacob Bernoulli developed the theory

[2]The first calculating machine based on modern principles must be credited, however, to Charles Babbage (1830).

[3]Life insurance per se is an ancient practice. The first crude life expectancy table was drawn up by Domitius Ulpianus circa A.D. 200.

of permutations and combinations. One-fourth of the work (published posthumously in 1713) consists of solutions of problems relating to games of chance. Bernoulli wrote other treatises on dice combinations and the problem of duration of play. He analyzed various card games (e.g., *Trijaques*) popular in his time and contributed to probability theory the famous theorem that by sufficiently increasing the number of observations, any preassigned degree of accuracy is attainable. Bernoulli's theorem was the first to express frequency statements within the formal framework of the probability calculus. Bernoulli envisioned the subject of probability from the most general point of view to that date. He predicted applications for the theory of probability outside the narrow range of problems relating to games of chance; the classical definition of probability is essentially derived from Bernoulli's work.

In 1708, Pierre Remond Montmort published his work on chance titled, *Essay d'Analyse sur les Jeux de Hazard*. This treatise was largely concerned with combinatorial analysis and dice and card probabilities. In connection with the game of Treize, or Rencontres, de Montmort was first to solve the matching problem (the probability that the value of a card coincides with the number expressing the order in which it is drawn). He also calculated the mathematical expectation of several intricate dice games: Quinquenove, Hazard, Esperance, Trois Dez, Passedix, and Rafle, inter alia. Of particular interest is his analysis of the card games le Her and le Jeu des Tas. Following de Montmort, Abraham de Moivre issued a work titled, *Doctrine of Chances*, that extended the knowledge of dice combinations, permutation theory, and card game analysis (specifically, the distribution of honors in the game of Whist) and, most important, that proposed the first limit theorem. In this and subsequent treatises, he developed further the concepts of matching problems, duration of play, probability of ruin, mathematical expectation, and the theory of recurring series. On the basis of his approximation for the sum of terms of a binomial expansion (1733), he is commonly credited with the invention of the normal distribution. De Moivre lived much of his life among the London coffeehouse gamblers.[4] It was likely this environment that led him to write what is virtually a gambler's manual.

Miscellaneous contributors to gambling and probability theory in the first half of the 18th century include Nicolas Bernoulli and Thomas Simpson, the latter responsible for introducing the idea of continuity into probability (1756). These mathematicians and their contemporaries analyzed card games and solved intricate dice problems. Little additional understanding of probability was furnished to the *orbis scientiarum*.

Daniel Bernoulli advanced the concepts of risk and mathematical expectation by the use of differential calculus. He also deserves credit, with de Moivre, for recognizing the significance of the limit theorem to probability theory. Leonhard Euler, far more renowned for contributions to other branches of mathematics, worked for some time on questions of probabilities and developed the theory of

[4]His mathematics classes were held at Slaughter's Coffee House in St. Martin's Lane; one enthusiastic student was Thomas Bayes.

partitions, a subject first broached in a letter from Leibnitz to Johann Bernoulli (1669). He published a memoir, *Calcul de la Probabilité dans le Jeu de Rencontre*, in 1751 and calculated various lottery sequences and ticket-drawing combinations. Jean le Rond d'Alembert was best known for his opinions contrary to the scientific theories of his time. His errors and paradoxes abound in 18th-century mathematical literature. According to d'Alembert, the result of tossing three coins differs from three tosses of one coin. He also believed that Tails are more probable after a long run of Heads, and promulgated the doctrine that a very small probability is practically equivalent to zero. This idea leads to the progression betting system that bears his name. His analyses of card and dice games were equally penetrating.

The Rev. Thomas Bayes (his revolutionary paper, "An Essay Towards Solving a Problem in the Doctrine of Chances," was published in 1763, two years after his death) contributed the theorem stating exactly how the probability of a certain "cause" changes as different events actually occur. Although the proof of his formula rests on an unsatisfactory postulate, he was the first to use mathematical probability inductively—that is, arguing from a sample of the population or from the particular to the general. Bayes' theorem has (with some hazard) been made the foundation for the theory of testing statistical hypotheses. Whereas Laplace later defined probability by means of the enumeration of discrete units, Bayes defined a continuous probability distribution by a formula for the probability between any pair of assigned limits. He did not, however, consider the metric of his continuum.

Joseph Louis Lagrange contributed to probability theory and solved many of the problems previously posed by de Moivre. Beguelin, George Louis Buffon,[5] and the later John Bernoulli published sundry articles on games of chance and the calculus of probability during the second half of the 18th century. Jean Antoine de Caritat, Marquis de Condorcet, supplied the doctrine of credibility: The mathematical measure of probability should be considered as an accurate measure of our degree of belief.

Quantitatively, the theory of probability is more indebted to Pierre Simon, Marquis de Laplace, than to any other mathematician. His great work, *Théorie Analytique des Probabilités*, was published in 1812 and was accompanied by a popular exposition, *Essai Philosophique sur les Probabilités*. These tomes represent an outstanding contribution to the subject, containing a multitude of new ideas, results, and analytic methods. A theory of equations in finite differences with one and two independent variables is proposed, the concomitant analysis being applied to problems of duration of play and random sampling. Laplace elaborated on the idea of inverse probabilities first considered by Bayes. In subsequent memoirs, he investigated applications of generating functions,

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[5]In his *Political Arithmetic*, Buffon considered the chance of the sun rising tomorrow. He calculated the probability as $1 - (1/2)^x$ where x is the number of days the sun is known to have risen previously. Another equally enlightening contribution from Buffon was the establishment of 1/10,000 as the lowest practical probability. More respectable is Buffon's needle problem: A needle of length l is randomly thrown onto a plane ruled with parallel lines separated by a distance d. The probability that the needle will intersect one of the lines is $2l/\pi d$ if $l < d$.

solutions and problems in birth statistics and the French lottery (a set of 90 numbers, five of which are drawn at a time), and generalized Montmort's matching problem. Although he employed overly intricate analyses and not always lucid reasoning, he provided the greatest advance in probabilistic methodology in history. Laplace can claim credit for the first scientifically reasoned deterministic interpretation of the universe.

In the mid-18th century, the empiricist philosopher and economist David Hume studied the nature of probability and concluded that chance events are possessed of a subjective nature. Although he initiated development of the logical structure of the causality concept, Hume was unable to grasp fully the significance of inductive inference, being insufficiently endowed mathematically to exploit the theory of probability in a philosophic framework.

By the early 19th century, the understanding of chance events had progressed beyond its previous, naïve level; it was now accepted that probability statements could be offered with a high degree of certainty if they were transformed into statistical statements. The analysis of games of chance had led to clarification of the meaning of probability and had solidified its epistemological foundations.

Following Laplace, Simeon Denis Poisson was the first to define probability as "the limit of a frequency" (*Rechercher sur le Probabilité des Jugements en Matière Criminelle et en Matière Civile*, 1837), the work that included the substantive "law of large numbers." Contemporaneously, the philosopher/priest Bernhard Bolzano promoted the view that probability represents a statistical concept of relative frequency. Augustus de Morgan, like Laplace, held that probability refers to a state of mind. Knowledge, he wrote, is never certain, and the measure of the strength of our belief in any given proposition is referred to as its probability. (Laplace had stated, "Chance is but the expression of man's ignorance.") The brilliant mathematician Karl Friedrich Gauss applied probability theory to astronomical research. After Gauss, most mathematicians chose to disregard a subject that had become a virtual stepchild in Victorian Europe. The English logician John Venn, the French economist Augustin Cournot, and the American logician Charles Sanders Peirce all expanded on Bolzano's relative frequency concept. Henri Poincaré wrote an essay on chance, as did Peirce. In *The Doctrine of Chances* (a title purloined from de Moivre *and* Bayes), Peirce in effect states that anything that can happen will happen; thus insurance companies must eventually fall bankrupt. This notion, if taken literally, raises a misleading signpost on a convoluted footpath.

For a time, the neglected orphan, probability theory, was fostered more by nonmathematicians—men who expanded the subject to include the modern statistical methods that may ultimately return far greater dividends. Among the major contributors are the Belgian demographer and anthropometrist Adolphe Quetelet[6]; Francis Galton, who investigated correlations and placed the laws

[6]In discussing inductive reasoning, Quetelet avered that if an event (e.g., the tide rising) has occurred m times, the probability that it will occur again is $(m + 1)/(m + 2)$. Note that the probability of unknown events ($m = 0$) is thereby defined as 1/2. Possibly, Buffon's logic of the previous century gains by comparison.

of heredity on a statistical basis (the initial concept of correlations is credited to the Frenchman Auguste Bravais); W.F.R. Weldon, English zoologist and biometrician; the American actuary Emory McClintock; and Josiah Williard Gibbs, who contributed to the foundations of statistical mechanics. One of the few pure statisticians of the era was Karl Pearson, responsible for introducing the method of moments and defining the normal curve and standard deviation (1893). In 1900, he independently devised the "chi-square" test[7] for goodness of fit. Pearson is the author of a fascinating treatise titled *The Scientific Aspect of Monte Carlo Roulette*. Few other qualified mathematicians were motivated to apply the fruits of mathematical study to the naughty practice of gambling.

During the latter half of the 19th century, advances in probability theory were associated in great measure with Russian mathematicians. Victor Bunyakovsky promoted the application of probability theory to statistics, actuarial science, and demography. Pafnutii Tchebychev (1821–1894) generalized Bernoulli's theorem for the law of large numbers and founded the influential Russian school of mathematical probability. Two of his students were A. Liapounov (central limit theorem) and A.A. Markov, credited with the formulation of enchained probabilities. Tchebychev, Liapounov, and Markov together promulgated use of the vastly significant concept of a random variable.

Probability theory was carried into the 20th century by two philosophers, Bertrand Russell and John Maynard Keynes, both of whom worked to establish the discipline as an "independent and deep science of great practical importance with its own distinctive methods of investigation." In his *Principia Mathematica* (1903), Lord Russell combined formal logic with probability theory. Keynes was a successful gambler who sharply attacked the frequency concept of probability and extended the procedures of inductive inference initiated by Bernoulli and Laplace. In related areas, the work of Vilfredo Pareto contributed to both mathematical economics (following Leon Walras) and sociology.

The first decade of the century witnessed the connection, first adumbrated by Émile Borel, between the theory of probability and the measure-theoretic aspects of the functions of a real variable. Borel's classic memoir, *Sur les Probabilités Dénombrables et leurs Applications Arithmétiques* (1909), marks the beginning of modern probability theory. In the 1920s, Borel's ideas were expanded by Khintchine, Kolmogorov, Eugene Slutsky, and Paul Lévy, among others. Rigorous mathematical developments were supplied by Richard von Mises who, in a 1921 paper, introduced the notion of a sample space representing all conceivable outcomes of an experiment (the glass to contain the fluid of probability theory). With R.A. Fisher, von Mises advanced the idea of statistical probability with a definite operational meaning. It was his intent to replace all the doubtful metaphysics of probability "reasoning" by sound mathematics.

Also in 1921, a mathematical theory of game strategy was first attempted by Borel, who was interested in gambling phenomena and subsequently applied some

[7] The "chi-square" test was originated by Helmert in 1875.

elementary probability calculations to the game of Contract Bridge. Beyond his real achievements, Borel struggled to apply probability theory to social, moral, and ethical concerns, and convinced himself (if few others) that the human mind cannot imitate chance.

The axiomatic foundations of probability theory, initiated by Serge Bernstein,[8] were finalized with the publication (1933) of A.N. Kolmogorov's postulates in *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Foundations of the Theory of Probabilities). This work established analogies between the measure of a set and the probability of an event, between a certain integral and the mathematical expectation, and between the orthogonality of functions and the independence of random variables.

Recent contributions to probability theory have become highly complex. The frontiers of the subject have advanced well beyond the earlier intimacy with games of chance. Fréchet, in the 1930s, generalized classical probability theory as an application of his abstract spaces. Kolmogorov and Khintchine, also in the 1930s, were largely responsible for the creation of the theory of stochastic processes. Other outstanding probability theorists—Y.W. Lindeberg and William Feller, for example—and contemporary students of the subject now wield such weapons as infinite probability fields, abstract Lebesgue integrals, and Borel sets, in addition to many specialized, esoteric, and formidable concepts applicable in formal analysis of the physical sciences. Probability theory and general mathematical developments have resumed the close coupling that existed from the mid-17th century to Laplace's day, but which was disrupted for more than a hundred years.

Game strategy as a mathematical discipline was firmly anchored by John von Neumann in 1928 when, in a creative mathematical paper on the strategy of Poker, he proved the minimax principle, the fundamental theorem of game theory. Later, in 1944, the concepts of game theory were expanded when von Neumann and Oskar Morgenstern published *The Theory of Games and Economic Behavior*. More recently, quantum strategies have been added to the arsenal of game theorists. The player implementing such strategies can increase his expectation over those of a player restricted to classical strategies. And Parrondo's Paradox, acclaimed as one of the most significant advances in game theoretic principles since the minimax process, has demonstrated how two losing games can be combined to produce a winning one.

With modern technology we can now provide complete solutions to most of the conventional games of chance (card games such as Poker and Bridge are excluded because of the Herculean burden of calculations inherent in a realistic model). Probability theory, statistics, and game-theoretic principles have seeped into population genetics, psychology, biology, cat-whisker technology, weather forecasting, and a host of other disciplines. Mathematicians are closing in on the higher-order abstractions. As Aldous Huxley invented Riemann-Surface Tennis to fill the leisure hours of the denizens of his brave new world, so we might turn

[8]Bernstein's "probability theory" (1927) introduces three axioms: that of comparability of probabilities, that of incompatible (disjoint) events, and that of combination of events.

to such games as Minkowski Roulette, transfinite-cardinal Bingo, Kaluza Craps, or Quaternion Lotteries to challenge our mathematically inclined gamblers.

The concept of mathematical probability, cultivated in a petri dish of dice, cards, and Chinese puzzles, is now acknowledged as one of the main conceptual structures that distinguishes the intellectual culture of modern civilization from that of its precursors. Gamblers can justifiably stand as the godfathers of this realm, having provoked the stimulating interplay of gambling and mathematics. Yet few gamblers today are sufficiently knowledgeable of probability theory to accept their patrimony.

Psychology and the graduated income tax have largely outmoded the grandiose gesture in gambling. Freud (not surprisingly) suggested a sexual element in gambling, and to emphasize the neurotic appeal, Gamblers Anonymous was organized in California in 1947, proffering sympathy to repentant, obsessive gamblers. The tax collector, hovering nearby, has discouraged the big winner from proudly publicizing his feat. "Bet-a-million" Gates and Arnold Rothstein no longer inspire adulation. Most tales of recent great coups likely have been apocryphal. "Breaking the bank at Monte Carlo" is a euphemism for closing a single gaming table. It was last accomplished at the Casino Société des Bains de Mer during the final days of 1957, with a harvest of 180 million francs. As in so many other human pursuits, quantity has supplanted quality, and while today's wagers are smaller, it is the masses who are now gambling. Authorities estimate that in the United States alone, with casino gambling legalized in 36 states, tens of millions of gamblers wager more than $75 billion every year.

The ancient and universal practice of mankind has become a modern and universal practice. Gambling—the desire to engage with destiny and chance—steadfastly reflects the human spirit. Casinos are spread across the globe: Las Vegas (first and foremost), Monte Carlo, Reno, Campione, Macao, the Casino d'Enghien, Geneva, Constanţa (for gambling commissars), the Urca casino in Rio de Janeiro, Viña del Mar Kursaal, Quintandinha, Mar del Plata, Rhodes, Corfu, Marrakech, San Juan, the Casino du Liban in Lebanon, Cannes, Nice, Deauville, Accra, Brighton, Estoril, Baden-Baden, Manila, and scores of others, glamorous and shoddy, legal and illegal.

To phrase it in the vulgate: You can lose your shirt anywhere in the world.

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Chapter | two

Mathematical Preliminaries

THE MEANING OF PROBABILITY

The word "probability" stems from the Latin *probabilis*: "truth-resembling"; thus the word itself literally invites semantic imprecision. Yet real concepts exist and are applicable in resolving or ordering certain questions. For our purpose, which is to categorize gambling phenomena, we adopt an operational definition of probability that avoids philosophic implications and Bergsonian vagueness and constrains the area of application. This restricted definition we term "rational probability" and credit it as the only valid mathematical description of likelihood or uncertainty.

> Rational probability is concerned only with mass phenomena or repetitive events that are subject to observation or to logical extrapolation from empirical experience.

That is, we must have, either in space or in time, a practically unlimited sequence of uniform observations or we must be able to adapt correlated past experience to the problem for it to merit the attention of rational probability theory.

The extension of probability theory to encompass problems in the moral sciences, as ventured in the rampant rationalism of the 18th century, is not deemed invalid, but simply another field of endeavor. The views of de Morgan and Lord Keynes we classify as "philosophic probability." Questions of ethics, conduct, religious suppositions, moral attributes, unverifiable propositions, and the like are, in our assumed context, devoid of meaning.

Similarly, "fuzzy logic," an offshoot of set theory that deals with degrees of truth, lies outside the domain of gambling theory.[1] Degrees of truth are not probabilities, although the two are often conflated.

Three principal concepts of probability theory have been expressed throughout the sensible history of the subject. First, the classical theory is founded on the indefinable concept of equally likely events. Second, the limit-of-relative-frequency theory is founded on an observational concept and a mathematical postulate. Third,

[1]Fuzzy logic has, in fact, been applied to Backgammon computers (q.v. Chapter 6).

the logical theory defines probability as the degree of confirmation of an hypothesis with respect to an evidence statement.

Our definition of rational probability theory is most consistent with and completed by the concept of a limiting relative frequency. If an experiment is performed whereby n trials of the experiment produce n_0 occurrences of a particular event, the ratio n_0/n is termed the relative frequency of the event. We then postulate the existence of a limiting value as the number of trials increases indefinitely. The probability of the particular event is defined as

$$P = \lim_{n \to \infty} \frac{n_0}{n} \tag{2-1}$$

The classical theory considers the mutually exclusive, exhaustive cases, with the probability of an event defined as the ratio of the number of favorable cases to the total number of possible cases. A weakness of this perspective lies in its complete dependence upon *a priori* analysis—a process feasible only for relatively unsophisticated situations wherein all possibilities can be assessed accurately; more often, various events cannot be assigned a probability *a priori*.

The logical theory is capable of dealing with certain interesting hypotheses; yet its flexibility is academic and generally irrelevant to the solution of gambling problems. Logical probability, or the degree of confirmation, is not factual, but *L*-determinate—that is, analytic; an *L* concept refers to a logical procedure grounded only in the analysis of senses and without the necessity of observations in fact.

Notwithstanding this diversity of thought regarding the philosophical foundations of the theory of probability, there has been almost universal agreement as to its mathematical superstructure. And it is mathematics rather than philosophy or semantic artifacts that we summon to support statistical logic and the theory of gambling. Specifically in relation to gambling phenomena, our interpretation of probability is designed to accommodate the realities of the situation as these realities reflect accumulated experience. For example, a die has certain properties that can be determined by measurement. These properties include mass, specific heat, electrical resistance, and the probability that the up-face will exhibit a "3." Thus we view probability much as a physicist views mass or energy. Rational probability is concerned with the empirical relations existing among these types of physical quantities.

THE CALCULUS OF PROBABILITY

Mathematics, qua mathematics, is empty of real meaning. It consists merely of a set of statements: "if …, then … ." As in Euclidean geometry, it is necessary only to establish a set of consistent axioms to qualify probability theory as a rigorous branch of pure mathematics. Treating probability theory, then, in a geometric sense, each possible outcome of an experiment is considered as the location of a point on a line. Each repetition of the experiment is the coordinate of the point in another dimension. Hence probability is a measure—like the geometric measure of volume. Problems in probability are accordingly treated as a geometric analysis of points in a multidimensional space. Kolmogorov has been largely responsible for providing this axiomatic basis.

Axioms and Corollaries

A random event is an experiment whose outcome is not known *a priori*. We can state

Axiom I: To every random event $\boldsymbol{A}$ there corresponds a number $P(\boldsymbol{A})$, referred to as the probability of $\boldsymbol{A}$, that satisfies the inequality

$$0 \leq P(A) \leq 1$$

Thus the measure of probability is a nonnegative real number in the range 0 to 1.

Now consider an experiment whose outcome is certain or whose outcomes are indistinguishable (tossing a two-headed coin). To characterize such an experiment, let $\boldsymbol{E}$ represent the collection of all possible outcomes; thence

Axiom II: The probability of the certain event is unity. That is,

$$P(\boldsymbol{E}) = 1$$

The relative frequency of such events is (cf. Eq. 2-1) $n_0/n = 1$.

Lastly, we require an axiom that characterizes the nature of mutually exclusive events (if a coin is thrown, the outcome Heads excludes the outcome Tails, and vice versa; thus, Heads and Tails are mutually exclusive). We formulate this axiom as follow:

Axiom III: The theorem of total probability. If events $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ are mutually exclusive, the probability of the alternative of these events is equal to the sum of their individual probabilities. Mathematically,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

where $\boldsymbol{A}_1 \cup \boldsymbol{A}_2$ denotes the union of events $\boldsymbol{A}_1$, and $\boldsymbol{A}_2$. Axiom III expresses the additive property of probability. It can be extended to include events not mutually exclusive. If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are such events, the total probability ($\boldsymbol{A}_1$, and/or $\boldsymbol{A}_2$ occurs) is

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2) \qquad (2\text{-}2)$$

where $P(\boldsymbol{A}_1\boldsymbol{A}_2)$ is defined as the *joint* or *compound probability* that both $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ occur.

Generalizing, Eq. 2-2 can be extended to any number of events–that is, the probability of occurrence, P_1, of at least one event among n events, $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$, is given by

$$P_1 = \sum P\{A_i\} - \sum P\{A_i A_j\} + \sum P\{A_i A_j A_k\} - \cdots$$

And Eq. 2-2 can be extended to any number of independent events:

$$P_1(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i A_j) + \sum_{i<j<k}^{n} P(A_i A_j A_k) - \cdots$$

From these axioms several logical corollaries follow.

Corollary I: The sum of the probabilities of any event $\boldsymbol{A}$ and its complement $\bar{\boldsymbol{A}}$ is unity. That is,

$$P(\boldsymbol{A}) + P(\bar{\boldsymbol{A}}) = 1$$

The complement of an event is, of course, the nonoccurrence of that event.

Characterizing an impossible event O, we can state the following:

Corollary II: The probability of an impossible event is zero, or

$$P(O) = 0$$

It is worth noting that the converse of Corollary II is not true. Given that the probability of some event equals zero, it does not follow that the event is impossible (there is a zero probability of selecting at random a prespecified point on a line).

Another important concept is that of "conditional probability." The expression $P(\boldsymbol{A}_2|\boldsymbol{A}_1)$ refers to the probability of an event $\boldsymbol{A}_2$, given (or conditional to) the occurrence of the event $\boldsymbol{A}_1$. We can write, for $P(\boldsymbol{A}_1) > 0$,

$$P(\boldsymbol{A}_2|\boldsymbol{A}_1) = \frac{P(\boldsymbol{A}_1\boldsymbol{A}_2)}{P(\boldsymbol{A}_1)} \tag{2-3}$$

By induction from Eq. 2-3, we can obtain the multiplication theorem (or the general law of compound probability):

Corollary III: The probability of the intersection of n events $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ is equal to the probability of the first event times the conditional probability of the second event, given the occurrence of the first event, times the conditional probability of the third event given the joint occurrence of the first two events, etc. In mathematical form,

$$P(A_1\, A_2 \ldots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1A_2\cdots A_{n-1})$$

It can be demonstrated that conditional probability satisfies Axioms I, II, and III. A special case of Corollary III occurs when the events $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ are independent; then the law of compound probability reduces to

$$P(A_1\, A_2 \ldots A_n) = P(A_1)P(A_2)\cdots P(A_{n-1})P(A_n)$$

Thus the necessary and sufficient condition for n events to be independent is that the probability of every n-fold combination that can be formed from the n events or their complements be factorable into the product of the probabilities of the n distinct components.

Finally, the concept of complete (or absolute) probability is expressed by

Corollary IV: If the random events $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ are pairwise exclusive, then the probability $P(\boldsymbol{X})$ of any random event $\boldsymbol{X}$ occurring together with one of the events $\boldsymbol{A}_i$ is given by

$$P(\boldsymbol{X}) = P(\boldsymbol{A}_1)P(\boldsymbol{X}|\boldsymbol{A}_1) + P(\boldsymbol{A}_2)P(\boldsymbol{X}|\boldsymbol{A}_2) + \cdots + P(\boldsymbol{A}_n)P(\boldsymbol{X}|\boldsymbol{A}_n)$$

$$\text{if } P(\boldsymbol{A}_i) > 0 \text{ for } i = 1, 2, \ldots, n$$

Combining the principles of complete and conditional probabilities leads to the following statement:

Corollary IV(a): If the mutually exclusive events $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ satisfy the assumptions of the complete probability theorem, then for an arbitrary event $\boldsymbol{X}$ associated with one of the events $\boldsymbol{A}_i$ we have, for $P(\boldsymbol{X}) > 0$,

$$P(\boldsymbol{A}_i|\boldsymbol{X}) = \frac{P(\boldsymbol{A}_i)P(\boldsymbol{X}|\boldsymbol{A}_i)}{P(\boldsymbol{A}_1)P(\boldsymbol{X}|\boldsymbol{A}_1) + P(\boldsymbol{A}_2)P(\boldsymbol{X}|\boldsymbol{A}_2) + \cdots + P(\boldsymbol{A}_n)P(\boldsymbol{X}|\boldsymbol{A}_n)} \qquad (2\text{-}4)$$

$$i = 1, 2, 3, \ldots, n$$

Equation 2-4 is known as Bayes' theorem, or the formula for *a posteriori* probability. It should be noted that no assumptions as to the probabilities of the respective $\boldsymbol{A}_i$'s are implied (a common error is to interpret Bayes' theorem as signifying that all $\boldsymbol{A}_i$'s have equal probabilities).

Together, the preceding axioms and corollaries constitute the foundation of the theory of probability as a distinct branch of mathematics.

An example may aid in clarifying these concepts. Given that the probability $P(K)$ of drawing at least one King in two tries from a conventional deck of 52 cards is $396/(52 \times 51) = 0.149$, that the probability $P(Q)$ of drawing a Queen in two tries is also 0.149, and that the joint probability (drawing both a King and a Queen) is $32/(52 \times 51) = 0.012$, what is the conditional probability $P(K|Q)$ that *one* of the two cards is a King, given that the other is a Queen?[2] What is the total probability $P(K|Q)$ that at least one card is either a King or a Queen? Are the two events—drawing a King and drawing a Queen—independent?

Applying Eq. 2-3 directly.

$$P(K|Q) = \frac{P(KQ)}{P(Q)} = \frac{32/(52 \times 51)}{396/(52 \times 51)} = \frac{8}{99} = 0.081$$

[2]Note that either of the two cards may be the postulated queen. A variation asks the probability of one card being a King, given the other to be a non-King ($\bar{K}$). In this case, $P(K|\bar{K}) = 0.145$.

and of course, the probability $P(Q|K)$ of one of the two cards being a Queen, given the other to be a King, is also 0.081. From Eq. 2-2, the total probability (drawing a King and/or a Queen) is

$$P(K \cup Q) = P(K) \cup P(Q) - P(KQ) = \frac{396 + 396 - 32}{52 \times 51} = 0.287$$

It is evident that the two events are not independent, since $P(KQ) \neq P(K)P(Q)$.

Permutations and Combinations

Gambling phenomena frequently require the direct extension of probability theory axioms and corollaries into the realm of permutational and combinatorial analysis. A *permutation* of a number of elements is any arrangement of these elements in a definite order. A *combination* is a selection of a number of elements from a population considered without regard to their order. Rigorous mathematical proofs of the theorems of permutations and combinations are available in many texts on probability theory. By conventional notation, the number of permutations of n distinct objects (or elements or *things*) considered r at a time without repetition is represented by P_n^r. Similarly, C_n^r represents the number of combinations of n distinct objects considered r at a time without regard to their order.

To derive the formula for P_n^r, consider that we have r spaces to fill and n objects from which to choose. The first space can be filled with any of the n objects (that is, in n ways). Subsequently, the second space can be filled from any of $(n-1)$ objects $(n-1)$ ways, the third space in $(n-2)$ ways, etc., and the rth space can be filled in $[n-(r-1)]$ ways. Thus

$$P_n^r = n(n-1)(n-2)\cdots(n-r+1) \tag{2-5}$$

For the case $r = n$, Eq. 2-5 becomes

$$P_n^n = n(n-1)(n-2)\cdots 1 \equiv n! \tag{2-6}$$

Combining Eqs. 2-6 and 2-5, the latter can be rewritten in the form

$$P_n^r = \frac{P_n^n}{P_{n-r}^{n-r}} = \frac{n!}{(n-r)!}$$

It is also possible, from Eq. 2-5, to write the recurrence relation

$$P_n^r = P_{n-1}^r + rP_{n-1}^{r-1}$$

Similar considerations hold when we are concerned with permutations of n objects that are not all distinct. Specifically, let the n objects be apportioned into m kinds of elements with n_1 elements of the first kind, n_2 elements of the second, etc., and $n = n_1 + n_2 + \cdots + n_m$. Then the number of permutations P_n of the n objects taken all together is given by

$$P_n = \frac{n!}{n_1!n_2!n_3!\cdots n_m!} \tag{2-7}$$

Illustratively, the number of permutations of four cards selected from a 52-card deck is

$$P_{52}^4 = \frac{52!}{(52-4)!} = 52 \times 51 \times 50 \times 49 = 6,497,400$$

Note that the order is of consequence. That is, the A, 2, 3, 4 of Spades differs from, say, the 2, A, 3, 4, of Spades. If we ask the number of permutations of the deck, distinguishing ranks but not suits (i.e., 13 distinct kinds of objects, each with four indistinct elements[3]), Eq. 2-7 states that

$$P_{52} = \frac{52!}{(4!)^{13}} = 9.203 \times 10^{49}$$

To this point we have allowed all elements of a population to be permuted into all possible positions. However, we may wish to enumerate only those permutations that are constrained by prescribed sets of restrictions on the positions of the elements permuted. A simple example of a restricted-position situation is the "problème des ménages,"[4] which asks the number of ways of seating n married couples around a circular table with men and women in alternate positions and with the proviso that no man be seated next to his spouse. This type of problem is equivalent to that of placing k Rooks on a "chessboard"—defined by mathematicians as an arbitrary array of cells arranged in rows and columns—in such a way that no Rook attacks any other (the Rook is a Chess piece that moves orthogonally). Applications of these concepts lie in the domain of combinatorial analysis. As noted briefly in Chapter 7, they are particularly useful and elegant in the solution of matching problems.

Turning to r combinations of n distinct objects, we observe that each combination of r distinct objects can be arranged in $r!$ ways—that is $r!$ permutations. Therefore, $r!$ permutations of each of the C_n^r combinations produce $r!C_n^r$ permutations:

$$r!C_n^r = P_n^r = \frac{n!}{(n-r)!}$$

[3]Observe that if we distinguish suits but not ranks (four distinct kinds of objects, each with 13 indistinct elements), we obtain a considerably smaller number of permutations:

$$\frac{52!}{(13!)^4} = 5.3645 \times 10^{28}$$

[4]So named by E. Lucas in *Théorie des Nombres*, Paris, 1891.

Dividing by $r!$, we obtain the expression for the number of combinations of n objects taken r at a time:

$$C_n^r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{2-8}$$

where $\binom{n}{r}$, the symbol for binomial coefficients, is the $(r+1)$st coefficient of the expansion of $(a+b)^n$. For values of r less than zero and greater than n, we define C_n^r to be equal to zero. (Eq. 2-8 can also be extended to include negative values of n and r; however, in our context we shall not encounter such values.) It is also apparent that the number of combinations of n objects taken r at a time is identical to the number of combinations of the n objects taken $n-r$ at a time. That is,

$$\binom{n}{r} = \binom{n}{n-r} \quad \text{or} \quad C_n^r = C_n^{n-r}$$

We can also derive this equivalence in the format

$$C_n^r = C_{n-1}^{r-1} + C_{n-1}^r \tag{2-9}$$

Equation 2-9 is known as Pascal's rule. By iteration it follows that

$$C_n^r = C_{n-1}^{r-1} + C_{n-2}^{r-1} + \cdots + C_{r-1}^{r-1} = C_{n-1}^r + C_{n-2}^{r-1} + \cdots + C_{n-1-r}^0$$

As an example, we might ask how many Bridge hands of 13 cards are possible from a 52-card population. Equation 2-8 provides the answer:

$$\binom{52}{13} = \frac{52!}{13!(52-13)!} = 635{,}013{,}559{,}600$$

Note that the ordering of the 13 cards is of no consequence. If it were, the resulting number of permutations would be greater by a factor of 13!

Probability Distributions

With a nonhomogeneous population (n objects consisting of m kinds of elements with n_1 elements of the first kind, n_2 elements of the second kind, etc.), the number of permutations of the n objects taken all together is given by Eq. 2-7. We can also determine the number of combinations possible through selection of a group of r elements from the n objects. Thence it is feasible to ask the probability $P_{k_1,k_2,\ldots,k_m}$ that if a group of r elements is selected at random (without replacement and without ordering), the group will contain exactly $k_1 \le n_1$

elements of the first kind, $k_2 \le n_2$ elements of the second kind, etc., and $k_m \le n_m$ elements of the mth kind. Specifically, it can be shown that

$$P_{k_1 k_2, \ldots, k_m} = \frac{\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{n_3}{k_3}\cdots\binom{n_m}{k_m}}{\binom{n}{r}} \qquad \begin{aligned} n &= n_1 + n_2 + \cdots + n_m \\ r &= k_1 + k_2 + \cdots + k_m \end{aligned} \tag{2-10}$$

Equation 2-10 represents the generalized *hypergeometric distribution*, the probability distribution for sampling without replacement from a finite population. It should be noted that in the limit (for arbitrarily large populations) sampling with or without replacement leads to the same results.

As an illustration of the hypergeometric distribution, we compute the probability that a Bridge hand of 13 cards consists of exactly 5 Spades, 4 Hearts, 3 Diamonds, and 1 Club. According to Eq. 2-10,

$$P_{5,4,3,1} \frac{\binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} = 0.0054$$

since the order of the cards within the Bridge hand is irrelevant.

Another distribution of consequence in practical applications of probability theory is the *binomial distribution*. If an event has two alternative results, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, so that $P(\boldsymbol{A}_1) + P(\boldsymbol{A}_2) = 1$, and the probability of occurrence for an individual trial is constant, the number of occurrences r of the result $\boldsymbol{A}_1$ obtained over n independent trials is a discrete *random variable*, which may assume any of the possible values 0, 1, 2, …, n. A *random variable* is simply a variable quantity whose values depend on chance and for which there exists a distribution function. In this instance the number r is a *binomial variate*, and its distribution $P(r)$ defines the binomial distribution.[5] Letting $p = P(\boldsymbol{A}_1)$ and $q = P(\boldsymbol{A}_2) = 1 - p$, we can readily derive the expression

$$P(r) = \binom{n}{r} p^r q^{n-r}, \quad r = 0, 1, 2, \ldots, n \tag{2-11}$$

[5]An apparatus that generates the binomial distribution experimentally is the *quincunx* (named after a Roman coin valued at five-twelfths of a lira), described by Sir Francis Galton in his book, *Natural Inheritance* (1889). It consists of a board in which nails are arranged in rows, n nails in the nth row, the nails of each row being placed below the midpoints of the intervals between the nails in the row above. A glass plate covers the entire apparatus. When small steel balls (of diameter less than the horizontal intervals between nails) are poured into the quincunx from a point directly above the single nail of the first row, the dispersion of the balls is such that the deviations from the center-line follow a binomial distribution.

Equation 2-11 is also referred to as the *Bernoulli distribution*, since it was first derived by Jacob Bernoulli in *Ars Conjectandi*. An application to the Kullback matching problem is noted in Chapter 7.

We can readily generalize from two possible outcomes of an event to the case of m mutually exclusive outcomes $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m$, each $\boldsymbol{A}_i$ occurring with probability $p_i = P(\boldsymbol{A}_i) \geq 0$ for each individual trial. The probability of obtaining r_1 instances of A_1, r_2 instances of A_2, etc. with n independent trials is known as the *multinomial distribution* and is determined in a fashion similar to Eq. 2-11:

$$P(r_1, r_2, \ldots, r_m) = \frac{n!}{r_1! r_2! \cdots r_m!} p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} \tag{2-12}$$

where $p_1 + p_2 + \cdots + p_m = 1$, and $r_1 + r_2 + \cdots + r_m = n$. Equation 2-12 specifies the *compound* or *joint distribution* of the number of outcomes for each $\boldsymbol{A}_i$.

When a chance event can occur with constant probability p on any given trial, then the number of trials r required for its first occurrence is a discrete random variable that can assume any of the values 1, 2, 3, …, ∞. The distribution of this variable is termed the *geometric distribution*. If the first occurrence of the event is on the rth trial, the first $r-1$ trials must encompass nonoccurrences of the event (each with probability $q = 1-p$). The compound probability of $r-1$ nonoccurrences is q^{r-1}; hence $r-1$ nonoccurrences followed by the event has probability $q^{r-1}p$. Accordingly, the geometric distribution is defined by

$$p(r) = pq^{r-1}, \quad r = 1, 2, 3, \ldots \tag{2-13}$$

By implication from this distribution, the *expected* number of trials is equal to the reciprocal of the constant probability p (cf. Eq. 6-18).

For the situation where events occur randomly in time, let $P(r, T)$ denote the probability that exactly r events occur in a specified time interval T. Defining α as the probability that one event occurs in the incremental interval ΔT (so the *average* number of events in the interval T *is* αT), we can determine $P(r, T)$ as

$$P(r, T) = \frac{e^{-\alpha T}(\alpha T)^r}{r!} \tag{2-14}$$

Equation 2-14 defines the Poisson distribution; it serves as an approximation to the binomial distribution when the number of trials is large.

Illustratively, if a gambler can expect five successes in ten trials, the probability that he will win *at least* twice is then

$$1 - [P(0, T) + P(1, T)] = 1 - \left[\frac{5^0 e^{-5}}{0!} + \frac{5e^{-5}}{1!}\right] = 1 - (0.007 + 0.034) = 0.959$$

Note that the single-trial probability of success does not enter the computation—only the *average* number of successes pertains.

The Poisson distribution provides substantial benefit in representing the time sequence of random independent events. Whereas a binomial distribution is symmetric and bounded at both ends, the Poisson distribution is asymmetric, being bounded at the lower end but not at the top.

Consider a telephone service that receives an average of one call per minute from a pool of millions of customers. The Poisson distribution then specifies the probability of receiving r calls in any particular minute. Since we are dealing here with only a single parameter, αT, knowledge of the average (mean) is sufficient in itself to describe the pertinent phenomenon.

Applications of the several probability distributions are multifold. The following numerical example is representative. From a deck of 52 cards, ten are dealt at random. The full deck is then reshuffled, and 15 cards are selected at random. We ask the probability $P(r)$ that exactly r cards are common to both selections ($r = 0, 1, 2, \ldots, 10$). From the 52 cards, 15 can be selected in $\binom{52}{15}$ distinct ways. If r of these have been selected on the previous deal, then $15 - r$ are in the nonoccurring category. Obviously, there are $\binom{10}{r}$ distinct ways of choosing r of the cards from the first deal and $\binom{52-10}{15-r}$ ways of selecting the remaining cards. Thus, according to Eq. 2-10 for the hypergeometric distribution,

$$P(r) = \frac{\binom{10}{r}\binom{52-10}{15-r}}{\binom{52}{15}}$$

For no matching cards, $r = 0$, $P(0) = 0.022$, and for $r = 5$, $P(5) = 0.083$. The most probable number of matches is 3; and $P(3) = 0.296$.

Mathematical Expectation

A parameter of fundamental value in the evaluation of games of chance is the *mathematical expectation*. Its definition is straightforward: If a random variable X can assume any of n values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, the mathematical expectation of X, $\mathrm{E}(X)$, is expressed by

$$\mathrm{E}(X) = p_1x_1 + p_2x_2 + \cdots + p_nx_n = \sum_{i=1}^{n} p_ix_i \tag{2-15}$$

The mathematical expectation of the number showing on one die is accordingly

$$\mathrm{E}(X) = \frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6) = 3.5 \tag{2-16}$$

since the possible values of X, the number showing, are 1, 2, 3, 4, 5, and 6, each with probability 1/6. For two dice, X can range between 2 and 12 with varying probabilities:

$$\begin{aligned}\mathrm{E}(X) = {} & \frac{1}{36}(2) + \frac{2}{36}(3) + \frac{3}{36}(4) + \frac{4}{36}(5) + \frac{5}{36}(6) + \frac{6}{36}(7) + \frac{5}{36}(8) \\ & + \frac{4}{36}(9) + \frac{3}{36}(10) + \frac{2}{36}(11) + \frac{1}{36}(12) = 7\end{aligned} \tag{2-17}$$

A useful theorem states that the mathematical expectation of the sum of several random variables $X_1, X_2, \ldots, X_n$ is equal to the sum of their mathematical expectations. That is,

$$\mathrm{E}(X_1 + X_2 + \cdots + X_n) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_n) \tag{2-18}$$

Thus the mathematical expectation of the total showing on n dice is $3.5n$. (The result of Eq. 2-17 is directly obtainable by letting $n = 2$.)

Similarly, the theorem can be proved that the mathematical expectation of the product of several independent variables $X_1, X_2, \ldots, X_n$ is equal to the product of their expectations. That is,

$$\mathrm{E}(X_1 X_2 \cdots X_n) = \mathrm{E}(X_1)\mathrm{E}(X_2)\cdots\mathrm{E}(X_n) \tag{2-19}$$

(Note that Eq. 2-19 appertains to independent variables, whereas Eq. 2-18 is valid for any random variable.) It can also be shown in simple fashion that

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

for any numerical constants a and b.

The condensed description of probability theory presented here cannot, of course, include all those tools required for the solution of gambling problems. However, most of the requisite information has been covered briefly and is of propaedeutic value; more comprehensive developments can be found by consulting the References and Bibliography at the end of this chapter.

STATISTICS

Statistics[6] and probability theory cannot always be separated into water-tight compartments. Statistics may be considered as the offspring of the theory of probability, as it builds on its parent and extends the area of patronymic jurisdiction. In this sense, probability theory enables us to deduce the probable composition of a sample, given the composition of the original population; by the use of statistics we can reverse this reasoning process to infer the composition

[6]The word "statistics" derives from the Latin *status*, meaning state (of affairs).

of the original population from the composition of a properly selected sample. Frequently, however, the objective of a statistical investigation is not of a purely descriptive nature. Rather, descriptive characteristics are desired in order to compare different sets of data with the aid of the characteristics of each set. Or we may wish to formulate estimates of the characteristics that might be found in related sets of data. In either case, it is evident that description is a preliminary stage, and further analysis is our principal goal.

As in the theory of probability, the meaning and end result of a statistical study are a set of conclusions. We do not predict the individual event, but consider all possible occurrences and calculate the frequency of occurrences of the individual events. Also, like probability theory, statistics is an invention of men rather than of nature. It, too, has a meaning ultimately based on empirical evidence and a calculus established on an axiomatic foundation. Attributing to statistics the inherent ability to describe universal laws leads to a profusion of artful and provocative fallacies.[7] As per G.K. Chesterton, we caution against using statistics as a drunk uses a lamppost for support rather than illumination.

Mean and Variance

Analysis of the conventional games of chance involves only the more elementary aspects of statistical theory, primarily those related to the concepts of variance, estimation, hypothesis testing, and confidence limits. Our treatment of the subject is therefore correspondingly limited.

The mathematical expectation of a random variable X is also known as the *mean* value of X. It is generally represented by the symbol μ; that is, $\mu = \mathrm{E}(X)$. Thus $\mathrm{E}(X - \mu) = 0$. Considering a constant c instead of the mean μ, the expected value of $X - c$ [that is, $\mathrm{E}(X - c)$] is termed the *first moment* of X taken about c. The mean (or center of mass of the probability function) depicts the long-run average result for an experiment performed an arbitrarily large number of times. This type of average refers to the arithmetical average of a distribution, defined according to Eq. 2-15. It should not be confused with the *mode* (that value of the distribution possessing the greatest frequency and hence the most probable value of the distribution), the *weighted average* (wherein each value of the random variable X is multiplied by a weighting coefficient before the arithmetical averaging process), the *median* (the sum of the frequencies of occurrence of the values of X above and below the median are equal; for symmetrical distributions, the mean and the median are identical), or the *geometric mean* (the positive nth root of the product of n random variables), among others.[8]

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[7]For a fascinating compendium of statistical traps for the unwary, see Darrell Huff, *How to Lie with Statistics*, W.W. Norton and Co., 1954.

[8]Any type of average is known to statisticians as a "measure of central tendency." Its etymological ancestor is the Latin word *havaria*, which originally described compensation funds paid to owners of cargo sacrificed to lighten the ship during heavy storms. Those whose merchandise survived transit provided indemnification to those less fortunate. Thus the concept of "average" arises from a type of primitive insurance.

In addition to the mean, another parameter is required to describe the distribution of values: a measure of spread or variability comparing various results of the experiment. The most convenient and commonly used measure of spread is the *variance*. Let X be a random variable assuming any of the m values x_i ($i = 1, 2, \ldots, m$) with corresponding probabilities $p(x_i)$ and with mean $\mu = \mathrm{E}(X)$. The variance, $\mathrm{Var}(X)$, then is defined by

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2] = \sum_{i=1}^{m} (x_i - \mu)^2 \, p(x_i)$$

Note that the units of $\mathrm{Var}(X)$ are squares of the units of X. Therefore, to recover the original units, the *standard deviation* of X, $\sigma(X)$, is defined as

$$\sigma(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\mathrm{E}[(X - \mu)^2]}$$

An invaluable theorem formulates the variance of X as the mean of the square of X minus the square of the mean of X—that is,

$$\sigma^2(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \tag{2-20}$$

The mean value of the up-face of a die, as per Eq. 2-16, equals 7/2. The mean of this value squared is

$$\mathrm{E}(X^2) = \frac{1}{6}(1)^2 + \frac{1}{6}(2)^2 + \frac{1}{6}(3)^2 + \frac{1}{6}(4)^2 + \frac{1}{6}(5)^2 + \frac{1}{6}(6)^2 = \frac{91}{6}$$

Thus the variance of the number showing on the die is, according to Eq. 2-20,

$$\sigma^2(X) = (91/6) - (7/2)^2 = 35/12 \quad \text{and } \sigma(X) = 1.71$$

For two dice, $\mathrm{E}(X) = 7$ and $\mathrm{E}(X^2) = 329/6$. Therefore the variance of the sum of numbers on two dice is

$$\sigma^2(X) = (329/6) - (7)^2 = 35/6 \quad \text{and } \sigma(X) = 2.415$$

The expected total showing on two dice is occasionally stated as 7 ± 2.415.

We can readily compute the mean and variance for each well-known probability distribution. The hypergeometric distribution (Eq. 2-10), with but two kinds of elements, simplifies to

$$P_k = \frac{\binom{n_1}{k}\binom{n - n_1}{r - k}}{\binom{n}{r}}, \quad \text{for } k \le r \text{ or } n_1 \tag{2-21}$$

Consider a deck of n cards—n_1 red and $n - n_1$ black. Then if r cards are drawn *without replacement*, the probability that the number X of red cards drawn is exactly k is given by Eq. 2-21. The mean of X is expressed by

$$\mu = \mathrm{E}(X) = \sum_{k=0}^{r} \frac{k\binom{n_1}{k}\binom{n - n_1}{r - k}}{\binom{n}{r}} = \frac{n_1 r}{n}$$

and the mean of X^2 is

$$\mathrm{E}(X^2) = \frac{n_1(n_1 - 1)r(r - 1)}{n(n - 1)} + \frac{n_1 r}{n}$$

Eq. 2-20 specifies the variance:

$$\sigma^2 = \frac{n_1 r(n - n_1)(n - r)}{n^2(n - 1)}$$

For the binomial distribution (Eq. 2-11), we can compute the mean as

$$\mu = \mathrm{E}(X) = np$$

and the mean of X^2 as

$$\mathrm{E}(X^2) = np + n(n - 1)p^2$$

Thus the variance associated with the binomial distribution is

$$\sigma^2 = np + n(n - 1)p^2 - (np)^2 = np(1 - p) = npq \tag{2-22}$$

The Poisson distribution (Eq. 2-14) can be shown to possess both mean *and* variance equal to the average number of events occurring in a specified time interval T. That is,

$$\mu = \sigma^2 = \alpha T$$

Consequently, the Poisson distribution is often written in the form

$$P(r) = \frac{e^{-\mu}\mu^r}{r!}$$

The Law of Large Numbers

Another highly significant theorem can be deduced from the definition of the variance by separating the values of the random variable X into those that lie within the interval $\mu - k\sigma$ to $\mu + k\sigma$ and those that lie without. The sum of the probabilities assigned to the values of X outside the interval $\mu \pm k\sigma$ is equal to the probability that X is greater than $k\sigma$ from the mean μ and is less than or equal to $1/k^2$. Thus

$$\frac{1}{k^2} \geq P(|X - \mu| > k\sigma) \tag{2-23}$$

This expression, known as Tchebychev's theorem, states that no more than the fraction $1/k^2$ of the total probability of a random variable deviates from the mean value by greater than k standard deviations.

A notable application of Tchebychev's inequality lies in determining the point of *stochastic convergence*—that is, the convergence of a sample probability to its expected value. If, in Eq. 2-23, we replace the random variable X by the sample probability p' (the ratio of the number of occurrences of an event to the number of trials attempted) and the mean μ by the single-trial probability of success p, Tchebychev's inequality becomes

$$P\left[|p' - p| > k\sqrt{pq/n}\right] \leq 1/k^2 \tag{2-24}$$

since $\sigma = \sqrt{pq/n}$. Specifying the value of k as $k = \varepsilon/\sqrt{pq/n}$, where ε is some fraction greater than zero, Eq. 2-24 assumes the form

$$P\left[|p' - p| > \varepsilon\right] \leq pq/n\varepsilon^2 \tag{2-25}$$

which is the *law of large numbers*. It declares that no matter how small an ε is specified, the probability P that the sample probability differs from the single-trial probability of success by more than ε can be made arbitrarily small by sufficiently increasing the number of trials n. Thus for an unbiased coin, the probability of the ratio of Heads (or Tails) to the total number of trials differing from 1/2 by greater than a specified amount approaches zero as a limit. We conventionally express this fact by the statement that the sample probability converges stochastically to 1/2.

The law of large numbers has frequently (and erroneously) been cited as the guarantor of an eventual Head–Tail balance. Actually, in colloquial form, the law proclaims that the difference between the number of Heads and the number of Tails thrown may be expected to increase indefinitely with an increasing number of trials, although by decreasing proportions. Its operating principle is "inundation" rather than "compensation" (cf. Theorems I and II of Chapter 3).

Confidence

In addition to a measure of the spread or variability of a repeated experiment, it is desirable to express the extent to which we have confidence that the pertinent parameter or specific experimental result will lie within certain limits. Let the random variable X possess a known distribution, and let us take a sample of size $r(x_1, x_2, \ldots, x_r)$ with which we will estimate some parameter θ. Then, if θ_1 and θ_2 are two statistical estimations of θ, the probability ξ that θ lies within the interval θ_1 to θ_2 is called the *confidence level*—that is,

$$P(\theta_1 \le \theta \le \theta_2) = \xi$$

The parameter θ, it should be noted, is not a random variable. It represents a definite, albeit unknown, number. However, θ_1 and θ_2 *are* random variables, since their values depend on the random samples.

As an example, consider a random variable X that follows a normal distribution with a known standard deviation σ but with an unknown expectation μ. We ask the range of values of μ that confers a 0.95 confidence level that μ lies within that range. From the definition of the normal distribution (Eq. 2-26), the probability that μ lies between

$$\theta_1 = \bar{x} - y\sigma/\sqrt{r} \quad \text{and} \quad \theta_2 = \bar{x} + y\sigma/\sqrt{r}$$

is given by

$$P\left(\bar{x} - y\sigma/\sqrt{r} \le \mu \le \bar{x} + y\sigma/\sqrt{r}\right) = \frac{1}{\sqrt{2\pi}} \int_{-y}^{y} e^{-x^2/2} dx$$

From tables of the normal probability integral we find that this probability is equal to 0.95 for $y = 1.96$. Thus, for an estimate $\bar{x}$ of μ from a small sample of size r, we can claim a 0.95 confidence that μ is included within the interval $\bar{x} \pm 1.96\sigma/\sqrt{r}$.

Let us postulate a Bridge player who, having received ten successive Bridge hands, nine of which contain no Aces, complains that this situation is attributable to poor shuffling. What confidence level can be assigned to this statement? The probability that a hand of 13 cards randomly selected contains at least one Ace is

$$P(1\ \text{Ace}) = 1 - \binom{48}{13}\binom{52}{13}^{-1} = 0.696$$

The binomial distribution (Eq. 2-11) provides the probability that of ten hands, only one will contain at least one Ace:

$$P(1\ \text{Ace in 10 hands}) = \binom{10}{1}(0.696)^1(1 - 0.696)^9 = 0.00015$$

Thus, with a confidence level of 99.985%, the Bridge player can justly decry the lack of randomness and maintain that as a consequence the null hypothesis [$P(1\ \text{Ace}) = 0.696$] does not hold in this game.

Estimation

In the field of economics, one aspect of mathematical statistics widely applied is that of estimation from statistical data. Statistical inference is a method of educing population characteristics on the basis of observed samples of information.[9] For example, we might be ignorant of the probability p that the throw of a particular coin will result in Heads; rather we might know that in 100 trials, 55 Heads have been recorded and wish to obtain from this result an estimate of p. In general, the parameter to be estimated can be a probability, a mean, a variance, etc.

There are several "good" estimates of unknown parameters offered by the discipline of statistics. A common procedure is that of the method of moments. To estimate k unknown parameters, this procedure dictates that the first k sample moments be computed and equated to the first k moments of the distribution. Since the k moments of the distribution are expressible in terms of the k parameters, we can determine these parameters as functions of the sample moments, which are themselves functions of the sample values.

Assuredly, the most widely used measure is the *maximum likelihood* estimate—obtainable by expressing the joint distribution function of the sample observations in terms of the parameters to be estimated and then maximizing the distribution function with respect to the unknown parameters. Solutions of the maximization equation yield the estimation functions as relations between the estimated parameters and the observations. Maximum likelihood estimates are those that assign values to the unknown parameters in a manner that maximizes the probability of the observed sample.

Statistical Distributions

Of the various distributions of value in statistical applications, two are summoned most frequently: the *normal* or *Gaussian distribution* and the *chi-square distribution*. The former constitutes a smooth-curve approximation to the binomial distribution (Eq. 2-11) derived by replacing factorials with their Stirling approximations[10] and defining a continuous variable X:

$$X = \frac{r - np}{\sqrt{np(1-p)}}$$

which is the deviation of the distribution measured in terms of the standard deviation [for the binomial distribution, $\sigma = \sqrt{npq}$ (cf. Eq. 2-22)]. We then obtain

$$P(X) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{2-26}$$

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[9]The dangers of statistical inference are well known in certain commercial situations. Consider the proposition: "Statistics demonstrate that cancer-prone individuals tend toward cigarette smoking."

[10] $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$.

for the normal distribution of the random variable X with zero mean and unit variance (x signifies the values assumed by X).

Related to the normal distribution and readily derived therefrom, the *chi-square* (χ^2) *distribution* is defined as the distribution of the sum of the squares of n independent unit normal variates (and is sometimes referred to as *Helmert's distribution*, after F.R. Helmert, its first investigator). For n degrees of freedom (i.e., $\chi^2 = x_1^2 + x_2^2 + \cdots + x_n^2$), we can determine that

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma(n/2)}(\chi^2)^{(n/2)-1}e^{-x^2/2} \tag{2-27}$$

where $\Gamma(n)$ is the gamma function. $\Gamma(n) = (n-1)!$ for integral values of n, and $\Gamma(1/2) = \sqrt{\pi}$.[11] Values of χ^2 are tabulated in mathematical handbooks and computer packages.

In those instances where it is desired to measure the compatibility between observed and expected values of the frequency of an event's occurrence, the χ^2 statistic is particularly useful. For n trials, let r_i represent the number of times the event x_i is observed, and let s_i be the expected number of occurrences in n trials ($s_i = p_i n$, where p_i is the single-trial probability of x_i); then χ^2 is defined by

$$x^2 = \sum_{i=1}^{k} \frac{(r_i - s_i)^2}{s_i} \tag{2-28}$$

for the k events $x_1, x_2, \ldots, x_k$ (in actuality, Eq. 2-28 represents a limiting distribution valid for large n; empirically, we usually require $s_i > 5$ for all i). A value of zero for χ^2 corresponds to exact agreement with expectation. If the n-trial experiment is repeated an arbitrarily large number of times, we obtain a distribution of χ^2 identical to that expressed by Eq. 2-27 with $k-1$ degrees of freedom.

The χ^2 test for goodness of fit is one of the most widely used methods capable of testing an hypothesis for the mathematical form of a single distribution, for a difference in the distribution of two or more random variables, and for the independence of certain random variables or attributes of variables. Its ready application to discrete distributions is not shared by other tests for goodness of fit, such as the Kolmogorov-Smirnov test (which applies as a criterion of fit the maximum deviation of the sample from the true distribution function).

Other distributions—some with widespread uses in statistical studies—can be derived by considering diverse functions of random variables. For example, if only a small sample of the total population is available, the sample standard deviation is not an accurate estimate of the true standard deviation σ. This defect can be circumvented by introducing a new variable based on the sample standard deviation. This new variate is defined as the quotient of two independent

[11]Generally, $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ for all $n > 0$.

variates and results in the *Student's t-distribution*,[12] an offshoot of the chi-square distribution. Its importance stems from the fact that it is functionally independent of any unknown population parameters, thus avoiding dubious estimates of these parameters.

In connection with elementary statistical logic and gambling theory, those distributions stated *ut supra* encompass the majority of significant developments. We shall introduce additional distributions only as required for special applications.

GAME THEORY

Nomenclature

The theory of games, instituted by John von Neumann, is essentially a mathematical process for the analysis of conflict among people or such groups of people as armies, corporations, or Bridge partnerships. It is applicable wherever a conflicting situation possesses the capability of being resolved by some form of intelligence—as in the disciplines of economics (where it commands a dominating authority), military operations, political science, psychology, law, and biology, as well as in games of Chess, Bridge, Poker, *inter alia*. Hence the word "*game*" is defined as the course (playing) of a conflicting situation according to an *a priori* specified set of rules and conventions. Games are distinguished by the number of contending *interests*, by the value of the winner's reward, by the number of moves required, and by the amount of information available to the interests.

Care should be exercised to avoid confusion with the colloquial meaning of the word *game*. Ping-Pong and Sumo Wrestling are "games" but lie outside the domain of game theory since their resolution is a function of athletic prowess rather than intelligence (despite the fact that a minimum intelligence is requisite to understand the rules). For social conflicts, game theory should be utilized with caution and reservation. In the courtship of the village belle, for example, the competing interests might not conform to the rules agreed upon; it is also difficult to evaluate the payoff by a single parameter; further, a reneging of the promised payoff is not unknown in such situations. Game theory demands a sacred (and rational) character for rules of behavior that may not withstand the winds of reality. The real world, with its emotional, ethical, and social suasions, is a far more muddled skein than the Hobbesian universe of the game theorist.

The number of players or competitors in a game are grouped into distinct decision-making units, or interests (Bridge involves four players, but two interests). With n interests, a game is referred to as an *n-person* game. It is assumed

[12]The distribution of the quotient of two independent variates was first computed by the British statistician William Sealy Gosset who, in 1908, submitted it under the pseudonym "Student" (he was an employee of a Dublin brewery that did not encourage "frivolous" research).

that the value of the game to each interest can be measured quantitatively by a number, called the *payoff*. In practice, the payoff is usually in monetary units but may be counted in any type of exchange medium. If the payoff is transferred only among the n players participating, the game is designated a *zero-sum game*. Mathematically, if ρ_i is the payoff received by the ith player (when the ith player loses, ρ_i is negative), the zero-sum game is defined by the condition that the algebraic sum of all gains and losses equals zero:

$$\sum_{i-1}^{n} \rho_i = 0$$

Instances where wealth is created or destroyed or a percentage of the wealth is paid to a nonparticipant are examples of *non–zero-sum* games.

Tic-Tac-Toe (which involves a maximum of nine moves) and Chess (a maximum possible 5950 moves) describe a type of game wherein only a finite number of moves are possible, each of which is chosen from a finite number of alternatives. Such games are termed *finite* games, and obviously the converse situation comprises *infinite* games.

The amount of information also characterizes a game. Competitions such as Chess, Checkers, or Shogi, where each player's move is exposed to his opponent, are games of *complete information*. A variant of Chess without complete information is *Kriegspiel* (precise knowledge of each player's move is withheld from his or her opponent). Bridge or Poker with all cards exposed would considerably alter the nature of that game. A variant of Blackjack with the dealt cards exposed is *Zweikartenspiel* (Double Exposure), proposed in Chapter 8.

Strategy

A system of divisions that selects each move from the totality of possible moves at prescribed instances is a *strategy*. A strategy may consist of personal moves (based solely on the player's judgment or his opponent's strategy or both), chance moves (determined through some random Bernoulli-trial method with an assessment of the probability of the various results), or, as in the majority of games, by a combination of the two. If player **A** has a total of m possible strategies, $A_1, A_2, \ldots, A_m$, and player **B** has a total of n strategies, $B_1, B_2, \ldots, B_n$, the game is termed an $m \times n$ game. In such a game, for **A**'s strategy A_i parried against **B**'s strategy B_j, the payoff is designated by a_{ij} (by convention, **A**'s gain is assigned a positive sign and **B**'s a negative sign). The set of all values of a_{ij} is called the *payoff matrix* and is represented by $\|a_{ij}\|$—expressed as a paradigm in Figure 2-1.

As an elementary illustration, consider **A** and **B** selecting either Heads or Tails simultaneously; if the two selections match, **A** wins one unit and conversely. This coin-matching payoff matrix is elaborated in Figure 2-2. It is evident that if **A** employs a *pure* strategy [e.g., selecting A_1 (Heads) continually], his opponent, **B**, can gain the advantage by continually selecting Tails.

		Player **B**			
		B_1	B_2	$\cdots$	B_n
	A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
Player **A**	A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
	$\cdot$	$\cdot$	$\cdot$	$\cdots$	
	A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

FIGURE 2-1 The general payoff matrix.

Intuitively, **A**'s (and **B**'s) best course of action is a *mixed* strategy—that is, alternating among the possible strategies according to some probability

	B_1 (Heads)	B_2 (Tails)
A_1 (Heads)	1	–1
A_2 (Tails)	–1	1

FIGURE 2-2 Coin-matching payoff matrix.

distribution. For the coin-matching payoff matrix, **A**'s optimal strategy S_A^* is to select A_1 or A_2 with probability 1/2. In mathematical form,

$$S_A^* \equiv \begin{pmatrix} A_1 & A_2 \\ p_1^* & p_2^* \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 1/2 & 1/2 \end{pmatrix} \tag{2-29}$$

In general, **A**'s optimal strategy for a given payoff matrix can be determined from the *minimax* principle, the quintessential statement of the theory of games (see Chapter 3). For each row of the matrix of Figure 2-1, a_{ij} will have a minimum value α_i. That is, for all possible values of j (with a given i), α_i is the lowest value in the ith row of the matrix:

$$\alpha_i = \min_j a_{ij}$$

Player **B** can always prevent **A** from winning more than α_i. Thus, **A**'s best strategy is to select the maximum value of α_i. Denoting this maximum by α, we have

$$\alpha = \max_i \alpha_i$$

Combining these two equations,

$$\alpha = \max_i \min_j a_{ij}$$

which states that the maximum of the minimum yield, or *maximin*, is **A**'s optimal strategy. The quantity α is the *lower value* of the game, since a profit of not less than α is ensured to **A** regardless of **B**'s strategy.

Considerations from the viewpoint of **B** are similar. The maximum value of α_{ij} for each column in Figure 2-1 is defined as

$$\beta_j = \max_i \; a_{ij}$$

and the minimum over all the β_js is

$$\beta = \min_j \; \beta_j$$

so that β can be written in the form

$$\beta = \min_j \; \max_i \; a_{ij}$$

which states that the minimum of the maximum yield, or *minimax*, is **B**'s optimal strategy. The quantity β is the *upper value* of the game, since a loss of not more than β is guaranteed to **B** regardless of **A**'s strategy.

Clearly, every payoff matrix has a value—and that value can be achieved by at least one optimal strategy for each player.

Solutions of Games with Saddle Points

In the example of Figure 2-2, $\alpha_1 = \alpha_2 = -1$. Therefore the lower value of the game is $\alpha = -1$. Similarly, $\beta = +1$. Any strategy of **A**'s is a maximin strategy, since **A** can never lose more than one unit, and similarly for **B**. If both **A** and **B** adhere to a mixed strategy, the *average value* γ of the game lies between the upper and lower values:

$$\alpha \leq \gamma \leq \beta$$

For the case considered here, $\gamma = 0$ if the mixed strategies are unpredictable.

In some instances, the lower value of the game is equal to the upper value:

$$\alpha = \beta = \gamma$$

and the game is said to possess a *saddle point*. Every game with a saddle point has a solution that defines the optimal strategies for all the players; the value of the game is simultaneously its lower and upper values. Further, if any player deviates from the indicated optimal strategy, while the other players adhere to theirs, the outcome of the game for the deviating player can only be less than the average value. A proven theorem of game theory states that every game with complete information possesses a saddle point and therefore a solution.

These considerations can be summarized as follows: For every finite game matrix $\|a_{ij}\|$, a necessary and sufficient condition for

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = \gamma$$

is that $\|\alpha_{ij}\|$ possesses a saddle point. That is, there exists a pair of integers i_0 and j_0 such that $a_{i_0 j_0}$ is simultaneously the minimum of its row and the maximum of its column. Thus, if a game matrix possesses a saddle point, the solution of the game is evident and trivial.

A 2×2 game has a saddle point if and only if the two numbers of either diagonal are *not* both higher than either of the other two numbers—a situation that occurs with probability 2/3. With a 3×3 game, the probability of encountering a saddle point decreases to 3/10. The probability $P_{m,n}$ that a randomly selected $m \times n$ game matrix exhibits a saddle point is

$$P_{m,n} = \frac{m!\, n!}{(m + n - 1)!}$$

Solutions of Games without Saddle Points

In the Heads–Tails matching game of Figure 2-2, let **A** select his strategies via a random device that assigns a probability p to Heads and $1 - p$ to Tails. Further, let us assume that **B** has secured knowledge of the nature of **A**'s random device. Then the mathematical expectation of **A**, when **B** selects strategy B_1 (Heads), is

$$\mathrm{E} = (1)(p) + (-1)(1 - p) = 2p - 1$$

For **B**'s choice of strategy B_2 (Tails),

$$\mathrm{E} = (-1)(p) + (1)(1 - p) = 1 - 2p$$

Clearly, if $p > 1/2$, **B** will select strategy B_2, and **A**'s expectation is negative. Conversely, for $p < 1/2$, **B** selects strategy B_1 and wins $1 - 2p$ units per game. It follows that if **A** selects A_1 and A_2 with equal probabilities 1/2, the expectation of both **A** and **B** is zero regardless of **B**'s strategy. Thus, **A**'s optimal strategy is to let $p = 1/2$ (as indicated in Eq. 2-29).

In general, for $m \times n$ game matrices, let **A** have a mixed strategy

$$S_A(m) = \begin{pmatrix} A_1\, A_2 \cdots A_i \cdots A_m \\ p_1\, p_2 \cdots p_i \cdots p_m \end{pmatrix}$$

which is interpreted to mean that strategy A_i is selected with probability p_i, where the p_is sum to 1. Similarly, a mixed strategy for **B** is designated by

$$S_B(n) = \begin{pmatrix} B_1\, B_2 \cdots B_j \cdots B_n \\ q_1\, q_2 \cdots q_j \cdots q_n \end{pmatrix}$$

where the q_js sum to 1. When **A** and **B** employ the mixed strategies $S_A(m)$ and $S_B(n)$, respectively, the mathematical expectation of **A** is given by

$$\mathrm{E}[S_A(m),\ S_B(n)] = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}p_i q_j$$

If it occurs that strategies $S_A^*(m)$ and $S_B^*(n)$, exist for **A** and **B**, respectively, such that

$$\mathrm{E}[S_A(m),\ S_B^*(n)] \le \mathrm{E}[S_A^*(m),\ S_B^*(n)] \le \mathrm{E}[S_A^*(m),\ S_B(n)]$$

then $S_A^*(m)$ and $S_B^*(n)$ are optimal strategies, and $\mathrm{E}[S_A^*(m),\ S_B^*(n)] = \gamma$ is the value of the game.

In solving games without saddle points, a preliminary step is to eliminate from the game matrix all duplicate strategies as well as those strategies that are totally unfavorable with respect to another strategy. If the remaining game generates an $m \times n$ matrix, the necessary and sufficient condition for γ to be the value of the game and for $S_A^*(m)$ and $S_B^*(n)$ to be the optimal strategies of **A** and **B**, respectively, is that

$$\mathrm{E}[S_A(i),\ S_B^*(n)] \le \gamma \le \mathrm{E}[S_A^*(m),\ S_B(j)]$$

for $1 \le i \le m$ and $1 \le j \le n$. Thus, for the solution of the game matrix $\|\alpha_{ij}\|$, there must exist values of p_i, q_j, and γ that satisfy the conditions

$$\begin{aligned}
a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m &\ge \gamma \\
a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m &\ge \gamma \\
&\cdots\cdots \\
a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m &\ge \gamma \\
a_{11}p_1 + a_{12}q_2 + \cdots + a_{1n}q_n &\le \gamma \\
a_{21}p_1 + a_{22}q_2 + \cdots + a_{2n}q_n &\le \gamma \\
&\cdots\cdots \\
a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n &\le \gamma
\end{aligned} \tag{2-30}$$

To illustrate the preceding concepts, consider the 4×4 matrix:

$$\|a_{ij}\| = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \left\|\begin{array}{cccc} 1 & -1 & -1 & 2 \\ -1 & -1 & 3 & 0 \\ -1 & 2 & -1 & 1 \\ -2 & 0 & -2 & 0 \end{array}\right\| \end{array}$$

Here, **A** would assign zero probability to strategy A_4, since, regardless of **B**'s strategy, **A**'s gain by selecting A_3 is always greater than the yield from A_4. Thus, A_3 *dominates* A_4, and we can eliminate A_4. Similarly, B_4 can be eliminated because it is dominated by B_1. The remaining game matrix is

$$\|a_{ij}\| = \left\| \begin{matrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{matrix} \right\|$$

Applying Eqs. 2-30 with the equality signs,

$$\begin{aligned} p_1 - p_2 - p_3 &= \gamma \qquad & q_1 - q_2 - q_3 &= \gamma \\ -p_1 - p_2 + 2p_3 &= \gamma \qquad & -q_1 - q_2 + 3q_3 &= \gamma \\ -p_1 + 3p_2 - p_3 &= \gamma \qquad & -q_1 + 2q_2 - q_3 &= \gamma \end{aligned}$$

Combining these equations with the conditions $p_1+p_2+p_3=1$ and $q_1+q_2+q_3=1$, elementary algebraic methods provide the solution:

$$\begin{aligned} p_1^* &= q_1^* = 6/13 \qquad & p_3^* &= 4/13;\; q_3^* = 3/13 \\ p_2^* &= 3/13;\; q_2^* = 4/13 \qquad & \gamma &= -1/13 \end{aligned}$$

Thus, **A**'s optimal strategy is to select $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{A}_3$ with probabilities 6/13, 4/13 and 3/13, respectively, thereby guaranteeing to **A** a loss of no more than 1/13.

Solving game matrices without saddle points often involves computational messiness. In some instances, a geometrical interpretation is fruitful, particularly for $2 \times n$ games. In general, geometrical (or graphical) solutions of $m \times n$ game matrices pose exceptional mathematical difficulties. More advanced analytic techniques exist but are unlikely to be required in connection with any of the conventional competitive games.

Equilibria

It is possible to generalize (to multiperson games) the concept of the solution of a two-person, zero-sum game by introducing the notion of an *equilibrium point* (John Nash is responsible for this idea, although it had been previously applied in the field of economics under the terminology of Cournot's duopoly point.) A Nash equilibrium (Ref: Nash, 1950) is a set of strategies and corresponding payoffs such that no player can benefit by changing his strategy while the strategies of all other players are held invariant. Every game with a finite number of players and a finite set of strategies has at least one mixed-strategy Nash equilibrium. For zero-sum games, the Nash equilibrium is also a minimax equilibrium.

A "weakness" of the Nash equilibrium is that it rests on the foundation of rational behavior (each player restricted to rational choices). Outside the rigid and enforced rules of formal games, more realistic behavior conforms to the quantal response equilibrium (QRE), which entertains the notion that players' responses to differences in expected payoffs are more orderly for large differences and more random for small differences (Ref. Nash, 1951).

In addition, a large body of work has developed to deal with infinite games, games with infinitely many strategies, continuous games, and "nonrational" games such as the Prisoner's Dilemma.[13] However, these situations lie beyond our scope of interest; here, we have outlined only the basic mathematical tools useful for solving problems in applied gambling theory and statistical logic.

RANDOM WALKS

One-Dimensional

Most practical gambling situations can be formalized as sequences of Bernoulli trials wherein the gambler wins one unit for each success and loses one unit for each failure—with respective probabilities p and q = 1 − p. We represent the gambler's capital x by a point on the real line; then from an initial capital $x = x_0$, x moves along the line at each play either to $x_0 + 1$ (probability p) or to $x_0 - 1$ (probability q). After an indefinite number of plays, the random walk terminates either at $x = 0$ or at $x = C$, where C is the adversary's capital. With a casino as the adversary, C is effectively infinite.

This construct describes the classic random walk and leads to the gambler's ruin theorem (Chapter 3).

To determine the probability that the gambler's capital returns to its initial value x_0, we note that this event can occur only at an even-numbered play. After $2n$ plays, it is both necessary and sufficient for the gambler to have experienced n wins and n losses. Therefore,

$$\text{Probability}(n \text{ wins and } n \text{ losses}) = P_{2n} = \binom{2n}{n} \mathrm{p}^n \mathrm{q}^n \sim \frac{(4\mathrm{pq})^n}{\sqrt{\pi n}}$$

(see footnote 10). For p = q = 1/2,

$$P_{2n} \sim 1/\sqrt{\pi n}$$

[13]Originally formulated in 1950 by Princeton mathematician Albert W. Tucker, the issue is framed by two prisoners, **A** and **B**, who, under separate police interrogation, must either Confess—and implicate the other—or Dummy Up (remain silent). If both Confess, each is sentenced to five years. If one Confesses, while the other Dummies Up, the implicating prisoner goes free, while the silent prisoner receives a sentence of ten years. If both Dummy Up, each is sentenced to one year.

A reasons that, should **B** confess, **A** would serve ten years if he Dummies Up and five years if he too Confesses. On the other hand, should **B** Dummy Up, **A** would serve one year if he likewise Dummies Up, but will go free if he Confesses. Ergo, his superior strategy is to Confess. **B** will reason similarly. "Rational" decision making will therefore prompt both **A** and **B** to Confess—and to serve five years in prison. Yet if both reached "irrational" conclusions and Dummied Up, each then would serve only one year.

The dilemma arises from the fact that *these individually rational strategies lead to a result inferior for both prisoners compared to the result if both were to remain silent*. No solution exists within the restricted confines of classical game theory.

Note that the series ΣP_{2n} diverges, but that the probability $P_{2n} \to 0$ with increasing n. Consequently, *the return to initial capital* x_0 *is a certain recurrent event, while the mean recurrence time is infinite.*

Defining r_j as the probability that a return to x_0 occurs at the jth play (not necessarily for the first time), then the recurrent event $x = x_0$ is uncertain if and only if

$$R = \sum_{j=0}^{\infty} r_j$$

is finite. In this case the probability ρ that $x = x_0$ ever recurs is given by

$$\rho = (R - 1)/R$$

For p $\neq$ 1/2, we have $R = \dfrac{1}{\sqrt{1 - 4pq}} = \dfrac{1}{|p - q|}$

Hence the probability that the accumulated numbers of successes and failures will ever be equal is expressed by

$$1 - |p - q| \tag{2-31}$$

Equation 2-31 also represents the probability of at least one return to x_0.

Biased Walks

Let u and v denote the relative frequencies of moving left or right (failure or success) at each play. We can then write the generating function f(x) as

$$f(x) = (ux^{-1} + vx)^n$$

for n plays. With $u = 1$, $v = 5$, and $n = 3$ for example, the generating function is

$$(x^{-1} + 5x)^3 = x^{-3} + 15x^{-1} + 75x + 125x^3$$

Thus there is one possible outcome at -3, 15 outcomes at -1, 75 at $+1$, and 125 at $+3$, with respective probabilities [dividing by $(1 + 5)^3$] 1/216, 5/72, 25/72, and 125/216.

Walk, Don't Walk

To the Biased Walk, we can add the possibility, with frequency w, of remaining in place (a tie game). Let $w = 3$ as an example; then our generating function (with $n = 3$) becomes

$$\begin{aligned} f(x) = (x^{-1} + 5x + 3)^3 = {} & x^{-3} + 9x^{-2} + 42x^{-1} + 117 \\ & + 210x + 225x^2 + 125x^3 \end{aligned}$$

Table 2-1 displays the seven possible outcomes for the three-plays Walk, Don't Walk game.

Table 2-1 Outcomes for Three-Plays Walk, Don't Walk

Final Position	−3	−2	−1	0	+1	+2	+3
Relative Frequency	1	9	42	117	210	225	125
Probablity ÷ by $(1+5+3)^3$	1/729	1/81	14/243	13/81	70/243	25/81	125/729

The number of outcomes for which success (unit moves rightward) minus failure (unit moves leftward) has the value $j-n$ is

$$\sum_{k=\max(0,j-n)}^{[j/2]} \binom{n}{j-k}\binom{j-k}{k} u^{n-j+k} v^{k} w^{j-2k}$$

where $[j/2]$ denotes the largest integer less than $j/2$. Designating the summand as Q, the number of successes can be expressed as

$$\sum_{j=n+1}^{2n} \sum_{k=j-n}^{[j/2]} Q$$

And the number of failures as

$$\sum_{j=0}^{n-1} \sum_{k=0}^{[j/2]} Q$$

The number of times the position remains unchanged is

$$\sum_{k=0}^{[n/2]} Q$$

Several modifications to the random walk can be implemented. One example: a random walk with persistence, wherein the *direction* of moves taken by the gambler is correlated with previous moves. Another modification correlates the extent of each move (rather than $+1$ or -1). With a little ingenuity, games can be concocted to embody these rules.

While random walks in higher dimensions offer no insights into classical gambling theory, it is of interest to note that, quite surprisingly, a two-dimensional random walker (on a lattice) also has unity probability of reaching any point (including the starting point) in the plane and will reach that point infinitely often as the number of steps approaches infinity. A three-dimensional random walker, on the other hand, has less than unity probability of reaching any particular point; his probability of returning to the starting point is ~0.34, and his expected number of returns ~0.538. In the fourth dimension, the (hyper)random walker will revisit his starting point with probability ~0.193. With yet higher dimensions, the (hyper)random walker will experience decreasing probabilities of returning to his starting point; in an n-dimensional hypercube, that probability is $\sim 1/2(n-1)$ for $n \geq 8$.

Change in Capital

Of equal concern is the amount Δ_n that the gambler's capital has increased or decreased after n plays. It can be shown, with adroit mathematical manipulations (Ref: Weisstein), that

$$\Delta_n = \begin{cases} \dfrac{(n-1)!!}{(n-2)!!} & \text{for } n \text{ even} \\ \dfrac{n!!}{(n-1)!!} & \text{for } n \text{ odd} \end{cases}$$

where $n!!$ is the double factorial.[14] The first several values of Δ_n for $n = 0$, 1, 2, … are 0, 1, 1, 3/2, 3/2, 15/8, 15/8, 35/16, 35/16, 315/128, 315/128, ….

QUANTUM GAMES

While quantum game theory explores areas closed off to its classical counterpart, it does not always secure a foothold in reality.

In classical game theory, a mixed strategy different from the equilibrium strategy cannot increase a player's expected payoff. Quantum strategies transcend such limitations and can prove more successful than classical strategies. A quantum coin, in contrast to a real coin that can be in one of only two states (Heads or Tails), enjoys the property of existing in a state that superimposes both Heads *and* Tails, and thus can be in an infinite number of states.

It has been shown that with quantum strategies available to both players, a two-person, zero-sum game does not necessarily possess an equilibrium solution, but does (always) have a mixed-strategy solution.

Quantum decoherence[15] often precludes implementing such strategies in the real world—although two-state quantum systems exist that are consonant with the superposition of states necessary for a quantum strategy.

Parrondo's principle (Chapter 4), Tic-Tac-Toe, Duels, and Truels, and the Monty Hall problem (Chapter 5) are examples of games amenable to the introduction of quantum theory.

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[14]An extension of the conventional factorial,

$$n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1, & n > 0 \text{ odd} \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2, & n > 0 \text{ even} \\ 1, & n = -1, 0 \end{cases}$$

Also $-1!! = 0!! = 1$.

[15]Decoherence is the mechanism whereby quantum systems interact with their environments to produce probabilistic behavior.

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Chapter | three

Fundamental Principles of a Theory of Gambling

DECISION MAKING AND UTILITY

The essence of the phenomenon of gambling is decision making. The act of making a decision consists of selecting one course of action, or strategy, from among the set of admissible strategies. A particular decision might indicate the card to be played, a horse to be backed, the fraction of a fortune to be hazarded over a given interval of play, or the time distribution of wagers. Associated with the decision-making process are questions of preference, utility, and evaluation criteria, *inter alia*. Together, these concepts constitute the *sine qua non* for a sound gambling-theory superstructure.

Decisions can be categorized according to the relationship between action and outcome. If each specific strategy leads invariably to a specific outcome, we are involved with decision making under *certainty*. Economic games may exhibit this deterministic format, particularly those involving such factors as cost functions, production schedules, or time-and-motion considerations. If each specific strategy leads to one of a set of possible specific outcomes with a known probability distribution, we are in the realm of decision making under *risk*. Casino games and games of direct competition among conflicting interests encompass moves whose effects are subject to probabilistic jurisdiction. Finally, if each specific strategy has as its consequence a set of possible specific outcomes whose *a priori* probability distribution is completely unknown or is meaningless, we are concerned with decision making under *uncertainty*.

Conditions of certainty are, clearly, special cases of risk conditions where the probability distribution is unity for one specific outcome and zero for all others. Thus a treatment of decision making under risk conditions subsumes the state of certainty conditions. There are no conventional games involving conditions of uncertainty[1] without risk. Often, however, a combination of uncertainty and risk arises; the techniques of statistical inference are valuable in such instances.

Gambling theory, then, is primarily concerned with decision making under conditions of risk. Furthermore, the making of a decision—that is, the process of selecting among n strategies—implies several logical avenues of development. One implication is the existence of an expression of preference or ordering of the strategies. Under a particular criterion, each strategy can be evaluated according to an individual's taste or desires and assigned a *utility* that defines a measure of the effectiveness of a strategy. This notion of utility is fundamental and must be encompassed by a theory of gambling in order to define and analyze the decision process.

While we subject the concept of utility to mathematical discipline, the nonstochastic theory of preferences need not be cardinal in nature. It often suffices to express all quantities of interest and relevance in purely ordinal terms. The essential precept is that preference precedes characterization. Alternative A is preferred to alternative B; therefore A is assigned the higher (possibly numerical) utility; conversely, it is incorrect to assume that A is preferred to B because of A's higher utility.

The Axioms of Utility Theory

Axiomatic treatments of utility have been advanced by von Neumann and Morgenstern originally and in modified form by Marschak, Milnor, and Luce and Raiffa (Ref.), among others. Von Neumann and Morgenstern (Ref.) have demonstrated that utility is a measurable quantity on the assumption that it is always possible to express a comparability—that is, a preference—of each *prize* (defined as the value of an outcome) with a probabilistic combination of other *prizes*. The von Neumann-Morgenstern formal axioms imply the existence of a numerical scale for a wide class of partially ordered utilities and permit a restricted specific utility of gambling.

Herein, we adopt four axioms that constrain our actions in a gambling situation. To facilitate formulation of the axioms, we define a lottery L as being composed of a set of *prizes* $A_1, A_2, \ldots, A_n$ obtainable with associated probabilities $p_1, p_2, \ldots, p_n$ (a prize can also be interpreted as the right to participate in another lottery). That is,

$$L = (p_1 A_1; p_2 A_2; \ldots; p_n A_n)$$

And, for ease in expression, we introduce the notation **p** to mean "is preferred to" or "takes preference over", **e** to mean "is equivalent to" or "is indifferent to",

[1] For decision making under conditions of uncertainty. Bellman (Ref.) has shown that for decision-making under conditions of uncertainty, one can approach an optimum moderately well by maximizing the ratio of expected gain to expected cost.

and **q** to mean “is either preferred to or is equivalent to.” The set of axioms delineating Utility Theory can then be expressed as follows.

Axiom I(a). Complete Ordering: Given a set of alternatives $A_1, A_2, \ldots, A_n$, a comparability exists between any two alternatives A_i and A_j. Either A_i **q** A_j or A_j **q** A_i or both.

Axiom I(b). Transitivity: If A_i **q** A_j and A_j **q** A_k, then it is implied that A_i **q** A_k. This axiom is requisite to consistency. Together, axioms $I(a)$ and $I(b)$ constitute a complete ordering of the set of alternatives by **q**.

Axiom II(a). Continuity: If A_i **p** A_j **p** A_k, there exists a real, nonnegative number r_j, $0 < r_j < 1$, such that the prize A_j is equivalent to a lottery wherein prize A_i is obtained with probability r_j and prize A_k is obtained with probability $1 - r_j$.

In our notation, A_j **e** $[r_j A_i; (1 - r_j)A_k]$. If the probability r of obtaining A_i is between 0 and r_j, we prefer A_j to the lottery, and contrariwise for $r_j < r \leq 1$. Thus r_j defines a point of inversion where a prize A_j obtained with certainty is equivalent to a lottery between a lesser prize A_i and a greater prize A_k whose outcome is determined with probabilities r_j and $(1 - r_j)$, respectively.

Axiom II(b). Substitutability: In any lottery with an ordered value of prizes defined by A_i **p** A_j **p** A_k, $[r_j A_i; (1 - r_j)A_k]$ is substitutable for the prize A_j with complete indifference.

Axiom III. Monotonicity: A lottery $[rA_i; (1 - r)A_k]$ **q** $[r_jA_i; (1 - r_j)A_k]$ if and only if $r \geqslant r_j$. That is, between two lotteries with the same outcome, we prefer the one that yields the greater probability of obtaining the preferred alternative.

Axiom IV. Independence: If among the sets of prizes (or lottery tickets) $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, A_1 **q** B_1, A_2 **q** B_2, etc., then an even chance of obtaining A_1 or A_2, etc. is preferred or equivalent to an even chance of obtaining B_1 or B_2, and so forth.

This axiom is essential for the maximization of expected cardinal utility.

Together, these four axioms encompass the principles of consistent behavior. Any decision maker who accepts them *can*, theoretically, solve any decision problem, no matter how complex, by merely expressing his basic preferences and judgments with regard to elementary problems and performing the necessary extrapolations.

Utility Functions

With Axiom I as the foundation, we can assign a number $u(L)$ to each lottery L such that the magnitude of the values of $u(L)$ is in accordance with the preference relations of the lotteries. That is, $u(L_i) \geq u(L_j)$ if and only if L_i **q** L_j. Thus we can assert that a utility function u exists over the lotteries.

Rational behavior relating to gambling or decision making under risk is now clearly defined; yet there are many cases of apparently rational behavior that violate one or more of the axioms. Common examples are those of the military hero or the mountain climber who willingly risks death and the “angel” who finances a potential Broadway play despite a negative expected return. Postulating that the probability of fatal accidents in scaling Mt. Everest is 0.10 if a Sherpa guide is employed and 0.20 otherwise, the mountain climber appears to prefer a survival probability of 0.9 to that of 0.8, but also to that of 0.999 (depending on his age and health)—or he would forego the climb. Since his utility function has a maximum at 0.9, he would seem to violate the monotonicity axiom. The answer lies in the "total reward" to the climber—as well as the success in attaining the summit, the payoff includes accomplishment, fame, the thrill from the danger itself, and perhaps other forms of mental elation and ego satisfaction. In short, the utility function is still monotonic when the payoff matrix takes into account the climber's "gestalt." Similar considerations evidently apply for the Broadway angel and the subsidizer of unusual inventions, treasure-seeking expeditions, or "wildcat" oil-drilling ventures.

Since utility is a subjective concept, it is necessary to impose certain restrictions on the types of utility functions allowable. First and foremost, we postulate that our gambler be a *rational being*. By definition, the rational gambler is logical, mathematical, and consistent. Given that all x's are y's, he concludes that all non-y's are non-x's, but does not conclude that all y's are x's. When it is known that U follows from V, he concludes that non-V follows from non-U, but not that V follows from U. If he prefers alternative A to alternative B and alternative B to alternative C, then he prefers alternative A to alternative C (transitivity). The rational being exhibits no subjective probability preference. As initially hypothesized by Gabriel Cramer and Daniel Bernoulli as "typical" behavior, our rational gambler, when concerned with decision making under risk conditions, acts in such a manner as to maximize the expected value of his utility.

Numerous experiments have been conducted to relate probability preferences to utility (Ref. Edwards, 1953, 1954). The most representative utility function, illustrated in Figure 3-1, is concave for small increases in wealth and

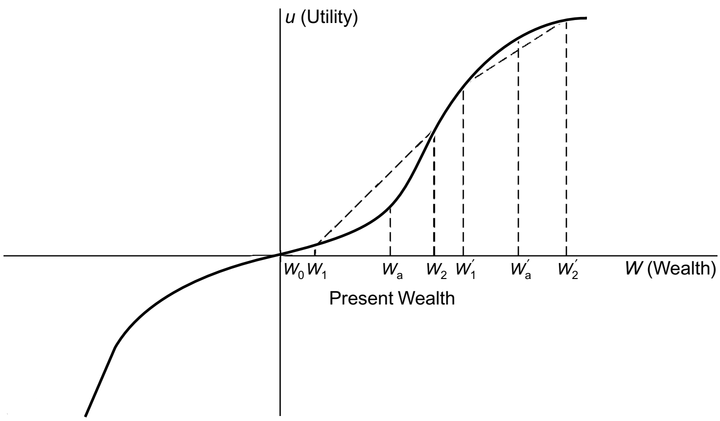

FIGURE 3-1 Utility as a function of wealth.

then convex for larger gains. For decreases in wealth, the typical utility function is first convex and then concave, with the point of inflection again proportional to the magnitude of present wealth. Generally, $u(W)$ falls faster with decreased wealth than it rises with increased wealth.

From Figure 3-1, we can determine geometrically whether we would accept added assets W_a or prefer an even chance to obtain W_2 while risking a loss to W_1. Since the line $\overline{[W_2, u(W_2)], [W_1, u(W_1)]}$ passes above the point $[W_a, u(W_a)]$, the expected utility of the fair bet is greater than $u(W_a)$; the lottery is therefore preferred. However, when we are offered the option of accepting W'_a or the even chance of obtaining W'_2 or falling to W'_1, we draw the line $\overline{[W'_2, u(W'_2)], [W'_1, u(W'_1)]}$, which we observe passes below the point $u(W'_a)$. Clearly, we prefer W'_a with certainty to the even gamble indicated, since the utility function $u(W'_a)$ is greater than the expected utility of the gamble. This last example also indicates the justification underlying insurance policies. The difference between the utility of the specified wealth and the expected utility of the gamble is proportional to the amount of "unfairness" we would accept in a lottery or with an insurance purchase.

This type of utility function implies that as present wealth decreases, there is a greater tendency to prefer gambles involving a large chance of small loss versus a small chance of large gain (the farthing football pools in England, for example, were patronized by the proletariat, not the plutocrats). Another implication is that a positive skewness of the frequency distribution of total gains and losses is desirable. That is, Figure 3-1 suggests that we tend to wager more conservatively when losing moderately and more liberally when winning moderately. The term "moderate" refers to that portion of the utility function between the two outer inflection points.

The Objective Utility Function

From the preceding arguments it is clear that the particular utility function determines the decision-making process. Further, this utility function is frequently subjective in nature, differing among individuals and according to present wealth. However, to develop a simple theory of gambling, we must postulate a specific utility function and derive the laws that pertain thereto. Therefore we adopt the objective model of utility, Figure 3-2, with the reservation that the theorems to be stated subsequently must be modified appropriately for subjective models of utility.

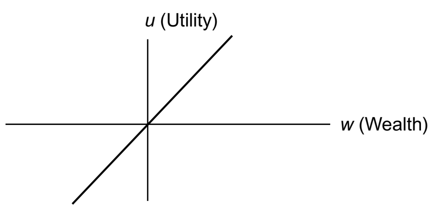

FIGURE 3-2 The objective utility function.

It has been opined that only misers and mathematicians truly adhere to objective utilities. For them, each and every dollar maintains a constant value regardless of how many other dollars can be summoned in mutual support. For those who are members of neither profession, we recommend the study of such behavior as a sound prelude to the intromission of a personal, subjective utility function.

The objective model assumes that the utility of wealth is linear with wealth and that decision making is independent of nonprobabilistic considerations. It should also be noted that in subjective models, the utility function does not depend on long-run effects; it is completely valid for single-trial operations. The objective model, on the other hand, rests (at least implicitly) on the results of an arbitrarily large number of repetitions of an experiment.

Finally, it must be appreciated that, within the definition of an objective model, there are several goals available to the gambler. He may wish to play indefinitely, gamble over a certain number of plays, or gamble until he has won or lost specified amounts of wealth. In general, the gambler's aim is not simply to increase wealth (a certainty of $1 million loses its attraction if a million years are required). Wealth per unit time or per unit play (income) is a better aim, but there are additional constraints. The gambler may well wish to maximize the expected wealth received per unit play, subject to a fixed probability of losing some specified sum. Or he may wish with a certain confidence level to reach a specified level of wealth. His goal dictates the amount to be risked and the degree of risk. These variations lie within the objective model for utility functions.

PROSPECT THEORY

Developed by cognitive psychologists Daniel Kahneman[2] and Amos Tversky (Ref.), *Prospect Theory* describes the means by which individuals evaluate gains and losses in psychological reality, as opposed to the von Neumann-Morgenstern rational model that prescribes decision making under uncertainty conditions (descriptive versus prescriptive approaches). Thus Prospect Theory encompasses anomalies outside of conventional economic theory (sometimes designated as "irrational" behavior).

First, choices are ordered according to a particular heuristic. Thence (subjective) values are assigned to gains and losses rather than to final assets. The value function, defined by deviations from a reference point, is normally concave for gains ("risk aversion"), convex for losses ("risk seeking"), and steeper for losses than for gains ("loss aversion"), as illustrated in Figure 3-3.

Four major principles underlie much of Prospect Theory:

- People are inherently (and irrationally) less inclined to gamble with profits than with a bankroll reduced by losses.
- If a proposition is framed to emphasize possible gains, people are more likely to accept it; the identical proposition framed to emphasize possible losses is more likely to be rejected.

[2] Daniel Kahneman shared the 2002 Nobel Prize for Economics.

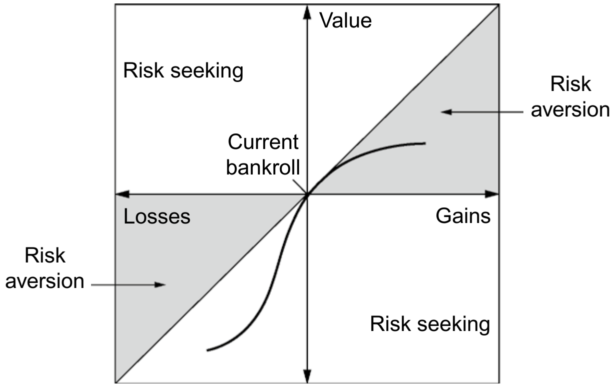

FIGURE 3-3 A Prospect Theory value function.

- A substantial majority of the population would prefer \$5000 to a gamble that returns either \$10,000 with 0.6 probability or zero with 0.4 probability (an expected value of \$6000).
- Loss-aversion/risk-seeking behavior: In a choice between a certain loss of \$4000 as opposed to a 0.05 probability of losing \$100,000 combined with a 0.95 probability of losing nothing (an expected loss of \$5000), another significant majority would prefer the second option.

One drawback of the original theory is due to its acceptance of intransitivity in its ordering of preferences. An updated version, Cumulative Prospect Theory, overcomes this problem by the introduction of a probability weighting function derived from rank-dependent Expected Utility Theory.

Prospect Theory in recent years has acquired stature as a widely quoted and solidly structured building block of economic psychology.

DECISION-MAKING CRITERIA

Since gambling involves decision making, it is necessary to inquire as to the "best" decision-making procedure. Again, there are differing approaches to the definition of best. For example, the Bayesian solution for the best decision proposes minimization with respect to average risk. The Bayes principle states that if the unknown parameter θ is a random variable distributed according to a known probability distribution f, and if $F_\Delta(\theta)$ denotes the risk function of the decision procedure Δ, we wish to minimize with respect to Δ the average risk

$$\int F_\Delta(\theta)\, df(\theta)$$

We could also minimize the average risk with respect to some assumed *a priori* distribution suggested by previous experience. This latter method, known as the restricted Bayes solution, is applicable in connection with decision processes wherein maximum risk does not exceed minimum risk by more than a specified amount. A serious objection to the use of the restricted Bayesian principle for gambling problems lies in its dependence on subjective probability assignments. Whenever possible, we wish to avoid subjective weightings.

Another method of obtaining a best decision follows the principle of "minimization of regret." As proposed by Leonard Savage (Ref. 1951), this procedure is designed for the player who desires to minimize the difference between the payoff actually achieved by a given strategy and the payoff that could have been achieved if the opponent's intentions were known in advance and the most effective strategy adopted. An obvious appeal is offered to the businessman whose motivation is to minimize the possible disappointment deriving from an economic transaction. The regret matrix is formed from the conventional payoff matrix by replacing each payoff a_{ij} with the difference between the column maximum and the payoff—that is, with $\max_i(a_{ij}) - a_{ij}$. For example, consider the payoff matrix,

$$\|a_{ij}\| = \begin{array}{c||ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 3 & 4 & 5 \\ A_2 & 4 & 6 & 8 \\ A_3 & 3 & 6 & 9 \end{array}$$

which possesses a saddle point at (A_2, B_1) and a game value of 4 to player **A** under the minimax criterion. The regret matrix engendered by these payoffs is

$$\begin{array}{c||ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & 2 & 4 \\ A_2 & 0 & 0 & 1 \\ A_3 & 1 & 0 & 0 \end{array}$$

While there now exists no saddle point defined by two pure strategies, we can readily solve the matrix (observe that we are concerned with row maxima and column minima since **A** wishes to minimize the regret) to obtain the mixed strategies (0, 1/4, 3/4) for **A** and (3/4, 0, 1/4) for his opponent **B**. The "regret value" to **A** is 3/4; the actual profit to **A** is

$$(3/4)[4(1/4) + 3(3/4)] + (1/4)[8(1/4) + 9(3/4)] = 4\tfrac{5}{8}$$

compared to a profit of 4 from application of the minimax principle.

Chernoff (Ref.) has pointed out the several drawbacks of the "minimization of regret" principle. First, the "regret" is not necessarily proportional to losses in utility. Second, there can arise instances whereby an arbitrarily small advantage in one state of nature outweighs a sizable advantage in another state. Third, it is possible that the presence of an undesirable strategy might influence the selection of the best strategy.

Other criteria include the Hurwicz pessimism–optimism index (Ref.), which emphasizes a weighted combination of the minimum and maximum utility numbers, and the "principle of insufficient reason," credited to Jacob Bernoulli. This latter criterion assigns equal probabilities to all possible strategies of an opponent if no *a priori* knowledge is given. Such sophistry leads to patently nonsensical results.

The objective model of utility invites an objective definition of best. To the greatest degree possible, we can achieve objectivity by adopting the minimax principle—that is, minimization with respect to maximum risk. We wish to minimize the decision procedure Δ with respect to the maximum risk $\sup_\theta F_\Delta(\theta)$—or, in terms of the discrete theory outlined in Chapter 2, $\min\limits_{\Delta} \max\limits_{\theta} F_\Delta(\theta)$.

The chief criticism of the minimax (or maximin) criterion is that, in concentrating on the worst outcome, it tends to be overly conservative. In the case of an intelligent opponent, the conservatism may be well founded, since a directly conflicting interest is assumed. With "nature" as the opposition, it seems unlikely that the resolution of events exhibits such a diabolical character. This point can be illustrated (following Luce and Raiffa, Ref.) by considering the matrix

$$\|a_{ij}\| = \begin{array}{c||cc} & B_1 & B_2 \\ \hline\hline A_1 & 0 & \lambda \\ A_2 & 1 & 1+\varepsilon \end{array}$$

Let λ be some number arbitrarily large and ε some positive number arbitrarily small. The minimax principle asserts that a saddle point exists at (A_2, B_1). While an intelligent adversary would always select strategy B_1, nature might not be so inconsiderate, and **A** consequently would be foolish not to attempt strategy A_1 occasionally with the possibility of reaping the high profit λ.

Although this fault of the minimax principle can entail serious consequences, it is unlikely to intrude in the great preponderance of gambling situations, where the opponent is either a skilled adversary or one (such as nature) whose behavior is known statistically. In the exceptional instance, for lack of a universally applicable criterion, we stand accused of conservatism. Accordingly, we incorporate the minimax principle as one of the pillars of gambling theory.

THE BASIC THEOREMS

With an understanding of utility and the decision-making process, we are in a position to establish the theorems that constitute a comprehensive theory of gambling. While the theorems proposed are, in a sense, arbitrary, they are intended to be necessary and sufficient to encompass all possible courses of action arising from decision making under risk conditions. Ten theorems with associated corollaries are stated.[3] The restrictions to be noted are that our theorems are valid only for phenomena whose statistics constitute a stationary time series and whose game-theoretic formulations are expressible as zero-sum games. Unless otherwise indicated, each theorem assumes that the sequence of plays constitutes successive independent events.

> *Theorem I:* If a gambler risks a finite capital over a large number of plays in a game with constant single-trial probabilities of winning, losing, and tying, then any and all betting systems lead ultimately to the same mathematical expectation of gain per unit amount wagered.

A betting system is defined as some variation in the magnitude of the wager as a function of the outcome of previous plays. At the ith play of the game, the player wagers β_i units, winning the sum $k\beta_i$ with probability p (where k is the payoff odds), losing β_i with probability q, and tying with probability $1 - p - q$.

It follows that a favorable (unfavorable) game remains favorable (unfavorable) and an equitable game ($pk = q$) remains equitable regardless of the variations in bets. It should be emphasized, however, that such parameters as game duration and the probability of success in achieving a specified increase in wealth subject to a specified probability of loss *are* distinct functions of the betting systems; these relations are considered shortly.

The number of "guaranteed" betting systems, the proliferation of myths and fallacies concerning such systems, and the countless people believing, propagating, venerating, protecting, and swearing by such systems are legion. Betting systems constitute one of the oldest delusions of gambling history. Betting-system votaries are spiritually akin to the proponents of perpetual-motion machines, butting their heads against the second law of thermodynamics.

The philosophic rationale of betting systems is usually entwined with the concepts of primitive justice—particularly the principle of retribution (*lex talionis*), which embodies the notion of balance. Accordingly, the universe is governed by an eminently equitable god of symmetry (the "great CPA in the sky") who ensures that for every Head there is a Tail. Confirmation of this idea is often distilled by a process of wishful thinking dignified by philosophic (or metaphysical) cant. The French philosopher Pierre-Hyacinthe Azaïs (1766–1845) formalized the statement that good and evil fortune are exactly balanced in that they produce for each person an equivalent result. Better known are such notions as Hegelian compensation, Marxian dialectics, and the Emersonian pronouncements on the divine or scriptural origin of earthly compensation. With less formal expression, the notion of Head–Tail balance extends back at least to the ancient Chinese doctrine of Yin-Yang polarity.

"Systems" generally can be categorized into multiplicative, additive, or linear betting prescriptions. Each system is predicated on wagering a sum of money determined by the outcomes of the previous plays, thereby ignoring the

[3] Proofs of the theorems were developed in the first edition of this book.

implication of independence between plays. The sum β_i is bet on the *i*th play in a sequence of losses. In multiplicative systems,

$$\beta_i = K f(\beta_{i-1}, \beta_{i-2}, \ldots, \beta_1)$$

where K is a constant ≥1 and plays $i - 1, i - 2, \ldots, 1$ are immediately preceding (generally, if play $i - 1$ results in a win, β_i reverts to β_1).

Undoubtedly, the most popular of all systems is the "Martingale"[4] or "doubling" or "geometric progression" procedure. Each stake is specified only by the previous one. In the event of a loss on the $i-1$ play, $\beta_i = 2\beta_{i-1}$. A win on the $i - 1$ play calls for $\beta_i = \beta_1$. (With equal validity or equal misguided intent, we could select any value for K and any function of the preceding results.)

To illustrate the hazards of engaging the Martingale system with a limited bankroll, we posit an equitable game ($p = q = 1/2$) with an initial bankroll of 1000 units and an initial wager of one unit, where the game ends either after 1000 plays—or earlier if the player's funds become insufficient to wager on the next play. We wish to determine the probability that the player achieves a net gain—a final bankroll exceeding 1000 units.

Consider an R × C matrix whose row numbers, R = 0 to 10, represent the possible lengths of the losing sequence at a particular time, and whose column numbers, C = 0 to 2000, represent the possible magnitude of the player's current bankroll, B_C. That is, each cell of the matrix, P(R, C), denotes the probability of a specified combination of losing sequence and corresponding bankroll. Matrices after each play are (computer) evaluated (Ref. Cesare and Widmer) under the following criteria: if $C < 2^R$, then B_C is insufficient for the next prescribed wager, and the probability recorded in P(R, C) is transferred to the probability of being bankrupt, P_B; otherwise, 1/2 the probability recorded in P(R, C) is transferred to $P(R + 1, C - 2^R)$ and the remaining 1/2 to $P(0, C + 2^R)$, since $p = q = 1/2$ for the next play.

The results for B_F, the final bankroll, indicate that

Prob.($B_F = 0$) = 0.399^- (bankrupt)
Prob.($B_F <$ 1000 units) = 0.001^-
Prob.($B_F >$ 1000 units) = 0.600^+
Prob.($B_F =$ 1000 units) = 0.000^+

It is perhaps surprising that with a bankroll 1000 times that of the individual wager, the player is afforded only a 0.6 probability of profit after 1000 plays. Under general casino play, with $p < q$, this probability, of course, declines yet further.

[4] From the French *Martigal*, the supposedly eccentric inhabitants of Martigues, a village west of Marseille. The Martingale system was employed in the 18th century by Casanova (unsuccessfully) and was favored by James Bond in *Casino Royale*.

Another multiplicative system aims at winning n units in n plays—with an initial bet of one unit. Here, the gambler wagers $2^n - 1$ units at the nth play. Thus every sequence of $n - 1$ losses followed by a win will net n units:

$$-1 - 3 - 7 - \cdots - (2^{n-1} - 1) + (2^n - 1) = n$$

In general, if the payoff for each individual trial is at odds of k to 1, the sequence of wagers ending with a net gain of n units for n plays is

$$-\sum_{1}^{n} \frac{(k+1)^{i-1} - k^{i-1}}{k^{i-1}} + \frac{(k+1)^n - k^n}{k^{n-1}} = n$$

The nth win recovers all previous $n-1$ losses and nets n units.

Additive systems are characterized by functions that increase the stake by some specific amount (which may change with each play). A popular additive system is the "Labouchère,"[5] or "cancellation," procedure. A sequence of numbers $a_1, a_2, \ldots, a_n$ is established, the stake being the sum of a_1 and a_n. For a win, the numbers a_1 and a_n are removed from the sequence and the sum $a_2 + a_{n-1}$ is wagered; for a loss, the sum $a_1 + a_n$ constitutes the a_{n+1} term in a new sequence and $a_1 + a_{n+1}$ forms the subsequent wager. Since the additive constant is the first term in the sequence, the increase in wagers occurs at an arithmetic rate. A common implementation of this system begins with the sequence 1, 2, 3—equivalent to a random walk with one barrier at +6 and another at –C, the gambler's capital.

In a fair game, to achieve a success probability P_s equal to 0.99 of winning the six units requires a capital:

$$C = \frac{6P_s}{1 - P_s} = 594$$

On the other hand, to obtain a 0.99 probability of winning the gambler's capital of 594, the House requires a capital of $6m$, where

$$m = \frac{99 \times .99}{1 - .99} = 9801$$

and $6m = 58{,}806$. Of course, the House can raise its winning probability as close to certainty as desired with sufficiently large m. For the gambler to bankrupt the House (i.e., win 58,806), he must win 9801 consecutive sequences.

Note that g gamblers competing against the House risk an aggregate of $594g$, whereas the House risks only $6g$. For unfavorable games, this imbalance is amplified, and the probability of the House winning approaches unity.

[5] After its innovator, Henry Du Pre Labouchère, a 19th-century British diplomat and member of parliament.

Linear betting systems involve a fixed additive constant. A well-known linear system, the “d’Alembert” or “simple progression” procedure, is defined by

$$\beta_i = \beta_{i-1} \pm K \begin{cases} + \text{ if } i-1 \text{ play results in a loss} \\ - \text{ if } i-1 \text{ play results in a win} \end{cases}$$

It is obviously possible to invent other classes of systems. We might create an exponential system wherein each previous loss is squared, cubed, or, in general, raised to the *n*th power to determine the succeeding stake. For those transcendentally inclined, we might create a system that relates wagers by trigonometric functions. We could also invert the systems described here. The “anti-Martingale” (or “Paroli”) system doubles the previous wager after a win and reverts to β_1 following a loss. The “anti-Labouchère” system adds a term to the sequence following a win and decreases the sequence by two terms after a loss, trying to lose six (effecting a high probability of a small loss and a small probability of a large gain). The “anti-d’Alembert” system employs the addition or subtraction of *K* according to a preceding win or loss, respectively.

Yet other systems can be designed to profit from the existence of imagined or undetermined biases. For example, in an even-bet Head–Tail sequence, the procedure of wagering on a repetition of the immediately preceding outcome of the sequence would be advantageous in the event of a constant bias; we dub this procedure the Epaminondas system in honor of the small boy (not the Theban general) who always did the right thing for the previous situation. The “quasi-Epaminondas” system advises a bet on the majority outcome of the preceding three trials of the sequence. Only the bounds of our imagination limit the number of systems we might conceive. No one of them can affect the mathematical expectation of the game when successive plays are mutually independent.

The various systems do, however, affect the expected duration of play. A system that rapidly increases each wager from the previous one will change the pertinent fortune (bankroll) at a greater rate; that fortune will thereby fall to zero or reach a specified higher level more quickly than through a system that dictates a slower increase in wagers. The probability of successfully increasing the fortune to a specified level is also altered by the betting system for a given single-trial probability of success.

A multiplicative system, wherein the wager rises rapidly with increasing losses, offers a high probability of small gain with a small probability of large loss. The Martingale system, for example, used with an expendable fortune of 2^n units, recovers all previous losses and nets a profit of 1 unit—insofar as the number of consecutive losses does not exceed $n-1$. Additive systems place more restrictions on the resulting sequence of wins and losses, but risk correspondingly less capital. The Labouchère system concludes a betting sequence whenever the number of wins equals one-half the number of losses plus one-half the number of terms in the original sequence. At that time, the gambler has increased his fortune by $a_1 + a_2 + \cdots + a_n$ units. Similarly, a linear betting system requires near equality between the number of wins and losses, but risks less in large wagers to recover early losses. In the d'Alembert system, for $2n$ plays involving n wins and n losses in any order, the gambler increases his fortune by $K/2$ units. Exactly contrary results are achieved for each of the "anti" systems mentioned (the anti-d'Alembert system decreases the gambler's fortune by $K/2$ units after n wins and n losses).

In general, for a game with single-trial probability of success p, a positive system offers a probability $p' > p$ of winning a specified amount at the conclusion of each sequence while risking a probability $1 - p'$ of losing a sum larger than $\beta_1 n$, where β_1 is the initial bet in a sequence and n is the number of plays of the sequence. A negative system provides a smaller probability $p' < p$ of losing a specified amount over each betting sequence but contributes insurance against a large loss; over n plays the loss is always less than $\beta_1 n$. The "anti" systems described here are negative. Logarithmic ($\beta_i = \log \beta_{i-1}$ if the $i - 1$ play results in a loss) or fractional [$\beta_i = (\beta_{i-1})/K$ if the $i - 1$ play is a loss] betting systems are typical of the negative variety. For the fractional system, the gambler's fortune X_{2n} after $2n$ plays of n wins and n losses in any order is

$$X_{2n} = X_0\left(1 - \frac{1}{K^2}\right)^n$$

The gambler's fortune is reduced, but the insurance premium against large losses has been paid. Positive systems are inhibited by the extent of the gambler's initial fortune and the limitations imposed on the magnitude of each wager. Negative systems have questionable worth in a gambling context but are generally preferred by insurance companies. Neither type of system, of course, can budge the mathematical expectation from its fixed and immutable value.

Yet another type of system involves wagering only on selected members of a sequence of plays (again, our treatment is limited to mutually independent events). The decision to bet or not to bet on the outcome of the nth trial is determined as a function of the preceding trials. Illustratively, we might bet only on odd-numbered trials, on prime-numbered trials, or on those trials immediately preceded by a sequence of ten "would have been" losses. To describe this situation, we formulate the following theorem.

> *Theorem II:* No advantage accrues from the process of betting only on some subsequence of a number of independent repeated trials forming an extended sequence.

This statement in mathematical form is best couched in terms of measure theory. We establish a sequence of functions Δ_1, $\Delta_2(\zeta_1)$, $\Delta_3(\zeta_1, \zeta_2)$, …, where Δ_i assumes a value of either one or zero, depending only on the outcomes $\zeta_1, \zeta_2, \ldots, \zeta_{i-1}$ of the $i - 1$ previous trials. Let ω:$(\zeta_1, \zeta_2, \ldots)$ be a point of the infinite-dimensional Cartesian space Ω_∞ (allowing for an infinite number of trials); let $F(n, \omega)$ be the nth value of i for which $\Delta_i = 1$; and let $\zeta'_n = \zeta_F$. Then $(\zeta'_1, \zeta'_2, \ldots)$ is the subsequence of trials on which the gambler wagers. It can be shown that the probability relations valid for the space of points $(\zeta'_1, \zeta'_2, \ldots)$ are equivalent to those pertaining to the points $(\zeta_1, \zeta_2, \ldots)$ and that the gambler therefore cannot distinguish between the subsequence and the complete sequence.

Of the system philosophies that ignore the precept of Theorem II the most prevalent and fallacious is "maturity of the chances." According to this doctrine, we wager on the nth trial of a sequence only if the preceding $n-1$ trials have produced a result opposite to that which we desire as the outcome of the nth trial. In Roulette, we would be advised, then, to await a sequence of nine consecutive "reds," say, before betting on "black." Presumably, this concept arose from a misunderstanding of the law of large numbers (Eq. 2-25). Its effect is to assign a memory to a phenomenon that, by definition, exhibits no correlation between events.

One final aspect of disingenuous betting systems remains to be mentioned. It is expressible in the following form.

> *Corollary:* No advantage in terms of mathematical expectation accrues to the gambler who may exercise the option of discontinuing the game after each play.[6]

The question of whether an advantage exists with this option was first raised by John Venn, the English logician, who postulated a coin-tossing game between opponents **A** and **B**; the game is equitable except that only **A** is permitted to stop or continue the game after each play. Since Venn's logic granted infinite credit to each player, he concluded that **A** indeed has an advantage. Lord Rayleigh subsequently pointed out that the situation of finite fortunes alters the conclusion, so that no advantage exists.

Having established that no class of betting system can alter the mathematical expectation of a game, we proceed to determine the expectation and variance of a series of plays. The definition of each play is that there exists a probability p that the gambler's fortune is increased by α units, a probability q that the fortune is decreased by β units ($\alpha \equiv k\beta$), and a probability $r = 1-p-q$ that no change in the fortune occurs (a "tie"). The bet at each play, then, is β units of wealth.

> *Theorem III:* For n plays of the general game, the mean or mathematical expectation is $n(\alpha p - \beta q)$, and the variance is $n[\alpha^2 p + \beta^2 q - (\alpha p - \beta q)^2]$.

This expression for the mathematical expectation leads directly to the definition of a *fair* or *equitable* game. It is self-evident that the condition for no inherent advantage to any player is that

$$\mathrm{E}(X_n) = 0$$

For the general case, a fair game signifies

$$\alpha p = \beta q$$

[6] In French, the privilege of terminating play arbitrarily while winning has been accorded the status of an idiom: *le droit de faire Charlemagne*.

To complete the definition we can refer to all games wherein $E(X_n) > 0$ (i.e., $\alpha p > \beta q$) as positive games; similarly, negative games are described by $E(X_n) < 0$ (or $\alpha p < \beta q$).

Comparing the expectation and the variance of the general game, we note that as the expectation deviates from zero, the variance decreases and approaches zero as p or q approaches unity. Conversely, the variance is maximal for a fair game. It is the resulting fluctuations that may cause a gambler to lose a small capital even though his mathematical expectation is positive. The variance of a game is one of our principal concerns in deciding whether and how to play. For example, a game that offers a positive single-play mathematical expectation may or may not have strong appeal, depending on the variance as well as on other factors such as the game's utility. A game characterized by $p = 0.505$, $q = 0.495$, $r = 0$, $\alpha = \beta = 1$, has a single-play expectation of $E(X) = 0.01$. We might be interested in this game, contingent upon our circumstances. A game characterized by $p = 0.01$, $q = 0$, $r = 0.99$, $\alpha = \beta = 1$, also exhibits an expectation of 0.01. However, our interest in this latter game is not casual but irresistible. We should hasten to mortgage our children and seagoing yacht to wager the maximum stake. The absence of negative fluctuations clearly has a profound effect on the gambler's actions.

These considerations lead to the following two theorems, the first of which is known as "the gambler's ruin." (The ruin probability and the number of plays before ruin are classical problems that have been studied by James Bernoulli, de Moivre, Lagrange, and Laplace, among many.)

> *Theorem IV:* In the general game (where a gambler bets β units at each play to win α units with single-trial probability p, lose β with probability q, or remain at the same financial level with probability $r = 1 - p - q$) begun by a gambler with capital z and continued until that capital either increases to $\alpha \geqslant z + \beta$ or decreases to less than β (the ruin point), the probability of ruin P_z is bounded by
>
> $$\lambda^z \frac{\lambda^{a-z-(\alpha-1)} - 1}{\lambda^{a-(\alpha-1)} - 1} \le P_z \le \lambda^{z-(\beta-1)} \frac{\lambda^{a-z} - 1}{\lambda^{a-(\beta-1)} - 1}$$
>
> where λ is a root of the exponential equation $p\lambda^{\alpha+\beta} - (1-r)\lambda^{\beta} + q = 0$. For values of a and z large with respect to α and β, the bounds are extremely tight.

If the game is equitable—a mathematical expectation equal to zero—we can derive the bounds on P_z as

$$\frac{a - z - (\alpha - 1)}{a - (\alpha - 1)} \le P_z \le \frac{a - z}{a - (\beta - 1)} \tag{3-1}$$

With the further simplification that $\alpha = \beta = 1$, as is the case in many conventional games of chance, the probability of ruin reduces to the form

$$P_z = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1} \quad \text{for } p \neq q \tag{3-2}$$

And the probability, $\bar{P}_z$, of the gambler successfully increasing his capital from z to a is

$$\bar{P}_z = 1 - P_z = \frac{(q/p)^z - 1}{(q/p)^a - 1}, \quad p \neq q \tag{3-3}$$

For a fair game, $p = q$, and we invoke l'Hospital's rule to resolve the concomitant indeterminate form of Eqs. 3-2 and 3-3:

$$P_z = \frac{a - z}{a}$$

and

$$\bar{P}_z = \frac{z}{a} \tag{3-4}$$

We can modify the ruin probability P_z to admit a third outcome for each trial, that of a tie (probability r). With this format,

$$p = \frac{\alpha}{1 - r} \quad \text{and} \quad q = \frac{\beta}{1 - r}$$

And Eq. 3-2 takes the form

$$P_z = \frac{(\beta/\alpha)^a - (\beta/\alpha)^z}{(\beta/\alpha)^a - 1}$$

It is interesting to note the probabilities of ruin and success when the gambler ventures against an infinitely rich adversary. Letting $a \to \infty$ in the fair game represented by Eq. 3-1, we again observe that $P_z \to 1$. Therefore, whenever $\alpha p \leq \beta q$, the gambler must eventually lose his fortune z. However, in a favorable game as $(a - z) \to \infty$, $\lambda < 1$, and the probability of ruin approaches

$$P_z \to \lambda^z$$

With the simplification of $\alpha = \beta = 1$, Eq. 3-2 as $a \to \infty$ becomes

$$P_z \to (q/p)^z$$

for $p > q$. Thus, in a favorable game, the gambler can continue playing indefinitely against an infinite bank and, with arbitrarily large capital z, can render his probability of ruin as close to zero as desired. We can conclude that with large initial capital, a high probability exists of winning a small amount even in games that are slightly unfavorable; with smaller capital, a favorable game is required to obtain a high probability of winning an appreciable amount.

As an example, for $p = 0.51$ and $q = 0.49$, and betting 1 unit from a bankroll of 100 units, the ruin probability is reduced to 0.0183. Betting 1 unit from a bankroll of 1000 confers a ruin probability of 4.223×10^{-18}.

Corollary: The expected duration D_z of the game is equivalent to the random-walk probability of reaching a particular position x for the first time at the nth step (Chapter 2, Random Walks). Specifically,

$$D_z = \left| \frac{z}{q-p} - \frac{a}{q-p} \cdot \frac{1-(q/p)^z}{1-(q/p)^a} \right| \qquad p \neq q \tag{3-5}$$

For $p = q$, Eq. 3-5 becomes indeterminate as before—and is resolved in this case by replacing $z/(q - p)$ with z^2, which leads to

$$D_z = z(a - z)$$

Illustratively, two players with bankrolls of 100 units each tossing a fair coin for stakes of one unit can expect a duration of 10,000 tosses before either is ruined. If the game incorporates a probability r of tying, the expected duration is lengthened to

$$\frac{D_z}{1-r}$$

Against the infinitely rich adversary, an equitable game affords the gambler an infinite expected duration (somewhat surprisingly). For a parallel situation, see Chapter 5 for coin tossing with the number of Heads equal to the number of Tails; see also Chapter 2 for unrestricted symmetric random walks, where the time until return-to-zero position has infinite expectation.

These considerations lead directly to formulation of the following theorem.

Theorem V: A gambler with initial fortune z, playing a game with the fixed objective of increasing his fortune by the amount $a - z$, has an expected gain that is a function of his probability of ruin (or success). Moreover, the probability of ruin is a function of the betting system. For equitable or unfair games, a "maximum boldness" strategy is optimal—that is, the gambler should wager the maximum amount consistent with his objective and current fortune. For favorable games, a "minimal boldness" or "prudence" strategy is optimal—the gambler should wager the minimum sum permitted under the game rules.

The maximum-boldness strategy dictates a bet of z when $z \leq a/(k + 1)$ and $(a - z)/k$ when $z > a/(k + 1)$. For $k = 1$ (an even bet), maximum boldness coincides with the Martingale, or doubling, system. (Caveat: If a limit is placed on the number of plays allowed the gambler, maximum boldness is not then necessarily optimal.) For equitable games, the stakes or variations thereof are immaterial; the probability of success (winning the capital $a - z$) is then simply the ratio of the initial bankroll to the specified goal, as per Eq. 3-4.

Maximizing the probability of success in attaining a specified increase in capital without regard to the number of plays, as in the preceding theorem, is but one of many criteria. It is also feasible to specify the desired increase $a - z$ and determine the betting strategy that minimizes the number of plays required to gain this amount. Another criterion is to specify the number of plays n and maximize the magnitude of the gambler's fortune after the n plays. Yet another is to maximize the rate of increase in the gambler's capital. In general, any function of the gambler's fortune, the desired increase therein, the number of plays, etc. can be established as a criterion and maximized or minimized to achieve a particular goal.

In favorable games ($p > q$), following the criterion proposed by John L. Kelly (Ref.) leads in the long run to an increase in capital greater than that achievable through any other criterion. The Kelly betting system entails maximization of the exponential rate of growth G of the gambler's capital, G being defined by

$$G = \lim_{n\to\infty} \frac{1}{n} \log \frac{x_n}{x_0}$$

where x_0 is initial fortune and x_n the fortune after n plays.

If the gambler bets a fraction f of his capital at each play, then x_n has the value

$$x_n = (1 + f)^w (1 - f)^\ell x_0$$

for w wins and l losses in the n wagers. The exponential rate of growth is therefore

$$\begin{aligned} G &= \lim_{n\to\infty} \left[\frac{w}{n} \log(1 + f) + \frac{n - w}{n} \log(1 - f) \right] \\ &= p \log(1 + f) + q \log(1 - f) \end{aligned}$$

where the logarithms are defined here with base 2. Maximizing G with respect to f, it can be shown that

$$G_{\max} = (1 - r) + p \log p + q \log q - (1 - r) \log(1 - r)$$

which, for $r = 0$ (no ties), reduces to $1 + p \log p + q \log q$, the Shannon rate of transmission of information over a noisy communication channel.

For positive-expectation bets at even odds, the Kelly recipe calls for a wager equal to the mathematical expectation. A popular alternative consists of wagering one-half this amount, which yields 3/4 the return with substantially less volatility. (For example, where capital accumulates at 10% compounded with full bets, half-bets still yield 7.5%.)

In games that entail odds of k to 1 for each individual trial with a win probability P_w, the Kelly criterion prescribes a bet of

$$P_w - (1 - P_w)/k$$

Illustratively, for k equals 3 to 1 and a single-trial probability $P_w = 0.28$, the Kelly wager is $0.28 - 0.72/3 = 4\%$ of current capital.

Breiman (Ref.) has shown that Kelly's system is asymptotically optimal under two criteria: (1) minimal expected time to achieve a fixed level of resources and (2) maximal rate of increase of wealth.

Contrary to the types of systems discussed in connection with Theorem I, Kelly's fractional (or logarithmic or proportional) gambling system is not a function of previous history. Generally, a betting system for which each wager depends only on present resources and present probability of success is known as a *Markov betting system* (the sequence of values of the gambler's fortune form a Markov process). It is heuristically evident that, for all sensible criteria and utilities, a gambler can restrict himself to Markov betting systems.

While Kelly's system is particularly interesting (as an example of the rate of transmission of information formula applied to uncoded messages), it implies an infinite divisibility of the gambler's fortune—impractical by virtue of a lower limit imposed on the minimum bet in realistic situations. There also exists a limit on the maximum bet, thus further inhibiting the logarithmic gambler. Also, allowable wagers are necessarily quantized.

We can observe that fractional betting constitutes a negative system (see the discussion following Theorem I).

Disregarding its qualified viability, a fractional betting system offers a unity probability of eventually achieving any specified increase in capital. Its disadvantage (and the drawback of any system that dictates an increase in the wager following a win and a decrease following a loss) is that any sequence of n wins and n losses in a fair game results in a bankroll B reduced to

$$\mathrm{B}\left(1 + \frac{1}{m}\right)^n \left(1 - \frac{1}{m}\right)^n = \mathrm{B}\left(1 - \frac{1}{m^2}\right)^n$$

when the gambler stakes $1/m$ of his bankroll at each play.

When the actual number of wins is *approximately* the expected number, proportional betting yields about half the arithmetic expectation. However, when the actual number of wins is only half the expectation, proportional betting yields a zero return. Thus the Kelly criterion acts as insurance against bankruptcy—and, as with most insurance policies, exacts a premium.

For favorable games replicated indefinitely against an infinitely rich adversary, a better criterion is to optimize the "return on investment"—that is, the net gain per unit time divided by the total capital required to undertake the betting process. In this circumstance we would increase the individual wager $\beta(t)$

as our bankroll $z(t)$ increased, maintaining a constant return on investment, R, according to the definition

$$R = \frac{(p-q)\beta(t)}{z(t)}$$

A more realistic procedure for favorable games is to establish a confidence level in achieving a certain gain in capital and then inquire as to what magnitude wager results in that confidence level for the particular value of $p > q$. This question is answered as follows.

Theorem VI: In games favorable to the gambler, there exists a wager β defined by

$$\begin{aligned}\beta &\simeq \left[\frac{-z}{\log \bar{Q}_z} - \frac{\bar{P}_z(\bar{Q}_z)^{a/z}z^2}{a[\log \bar{Q}_z]^2\ \bar{Q}_z^{\,a/z}\bar{P}_z - z\bar{Q}_z[\log \bar{Q}_z]^2}\right]\log(q/p)^{-1} \\ &\simeq \frac{-z}{\log \bar{Q}_z}\log(q/p)^{-1} \quad \text{for } \bar{P}_z > 0.5 \text{ and } a \gg z,\end{aligned} \tag{3-6}$$

such that continually betting β units or less at each play yields a confidence level of $\bar{P}_z$ or greater of increasing the gambler's fortune from an initial value z to a final value a. (Here, $\bar{Q}_z \equiv 1 - \bar{P}_z$.)

Figure 3-4 illustrates Eq. 3-6 for $z = 100$, $a = 1000$, and four values of the confidence level $\bar{P}_z$. For a given confidence in achieving a fortune of 1000

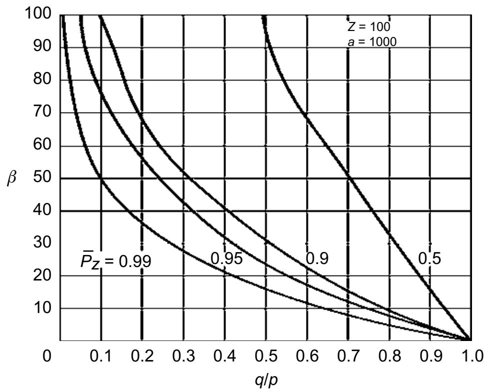

FIGURE 3-4 Optimal wager for a fixed confidence level.

units, higher bets cannot be hazarded until the game is quite favorable. A game characterized by $(q/p) = 0.64$ ($p = 0.61$, $q = 0.39$, $r = 0$), for example, is required before ten units of the 100 initial fortune can be wagered under a 0.99 confidence of attaining 1000. Note that for $a \gg z$, the optimal bets become virtually independent of the goal a. This feature, of course, is characteristic of favorable games.

Generally, with an objective utility function, the purpose of increasing the magnitude of the wagers in a favorable game (thereby decreasing the probability of success) is to decrease the number of plays required to achieve this goal.

We can also calculate the probability of ruin and success for a specific number of plays of a game:

Theorem VII: In the course of n resolved plays of a game, a gambler with capital z playing unit wagers against an opponent with capital $a-z$ has a probability of ruin given by

$$P_{z,n} = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1} - \frac{2\sqrt{pq}(q/p)^{z/2}}{a} \sum_{j=1}^{a-1} \frac{\sin(\pi j/a)\sin(\pi zj/a)[\cos(\pi j/a)]^n}{1 - 2\sqrt{pq}\cos(\pi j/a)} \tag{3-7}$$

where p, and q are the gambler's single-trial probabilities of winning, and losing, respectively.

As the number of plays $n \to \infty$, the expression for $P_{z,n}$ approaches the P_z of Eq. 3-2. Equation 3-7 is illustrated in Figure 3-5, which depicts $P_{z,n}$ versus

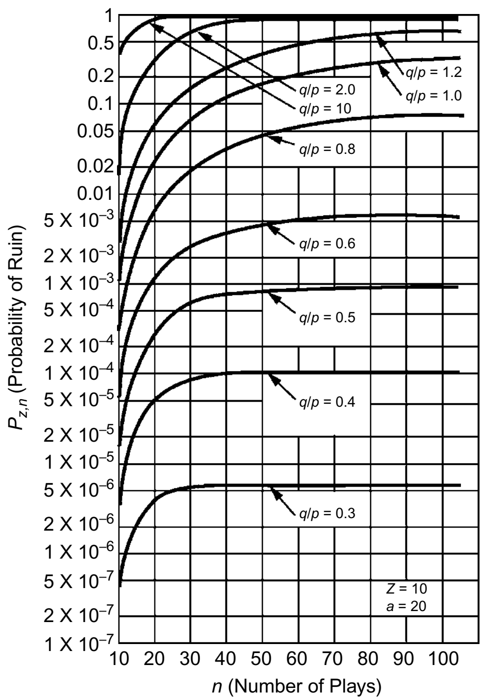

FIGURE 3-5 Ruin probability in the course of *n* plays.

n for an initial gambler's capital of $z = 10$, an adversary's capital $a - z = 10$, and for several values of q/p.

An obvious extension of Theorem VII ensues from considering an adversary either infinitely rich or possessing a fortune such that n plays with unit wagers cannot lead to his ruin. This situation is stated as follows.

Corollary: In the course of n resolved plays of a game, a gambler with capital z, wagering unit bets against an (effectively) infinitely rich adversary, has a probability of ruin given by

$$P_{z,n} = q^z \left[1 + zpq + \frac{z(z+3)/(pq)^2}{2!} + \cdots + \frac{z + (z+k+1)\cdots(z+2k-1)(pq)^k}{k!} \right] \tag{3-8}$$

where $k = \dfrac{n-z}{2}$ or $\dfrac{n-z-1}{2}$, as $|n - z|$ is even or odd.

Figure 3-6 illustrates Eq. 3-8 for $z = 10$ and $p = q = 1/2$. For a large number of plays of an equitable game against an infinitely rich adversary, we can derive the approximation

$$P_{z,n} \simeq 1 - \frac{2}{\sqrt{\pi}} \int_0^{z\sqrt{3/(6n+4)}} e^{-\zeta^2} d\zeta \tag{3-9}$$

which is available in numerical form from tables of probability integrals. Equation 3-9 indicates, as expected, that the probability of ruin approaches unity asymptotically with increasing values of n.

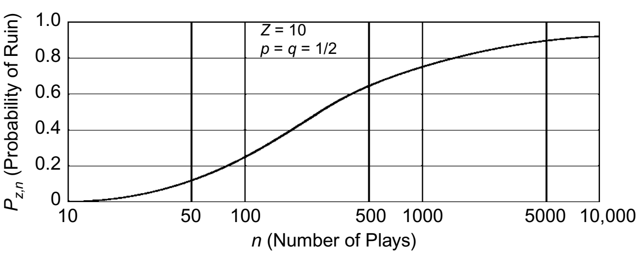

FIGURE 3-6 Ruin probability in the course of n plays of a "fair" game against an infinitely rich adversary.

Equations 3-8 and 3-7 are attributable to Lagrange. A somewhat simpler expression results if, instead of the probability of ruin over the course of n plays, we ask the expected number of plays $\mathrm{E}(n)_z$ prior to the ruin of a gambler with initial capital z:

$$\mathrm{E}(n)_z = \frac{z}{(q-p)} - \frac{a[1-(q/p)^z]}{(q-p)[1-(q/p)^a]}, \quad p \neq q \tag{3-10}$$

For equitable games, $p = q$, and this expected duration simplifies to

$$E(n)_z = z(a - z)$$

For example, a gambler with 100 units of capital wagering unit bets against an opponent with 1000 units can expect the game to last for 90,000 plays. Against an infinitely rich opponent, Eq. 3-10, as $a \to \infty$, reduces to

$$E(n)_z \to \frac{z}{q - p}$$

for $p < q$. In the case $p > q$, the expected game duration versus an infinity of Croesuses is a meaningless concept. For $p = q$, $E(n)_z \to \infty$.

We have previously defined a favorable game as one wherein the single-trial probability of success p is greater than the single-trial probability of loss q. A game can also be favorable if it comprises an ensemble of subgames, one or more of which is favorable. Such an ensemble is termed a *compound game*. Pertinent rules require the player to wager a minimum of one unit on each subgame; further, the relative frequency of occurrence of each subgame is known to the player, and he is informed immediately in advance of the particular subgame about to be played (selected at random from the relevant probability distribution). Blackjack (Chapter 8) provides the most prevalent example of a compound game.

Clearly, given the option of bet-size variation, the player of a compound game can obtain any positive mathematical expectation desired. He has only to wager a sufficiently large sum on the favorable subgame(s) to overcome the negative expectations accruing from unit bets on the unfavorable subgames. However, such a policy may face a high probability of ruin. It is more reasonable to establish the criterion of minimizing the probability of ruin while achieving an overall positive expectation—i.e., a criterion of survival. A betting system in accord with this criterion is described by Theorem VIII.

Theorem VIII: In compound games there exists a critical single-trial probability of success $p^* > 1/2$ such that the optimal wager for each subgame $\beta^*(p)$, under the criterion of survival, is specified by

$$\beta^*(p) = \begin{cases} 1 & \text{if } p < p^* \\ \dfrac{\log[(1-p)/p]}{\log[(1-p^*)/p^*]} & \text{if } p \geq p^* \end{cases} \tag{3-11}$$

As a numerical illustration of this theorem, consider a compound game where two-thirds of the time the single-trial probability of success takes a value of 0.45 and the remaining one-third of the time a value of 0.55. Equation 3-11 prescribes the optimal wagers for this compound game as

$$\beta^* = \begin{cases} 1 & \text{for } p = 0.45 \\ \dfrac{\log(0.45/0.55)}{2\log 0.9777} = 4.45 & \text{for } p = 0.55 \end{cases}$$

Thus the gambler who follows this betting prescription achieves an overall expectation of

$$\frac{2}{3}(0.45 - 0.55) + \frac{1}{3}[4.45(0.55 - 0.45)] = +0.082$$

while maximizing his probability of survival (it is assumed that the gambler initially possesses a large but finite capital pitted against essentially infinite resources).

Much of the preceding discussion in this chapter has been devoted to proving that no betting system constructed on past history can affect the expected gain of a game with independent trials. With certain goals and utilities, it is more important to optimize probability of success, number of plays, or some other pertinent factor. However, objective utilities focus strictly on the game's mathematical expectation. Two parameters in particular relate directly to mathematical expectation: strategy and information. Their relationship is a consequence of the fundamental theorem of game theory, the von Neumann minimax principle (explicated in Chapter 2), which can be stated as follows.

Theorem IX: For any two-person rectangular game characterized by the payoff matrix $\|a_{ij}\|$, where **A** has m strategies $A_1, A_2, \ldots, A_m$ selected with probabilities $p_1, p_2, \ldots, p_m$, respectively, and **B** has n strategies $B_1, B_2, \ldots, B_n$ selected with probabilities $q_1, q_2, \ldots, q_n$, respectively ($\sum p_i = \sum q_j = 1$), the mathematical expectation $E[S_A(m), S_B(n)]$ for any strategies $S_A(m)$ and $S_B(n)$ is defined by

$$\mathrm{E}[S_A(m),\ S_B(n)] = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}p_iq_j$$

Then, if the $m \times n$ game has a value γ, a necessary and sufficient condition that $S_A^*(m)$ be an optimal strategy for **A** is that

$$\mathrm{E}[S_A^*(m),\ S_B(n)] \geq \gamma$$

for all possible strategies $S_B(n)$. Similarly, a necessary and sufficient condition that $S_B^*(n)$ be an optimal strategy for **B** is that

$$\mathrm{E}[S_A(m),\ S_B^*(n)] \leq \gamma$$

Some of the properties of two-person rectangular games are detailed in Chapter 2. If it is our objective to minimize our maximum loss or maximize our minimum gain; the von Neumann minimax theorem and its associated corollaries enable us to determine optimal strategies for any game of this type.

Further considerations arise from the introduction of information into the game format since the nature and amount of information available to each player can affect the mathematical expectation of the game. In Chess, Checkers, or Shogi, for example, complete information is axiomatic; each player is fully aware of his opponent's moves. In Bridge, the composition of each hand is information withheld, to be disclosed gradually, first by imperfect communication and then by sequential exposure of the cards. In Poker, concealing certain components of each hand is critical to the outcome. We shall relate, at least qualitatively, the general relationship between information and expectation.

In the coin-matching game matrix of Figure 2-2, wherein each player selects Heads or Tails with probability $p = 1/2$, the mathematical expectation of the game is obviously zero. But now consider the situation where **A** has in his employ a spy capable of ferreting out the selected strategies of **B** and communicating them back to **A**. If the spy is imperfect in applying his espionage techniques or the communication channel is susceptible to error, **A** has only a probabilistic knowledge of **B**'s strategy. Specifically, if p is the degree of knowledge, the game matrix is that shown in Figure 3-7, where (i, j) represents the

	B_1(H)	B_2(T)
A_1(H,H)	1	−1
A_2(H,T)	$2p - 1$	$2p - 1$
A_3(T,H)	$1 - 2p$	$1 - 2p$
A_4(T,T)	−1	1

FIGURE 3-7 Coin-matching payoff with spying.

strategy of **A** that selects i when **A** is informed that **B**'s choice is H (Heads) and j when **A** is informed that **B** has selected T (Tails). For $p > 1/2$, strategy A_2 dominates the others; selecting A_2 consistently yields an expectation for **A** of

$$E = \frac{1}{2}(2p - 1) + \frac{1}{2}(2p - 1) = 2p - 1$$

With perfect spying, $p = 1$, and **A**'s expectation is unity at each play; with $p = 1/2$, spying provides no intelligence, and the expectation is, of course, zero.

If, in this example, **B** also has recourse to undercover sleuthing, the game outcome is evidently more difficult to specify. However, if **A** has evidence that **B** intends to deviate from his optimal strategy, there may exist counter-strategies that exploit such deviation more advantageously than **A**'s maximin strategy. Such a strategy, by definition, is constrained to yield a smaller payoff against **B**'s optimal strategy. Hence **A** risks the chance that **B** does not deviate appreciably from optimality. Further, **B** may have the option of providing false information or "bluffing" and may attempt to lure **A** into selecting a nonoptimal strategy by such means. We have thus encountered a *circulus in probando*.

It is not always possible to establish a precise quantitative relationship between the information available to one player and the concomitant effect on his expectation. We can, however, state the following qualitative theorem.

> *Theorem X:* In games involving partnership communication, one objective is to transmit as much information as possible over the permissible communication channel of limited capacity. Without constraints on the message-set, this state of maximum information occurs under a condition of minimum redundancy and maximum information entropy.

Proofs of this theorem are given in many texts on communication theory.[7] The condition of maximum information entropy holds when, from a set of n possible messages, each message is selected with probability $1/n$. Although no conventional game consists solely of communication, there are some instances (Ref. Isaacs), where communication constitutes the prime ingredient and the principle of this theorem is invoked. In games such as Bridge, communication occurs under the superintendence of constraints and cost functions. In this case, minimum redundancy or maximum entropy transpires if every permissible bid has equal probability. However, skewed weighting is obviously indicated, since certain bids are assigned greater values, and an ordering to the sequence of bids is also imposed.

Additional Theorems

While it is possible to enumerate more theorems than the ten advanced herein, such additional theorems do not warrant the status accruing from a fundamental character or extensive application in gambling situations. As individual examples requiring specialized game-theoretic tenets arise in subsequent chapters, they will be either proved or assumed as axiomatic.

For example, a "theorem of deception" might encompass principles of misdirection. A player at Draw Poker might draw two cards rather than three to an original holding of a single pair in order to imply a better holding or to avoid the adoption of a pure strategy (thereby posing only trivial analysis to his opponents). A bid in the Bridge auction might be designed (in part) to minimize the usefulness of its information to the opponents and/or to mislead them.

There are also theorems reflecting a general nature, although not of sufficient importance to demand rigorous proof. Illustrative of this category is the "theorem of pooled resources":

> If g contending players compete (against the House) in one or more concurrent games, with the ith player backed by an initial bankroll z_i ($i = 1, 2, \ldots, g$), and if the total resources are pooled in a fund available to all, then each player possesses an effective initial bankroll z_{eff} given by
>
> $$z_{\text{eff}} = \sum_{i}^{g} z_i - \varepsilon \tag{3-12}$$
>
> under the assumption that the win–loss–tie sequences of players i and j are uncorrelated.

The small quantity ε is the quantization error—that is, with uncorrelated outcomes for the players, the g different wagers can be considered as occurring sequentially in time, with the proviso that each initial bankroll z_i can be replenished only once every gth play. This time restriction on withdrawing

[7] For example, Arthur E. Laemmel, *General Theory of Communication*, Polytechnic Institute of Brooklyn, 1949.

funds from the pool is a quantization error, preventing full access to the total resources. As the correlation between the players' outcomes increases to unity, the value of z_{eff} approaches $\sum z_i/g$.

Another theorem, "the principle of rebates," provides a little-known but highly useful function, the Reddere integral,[8] E(c), which represents the expected excess over the standard random variable (the area under the standard normal curve from 0 to c):

$$\mathrm{E}(c) \sim \int_c^{\alpha} (t - c)\, N(t)\, dt$$

where $N(t)$ is the pdf for the standard normal distribution. E(c) can be rewritten in the form

$$\mathrm{E}(c) \sim N(c) - c\int_c^{\infty} N(t)\, dt$$

which is more convenient for numerical evaluation. (This integral is equivalent to 1 minus the cumulative distribution function [cdf] of c.) According to the rebate theorem, when the "standard score" (the gambler's expectation after n plays of an unfavorable game divided by $\sqrt{n}$) has a Reddere-integral value equal to itself, that value, R_e, represents the number for which the player's break-even position after n plays is R_e standard deviations above his expected loss.

The Reddere integral is evaluated in Appendix Table A for values of t from 0 to 3.00. Applicable when the gambler is offered a rebate of his potential losses, examples for Roulette and Craps are detailed in Chapters 5 and 6, respectively.

Finally, while there are game-theoretic principles other than those considered here, they arise—for the most part—from games not feasible as practical gambling contests. Games with infinite strategies, games allowing variable coalitions, generalized n-person games, and the like, are beyond our scope and are rarely of concrete interest.

In summation, the preceding ten theorems and associated expositions have established a rudimentary outline for the theory of gambling. A definitive theory depends, first and foremost, on utility goals—that is, upon certain restricted forms of subjective preference. Within the model of strictly objective goals, we have shown that no betting system can alter the mathematical expectation of a game; however, probability of ruin (or success) and expected duration of a game *are* functions of the betting system employed. We have, when apposite, assumed that the plays of a game are statistically independent and that no bias exists in the game mechanism. It follows that a time-dependent bias permits application of a valid betting system, and any bias (fixed or varying) enables an improved strategy.

Statistical dependence between plays also suggests the derivation of a corresponding betting system to increase the mathematical expectation. In Chapter 7

[8] Developed by Peter Griffin (Ref.) as the unit normal linear loss integral.

we outline the theory of Markov chains, which relates statistical dependence and prediction.

Prima facie validation of the concepts presented here is certified by every gambling establishment in the world. Since the House is engaged in favorable games (with the rarest of exceptions), it wishes to exercise a large number of small wagers. This goal is accomplished by limiting the maximum individual wager; further, a limit on the lowest wager permitted inhibits negative and logarithmic betting systems. In general, casino games involve a redistribution of wealth among the players, while the House removes a certain percentage, either directly or statistically. In the occasional game where all players contend against the House (as in Bingo), the payoff is fixed per unit play. The House, of course, has one of the most objective evaluations of wealth that can be postulated. No casino has ever been accused of subjective utility goals.

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Chapter | four

Parrondo's Principle

THE BASIC PRINCIPLE

Widely acclaimed as the most significant advance in game-theoretic principles since the minimax process, "Parrondo's Paradox"[1] states that two games, each with a negative expectation, can be combined via deterministic or nondeterministic mixing of the games to produce a positive expectation. That is, *a player may alternate regularly or randomly between two losing games to change the overall outcome to a winning one*.

This counterintuitive principle can be illustrated as follows with games **A** and **B**: Game **A** is defined by tossing a biased coin C_1 (or taking a biased random walk) that offers a probability of winning (Heads), $P_1 = 1/2 - \alpha$, and a probability of losing (Tails), $1 - P_1 = 1/2 + \alpha$, where $0 \leq \alpha < 1/2$ represents the bias against the player. Obviously, the condition for **A** to be fair is $P_1 = 1/2$.

The expected value of game **A** is

$$\mathrm{E}(\mathbf{A}) = \frac{P_1 - (1 - P_1)}{P_1 + (1 - P_1)} = (1/2 - \alpha) - (1/2 + \alpha) = -2\alpha$$

Game **B** entails two biased coins, C_2 and C_3, and is dependent on the player's present capital. Coin C_2 presents a probability of winning (Heads), $P_2 = x - \alpha$, with probability of losing (Tails), $1 - P_2 = 1 - x + \alpha$, and is engaged when the player's capital, F(t), equals 0 mod m units, $m \geq 2$. With F(t) *not* a multiple of m, the player engages coin C_3—with probability of winning, $P_3 = y - \alpha$, and probability of losing, $1 - P_3 = 1 - y + \alpha$, where $y = [1 - x - \sqrt{x(1 - x)}]/(1 - 2x)$. (Here y is chosen so that $xy^2 = (1 - x)(1 - y)^2$.) Although any value of x (between 0 and 1/2) can be called on to demonstrate Parrondo's principle, only $x = a^2/(a^2 + b^2)$, for positive integers a and b with $a < b$, furnishes rational values of x and y (and

[1] After Juan Parrondo, a physicist at the Universidad Complutense de Madrid. The seminal paper expounding on Parrondo's innovation is due to Gregory Harmer and Derek Abbott (Ref. 1999).

thereby satisfies the anal-retentive).[2] Parrondo adopted $x = 1/10$ for the original construct, and here we follow that precedent. Further, being engaged in a modulo-dependent game, **B** requires $m \geq 3$ since a modulo-2 rule would allow the symmetry $\mathrm{F}(t) \rightarrow -\mathrm{F}(t)$ and thereby preclude the "losing + losing = winning" effect.

The wager for both **A** and **B** is one unit.

The combination of games **A** and **B** is illustrated in diagrammatic form in Figure 4-1 for the case $m = 3$.

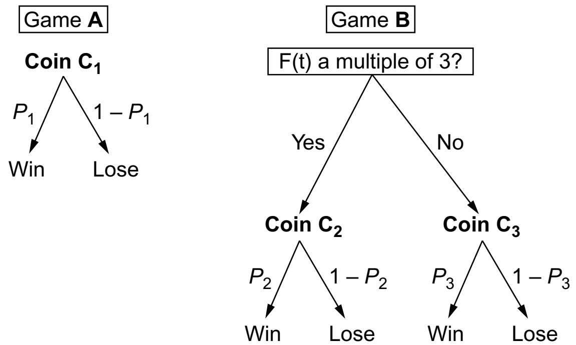

FIGURE 4-1 The basic Parrondo composite game.

The condition for game **B** to be fair is $(1 - P_2)(1 - P_3)^2 = P_2 P_3^2$ since coin C_3 is engaged twice as often as coin C_2 (thus the probability of winning equals the probability of losing). Therefore, with the given values of P_1, P_2, and P_3, game **A** shows a negative expectation for $1/2 > \alpha > 0$, while with the same α, and for $m = 2$ or 3, game **B** also has a negative expectation—although it is a winning game for $m \geq 4$. Of the two subgames (with coins C_2 and C_3), coin C_3 offers a positive expectation. (For the Parrondo principle to be in effect, all three coins cannot be negatively biased.)

To calculate the frequency of using coin C_2, we define the variable $\mathrm{G}(t) = \mathrm{F}(t) \bmod m$. For $m = 3$, $\mathrm{G}(t)$—which can take only three values, 0, 1, and 2, with assigned probabilities 5/13, 2/13, and 6/13, respectively—is a Markov chain with three states and a transition matrix

$$\Pi = \begin{vmatrix} 0 & \tfrac{1}{10} - \alpha & \tfrac{9}{10} + \alpha \\ \tfrac{1}{4} + \alpha & 0 & \tfrac{3}{4} - \alpha \\ \tfrac{3}{4} - \alpha & \tfrac{1}{4} + \alpha & 0 \end{vmatrix}$$

The entries for each row sum to 1, and the probability distribution $\boldsymbol{P}(t)$—i.e., the vector whose components are the probabilities $P_i(t)$ of $G(t) = i$ for $i = 0, 1, 2$—obeys the evolution equation

$$\boldsymbol{P}(t + 1) = \prod \boldsymbol{P}(t)$$

[2]The mathematical observation is credited to Stewart Ethier.

and, for $t \gg 1$, approaches a stationary distribution. That is, $P(t)$ is the eigenvector of the matrix Π with eigenvalue 1. Specifically, the probability P_0 of $G(t) = 0$ is computed as

$$P_0 = (5/13) - (440/2197)\,\alpha + O(\alpha^2)$$

which is the probability of using coin C_2. The probability of winning, P_{win}, is then

$$P_{win} = (1 - P_0)(1/10 - \alpha) + P_0(3/4 - \alpha) = 1/2 - (147/169)\alpha + O(\alpha^2)$$

which, for $\alpha > 0$ small yields a negative expectation, E(**B**):

$$E(\mathbf{B}) = (1)P_{win} + (-1)P_{lose} = -(294/169)\alpha + O(\alpha^2)$$

Combining games **A** and **B** to form game **C**, the probability of using coin C_3 when playing the combined game is P'_0—that is, P_0 times the probability that we are playing game **B**:

$$P'_0 = 245/709 - (48{,}880/502{,}681)\alpha + O(\alpha^2)$$

and the probability of winning, P'_{win}, is then

$$\begin{aligned} P'_{win} &= (1 - P'_0)(5/8 - \alpha) + P'_0(3/10 - \alpha) \\ &= 727/1418 - (486{,}795/502{,}681)\alpha + O(\alpha^2) \end{aligned}$$

Thus, for $\alpha > 0$ small, we have $P'_{win} > 0.5$, and the principle holds. The expectation, E(**C**), of the composite game is

$$E(\mathbf{C}) = 36/1418 - 2(486{,}795/502{,}681)\alpha + O(\alpha^2)$$

With $\alpha = 0.005$, game **A** shows an expectation of -0.01, game **B** an expectation of -0.0087, and the composite game **A** + **B** an expectation of $+0.0157$.

(While it may not be immediately apparent, games **A** and **B** are *not* independent, being linked through available capital; changes in capital, however, *may* change the probabilities of subsequent games.)

At each discrete-time step, either game **A** or game **B** will be played. The pattern that selects the particular game is defined as the switching strategy—which can be a random-selection process with prescribed probabilities for **A** and **B**, a periodic pattern, or even a chaotic pattern (following a random-number sequence). In general, the shorter the switching period, the larger the return.

With the values of P_1, P_2, and P_3 as specified, with $\alpha = 0$ (reducing **A** and **B** to fair games), and with $m = 3$, the random-selection strategy is optimized when game **A** is selected with probability 0.4145 and game **B** with probability 0.5854 (although it produces a positive expectation of only 0.026).

Chaotic switching yields a higher rate of winning than that achieved by random switching. However, periodic switching produces the highest rate of winning since such a strategy favors playing the higher-winning subset in game **B**.

The pattern **ABBAB** offers by far the best strategy for the same parameters: an expectation of 0.0756, exceeding the random-selection strategy by a factor of almost 3. By comparison, the period-3 pattern, **ABB**, yields an expectation of 0.068. For $m = 4$, the pattern **AB** is unsurpassed (Ref. Ekhad and Zeilberger).

Parrondo has run a computer simulation averaged over one million trials, playing game **A** a times and then switching to game **B** for b plays—a combination designated $[a,b]$. Figure 4-2 shows the results for several switching sequences.

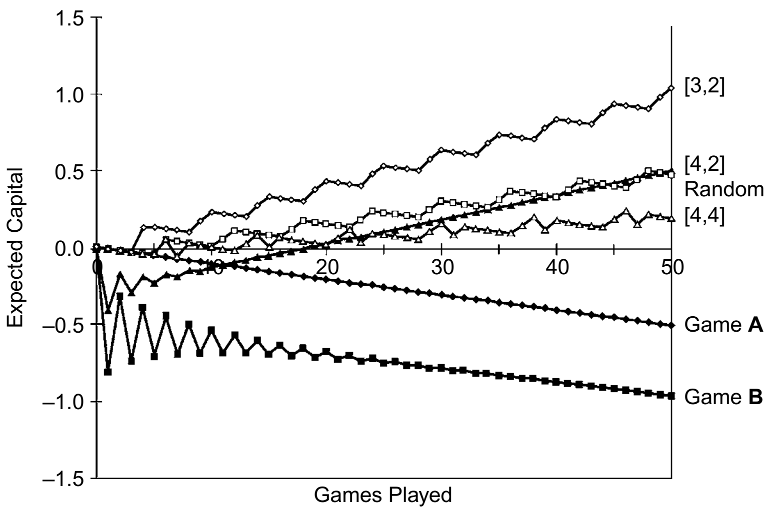

FIGURE 4-2 Capital-dependent games. Expected capital appreciation (depreciation) over 50 games ($\alpha = 0.005$).

Alternating between the games produces a ratchet-like effect. If we envision a hill whose slope is related to a coin's bias, then winning means moving uphill. In the single-coin game **A**, the slope is smooth; in the two-coin game **B**, the slope has a sawtooth profile. Alternating between **A** and **B** is akin to switching between smooth and sawtooth profiles—whatever gain occurs is trapped by the switch to the other game before subsequent repetitions of the original game can fulfill its negative expectation.

Etiology

The inspiration for Parrondo's principle arose from the phenomenon known as Brownian motion—a random motion of molecular particles that manifests itself as noise. In 1992, Ajdari and Prost (Ref.) conceptualized a microscopic engine that, in operation, pushes particles to the left, but, if turned on and off repeatedly, moves particles to the right. This engine, known as a "flashing ratchet," operates by exploiting the random (Brownian) motion of the particles.

Illustratively, consider a collection of electrically charged particles distributed randomly along a gradual slope. Assuming the particles to be in Brownian motion, they move left or right with equal probability, but with a slight drift down the slope. If we then superimpose along the slope an electric potential whose profile resembles the teeth of a saw, we will produce periodic energy peaks that rise steeply on the left side of the tooth and descend gradually along its right side. When the current is again switched off, the particles have a higher probability of crossing the peak of the potential to its immediate right than the one to its immediate left—since the troughs of the potential are offset to the right as the particles drift downward. With the current switched on a second time, many of the particles that had drifted left will be swept back into the trough from which they started. However, some of the particles that had drifted right will have traveled far enough to reach the adjacent trough. If the electric potential is switched on and off, or "flashed," at the appropriate frequency, the charged particles in the flashing ratchet will ascend the slope, apparently defying gravity.[3]

Since the motions of molecular particles are analogous to the randomness of a coin-tossing sequence, Parrando was able to translate the mechanics of the flashing ratchet into the context of game theory.

One startling result of Parrondo's principle suggests that a stock-market investor owning two "losing" stocks (i.e., stocks with declining prices) can—if he contrives to sell one stock during a brief uptick and shift those funds to another declining stock—overcome the general losing trend in both stocks. (Practical considerations—transaction fees, monotonically decreasing prices across the board—inhibit the operation of the Parrondo principle in this field.)

Other disciplines amenable to application of the Parrondo principle—extracting benefits from a detrimental situation—lie in the economic, genetic, sociological, and ecological realms. Certain biological processes may have long exploited the process to channel random forces toward the assembly of complex amino acids and thus create order out of disorder—and the development of complex life forms. Revolutionary changes in our understanding of the workings of nature may emerge.

Parrondo's Domain

In general, for Parrondo's principle to apply, three conditions need to be present:

1. An element of chance, as in Brownian motion, stock-market fluctuations, dice or coin throws
2. A ratchet-like asymmetry to the operating principles
3. Two basic dynamic processes that alternate regularly or randomly

[3]The flashing ratchet has been accused of violating the second law of thermodynamics by giving rise to more order (directed motion) than is fed into it. The defense brief holds that an outside source is here acting on the system—viz. the energy required to switch the potential on and off. Maxwell's demon was not called to testify.

R.D. Astumian (Ref.)[4] has created a simple game to illustrate Parrondo's principle. The playing field consists of five squares in a row, labeled as shown in Figure 4-3. Initially, a counter is placed on the START square and is then moved one square to the left or right in accordance with certain probabilistic rules.

Lose	Left	Start	Right	Win

FIGURE 4-3 A simple Astumian game.

Let λ be the probability of moving left from the START position and μ be the probability of moving left again (from the LEFT position). Finally, let ρ be the probability of moving right from the RIGHT position. These three numbers completely specify the game (designated by $[\lambda, \mu, \rho]$), which, it should be noted, is *symmetric* if and only if $\lambda = 1/2$ and $\rho = \mu$; the game is *fair* if and only if $\lambda\mu = (1 - \lambda)\rho$, which includes, but is not limited to, the symmetric case.

Consider two Astumian games, **A** and **B**, described by

$$\mathbf{A} = [1, p, 1] \text{ and } \mathbf{B} = [p, p, 0]$$

Both games, when played separately, are losing. In **A**, the initial move is leftward with certainty. In **B**, the counter can never reach the WIN square since $\rho = 0$. Yet by choosing a value of p sufficiently small, the composite game, **A** + **B** (each selected with probability 0.5), will have a winning probability arbitrarily close to 1.

After two moves of the **A** + **B** game, the counter will be in one of three positions (averaging over the four possible sequences **AA, AB, BA**, and **BB**):

$$\text{LOSE with probability } 0.5p(1 + p)$$

$$\text{WIN with probability } 0.25(1 - p)$$

$$\text{START with probability } 0.25(1 - p)(3 + 2p)$$

"Renormalizing" (to exclude those sequences that return to START), the probabilities for playing the game to WIN or LOSE are

$$\text{LOSE with probability } \frac{2p(1 + p)}{1 + p + 2p^2}$$

$$\text{WIN with probability } \frac{1 - p}{1 + p + 2p^2}$$

and, as noted, as $p \to 0$, the WIN probability approaches 1.

[4] The original game failed to satisfy condition 2. We present here the corrected version (Ref. Martin and von Baeyer).

If the **A** + **B** game is played by using one outcome of a die ($p = 1/6$), wins outnumber losses by 15 to 7. It remains a winning game for all $p < 0.25(\sqrt{17} - 3) = 0.281$.

Astumian games can also illustrate the Parrondo principle by combining two slow, losing games into a fast, winning one. Consider the two Astumian games described by

$$\mathbf{\Gamma} = [1 - p, p, 1 - 2p] \quad \text{and} \quad \mathbf{\Delta}_n = [p, p, p^n]$$

where n is a positive integer, and $0 < p < 1/2$. We can calculate the renormalized probabilities for game $\mathbf{\Gamma}$ as

$$\text{LOSE with probability} \frac{1 - p}{2 - 3p}$$

$$\text{WIN with probability} \frac{1 - 2p}{2 - 3p}$$

As p approaches 1/2, the probability of losing approaches 1.

For game Δ_n, the renormalized probabilities are

$$\text{LOSE with probability} \frac{p^2}{p^2 + p^n - p^{n+1}}$$

$$\text{WIN with probability} \frac{p^n - p^{n+1}}{p^2 + p^n - p^{n+1}}$$

As n increases, the probability of losing approaches 1 independent of p.

For the randomly combined game, $\mathbf{\Gamma} + \Delta_n$, the renormalized probabilities of losing and winning are

$$\text{LOSE with probability} \frac{2p}{1 + p^n}$$

$$\text{WIN with probability} \frac{1 - 2p + p^n}{1 + p^n}$$

With large values of n, the probability of losing in this combined game is less than 1/2 for $p < 0.25$. Ergo, Parrondo's principle pertains for $\mathbf{\Gamma} + \mathbf{\Delta}_n$.

Both $\mathbf{\Gamma}$ and $\mathbf{\Delta}_n$ are games that are slow to resolve themselves into either LOSE or WIN outcomes. In game $\mathbf{\Gamma}$, for $n = 2$ and $p = 0.01$, the counter will be in the START square 98.03% of the time:

$$[\mathbf{\Gamma}] \text{ START with probability } = 1 - 2p + 3p^2 = 0.9803$$

And, for game $\mathbf{\Delta}_n$ with the same parameters, the counter will be in the START square 99.98% of the time:

$$[\mathbf{\Delta}_n] \text{ START with probability } = 1 - p^2 - p^n + p^{n+1} = 0.9998$$

In the composite game $\mathbf{\Gamma} + \mathbf{\Delta}_n$, the counter remains in the START square less than 75% of the time:

$$[\mathbf{\Gamma} + \mathbf{\Delta}_n] \text{ START with probability } = \frac{3 - p^n}{4} = 0.749975$$

After ten moves, the composite game will reach a resolution more than 76% of the time, whereas game $\mathbf{\Gamma}$, the faster of the two individual games, will resolve itself less than 10% of the time.

Astumian games, although simple in format, nonetheless reflect the fundamental concept of how randomness may be converted to directed motion.

MULTIPLAYER GAMES

A corollary to the Parrondo principle states that while random or periodic alternations of games **A** and **B** are winning, a strategy that chooses that game with the higher average return is losing (Ref. Dinis and Parrondo, 2002). Again, this principle runs counterintuitively.

Consider a large number N of players. At each turn of the game, a subset of ϕN players ($0 < \phi < 1$), each with a known capital, selects either **A** or **B**—as defined initially—with the goal of maximizing expected average earnings. (The remaining $(1 - \phi)$N players are inactive during this turn.) Let $p_0(t)$ equal the fraction of players whose capital is a multiple of 3 at turn t. Then the average fractions of winning players in games **A** and **B**, respectively, are

$$P_A = P_1$$

$$P_B = p_0(t)\,P_2 + [1 - p_0(t)]P_3$$

Following the stated maximization goal dictates the strategy:

$$\text{play } \mathbf{A} \text{ if } p_0(t) \geq (P_1 - P_3)/(P_2 - P_3) = \frac{5}{13}$$

$$\text{play } \mathbf{B} \text{ if } p_0(t) < \frac{5}{13}$$

Note that if game **A** is played continually, $p_0(t)$ tends to 1/3 since the capital F(t) constitutes a symmetric and homogeneous random walk. On the other hand, if game **B** is played continually, $p_0(t)$ tends to 5/13—and, insofar as $p_0(t)$ does not exceed 5/13, the maximization strategy continues to choose **B.** However, though this choice, by definition, maximizes the gain at each turn, it drives $p_0(t)$ toward 5/13—i.e., toward values of $p_0(t)$ where the gain is small.

As an example, for $\phi = 1/2$, $\alpha = 0$, and $p_0(0) < 5/13$, the maximization strategy chooses **B** forever—and the average capital barely appreciates.

Both the random and the periodic strategies choose game **A** even for $p_0 < 5/13$. While these strategies will not produce earnings at each specific turn, they will drive p_0 away from 5/13, whence the average capital will increase at a rate faster than that for the maximization (or "greedy") strategy.

These concepts are illustrated in Figure 4-4, showing the appreciation (or depreciation) of average capital for $\phi = 0.675$ and $\alpha = 0.005$.

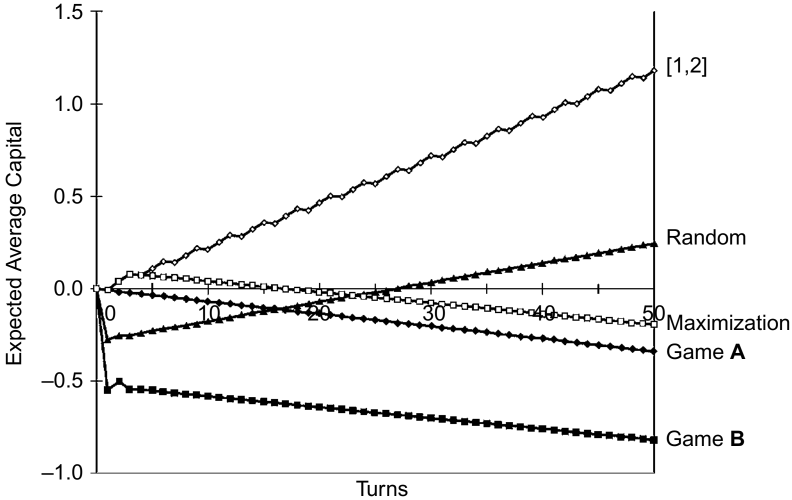

FIGURE 4-4 Multiplayer games. Expected appreciation (depreciation) of average capital over 50 turns ($\gamma = 0.675$, $\alpha = 0.005$).

With these values of ϕ and α, **B** is a losing game, and since the maximization strategy here dictates playing **B** forever, the average capital decreases. Part of the "paradox" herein stems from the fact that neither the random nor the periodic strategy involves any information about the state of the system—whereas the strategy of choosing the game with the higher average return obviously does so. Dinis and Parrondo (Ref. 2003) describe the maximization strategy as "killing the goose that laid the golden eggs." To increase earnings above this strategy, short-term profits must be sacrificed for higher future returns.

HISTORY-DEPENDENT PARRONDO GAMES

Parrondo's original concept (derived from capital-dependent games) has since spawned history-dependent games and cooperative games (multiplayer rather than two opposing interests). The former encompasses those games whose outcomes are a function of previous game-states.[5]

[5]Capital-dependent games under modulo-arithmetic rules are not consonant with many processes such as biology and biophysics.

Table 4-1 The Parrondo History-Dependent Game

State at Time (t – 2)	State at Time (t – 1)	Coin Used at Time t	Probability of Winning at Time t	Probability of Losing at Time t
Lose	Lose	C_1	P_1	$(1 - P_1)$
Lose	Win	C_2	P_2	$(1 - P_2)$
Win	Lose	C_3	P_3	$(1 - P_3)$
Win	Win	C_4	P_4	$(1 - P_4)$

Again, the underlying desideratum is "losing + losing = winning." Game **A** is defined, as before, by the probability $P = 1/2 - \alpha$ of increasing the capital F(t). Game **B′** is now defined by the probabilities of four biased coins, where the specific coin used at a given time t depends on the outcomes of the two immediately preceding time steps ("the game history"), as illustrated in Table 4-1. Or, expressed in diagrammatic form—Figure 4-5.

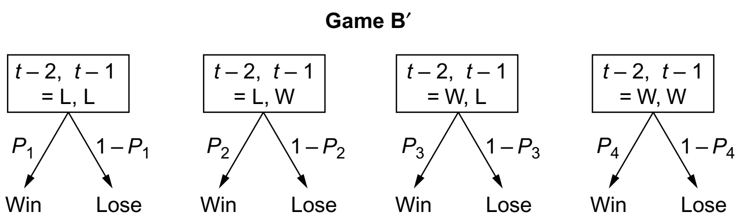

FIGURE 4-5 The basic history-dependent game.

Probabilities for game **B′** are chosen, as before, to maintain rational numbers throughout the analysis:

$$P_1 = x - \alpha$$

$$P_2 = P_3 = 1/4 - \alpha \qquad (4\text{-}1)$$

$$P_4 = 8/5 - x - \alpha$$

The restriction $P_2 = P_3$ is effected to enable the different regions of the parameter space to be constructed as a three-dimensional figure.

Following Parrondo, we adopt the value $x = 9/10$ ($P_1 = 9/10 - \alpha$, $P_4 = 7/10 - \alpha$). Figure 4-6 shows the appreciation (depreciation) of capital F(t) with these probabilities and with $\alpha = 0.003$ (averaged over the four initial conditions). We have assumed that the player tosses a fair coin twice, thereby establishing a "history," but that the results of these tosses are not figured into the capital appreciation.

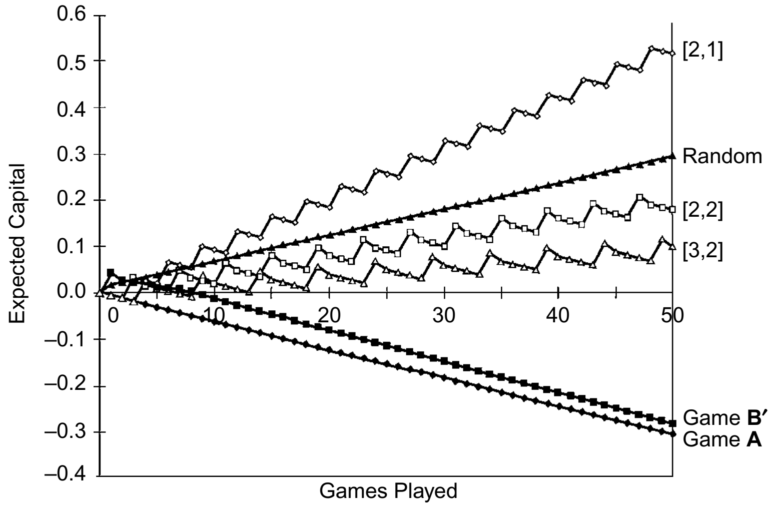

FIGURE 4-6 History-dependent games. Expected capital appreciation (depreciation) over 50 games ($\alpha = 0.003$).

Similar to the analysis for capital-dependent games, history-dependent games can be represented as a discrete-time Markov chain with four states and a transition matrix. Figure 4-7 illustrates the Markov chain formed by the vector

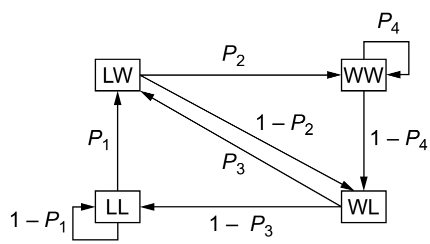

FIGURE 4-7 The discrete-time Markov chain for game **B′**.

$G'(n) = [F(n-1) - F(n-2), F(n) - F(n-1)]$, whose four states are $[-1, -1]$, $[-1, +1]$, $[+1, -1]$, and $[+1, +1]$, where $+1$ indicates a win and -1 a loss. The corresponding transition matrix is

$$\mathbf{II} = \begin{vmatrix} 1-P_1 & P_1 & 0 & 0 \\ 0 & 0 & 1-P_2 & P_2 \\ 1-P_3 & P_3 & 0 & 0 \\ 0 & 0 & 1-P_4 & P_4 \end{vmatrix}$$

where columns and rows represent the four states LL, LW, WL, and WW in succession.

The stationary probabilities for this Markov chain can be computed as

$$\mathbf{P} = \frac{1}{Z}\,[(1-P_3)(1-P_4),\ P_1(1-P_4),\ P_1(1-P_4), P_1P_2] \tag{4-2}$$

where $Z = P_1P_2 + (1 + 2P_1 - P_3)(1 - P_4)$ is a normalization constant. With the probabilities of Eq. 4-1, and with $x = 9/10$ and $\alpha = 0$,

$$\mathbf{P} = (1/22)\,[5,\ 6,\ 6,\ 5]$$

which yields a probability of winning for game $\mathbf{B'}$,

$$P_{win} = (5/22)(9/10) + 2(6/22)(1/4) + (5/22)(7/10) = 1/2$$

as expected for $\mathbf{B'}$ with the bias removed (a fair game).

With the stationary probabilities of Eq. 4-2, we can derive

$$P_{win} = \Sigma \Pi_i P_i = P_1(1 + P_2 - P_4)/Z$$

Setting $P_{win} < 1/2$ for games $\mathbf{A}$ and $\mathbf{B'}$ (losing) and $P_{win} > 1/2$ for the composite randomized game, $\mathbf{A} + \mathbf{B'}$ (winning), we have the following constraints:

$$\begin{aligned}
&\frac{1-P}{P} > 1 && \text{[for game } \mathbf{A} \text{ to lose]}\\
&\frac{(1-P_3)(1-P_4)}{P_1P_2} > 1 && \text{[for game } \mathbf{B'} \text{ to lose]}\\
&\frac{(2-P_3-P)(2-P_4-P)}{(P_1+P)(P_2+P)} < 1 && \text{[for game } \mathbf{A}+\mathbf{B'} \text{ to win]}
\end{aligned} \tag{4-3}$$

where, in the second condition of Eq. 4-3, P_i is replaced by $(P_i + P)/2$, and the inequality reversed to obtain the third condition—viz., that $\mathbf{A} + \mathbf{B'}$ is winning instead of losing. With the probabilities of Eq. 4-1 (and $x = 9/10$), the first two conditions are satisfied for $\alpha > 0$, while the third condition—and thus Parrondo's principle—applies in the range $0 < \alpha < 1/168$.

A logical generalization of these results is the situation wherein *both* games to be combined are history-dependent (Ref. Kay and Johnson). Thus we define game $\mathbf{B'}$ as before and game $\mathbf{B''}$ similar to $\mathbf{B'}$, but with Q_i replacing P_i for the probability of winning at time t with coin C'_i. Analogous to the second condition of Eq. 4-3, we have

$$\begin{aligned}
&\frac{(1-P_4)(1-P_3)}{P_1P_2} > 1\\
&\frac{(1-Q_4)(1-Q_3)}{Q_1Q_2} > 1
\end{aligned} \tag{4-4}$$

for $\mathbf{B'}$ and $\mathbf{B''}$ to be losing. For the composite game $\mathbf{B'} + \mathbf{B''}$, we define R_i as the probability of winning at time t. Then, for $\mathbf{B'} + \mathbf{B''}$ to be winning, we must have

$$\frac{(1-R_4)(1-R_3)}{R_1R_2} < 1$$

Randomly mixing the two games (with equal probability), this latter condition becomes

$$\frac{(2 - P_4 - Q_4)(2 - P_3 - Q_3)}{(P_1 + Q_1)(P_2 + Q_2)} < 1$$

And, since $\mathbf{B}'$ and $\mathbf{B}''$ must be fair for $\alpha = 0$, Eqs 4-4 with equality holding can be substituted for P_1 and Q_1. Therefore,

$$\frac{(2 - P_4 - Q_4)(2 - P_3 - Q_3)}{(P_2 + Q_2)\left[(1 - P_4)(1 - P_3)/P_2 + (1 - Q_4)(1 - Q_3)/Q_2\right]} < 1 \qquad (4\text{-}5)$$

for the combined game to be winning.

Using Parrondo's probabilities, Eqs 4-1, with the added restriction that $Q_2 = Q_3$, the condition of Eq. 4-5 becomes

$$Q_4 = \frac{(P_4 - 1)Q_2}{P_2} \gtrless 1 \quad \text{as } Q_2 \gtrless P_2$$

Without the restrictions $P_2 = P_3$ and $Q_2 = Q_3$, the more general condition for a winning composite game, $\mathbf{B}' + \mathbf{B}''$, becomes

$$Q_4 = \frac{(P_4 - 1)Q_2}{P_2} \gtrless 1 \quad \text{as } Q_3 \gtrless \frac{(P_3 - 1)Q_2}{P_2}$$

Periodic combinations of history-dependent games (as well as random combinations) have been investigated by Kay and Johnson (Ref). As with capital-dependent games, capital appreciation is greater with more rapid switching rates.

The parameter space for history-dependent games (the region where the combined game is winning) is substantially greater than for capital-dependent games—by a factor of about 55 (varying somewhat with the modulus and the playing frequency of each game). However, history-dependent games experience a lower rate of return.

Quantum games, spin systems, biogenesis, molecular transport, stock-market modeling, and noise-induced synchronization are but some of the many areas where these concepts have been applied. In general, it would seem that the phenomena amenable to analysis through history-dependent games are more extensive than those of capital-dependent games.

All-History Games

As well as the case where the specific coin played at time t depends on the outcome of the two immediately preceding time steps, we can generalize to any number of preceding time steps. That is, we consider a Markov process of order n that determines a new state based on the previous n states.

In some history-dependent games, there may be no (finite) limit on the number of previous states that can influence the new state. [A prominent example in game theory concerns computations of Grundy functions, wherein each function g(x) is dependent on g(0), g(1), ..., g($x - 1$).]

In determining the state at time $t - r$, it is *not necessary to randomize the behavior of* $t - r$ *as a function of* r, since that state is already different whenever r is increased by 1—thereby doubling the size of the truth table[6] and introducing a proportionate number of new coins.

The all-history–dependent game extends the columns of Table 4-1 backward to $t - r$ and downward to encompass 2^r terms and 2^r coins engaged at time t and as many linear probabilities (P_i) as coins, as shown in Table 4-2 for $r = 4$.

Table 4-2 The All-History–Dependent Game

State at Time ($t-4$)	State at Time ($t-3$)	State at Time ($t-2$)	State at Time ($t-1$)	Coin Used at Time t	Probability of Winning at Time t	Probability of Losing at Time t
Lose	Lose	Lose	Lose	C_1	P_1	$(1-P_1)$
Lose	Lose	Lose	Win	C_2	P_2	$(1-P_2)$
Lose	Lose	Win	Lose	C_3	P_3	$(1-P_3)$
Lose	Lose	Win	Win	C_4	P_4	$(1-P_4)$
Lose	Win	Lose	Lose	C_5	P_5	$(1-P_5)$
Lose	Win	Lose	Win	C_6	P_6	$(1-P_6)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Win	Win	Win	Lose	C_{15}	P_{15}	$(1-P_{15})$
Win	Win	Win	Win	C_{16}	P_{16}	$(1-P_{16})$

The array of Wins and Losses in Table 4-2 is isomorphic to a truth table with WIN and LOSE replacing the binary digits 0 and 1. Further, instead of the usual situation wherein multiple input states produce a single output state, Parrondo games entail binary input variables that are still binary, but outputs that are probabilities. This circumstance increases the number of possible Parrondo-type games hyperexponentially—viz., 2^{2^r}—depending on which subset of the 2^r coins shows Heads.

In the example of capital-dependent games, each input state determines a coin, and that coin, rather than yielding Win or Lose directly, instead offers a *probability* of Winning or Losing.

Finally, it should be noted that the process described here is ergodic—that is, instead of using r units of time separately, we can achieve the same result by tossing r coins simultaneously.

[6] A truth table, in Boolean logic (or in switching theory), consists of r binary input states, each of which yields one (deterministic) binary output and 2^{2^r} different truth tables.

Negative History

The question can be posed as to whether it is meaningful to consider that the coin used at time t can depend on future states $t + 1$ and $t + 2$.

Illustratively, if we express the Fibonacci recursion in terms of future events, we have

$$f(n) = f(n+2) - f(n+1)$$

But the same values of f(n), as a difference of two events, can occur in infinitely many ways. So without precognition *and* without knowledge of both the recursion and two consecutive future values of f(·), it is not possible to reverse the process back to the present.

QUANTUM PARRONDO GAMES

A protocol has been developed for a quantum version of history-dependent Parrondo games (Ref. Flitney, Ng, and Abbott). The probabilistic element is replaced by a superposition that represents all possible results in parallel. Game **A** becomes an arbitrary SU(2) operation on a qubit. Game **B** consists of four SU(2) operations controlled by the results of the previous two games. Superposition of qubits can be used to couple **A** and **B** and produce interference leading to payoffs quite different from those of the classical case. For example, in the conventional Parrondo game, the sequence **AAB** yields an effective payoff of $1/60 - 28\varepsilon/15$ (positive for $\varepsilon < 1/112$). In the quantum version with the same sequence, a higher payoff results for all values of $\varepsilon > 1/120$.

COOPERATIVE GAMES

Inspired, in turn, by history-dependent games, another family of Parrondo games, introduced by Raúl Toral (Ref. 2002) entails an ensemble of N players. At time t, one of the N players is selected at random. That player then chooses either game **B** or game **A′** with probability 1/2, where the latter game consists of transferring one unit of his capital to another randomly selected player (from the remaining $N - 1$). By definition, **A′** is a fair game (relative to the ensemble) since total capital is not changed but merely redistributed among the players. When game **B** is selected, a player risks his own capital (not that of the ensemble) to determine which coin to toss.

Figure 4-8 delineates the average capital per player versus time for the games **A′** and **B** and the randomly mixed composite game **A′** + **B**, where **B** is defined here by $P_2 = 1/10 - \alpha$, $P_3 = 3/4 - \alpha$, and $\alpha = 0.01$. (Time is measured in units of games per player—that is, at time t, each of the N players has wagered t times on average for a total of Nt individual games. Here, $N = 200$.)

This (again counterintuitive) result proves that the redistribution of capital can turn a losing game into a winning one, increasing the capital of the full ensemble

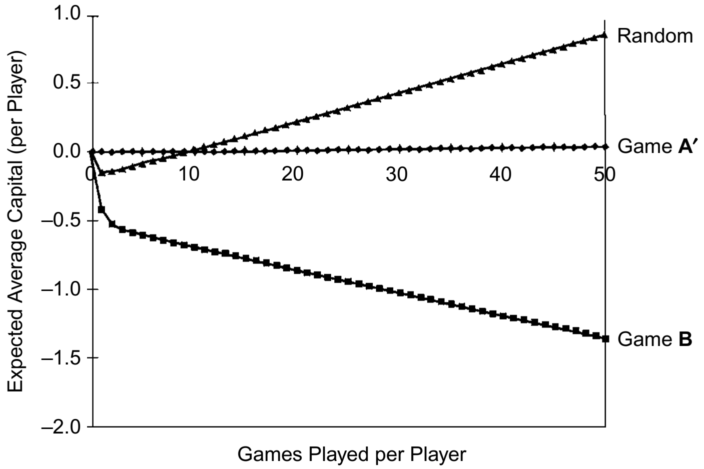

FIGURE 4-8 Cooperative games with capital dependence. Expected average capital appreciation (depreciation) per player over 10,000 games (200 players, $\alpha = 0.01$).

of players. In fact, the average return from playing **A′** + **B** is almost twice that from playing **A** + **B** with the same parameters (since game **A′**, wherein the capital of two players is changed by one unit, is equivalent to two games of **A**).

Replacing capital-dependent game **B** with history-dependent game **B′** (a losing game unto itself, defined by the probabilities of Eq. 4-1 and with $\alpha = 0.01$), we again observe the Parrondo principle for the composite game **A′** + **B′**, as per Figure 4-9.

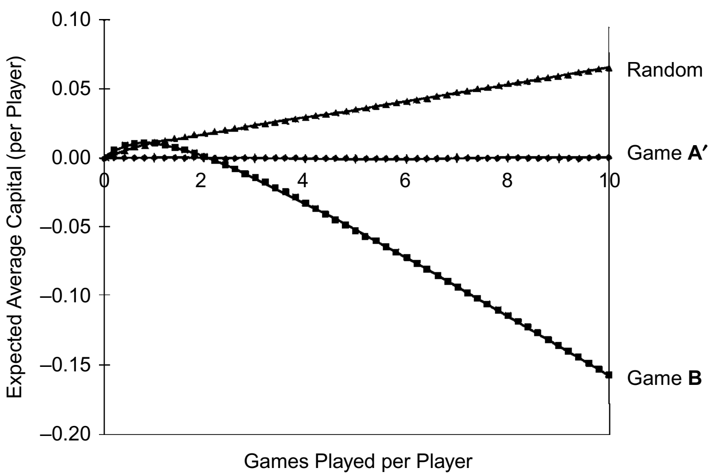

FIGURE 4-9 Cooperative games with history dependence. Expected average capital appreciation (depreciation) per player over 2000 games (200 players, $\alpha = 0.01$).

To illustrate a further reach of the Parrondo principle, we assume that the players are arrayed (conventionally, in a ring) so that each layer has two immediate neighbors. Denoting player i's capital after game t by $F_i(t)$, the total capital of the group is $\Sigma F_i(t)$. We can replace game **A′** by game **A″**, wherein the randomly selected player i transfers one unit of his capital to either of his immediate neighbors with a probability proportional to the difference in capital between player i and his neighbor. That is,

$$\text{Prob.}(i \rightarrow i \pm 1) \propto \max[F_i - F_{i\pm1}, 0]$$

with $\text{Prob.}(i \rightarrow i+1) + \text{Prob.}(i \rightarrow i-1) = 1$ (and with $F_0 = F_{200}$ and $F_{201} = F_1$). If i is the poorest of the three players, no transfer takes place, since the redistribution of capital in **A″** always flows from richer to poorer players. The combined game **A″** + **B** is illustrated in Figure 4-10.

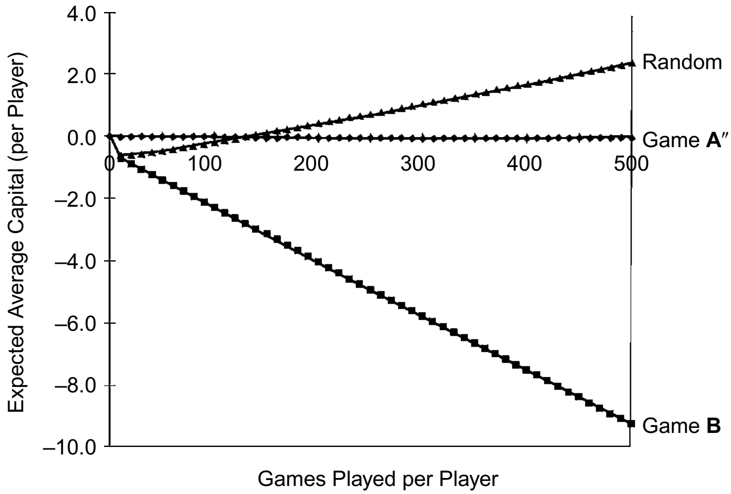

FIGURE 4-10 Cooperative games with redistribution from richer to poorer neighbors. Expected average capital appreciation (depreciation) per player over 100,000 games (200 players, $\alpha = 0.01$).

In each of these randomly mixed games, the expected average capital per player increases linearly with the number of games per player after a short initial nonlinear period. In each case, we can note that the redistribution of capital (without altering the total capital per se) actually increases the total capital available when combined with other losing games—ergo, redistribution of capital as defined here is beneficial for all N players.

In another version of this cooperative game (Ref. Toral, 2001), the N players compete, in turn, against a banker who collects or pays out the wager. Here, in game **B″**, the probability of the ith player winning at time t depends upon the state of his two neighbors, players $i+1$ and $i-1$. Specifically, this probability is given by

P_1, if $i+1$ and $i-1$ are both losers
P_2, if $i+1$ is a winner and $i-1$ is a loser
P_3, if $i+1$ is a loser and $i-1$ is a winner
P_4, if $i+1$ and $i-1$ are both winners

Whether a player is a winner or loser depends solely on the result of his most recent game. In game **A**, the selected player wins with probability P. Computer simulation has shown that there are sets of these probabilities wherein the alternate playing of **A** and **B″** yields positive-expectation results, although **A** and **B″** separately are losing or fair games (as per the Parrondo principle). For example, with game **A** (fair) defined by $P = 0.5$, and game **B″** (losing) defined by $P_1 = 1.0$, $P_2 = P_3 = 0.16$, and $P_4 = 0.7$, the average capital per player versus t for the composite game **A** + **B″** is illustrated in Figure 4-11. To initialize this

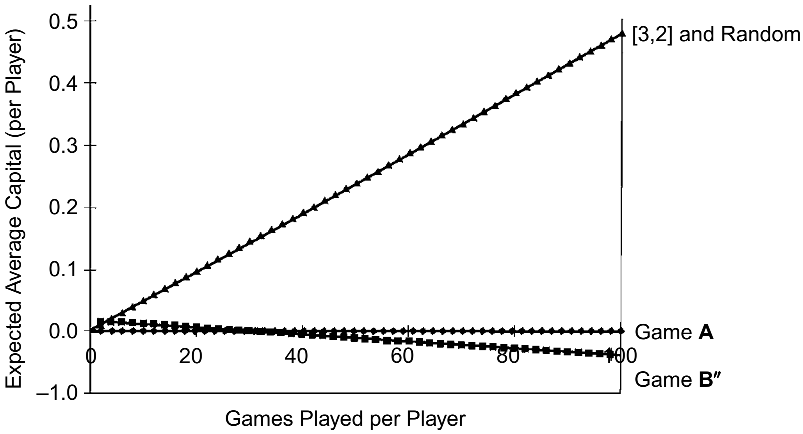

FIGURE 4-11 Cooperative games with neighbor dependence. Expected average capital appreciation (depreciation) per player over 20,000 games (200 players) (*solid* triangles are hidden behind the open triangles).

game, we assume that each player tosses a fair coin to determine his status as a winner or loser. Random mixing and alternating with an **AAB″B″** pattern yield virtually identical results.

Cooperative games introduce the novel concept that combining two losing (or nonwinning) games to produce a winning game results from interactions among the players.

THE ALLISON MIXTURE

Another form of Parrondo's principle, designated the Allison mixture, is formulated as follows.

Consider two sequences of binary random numbers, S_1 and S_2, with respective means μ_1 and μ_2. (By definition, each sequence has a zero autocorrelation.) We now generate a third sequence by randomly scrambling the first two. For this purpose, we use two biased coins: C_1, tossed when we are in S_1 and which indicates "shift to S_2" with probability p_1 and "don't shift" with probability $1 - p_1$; and C_2, tossed when we are in S_2 and which indicates "shift to S_1" with probability p_2 and don't shift with probability $1 - p_2$.

Specifically, we begin (arbitrarily) at position n in S_1 and then toss coin C_1 to determine either a move to the $n + 1$ position in S_2 *or* to the $n + 1$ position in S_1. When in S_2, we toss coin C_2 to determine similar actions: to shift back to S_1 or move on to the next position in S_2, and so forth.

It has been proved (Ref. Allison, Pearce, and Abbott) that the autocorrelation ρ for the new sequence is

$$\rho = \frac{p_1 p_2 (\mu_1 - \mu_2)^2 (1 - p_1 - p_2)}{(\mu_2 p_1 + \mu_1 p_2)[p_2(1 - \mu_1) + p_1(1 - \mu_2)]}$$

Thus, a random mixing of two random sequences has produced, counterintuitively, a sequence with less randomness—presuming that the two means *and* the two sequence-shifting probabilities are not equal.

Random sequences S_1 and S_2 contain maximal information in the Shannon sense; when $\mu_1 \neq \mu_2$, the back-and-forth shifting process is irreversible, the generated sequence contains redundancy, and we therefore lose information (i.e., $\rho \neq 0$). Further, when $p_1 + p_2 \neq 1$ (biased coins), switching between S_1 and S_2 leads to memory persistence and concomitant correlation in the generated sequence.

Both the Allison mixture and the Parrondo effect require an asymmetry to interact with random phenomena. (The Allison mixture is also valid for sequences other than binary.)

Finally, it should be noted that, for all the various games discussed herein, Parrondo's principle can be reversed to effect "winning + winning = losing." The nature of such a reversal renders it a matter of primarily academic interest.

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Chapter | five

Coins, Wheels, and Oddments

Coin matching and finger flashing were among the first formal games to arise in the history of gambling. The class of Morra games extends back to the pre-Christian era, although not until comparatively recent times have game-theoretic solutions been derived. While records do not reveal the nature of ancient coin games, even the first hand-hewn and noticeably unsymmetric metal currency was fashioned with its two sides distinguishable. With a form of Heads and Tails presenting itself in the medium of exchange, only a small degree of sophistication is required to propose a simple class of wagers.

BIASED COINS

Although coin-tossing is regarded as the epitome of fairness in deciding between two (equal) alternatives, there can be subtle factors that alter its equitableness. The Lincoln penny (Lincoln's head = Heads; the Lincoln Memorial = Tails), as one example, is markedly imbalanced. When the coin is spun about its vertical axis, it will settle with Tails upward about 75% of the time (Ref. Diaconis, Holmes, and Montgomery).

Possible bias in a coin is examined by the Tchebychev inequality in the form

$$P[|p' - p| > \varepsilon] \leq \frac{pq}{\varepsilon^2 n} \tag{5-1}$$

(cf. Eq. 2-25). Equation 5-1 expresses the probability P that the success ratio p' taken over n trials differs from the single-trial probability p by more than any small amount $\varepsilon > 0$. This probability can be made arbitrarily small by adjusting ε and n, and with any stated accuracy we can calculate the values of stochastic convergence of the probabilities of Heads and Tails.

An interesting application of Bayes's theorem (Eq. 2-7) arises in connection with a possibly biased coin. Consider, for example, that 50 throws of a coin have produced only five Heads. If the coin were true, we know the probability of such an occurrence to be

$$p = \binom{50}{5}(1/2)^5(1/2)^{45} = 1.88\times 10^{-9}$$

Let P_b be the extent of our *a priori* knowledge that the coin may be biased and P_t be the complementary probability that the coin is true. If the coin is indeed biased in favor of Tails, the event of five Heads in 50 trials has occurred with some probability P_1. Then, according to Bayes's theorem, the probability P_e that the coin is biased in view of the experimental evidence is

$$P_e = \frac{P_1}{P_1 + (P_t/P_b)(1.88\times 10^{-9})}$$

If the coin were selected from a conventional sample and if our suspicion as to its bias were quite small (P_t/P_b quite large), then P_e (which is greater than P_b) corrects our estimate of the coin. If, initially, we had suspected the coin with greater conviction, then P_e approaches unity, and the evidence implies near-certainty of bias. Postulating that the coin is biased toward Tails by a 9:1 ratio, we can compute P_1 as

$$P_1 = \binom{50}{5}(0.1)^5(0.9)^{45} = 0.185$$

A more pragmatic interpretation of the problem might state "in the neighborhood of" rather than "exactly" five Heads in 50 trials. In this case, the probability P_e of a biased coin is correspondingly greater.

When a coin used to generate a Head–Tail sequence is under suspicion of entertaining a bias, a logical betting procedure consists of wagering that the immediately preceding outcome will be repeated (cf. the Epaminondas system suggested in Chapter 3). If p and q represent the true probabilities of obtaining Heads and Tails, respectively, this system yields a single-trial winning probability of $p^2 + q^2$. The mathematical expectation of gain is, accordingly, $(2p - 1)^2$.

The question of the behavior of a typical coin was at one time of sufficient interest to generate some rather tedious experiments. The naturalist Buffon performed 4040 throws of a coin that resulted in 2048 Heads and 1992 Tails; the deviation of 28 from a perfect balance falls within one standard deviation of $\sqrt{pqn} = \sqrt{4040/4} = 31.8$, assuming probabilities of $p = q = 1/2$.

J.E. Kerrich, while interned during World War II, recorded ten series of coin throws, each consisting of 1000 throws. No series produced less than 476 or more than 529 Heads. The coin-tossing record is probably held by the statistician Karl Pearson who, in 24,000 tosses, obtained 12,012 Heads. In view of these dreary experiments and other empirical evidence, and in light of modern numismatic techniques, we can assume that a coin selected at random from those in circulation has a vanishingly small probability of significant bias.

J.B. Keller (Ref.) has analyzed the mechanics of coin tossing as governed by the laws of physics and the initial conditions: orientation, initial upward velocity u, and initial upward angular velocity ω. He has shown that

1. With random initial conditions, the Heads-or-Tails outcome is random—albeit influenced by minute changes in the initial conditions.
2. Nonetheless, a definite probability is associated with each outcome. Specifically, the probabilities of Heads, P_H, and Tails, P_T, devolve approximately from any continuous probability density, $p(u, \omega)$, of the initial values. This approximation improves as $p(u, \omega)$ is translated to sufficiently large values of u and ω.
3. To calculate the probability for each outcome, let H (T) represent the set of all pairs (u, ω) of nonnegative values for which the coin ends up Heads (Tails). P_H and P_T converge to a limit because the sets H and T consist of strips that become very narrow at infinity. Thus H and T occupy fixed fractions of the area of any disk that is shifted to infinity. These fractions give the limiting values of P_H and P_T.

(Typical coin tosses were observed under stroboscopic illumination. It was found that $\omega \sim 38$ revolutions/second—or about 19 revolutions per toss.)

Often an issue is resolved by tossing a coin "two out of three" or "three out of five," and so on—in general n out of $2n - 1$ ("majority rule"). If the coin has single-trial probability $p > 0.5$ of Heads, the probability $\mathrm{P}<\mathrm{H}>$ that Heads prevails in $2n - 1$ tosses is given by

$$\mathrm{P}<\mathrm{H}> = \mathrm{p}^n \left[1 + \sum_{i=1}^{n-1} \binom{n}{i} 2^{i-1} \mathrm{q}^i\right], \qquad n \geq 2$$

With $p = 0.51$, the probability that Heads wins three out of five is, accordingly,

$$(0.51)^3[1+3\times0.49+2\times3(0.49)^2] = 0.519$$

Thus a slight bias does not significantly alter the outcome in a short sequence of tosses. We can add the truism that, with an unbiased coin, tossing the coin once or invoking a majority-rule sequence are equivalent choices.

John von Neumann pointed out that even a biased coin can be used to obtain a fair result since the probability of HT remains equal to the probability of TH. He offered the following procedure:

1. Toss the coin twice.
2. If HH or TT results, start over.
3. If HT or TH results, use the first toss to decide between two alternatives.

STATISTICAL PROPERTIES OF COINS

Probability Distribution of Heads and Tails

As an historical aside, we note that Roberval (a mathematician contemporary to Pascal and Fermat) argued for the probability of obtaining at least one Heads in two throws of an unbiased symmetric coin to be 2/3 (instead of 3/4). Rather than considering four cases (HH, HT, TH, and TT), Roberval admitted only three (H, TH, and TT), since, he maintained, if H occurs on the first trial, it is unnecessary to throw the coin again. D'Alembert advanced similar arguments at a later date, claiming that in two throws of a coin, the three possible outcomes are two Heads, one Heads and one Tails, and two Tails. This type of "reasoning" illustrates a weakness of the initial formulation of probability theory.

For n throws of a coin, the probability distribution of the number of Heads and Tails is described by the binomial expression, Eq. 2-1. The binomial coefficients are represented in tabular form by Pascal's triangle, Figure 5-1. In the format shown, each a_b term indicates that in n throws of a fair coin, a gambler

$$
\begin{array}{c}
1_1 \quad 1_{-1} \\
1_2 \quad 2_0 \quad 1_{-2} \\
1_3 \quad 3_1 \quad 3_{-1} \quad 1_{-3} \\
1_4 \quad 4_2 \quad 6_0 \quad 4_{-2} \quad 1_{-4} \\
1_5 \quad 5_3 \quad 10_1 \quad 10_{-1} \quad 5_{-3} \quad 1_{-5} \\
1_6 \quad 6_4 \quad 15_2 \quad 20_0 \quad 15_{-2} \quad 6_{-4} \quad 1_{-6} \\
1_7 \quad 7_5 \quad 21_3 \quad 35_1 \quad 35_{-1} \quad 21_{-3} \quad 7_{-5} \quad 1_{-7} \\
1_8 \quad 8_6 \quad 28_4 \quad 56_2 \quad 70_0 \quad 56_{-2} \quad 28_{-4} \quad 8_{-6} \quad 1_{-8} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

FIGURE 5-1 Pascal's triangle.

betting one unit on Heads can expect a gain of b units a times out of the 2^n possible outcomes. Illustratively, for $n = 5$ throws, out of every 32 such trials the gambler can expect to win 5 units once, 3 units five times, and 1 unit ten times, and to lose 1 unit ten times, 3 units five times, and 5 units once. Pascal's triangle is readily extended by noting that each a_b term is the sum of the two terms above it.

Runs of Heads and Tails

In addition to the probability distribution of the number of Heads (Tails) in n throws of a coin, the statistical properties of a random sequence of Heads (Tails) also embrace the number of runs of various lengths. By definition, a run of length m comprises a sequence of m consecutive Heads (Tails) immediately preceded and followed by the occurrence of Tails (Heads). We are interested both in the probability $P_n^m(k)$ of observing a specified number k of runs of length m and in $\bar{k}$, the expected value of k (the mean number of runs):

$$\bar{k} = \mathrm{E}(k) = \sum_{k=0}^{\infty} k P_n^m(k) \tag{5-2}$$

(cf. Eq. 2-15). The summation consists of a finite number of terms, since $P_n^m(k) = 0$ for $k > (n-2)/m$.

The probability that a sequence commences with a run of m Heads or Tails (bookended by Tails or Heads, respectively) is clearly $q^2p^m + p^2q^m$, where p represents the probability of throwing Heads at each trial, and $q = 1 - p$ the probability of throwing Tails. There are $n - m - 1$ ways of positioning the run of m Heads or Tails within the entire sequence of length n, and the events of a sequence containing a run of m Heads or Tails beginning at each of the $n - m - 1$ positions are not mutually exclusive. Rather, the probability of each such event is weighted according to the number of runs occurring in the sequence. This weighting conforms to that of Eq. 5-2, which can then be written explicitly as

$$\bar{k} = (n - m - 1)(p^2 q^m + q^2 p^m) \tag{5-3}$$

As the number of throws becomes large, the number of runs of length m becomes approximately normally distributed. For $n \gg m$ and $p = q = 1/2$, we can write

$$\bar{k} \simeq \frac{n}{2^{m+1}}$$

as the expected number of runs of length m in a sequence of n throws of a fair coin.

Arbitrarily accurate determination of the probability $P_n^m(k)$ can be achieved by computing its higher moments. First, we derive the expression for the variance σ_m^2 of $P_n^m(k)$. We observe that the probability for a sequence to include two distinct runs of length m at two specified positions in the sequence is $(p^2q^m + q^2p^m)^2$. Further, for $n \gg m$, there are $n(n-1)/2$ ways of situating the two runs among the n positions in the sequence so that $[n(n-1)/2](p^2q^m + q^2p^m)^2$ represents the sum of the probabilities of all sequences containing at least two runs of length m. Similar to the reasoning leading to Eq. 5-3, each of these probabilities is to be weighted according to the number of ways a group of two runs can be selected from k runs—that is, $k(k-1)/2$. Therefore,

$$\frac{1}{2}\sum_{k=0}^{\infty} k(k-1) P_n^m(k) = \frac{n(n-1)}{2}(p^2 q^m + q^2 p^m)^2 \tag{5-4}$$

The left-hand side of Eq. 5-4 can be rewritten in the form

$$\sum_{k=0}^{\infty} k(k-1) P_n^m(k) = \sum_{k=0}^{\infty} k^2 P_n^m(k) - \sum_{k=0}^{\infty} k P_n^m(k) \tag{5-5}$$

The first term of the right-hand side of Eq. 5-5 is equivalent to $\sigma_m^2 + k^2$ (cf. Eq. 2-20). Thence, combining Eqs. 5-2, 5-3, 5-4, and 5-5,

$$\sigma_m^2 = \bar{k} = n(p^2 q^m + q^2 p^m)$$

for $n \gg m$.

By extending the procedure used to determine the variance, higher-order moments of $P_n^m(k)$ can also be derived. In general, for large n, the rth moment can be developed from the expression

$$\sum_{k=0}^{\infty} \binom{k}{r} P_n^m(k) = \binom{n}{r}(p^2 q^m + q^2 p^m)^r$$

R. von Mises (Ref.) has shown that since the moments remain finite as $n \to \infty$, the probability $P_n^m(k)$ asymptotically approaches a Poisson distribution:

$$P_n^m(k) \to \left(\frac{K^k}{k!}\right) e^{-K} \equiv \varphi(k) \tag{5-6}$$

where

$$K = \lim_{n \to \infty} n(p^2 q^m + q^2 p^m) \tag{5-7}$$

That is, the probability that n trials will exhibit k runs of length m approaches $\varphi(k)$ for large values of n, where K is the expected value of k.

Nearly a century ago, the German psychologist Karl Marbe developed a theory of probability based on the concept that long runs contradict the premise of independent trials. Searching for runs in the sequences of newborn males and females, he investigated birth records in four cities, each record covering 49,152 parturitions. The longest run he discovered consisted of 17 consecutive occurrences of the same sex. Marbe concluded that as a run of girls (say) continues, the probability of the next birth being another girl decreases from its initial value (of about 1/2).[1]

From Eqs. 5-6 and 5-7 we can obtain an assessment of Marbe's conclusions. With $m = 18$, $n = 49{,}152$, and, $p = q = 1/2$, we have

$$K = 0.09375 \quad \text{and} \quad \varphi(0) = 0.9105$$

Hence, in each of his four sequences, Marbe observed an event that had a probability of 0.9105—a situation hardly so unusual as to warrant overturning deep-rooted theories.

[1] Modern statisticians, confronted with the birth of 17 consecutive girls, might conclude that the probability of the next birth being a girl is greater than 1/2 and would search for some nonprobabilistic factor such as the town's drinking water.

Comparative Runs

In a sequence of Bernoulli trials, let **A** define the event that a run of s consecutive successes (Heads) occurs before a run of f consecutive failures (Tails). Then the probability of **A,** $P\{\mathrm{A}\}$, can be written as

$$P\{\mathbf{A}\} = p\alpha + q\beta \tag{5-8}$$

where p and q are the single-trial probabilities of success and failure, respectively, and α and β are the conditional probabilities of **A** under the hypotheses, respectively, that the first trial results in success or in failure. If success, then **A** can occur in $s - 1$ mutually exclusive ways: the succeeding $s - 1$ trials result in successes (probability p^{s-1}) or the first failure occurs at the βth trial, where $2 \leq \beta < s$. The probability of the latter event is $p^{\beta-2}q$, and the conditional probability of **A**, given this event, is β. Applying Eq. 2-3, the formula for compound probabilities,

$$\alpha = p^{s-1} + q\beta(1 + p + \cdots + p^{s-2}) = p^{s-1} + \beta(1 - p^{s-1})$$

and, similarly,

$$\beta = p\alpha(1 + q + \cdots + q^{f-2}) = \alpha(1 - q^{f-1})$$

Solving explicitly for α and β, combined with Eq. 5-8,

$$P\{\mathbf{A}\} = p^{s-1}\frac{1 - q^f}{p^{s-1} + q^{f-1} - p^{s-1}q^{f-1}} \tag{5-9}$$

To determine the probability of the converse event, $P\{\bar{\mathbf{A}}\}$, that a run of f consecutive failures occurs before a run of s consecutive successes, we interchange p with q and s with f:

$$P\{\bar{\mathbf{A}}\} = q^{f-1}\frac{1 - p^s}{p^{s-1} + q^{f-1} - p^{s-1}q^{f-1}}$$

As a numerical example, in a sequence of fair-coin tosses, the probability that a run of two Heads occurs before a run of three Tails is given by Eq. 5-9 as 0.7. A run of four Heads before a run of six Tails has probability of 21/26. With $p = 0.395$, the probability of a run of two Heads is equal to that of a run of three Tails, and, with $p = 0.418$, the probability of a run of four Heads is equal to that of a run of six Tails.

Three Outcomes and Three Kinds of Runs

In lieu of a three-sided coin, several schemes can be implemented to generate three equiprobable outcomes, **A**, **B**, and **C**. Three examples:

1. Toss a conventional coin for each of three pairs: **A** versus **B**, **B** versus **C**, and **C** versus **A**. If any outcome wins twice, it will have occurred with probability 1/3. (If no outcome wins twice, toss the coin three times again.)
2. Toss a coin until H appears. There is a 1/3 probability that this event will occur on an even-numbered toss:

$$(1/2)^2 + (1/2)^4 + (1/2)^6 + \cdots = \frac{(1/2)^2}{1-(1/2)^2} = \frac{1}{3}$$

3. Toss a coin twice. Let HH equal the outcome **A**, HT the outcome **B**, and TH the outcome **C**. In the event of TT, toss the coin twice again.

For three possible outcomes with respective probabilities a, b, and c ($a + b + c = 1$), the probability, $P(\alpha, \beta)$, that a run of α consecutive **A'**s will occur before a **B**-run of length β is expressed by

$$P(\alpha, \beta) = a\mathrm{u} + b\mathrm{v} + c\mathrm{w}$$

where u, v, w are determined by the equations

$$\mathrm{u} = (a^{\alpha-1} + b\mathrm{v} + c\mathrm{w})\frac{1-a^{\alpha-1}}{1-a}$$
$$\mathrm{v} = (a\mathrm{u} + c\mathrm{w})\frac{1-b^{\beta-1}}{1-b}$$
$$\mathrm{w} = \frac{a\mathrm{u} + b\mathrm{v}}{1-c}$$

With $a = b = c = 1/3$, the probability that a run of $\alpha = 4$ **A**'s will occur before a run of $\beta = 6$ **B**'s is 0.90.

Similarly, the probability that an **A**-run of length α will occur before a **B**-run of length β *or* a **C**-run of length γ is given by the same expression, but with w redefined by

$$\mathrm{w} = (a\mathrm{u} + b\mathrm{v})\frac{1-c^{\gamma-1}}{1-c}$$

Heads–Tails Sequences

While extended sequences of coin tosses conform to the Bernoulli distribution, certain subsequences can exhibit properties that seem counterintuitive

(Ref. Weisstein). For example, using the notation P(X||Y) to designate the probability of X occurring before Y, we have

P(TTH || THT) = 2/3
P(THH || HHT) = 3/4
P(THTH || HTHH) = 9/23
P(TTHH || HHH) = 7/19
P(THHH || HHH) = 7/15

A *minimum* of $2^n + n - 1$ tosses is required to include all possible subsequences of length n. The *average number* of tosses required is conjectured (not yet proven) to be $2^n(\gamma + n \ln 2)$, where γ is Euler's constant (0.57721 …).

Most Probable Number of Heads

In a sequence of $2n$ tosses of a fair coin, the most probable number of Heads is n, and the corresponding probability is

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \tag{5-10}$$

With ten tosses, $n = 5$ Heads will occur, according to Expression 5-10, with probability 63/256 = 0.246.

Leads of Heads or Tails

Other questions that arise in connection with a sequence of coin throws concern the probability of one player (betting, say, on Heads) remaining ahead of his opponent and the number of times during a sequence that the players are even. This type of problem is best attacked by considering the number of ways or paths by which a particular proportion of Heads and Tails can be obtained. A path from the origin to a point (n, s)—for any integer s and any positive integer n—is a polygonal line whose vertices have abscissas 0, 1, …, n and ordinates $s_0, s_1, s_2, \ldots, s_n$ such that $s_i - s_{i-1} = \pm 1$. The quantity $\{s_1, s_2, \ldots, s_n\}$ is therefore the representation of a path.

In the illustration of Figure 5-2, n increases one unit with each coin toss, and s increases one unit with each throw of Heads and decreases one unit with each

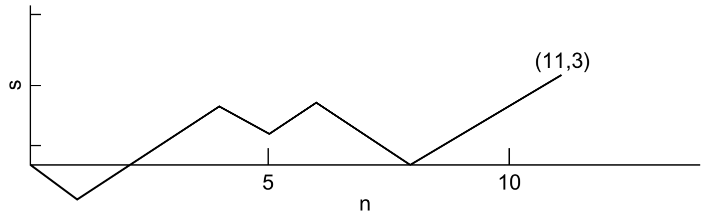

FIGURE 5-2 One-dimensional random-walk path.

occurrence of Tails. The path shown indicates a sequence of THHHTHTTHHH; the point (11, 3) relates that the sequence of length 11 has resulted in a net gain of three Heads. In general, the number of paths $N(n, s)$ by which we can arrive at a point (n, s) is given by

$$N(n,s) = \binom{n}{(n+s)/2} \tag{5-11}$$

which is valid for all values of n and s, since $n + s$ is always even.

For $s = 0$, we can rewrite Eq. 5-11 in the more convenient form

$$N(2n,0) = \binom{2n}{n}$$

which designates the number of paths from the origin to a point $2n$ on the abscissa. Then, if $P_{2t,2n}$ is the probability that in the interval from 0 to $2n$ along the abscissa the path has $2t$ vertices on the positive side (including zero) and $2n - 2t$ vertices on the negative side, it can be shown that

$$P_{2t,2n} = \binom{2t}{t}\binom{2n-2t}{n-t}2^{-2n} \tag{5-12}$$

That is, Eq. 5-12 defines the probability that the player betting on Heads, say, leads during exactly $2t$ out of the $2n$ trials. For large values of n, we can apply Stirling's formula to derive the approximation

$$P_{2t,2n} \simeq \frac{1}{\pi\sqrt{t(n-t)}}$$

The ratio $2t : 2n$ describes the fraction of the time spent on the positive side, and the probability that this fraction lies between 1/2 and k, $1/2 < k < 1$, is

$$\sum_{(n/2)<t<kn} P_{2t,2n} \simeq \frac{1}{\pi}\sum_{(n/2)<t<kn}\frac{1}{\sqrt{t(n-t)}}$$

The quantity $1/\sqrt{t(n-t)}$ can be written in the form

$$\frac{1}{n}\left[\frac{t}{n}\left(1-\frac{t}{n}\right)\right]^{-1/2}$$

And we note that the summation of the bracketed expression is the Riemann sum approximating the integral

$$\frac{1}{\pi}\int_{\frac{1}{2}}^{k}\frac{d\xi}{\sqrt{\xi(1-\xi)}} = \frac{2}{\pi}\arcsin k^{1/2} - \frac{1}{2} \tag{5-13}$$

Or, since the probability that $t/n \leq 1/2$ approaches 1/2 as $n \to \infty$,

$$\frac{1}{\pi}\int_0^k \frac{d\xi}{\sqrt{\xi(1-\xi)}} = \frac{2}{\pi}\arcsin k^{1/2} \tag{5-14}$$

Equation 5-14 constitutes the first arc sine law. It states that for a given value of k, $0 < k < 1$, and for very large values of the sequence length n, the probability that the fraction of time spent on the positive side t/n is less than k approaches $(2/\pi)$ arcsin $k^{1/2}$.

Values of t/n are most likely to be close to zero or unity and least likely to be near 1/2. For example, we can observe (from Eq. 5-12) that in $2n = 10$ throws of a coin, the less fortunate player has a probability of $2P_{0,10} = 0.492$ of never being in the lead. The probability that the player betting on Heads in a sequence of ten throws leads throughout is 0.246; the probability that Heads leads throughout 20 throws is 0.176; and over a trial of 30 throws, the probability of Heads being continually in the lead is 0.144. By contrast, the probability that Heads leads six times out of ten throws is 0.117; the probability that Heads leads ten times out of 20 throws is 0.061; and the probability of Heads leading 16 times out of 30 is only 0.041. Figure 5-3 illustrates the arc sine law for large

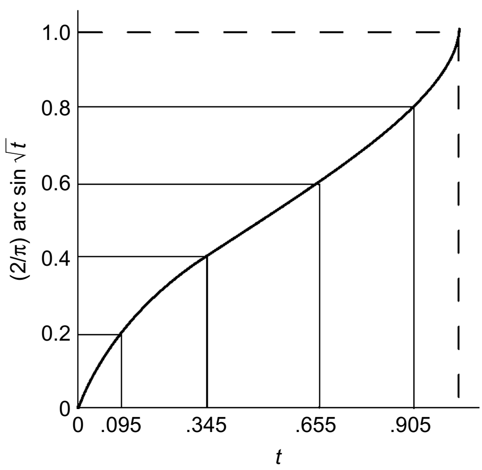

FIGURE 5-3 The arc sine law.

values of n. With probability 0.05, the more fortunate player will be in the lead 99.89% of the time, and with probability 0.5 he will lead 85.36% of the time. The lead changes sides less frequently than one would expect intuitively.

A rephrasing of the arc sine law implies that an increasingly large number of coin throws is required before the path from the origin crosses the abscissa (see Figure 5-2). Defining a "tie" as the condition that n Heads and n Tails have occurred out of $2n$ throws, the number of ties is proportional to $\sqrt{2n}$.

Specifically, if P_{2n}^u is the probability that the sequence of Heads and Tails reaches a tie exactly u times in $2n$ throws, we can show that

$$P_{2n}^u = \binom{2n-u}{n} 2^{u-2n} \tag{5-15}$$

which, for extremely large values of n, becomes (from Stirling's formula)

$$P_{2n}^u \simeq \sqrt{n\pi}\, e^{-u^2/4n} \tag{5-16}$$

Note that the number of ties, given by Eqs. 5-15 and 5-16, is stochastically twice the number of times the lead changes sides.

According to Eq. 5-15, P_{2n}^u is a monotonically decreasing function of n except for the equal values P_{2n}^0 and P_{2n}^1 (that is, $P_{2n}^0 = P_{2n}^1 > P_{2n}^2 > P_{2n}^3 > \cdots$). Thus, the most probable number of ties in a sequence of Heads and Tails of any length is 0 or 1. On the other hand, the expected number of ties in a sequence of length $2n$ is given by

$$\mathrm{E}(u) = \binom{2n}{n} (2n+1)2^{-2n} - 1$$

and for large values of n,

$$\mathrm{E}(u) \simeq 2\sqrt{n/\pi} \tag{5-17}$$

Hence, in a sequence of $2n = 20$ throws, the expected number of ties is 2.7, while for a sequence of a million throws, the expected number is fewer than 1130 and the most probable number of ties remains 0 or 1.

Also, we can sum Eq. 5-17 over k values of $u (u = 0, 1, 2, \ldots, k-1)$, obtaining, similarly to Eq. 5-13, the integral (from the Riemann sum)

$$P(k) = \sqrt{\frac{2}{\pi}} \int_0^k e^{-\zeta^2/2} d\zeta \tag{5-18}$$

That is, for a given value of k, $0 < k < \sqrt{2n}$, the probability that over a sequence of length $2n$ the number of ties is fewer than $k\sqrt{2n}$ is formulated by Eq. 5-18 as n becomes large. Illustratively, the probability that fewer than $\sqrt{2n}$ ties occur in a sequence of length $2n$ (n large) is 0.683; the probability that the number of ties is fewer than $2\sqrt{2n}$ is 0.9545; and the probability of fewer than $4\sqrt{2n}$ ties occurring is 0.999936.

Postulating a coin-tossing game that concludes at the appearance of the first tie, we can compute the expected game duration. From Eq. 5-15, the probability that the first tie occurs at play $2n$ is equal to

$$\begin{aligned} P_{2n-2}^0 - P_{2n}^0 &= \binom{2n-2}{n-1} 2^{2n+2} - \binom{2n}{n} 2^{-2n} \\ &= \frac{1}{2n} \binom{2n-2}{n-1} 2^{-2n+2} \end{aligned}$$

The expected duration E(v) is therefore

$$\mathrm{E}(v) = \sum_{n=1}^{\infty} 2n[P_{2n-2}^0 - P_{2n}^0] = \sum_{n=1}^{\infty} \binom{2n-2}{n-1} 2^{-2n+2} = \infty$$

as can be verified by Stirling's formula. It should be noted that although E(v) is infinite, the duration v is finite (a tie will occur in a finite number of plays with unity probability).

Finally, we can inquire as to the probability distribution of the maximum value assumed by the path between the origin and the point (2n, s). It can be shown that over a sequence of length 2n, the player betting on Heads has probability ${}_\eta P_{2n}$ of reaching his first maximum lead at the ηth play, where

$${}_\eta P_{2n} = \binom{2t}{t}\binom{2n \quad 2t}{n-t} 2^{-2n-1}, \text{ for } \begin{cases} \eta = 2t, & t = 1,2,\ldots,n \\ \eta = 2t+1, & t = 0,1,2,\ldots,n-1 \end{cases} \tag{5-19}$$

Quite remarkably, Eq. 5-19 displays values exactly 1/2 of those generated by Eq. 5-12. Thus the probability that Heads has its first maximum lead at either $2t$ or $2t + 1$ is identical to the probability that Heads leads by the fraction t/n of the time over a sequence of length 2n. Therefore, the approximation of the arc sine law applies, so that the maxima for Heads tend to occur near the beginning or the end of a sequence independent of its length.

Waiting Times Between Heads

In a sequence of coin flips, the average waiting time from one Heads to the next is—obviously—two flips (the first flip is not counted).

More interestingly, if a point is randomly selected between any two adjacent flips in the sequence, the average waiting time between the first Heads preceding that point and the first subsequent Heads is three flips. [The average waiting time for both the first Heads preceding and that succeeding the selected point is two; the initial Heads, however, is not counted in the definition of waiting time.]

In general, with p representing the probability of Heads, the average waiting time between Heads immediately straddling a randomly selected point is

$$2/\mathrm{p} - 1$$

Ballot Problems

An historical foreshadowing of the arc sine laws was contained in a lemma of the French mathematician J.L.F. Bertrand (Ref.). The first of a series of statements on the "ballot problem," Bertrand's lemma (1887) stated: "If in a ballot, candidate **A** scores α votes and candidate **B** scores $\beta < \alpha$ votes, then the probability that throughout the counting there are always more votes for **A** than for **B** equals $(\alpha - \beta)/(\alpha + \beta)$." An interesting generalization was subsequently contributed by Dvoretzky and Motzkin (Ref.): "If ω is an integer satisfying $0 < \omega < \alpha/\beta$, then the probability that throughout the counting the number r

of votes registered for **A** is always greater than ω times the number s of votes registered for **B** equals $(\alpha - \omega\beta)/(\alpha + \beta)$; and the probability that always $r \geq \omega s$ equals $(\alpha + 1 - \omega\beta)/(\alpha + 1)$."

It is implied that the counting procedure selects each ballot from a homogeneous mixture of all ballots. Thus, application of the theorems to the eccentric election-night rituals in the United States would likely lead to spurious conclusions in those (frequent) cases where the ballots are not counted in random order—urban and rural units, for example, are tallied in blocks and do not necessarily contribute similar voting proportions for candidates **A** and **B**.

COIN MATCHING

The simplest game of coin matching consists of two players simultaneously selecting Heads or Tails and comparing their choices. If the choices agree, **A** wins one unit; otherwise, **B** wins one unit—as depicted in the payoff matrix, Figure 2-2. Optimal strategies for both **A** and **B** consist of selecting their two pure strategies, each with probability 1/2; the mathematical expectation of the game then obviously equals zero.

Opportunity for profit arises if the payoff matrix is asymmetric or one player secures *a priori* knowledge of his opponent's strategy. We can illustrate two schemes for contriving an imbalance that, with only superficial analysis, appear rather deceptive. First, consider a variation of the simple coin-matching game wherein **A** pays 1-1/2 units to **B** if the coins match at Heads and 1/2 unit if the match occurs at Tails; for nonmatches, **B** pays one unit to **A**. The payoff matrix then becomes

$$
\begin{array}{c c}
 & \begin{array}{cc} B_1 & B_2 \\ \text{(Heads)} & \text{(Tails)} \end{array} \\
\begin{array}{c} A_1 \\ \text{(Heads)} \\ A_2 \\ \text{(Tails)} \end{array} &
\left\| \begin{array}{cc} -1\frac{1}{2} & 1 \\ & \\ 1 & -\frac{1}{2} \\ & \end{array} \right\|
\end{array}
$$

Applying Eqs. 2-30 for the 2×2 game where **A** selects strategy A_1 with frequency p_1 and A_2 with frequency $1 - p_1$,

$$a_{11}p_1 + a_{21}(1 - p_1) = \gamma$$

$$a_{12}p_1 + a_{22}(1 - p_1) = \gamma$$

Solving for p_1,

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{-\frac{1}{2} - 1}{-1\frac{1}{2} + \left(-\frac{1}{2}\right) - 1 - 1} = \frac{3}{8} \tag{5-20}$$

Thus **A**'s optimal strategy is specified as

$$S_A^* = \begin{pmatrix} A_1 & A_2 \\ 3/8 & 5/8 \end{pmatrix}$$

The value of the game is $\gamma = 1/16$ (i.e., **A** wins an average of 1/16 per play) regardless of **B**'s strategy. **B** can restrict his average loss to 1/16 by playing his optimal mixed strategy S_B^*, which similarly dictates selection of B_1 with probability 3/8 and B_2 with probability 5/8. (Note that if each strategy for both players is selected with probability 1/2, the value of the game reverts to 0.) For the game to be equitable, **A** must pay 1/16 unit per play to **B**.

A second, more sophisticated coin-tossing game with an unsymmetric payoff matrix can be represented as follows: **A** flips a coin, the outcome being withheld from **B**. If the coin shows Heads, **A** must declare "Heads" and demand payment of one unit from **B**. If it shows Tails, **A** is permitted two choices—he may declare "Heads" and demand one unit from **B**, or he may declare "Tails" and pay one unit to **B**. For each play wherein **A** declares "Heads," **B** has two pure strategies—he may believe **A**'s declaration of "Heads" and pay one unit or he may demand verification of the outcome. **A** is then compelled to reveal the coin to **B**; if it indeed shows Heads, **B** must pay two units. However, if it develops that **A** is bluffing and the coin shows Tails, **B** receives two units from **A**.

To determine the payoff matrix $\|a_{ij}\|$, we consider the four possible combinations of **A**'s being truthful or bluffing and **B**'s believing or demanding verification. If **A** bluffs and **B** believes (a_{11}), **A** receives one unit regardless of whether the coin shows Heads or Tails. Therefore

$$a_{11} = (1/2)(1) + (1/2)(1) = 1$$

If **A** bluffs and **B** demands verification (a_{12}), **A** receives two units when the coin shows Heads and pays two units when it discloses Tails. Thus,

$$a_{12} = (1/2)(2) + (1/2)(-2) = 0$$

For the pure strategy of **A**'s being truthful and **B**'s believing (a_{21}), **A** receives one unit in the case of Heads and pays one unit in the case of Tails. That is,

$$a_{21} = (1/2)(1) + (1/2)(-1) = 0$$

Finally, when **A** is truthful and **B** demands verification (a_{22}), **A** receives two units for Heads and pays one unit for Tails. Thus,

$$a_{22} = (1/2)(2) + (1/2)(-1) = 1/2$$

The game matrix is therefore represented by

$$\begin{array}{c|cc|}
 & \begin{array}{c} B_1 \\ \text{(Believe)} \end{array} & \begin{array}{c} B_2 \\ \text{(Disbelieve)} \end{array} \\
\begin{array}{c} A_1 \\ \text{(Bluff)} \end{array} & 1 & 0 \\
\begin{array}{c} A_2 \\ \text{(Truthful)} \end{array} & 0 & \frac{1}{2}
\end{array}$$

From Eq. 5-20 we can compute **A**'s selection frequency for A_1 as

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{1/2}{1 + (1/2)} = \frac{1}{3}$$

Hence **A**'s optimal strategy is given by

$$S_A^* = \begin{pmatrix} A_1 & A_1 \\ 1/3 & 2/3 \end{pmatrix}$$

and the value of the game is $\gamma = 1/3$. Both A_1 and A_2 are maximin strategies. If **A** uses either one as a pure strategy, his average yield is zero. The mixed strategy provides the positive yield of 1/3 unit per play. Similar considerations to ascertain **B**'s optimal strategy show that **B** should select B_1 with probability 1/3 and B_2 with probability 2/3. If **B** selects either B_1 or B_2 as a pure minimax strategy, his average yield is $-1/2$ instead of $-1/3$. For an equitable game, **A** must pay 1/3 unit per play to **B**.

A second method of increasing expected gain is through analyzing the mechanism by which an opponent selects his mixed strategies. Of course, if his set of strategies is derived from a random device, further analysis is of no avail. However, if the opponent is generating a sequence of Heads and Tails from the depths of his mind, there almost always exists a pattern, however complex.[2] To reveal such a pattern, Fourier analysis, individual symbol correlation, and pair correlation, are techniques that might prove advantageous. Generally, we attempt to predict the probability of Heads (Tails) occurring as the $(n + 1)$st term of a sequence, given the previous n terms. A tentative step toward this goal is shown in Figure 5-4.

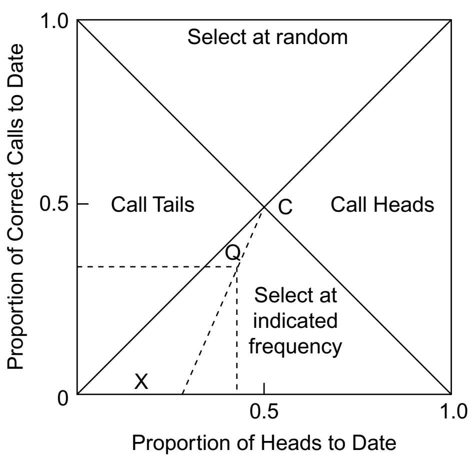

FIGURE 5-4 Heads–Tails predictive diagram.

[2] In Edgar Allan Poe's *The Purloined Letter*, C. Auguste Dupin describes a boy who could analyze his opponent's facial expression to predict the guessing of odd or even.

The point Q is determined by the proportion of correct calls to date and the proportion of Heads to date. If Q falls in the left or right triangles of the diagram, we select the pure strategies "always Tails" or "always Heads," respectively. If Q falls in the upper triangle, a mixed strategy is advised, with Heads and Tails equally likely to be chosen. If Q falls in the lower triangle, a mixed strategy is determined as follows: A line between the diagram center C and the point Q intercepts the abscissa at a point X. The proper frequency ratio of Heads and Tails is then X:$(1-X)$. For example, if $X = 0.2$, correct strategy dictates selecting Heads with probability 0.2 and Tails with probability 0.8.

On a more complex level, mechanisms have been constructed to analyze the sequence chosen by a human opponent and thereby achieve positive mathematical expectations in matching games. One of the first such machines was the Sequence Extrapolating Robot (SEER) (Ref. Hagelbarger), which could recognize any of the four periodic sequences:

HHHH ⋯
TTTT ⋯
HTHTHT ⋯
HHTTHHTT ⋯

A certain number of plays is required for SEER to acknowledge each sequence when it occurs, and a lesser number to assimilate any phase change in the sequence instigated by its opponent. Against human opposition, SEER usually emerged victorious since individual patterns tend to be not random but a function of emotions and previous training and experience. To increase its competitive élan, the machine selects a correlated output only when it is winning; when losing, it plays randomly. It is, of course, possible to outwit the machine from knowledge of its analytic procedure.[3]

COIN GAMES

The St. Petersburg Paradox

There exists an apocryphal folktale that the high-class gambling casinos of the 18th century employed a resident statistician for the purpose of establishing odds and entrance fees for any game whatsoever that a patron might wish to propose.

As usually recounted, a university student entered the casino in St. Petersburg and suggested a game celebrated henceforth as the St. Petersburg paradox.[4] The student offered to pay a fixed fee for each trial—defined as a sequence of coin tosses until Heads appears. Another trial and another fee might be instituted following each occurrence of Heads. The student was to receive 2^n rubles for each

[3] The obligatory umpire machine was constructed, which linked SEER with a machine built by C.E. Shannon, pitting them together in a Wellsian struggle. The Shannon machine triumphed with a ratio of 55 wins to 45 losses.

[4] Inquiries among Russian mathematicians have elicited the intelligence that this game was *not* subsequently renamed the Leningrad paradox.

trial, where n is the throw marking the first occurrence of Heads following a run of $n - 1$ Tails. According to this tale, the casino's statistician quoted some particular entrance fee per trial, and the student proceeded to break the bank, since the game possesses an infinite expected gain. (Actually, the game was originally proposed by Nicolas Bernoulli[5] and used by his cousin Daniel to formulate the concept of moral expectation; see Chapter 11.)

The probability of throwing $n - 1$ Tails followed by Heads is $(1/2)^n$. With the appearance of Heads at the nth throw, the trial concludes, and the student receives 2^n rubles. Thus the student's expectation is

$$\mathrm{E} = \sum_n \left(\frac{1}{2}\right)^n \cdot 2^n = 1 + 1 + 1 + \cdots$$

that is, infinity. Therefore, the only "fair" entrance fee would be an infinite sum of rubles.

Yet, in his mathematics classes, this logic might earn a failing grade. First, the expectation is infinite only if the casino has infinite wealth; otherwise (and in reality), the student's possible gain is limited. Second, the casino's finite wealth implies a finite "fair" entrance fee. Third, with a finite sum for his own fortune, the student would have a high probability of ruin before achieving any especially large gain.

One approach to resolving the "paradox" of infinite expectation renders the game "fair" in the classical sense by imposing an entrance fee that is not constant per trial, but rather is a function of the number of trials (Bertrand's solution). Letting e_n be the accumulated entrance fee and S_n the accumulated gain after n trials, then the game is equitable if, for every $\varepsilon > 0$,

$$P\left[\left|\frac{S_n}{e_n} - 1\right| > \varepsilon\right] \to 0$$

analogous to the law of large numbers (Eq. 2-25). From this expression it can be shown that

$$e_n = n \log_2 n$$

Thus, if the student had 64 rubles, he would fairly be allowed 16 trials. Note that two students each with 64 rubles (representing a total purchasing power of 32 trials) would differ from one student with 128 rubles (representing a purchasing power of 26.94 trials).

Variable entrance fees are, for several reasons, undesirable and unfeasible in gambling operations. A better approach is to consider a casino with finite capital N. The probability p_n that Heads first appears on the nth throw of a

[5] D'Alembert, Condorcet, Bertrand, and Poisson, among many, undertook explanations of the game. D'Alembert wrote to Lagrange: "Your memoir on games makes me very eager that you should give us a solution of the Petersburg problem, which seems to me insoluble on the basis of known principles."

sequence remains $(1/2)^n$. However, the payoff a_n at the nth throw is now constrained by

$$a_n = 2^n \text{ if } 2^n \leq N$$

$$a_n = N \text{ if } 2^n > N$$

Therefore, if $[n_0]$ is the highest value of n such that $2^n \leq N$, the mathematical expectation of the game is

$$\text{E} = \sum_{n=1}^{[n_0]} (1/2)^n 2^n + \sum_{n=[n_0]+1}^{\infty} (1/2)^n N \tag{5-21}$$

The second term in Eq. 5-21 is a geometric series whose sum is $N/(2)^{[n_0]}$. If the casino possesses a million rubles, $[n_0] = 19$ and

$$\text{E} = [n_0] + \frac{1{,}000{,}000}{2^{19}} = 19 + 1.91^- = 20.91^-$$

Hence the casino should demand 20.91 rubles from the student as an entrance fee. For a billion-ruble bank, the "fair" entrance fee is 30.86^+ rubles, and for a trillion-ruble bank, 40.82^- rubles. When the game is contested between two poor students, the one acting as banker exacts a "fair" entrance fee of 7.656 rubles if holding a fortune of 100 rubles. With a capitalization of only 10 rubles, the payment per trial should be 4.25 rubles.

If the casino imposes a House Take of 2.5%—i.e., paying $(1.95)^n$ for tossing n Heads—the player's expectation against an infinite bank becomes

$$\text{E} = \sum_{i=1}^{\infty} \frac{(1.95)^i}{2^i} = \frac{(1.95)/2}{1-(1.95)/2} = 39$$

and the casino needs no recourse to an especially large bankroll.

The expected duration of one trial of the St. Petersburg game is readily seen to be

$$\sum_{i=0}^{\infty} \frac{i}{2^i} = 2 \text{ throws}$$

Thus, on the average, every third toss of the coin initiates a new trial and decrees a fresh payment. Consequently, the player can avoid a high ruin-probability only with a large bankroll.

[The entropy of the St. Petersburg-game distribution also equals two bits (although its mean is infinite).]

Economists resolve the "paradox" of infinite mathematical expectation by substituting Expected Utility, E(U). Daniel Bernoulli introduced this concept and underscored the diminishing marginal utility of money, suggesting the logarithmic function as a practical measure of utility. Thus the Expected Utility of the St. Petersburg game becomes finite:

$$\mathrm{E(U)} = \sum_{i=1}^{\infty} \frac{\ln(2^{i-1})}{2^i} < \infty$$

If the casino also restricts itself to finite expected values, the paradox disappears for concave utility functions (i.e., when the player is risk averse [Chapter 3]—although not necessarily in the framework of Cumulative Prospect Theory).

St. Petersburg Revisited

Our posited student, were he a mathematics prodigy, might have proposed the more subtle game of being paid n rubles, where n now defines the first play of the sequence when the number of Heads exceeds the number of Tails. Here, too, an infinite-expectation game arises if the casino is granted infinite wealth. Since Heads can first exceed Tails only on an odd value of n, let $n = 2k + 1$. The probability that Heads first exceeds Tails at the nth throw of the coin (i.e., that over $n - 1$ throws the s versus n path of Figure 5-2 contains $n - 1$ vertices on or below the abscissa) is given by

$$p_{2k+1} = \frac{\binom{2k-1}{k}\frac{2}{k+1}}{2^{2k+1}}, \quad k > 0; \quad p_1 = \frac{1}{2}$$

and the payoff is $2k + 1$ rubles. Therefore, the expectation may be expressed as

$$E = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\binom{2k-1}{k}(2k+1)}{(k+1)2^{2k}} = \sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}}{2^{2k+1}} = \sum_{k} f(k)$$

The ratio of the kth term and the $(k + 1)$st term is

$$\frac{f(k)}{f(k+1)} = \frac{\binom{2k+1}{k}2^{2k+3}}{\binom{2k+3}{k+1}2^{2k+1}} = \frac{2k+4}{2k+3}$$

Thus,

$$\lim_{k-\infty} \frac{f(k)}{f(k+1)} \to 1$$

and the expectation is a divergent series.

As with the St. Petersburg game, we could erroneously conclude that the fair entrance fee is infinite. However, for a finite capital N possessed by the casino, the payoff a_n is, in reality,

$$a_n = n = 2k + 1, \quad \text{if } N \geq 2k + 1$$
$$a_n = N, \quad \text{if } N < 2k + 1$$

Similar to Eq. 5-21, we can write the true expectation as

$$\mathrm{E} = \sum_{k=0}^{[k_0]} \frac{\binom{2k+1}{k}}{2^{2k+1}} + \sum_{k=[k_0]+1}^{\infty} \frac{\binom{2k-1}{k} N}{(k+1)2^{2k}} \tag{5-22}$$

where $[k_0]$ is the greatest value of k for which $2k + 1 \leq N$.

Equation 5-22 can be simplified via Stirling's formula and by the further approximation $[1 - (1/x)]^x \sim e$ to the form

$$\sum_{k=0}^{[k']} \frac{\binom{2k+1}{k}}{2^{2k+1}} + 2\left\{\sqrt{[k_0]} + \sqrt{[k']}\right\} + \left\{\frac{e}{\pi[k_0]}\right\}^{\frac{1}{2}}$$

where $[k']$ is an integer (greater than 10) selected to reduce the error of the approximation to less than 1%. For values of k less than $[k']$, exact numerical calculations are required.

When played against another impoverished student with a limited fortune of $N = 10$ rubles, the fair entrance fee becomes, by these approximations, 6.36 rubles per trial. For $N = 100$, the first student should pay 23.3 rubles to begin a sequence of plays. Against casino bankrolls of 1 million and 1 billion rubles, fair entrance payments are 2580 and 81,500 rubles, respectively.

HEADS–TAILS INTERPLAY

Probability of an Even Number of Heads

A plethora of problems and puzzles can be formulated through sequences of coin throws. As one example, we ask the probability P_n that an even number of Heads is observed in n tosses of a coin. Let p equal the probability that the coin falls Heads, q that it falls Tails, and note that the distribution of the number of Heads is binomial. Then,

$$P_n = \sum_i \binom{n}{2i} p^{2i} q^{n-2i} = \frac{1}{2}[(p+q)^n + (q-p)^n] = \frac{1}{2}[1 + (q-p)^n]$$

For a fair coin, $p = q = 1/2$, and the probability of obtaining an even number of Heads in n tosses is evidently 1/2 (independent of n).

Equal Number of Heads and Tails

Let E be the expected number of tosses to obtain an equal number of Heads and Tails, assuming the first toss is Tails.

First, consider a one-dimensional random walk (Chapter 2) with probability p_{H} of tossing Heads—and moving one step to the right—and probability $1 - p_{\mathrm{H}}$ of tossing Tails—moving one step to the left. The expected change in position per move is

$$p_{\mathrm{H}}(+1) + (1 - p_{\mathrm{H}})(-1) = 2p_{\mathrm{H}} - 1$$

The first toss (postulated) resulted in moving one step leftward. Ergo, the expected number of moves, E, to net one step rightward is the reciprocal of the expected change in position per move. That is,

$$\mathrm{E} = \frac{1}{(2p_{\mathrm{H}} - 1)}$$

Note that for $p_{\mathrm{H}} = 0.5$, the expectation is infinite.

Probability of No Successive Heads

Calculating the probability of realizing a sequence of coin tosses that does not include two successive Heads prompts the use of an interesting mathematical formulation. First, we note that f_n, the number of ways of tossing the coin n times without the appearance of successive Heads, can be decomposed into two mutually exclusive terms: f_{n-1}, those in which the nth throw is Tails and the previous $n - 1$ throws have exhibited no successive Heads, and f_{n-2}, those in which the nth throw is Heads, the $(n - 1)$st throw is Tails, and the previous $n - 2$ throws occurred without successive Heads. Thus

$$f_n = f_{n-1} + f_{n-2} \tag{5-23}$$

Equation 5-23 is readily identified as the Fibonacci sequence, the nth term of which corresponds to $n - 2$ throws of the coin. Since all 2^n sequences of n throws are equiprobable, the probability P_n that two or more successive Heads do not appear in n throws is simply

$$P_n = \frac{f_{n+2}}{2^n}$$

or, expressed as a summation,

$$P_n = 1 - \sum_{i=0}^{n-1} 2^{-(i+1)} f_i$$

where f_i is the ith term of the Fibonacci sequence $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, \ldots$.

For ten throws, $P_{10} = 144/2^{10} = 0.141$. By twenty throws, P_{20} has fallen off to $17{,}711/2^{20} = 0.0169$.

Fibonacci sequences can be generalized to "multibonacci" sequences: n flips of the coin without the appearance of m (or more) successive Heads. The number of ways for this event to occur is

$$f_n = f_{n-1} + f_{n+2} + \cdots + f_{n-m}$$

Hence, the probability $P_n(m)$ that m (or more) consecutive Heads do not appear in n throws is

$$P_n(m) = \frac{f_{n+m}}{2^n}$$

or

$$P_n(m) = 1 - \sum_{i=0}^{n-m+1} 2^{-(i+1)} f_i$$

For $m = 3$ and $n = 10$, the "Nebonacci" number is 504, and the probability that three (or more) consecutive Heads do not appear in ten throws is

$$P_{10}(3) = \frac{504}{2^{10}} = 0.492^+$$

Thus ten throws are required to obtain a probability greater than 1/2 (specifically, 0.508) for the occurrence of three (or more) consecutive Heads.

For $m = 4$ and $n = 22$, the "Pronebonacci" number is 2,044,428. Dividing by 2^{22}:

$$P_{22}(4) = 0.487^+$$

Thus 22 throws are required to obtain a probability greater than 1/2 (specifically, 0.513) for the occurrence of four (or more) consecutive Heads.

Some multibonacci numbers are listed in Table 5-1.

Table 5-1 Multibonacci Numbers for $n = 0$ to 10 and $m = 2$ to 7

	$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
Fibonacci	2	1	2	3	5	8	13	21	34	55	89	144
Nebonacci	3	1	2	4	7	13	24	44	81	149	274	504
Pronebonacci	4	1	2	4	8	15	29	56	108	208	401	773
Propronebonacci	5	1	2	4	8	16	31	61	120	236	464	912
Terpronebonacci	6	1	2	4	8	16	32	63	125	248	492	976
Quaterpronebonacci	7	1	2	4	8	16	32	64	128	256	512	1024

Consecutive Heads

Let E_{nH} be the expected number of coin tosses until n consecutive Heads appear. For $n = 1$, the probability that the first toss is Heads equals 1/2. And, with probability 1/2, Tails appears, whence the first toss has been inconsequential. Therefore,

$$E_{1H} = \frac{1}{2} \cdot 1 + (1 + E_{1H})$$

and $E_{1H} = 2$.

For $n = 2$, we are concerned with the $(E_{1H}+1)$st toss, which has probability 1/2 of showing Heads and probability 1/2 of (effectively) returning the sequence to its beginning. Thus

$$E_{2H} = \frac{1}{2}(E_{1H} + 1) + \frac{1}{2}(E_{1H} + 1 + E_{2H}) = \frac{3}{2} + \frac{1}{2}(3 + E_{2H})$$

So $E_{2H} = 6$.

In general, the expected number of tosses for n consecutive Heads to appear is

$$\mathrm{E}_{nH} = \sum_{1}^{n} \frac{(i + \mathrm{E}_{nH})}{2^i} + \frac{n}{2^n}$$

For $n = 3$, $\mathrm{E}_{3\mathrm{H}} = 14$; for $n = 4$, $\mathrm{E}_{4\mathrm{H}} = 30$. Further, given the expected number of tosses until i consecutive Heads appear, the expected number for $i+1$ consecutive Heads is

$$\mathrm{E}_{(i+1)\mathrm{H}} = \mathrm{E}_{i\mathrm{H}} + 2$$

For the expected number of tosses until the appearance of Heads followed by Tails, we can write

$$\mathrm{E}_{\mathrm{HT}} = 1 + \frac{\mathrm{E(H)} + \mathrm{E(T)}}{2} \quad \text{and} \quad \mathrm{E(T)} = 1 + \frac{\mathrm{E(T)}}{2} = 2$$

where E(H) is the expected number of tosses, given that the previous toss was Heads; and E(T) is the expected number, given that the previous toss was Tails. Hence $\mathrm{E}_{\mathrm{HT}} = 4$.

Heads Before Tails

In a sequence of coin tosses, we wish the probability $P(h\|t)$ that h Heads will appear before t Tails. Since this event will occur at most $h + t - 1$ times, we can write

$$P(h\|t) = \sum_{i=h}^{h+t-1} \binom{h+t-1}{i} p^i q^{h+t-1-i}$$

As one example, with $h = 2$, $t = 3$, and $p = q = 1/2$ for a fair coin, we have

$$P(2\|3) = \left(\frac{1}{2}\right)^4 \sum_{2}^{4} \binom{4}{i} = \frac{11}{16}$$

And the probability that $h = 4$ Heads will appear before $t = 6$ Tails is

$$P(4\|6) = (1/2)^9 \sum_{4}^{9} \binom{9}{i} = 191/256 = 0.746^+$$

We can also determine that value of p for which the probability of two Heads appearing equals that of three Tails:

$$\sum_{2}^{4} \binom{4}{i} p^i (1-p)^{4-i} = 0.5$$

which occurs for $p = 0.279^+$.

As a further example, we calculate that value of p for which there is an equal probability for the appearance of four Heads or six Tails:

$$\sum_{4}^{9} \binom{9}{i} p^i (1-p)^{9-i} = 0.5$$

which yields $p = 0.393^+$.

All Heads

A total of N fair coins are flipped, and those showing Tails are collected and flipped again. This process is continued until all N coins show Heads. We wish the expected number of flips, F_N, required to achieve this end (all Heads).

With a single coin, we have

$$F_1 = 1/2 + (1/2)(F_1 + 1)$$

since Heads will occur with probability 1/2, and no change will have occurred with probability 1/2. Thus $F_1 = 2$.

After the first flip of N coins, we have probability $\binom{N}{h}2^{-N}$ that h coins show Heads. The remaining $N - h$ coins require F_N flips (Ref. Singleton), where

$$F_N = 1 + \frac{1}{2^N - 1}\left\{1 + \sum_{i=1}^{N-1}\binom{N}{i} F_{N-i}\right\}, \quad N \geq 2$$

Table 5-2 lists some of the values for F_N.

Table 5-2 Expected Number of Coin Flips, F_N, to Obtain All Heads for Various Values of N

N	F_N	N	F_N	N	F_N
1	2	6	4.035	15	5.287
2	8/3	7	4.241	20	5.690
3	22/7	8	4.421	25	6.002
4	368/105	9	4.581	100	7.984
5	2470/651	10	4.726		

Evermore Trials

A and **B** alternately flip a fair coin, with **A** flipping once, then **B** twice, then **A** three times, **B** four times, and so on. The first player to flip Heads wins the stake. **A**'s probability of winning is given by

$$\sum_{1}^{\infty} \frac{2^{k-1} - 1}{2^{k(2k-1)}} = 1/2 + 0.1093 + 0.0009 + \cdots = 0.61$$

Hence the game has a value to **A** of 0.220.

To generalize, let the (constant) probability of the successful event equal p, and let **A** first be accorded a single trial. If **A** fails, B is then given $n+1$ trials. If **B** fails, then **A**, in turn, tries with $2n+1$ trials, and so forth. **A**'s probability of success (as a function of n) is expressible as

$$\begin{aligned} P_n &= p + (1-p)^{2+n} - (1-p)^{3+3n} + (1-p)^{4+6n} - (1-p)^{5+10n} \\ &\quad + (1-p)^{6+15n} - (1-p)^{7+21n} + \cdots \\ &= p + \sum_{k=1}^{\infty} (-1)^{k+1} (1-p)^{k+1+[k(k+1)n]/2} \end{aligned} \tag{5-24}$$

For $n = 2, 3, 4, \ldots$, $p_2 = 0.561$, $p_3 = 0.521$, $p_4 = 0.516$, …, with respective expectations for **A** of 0.121, 0.052, 0.031, … . As n increases, the game remains favorable to **A**, albeit with an expectation approaching 0.

Further, we can pose the question: what value of p renders this procedure a fair game? Setting p_n equal to 1/2 (in Eq. 5-24) and solving for p (as a function of n), we have

$$p_3 = 0.4479; \quad p_4 = 0.4802; \quad p_5 = 0.4912; \quad p_{10} = 0.4998$$

(for $n = 1$ and $n = 2$, there is no solution; the game remains favorable to **A** for all values of p). As n increases, P_n rapidly approaches 1/2.

Throwing Different Numbers of Coins

Another facet of coin problems with unexpected ramifications arises when two players **A** and **B**, possessed of $n+1$ and n coins, respectively, each throw their coins simultaneously, observing the number that turn up Heads. We seek the probability P_A that **A**, owning the additional coin, has obtained more Heads than **B**. This probability can be formulated by the double sum

$$P_A = \frac{1}{2^{2n+1}} \sum_{i=1}^{n+1} \sum_{j=0}^{n} \binom{n+1}{i+j} \binom{n}{j} \tag{5-25}$$

A readily verifiable relationship for binomial coefficients can be invoked:

$$\sum_{j=0}^{n} \binom{n+1}{i+j} \binom{n}{j} = \binom{2n+1}{n+i}$$

Applying this equivalence to Eq. 5-25, the probability that **A** achieves more Heads than **B** is determined explicitly to be

$$P_A = \frac{1}{2^{2n+1}} \sum_{i=0}^{n+1} \binom{2n+1}{n+i} = \frac{2^{2n}}{2^{2n+1}} = \frac{1}{2} \tag{5-26}$$

Note that although P_A is not a function of the number of coins n, the probability P_B that **B** achieves more Heads than **A** *is* a function of n. Letting P_e be the probability that **A** and **B** throw the same number of Heads, we can evaluate P_e as

$$P_e = \frac{1}{2^{2n+1}} \sum_{j=0}^{n} \binom{n+1}{j} \binom{n}{j} = \binom{2n+1}{n} 2^{-2n-1} \qquad (5\text{-}27)$$

The difference between unity and the sum of Eqs. 5-26 and 5-27 defines the probability P_B:

$$P_B = \left[2^{2n} - \binom{2n+1}{n} \right] 2^{-2n-1}$$

With ten coins (to **A**'s 11), **B**'s probability of throwing more Heads than **A** is 0.332, while the probability that **A** and **B** throw an equal number of Heads is 0.168. For large values of n, P_e approaches 0 and P_B approaches 1/2.

As a first generalization, let **A** be accorded m more coins than the n possessed by **B**. For the probability that **A** obtains more Heads with $n + m$ coins than **B** with n coins, we state the recursive relationship

$$P_A(n,m) = P_A(n,m-1) + \frac{1}{2^{2n+m}} \binom{2n+m-1}{n}, \quad m \geq 1$$

$$P_A(n,0) = \frac{1}{2} - \frac{1}{2^{2n+1}}$$

or, expressed as a summation,

$$P_A(n,m) = \frac{1}{2} + \frac{1}{2^{2n+m}} \sum_{i=1}^{m-1} 2^{m-1-i} \binom{2n+i}{n}, \quad m \geq 2 \qquad (5\text{-}28)$$

Only the value $P_A(n, 1)$ is not a function of n. The probability $P_e(n, m)$ that **A** and **B** produce the same number of Heads is, in the general instance,

$$P_e(n,m) = \frac{1}{2^{2n+m}} \sum_{i=0}^{n} \binom{n+m}{i} \binom{n}{i} = \frac{1}{2^{2n+m}} \binom{2n+m}{n} \qquad (5\text{-}29)$$

The probability that **B** obtains more Heads than **A**, although throwing fewer coins, is given by $1 - P_A(n, m) - P_e(n, m)$.

In the simplest case, **A** and **B** each toss a fair coin n times. The probability that each obtains the same number of Heads (Tails) is given by Eq. 5-29 with $m = 0$:

$$\sum_{i=0}^{n} \binom{n}{i}^2 \frac{1}{2^{2n}} = \binom{2n}{n} \frac{1}{2^{2n}}$$

With $n = 10$ tosses, this probability equals 0.176^+. For large n, the probability is approximated by $(1/\pi n)^{\frac{1}{2}}$.

A further generalization consists of characterizing **A**'s coins by a probability p_a of turning up Heads and a probability $q_a = 1 - p_a$ of turning up Tails, and **B**'s coins by probabilities p_b and $q_b = 1 - p_b$ of showing Heads and Tails, respectively. The probability that **A** achieves the greater number of Heads can then be formulated by the double sum

$$P_A(n,m) = \sum_{j=0}^{n} \sum_{i=j+1}^{n+m} \binom{n}{j} \binom{m+n}{i} p_a^i p_b^j q_a^{n+m-i} q_b^{n-j}$$

which reduces to Eq. 5-28 for $p_a = p_b = 1/2$.

THE TWO-ARMED BANDIT

A coin problem in the domain of sampling statistics is that of the "Two-armed Bandit" (so-called because it can be mechanized as a slot machine with two levers). Presented are two coins, **A** and **B**, that possess probabilities p_a and p_b, respectively, of exhibiting Heads. The player is accorded no knowledge whatsoever of these probabilities; yet, at each step in a sequence of coin tosses, he must select one of the two coins and wager that its outcome is Heads. Over the sequence of n tosses he receives one unit payment for each Heads and loses one unit for each Tails. We wish to determine a judicious rule for the coin-selection procedure.

While no rigorous proof of an optimal strategy has been achieved, Robbins (Ref.) has proposed the principle of "staying on a winner" and has shown it to be uniformly better than a strategy of random selection. According to this principle, the player chooses coin **A** or **B** at random for the first toss and thereafter switches coins only upon the occurrence of Tails. That is, for $j = 1, 2, \ldots, n$, if the jth toss results in Heads, the same coin is thrown at the $(j + 1)$st trial; otherwise, the alternate coin is thrown. Accordingly, the probability P_j of obtaining Heads at this jth toss can be expressed by the recursive relationship

$$P_{j+1} = P_j(p_a + P_b - 1) + p_a + p_b - 2p_a p_b$$

and we can derive the limiting expression for P_j explicitly as

$$\lim_{j\to\infty} P_j = \frac{p_a + p_b - 2p_a p_b}{2 - (p_a + p_b)} = q_a + \frac{q_b^2}{1 - q_a} \qquad (5\text{-}30)$$

where

$$q_a = \frac{p_a + p_b}{2} \quad \text{and} \quad q_b = \frac{|p_a - p_b|}{2} \qquad (5\text{-}31)$$

Over the n tosses he is permitted, the player wishes to achieve the greatest possible expected value of the sum $S_n = x_1 + x_2 + \cdots + x_n$, where $x_j = 1$ or -1, as Heads or Tails occurs at the jth toss. Hence, his expectation per toss from holding to the "staying on a winner" rule is, applying Eqs. 5-30 and 5-31,

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{S_n}{n}\right) = 2\left(\frac{q_a - p_a p_b}{1 - q_a}\right) - 1 \tag{5-32}$$

If the player knew the greater of the two probabilities p_a and p_b, he would then achieve the expectation

$$\mathrm{E}\left(\frac{S_n}{n}\right) = 2\max(p_a, p_b) - 1 = 2(q_a + q_b) - 1 \tag{5-33}$$

Robbins has proposed the difference in these expectations as a measure of the asymptotic loss per toss owing to ignorance of the true probabilities. Ergo, the loss $L(\mathbf{A}, \mathbf{B})$ associated with the "staying on a winner" rule is given by Eq. 5-33 minus Eq. 5-32, or

$$L(\mathbf{A},\mathbf{B}) = 2q_b\left(1 - \frac{q_b}{1 - q_a}\right) \geq 0$$

It can be shown that this loss assumes its maximum value of $2(3 - 2^{3/2}) \simeq 0.343$ when $p_a = 0$ and $p_b = 2 - 2^{1/2} \simeq 0.586$ (or vice versa). Consequently, applying the "staying on a winner" rule entails an average loss per toss of 0.343 unit owing to ignorance of the true state of "nature."

By adopting the rule of selecting a coin at random for each throw, or selecting a coin at random and continuing to use it, or alternating between coins, the player sustains an average loss per toss, $\bar{L}(\mathbf{A}, \mathbf{B})$, equal to

$$\bar{L}(\mathbf{A},\mathbf{B}) = 2q_b = |p_a - p_b| \tag{5-34}$$

Equation 5-34 exhibits a maximum value of 1 when $p_a = 0$ and $p_b = 1$ (or vice versa). We can also prove that $L(\mathbf{A}, \mathbf{B}) \leq \bar{L}(\mathbf{A}, \mathbf{B})$ for all values of p_a and p_b. Further, no other strategy will yield a loss function less than $L(\mathbf{A}, \mathbf{B})$.

For the special case where it is given that $p_a = 1 - p_b$, the "Two-armed Bandit" problem reduces to one of sequential decisions with a memory including all known history of the coin. The obvious strategy in this instance is to wager on Heads or Tails as either claims a majority of the previous tosses.

More extensive developments of the nature of adaptive competitive decisions have been published by J.L. Rosenfeld (Ref.).

NONTRANSITIVE SEQUENCES

The graph shown in Figure 5-5 defines a two-person game (Refs. Berlekamp, Conway, and Guy; Penney) wherein **A** selects any sequence of three Heads or

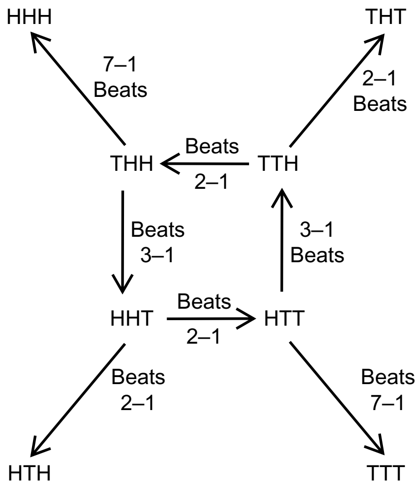

FIGURE 5-5 A nontransitive game.

Tails, and **B** then selects a different sequence of length three. A coin is then tossed repeatedly until either **A**'s selection or **B**'s selection appears, whence **A** or **B**, respectively, wins the game.

Note, as one example, the nontransitive relationship:

THH beats HHT beats HTT beats TTH beats THH

Following the strategy indicated, the probability that **B**'s selection beats **A**'s selection is 0.667, 0.750, or 0.875. If **A** selects his sequence at random, the value of the game is 23/48 to **B**.

Since the eight possible H-T sequences of length three are equiprobable, the fact that a particular sequence may be likely to precede another sequence falls into the surprisingly large category of counterintuitive probabilistic phenomena.

A game ensuing from that depicted in Figure 5-5 has each player selecting his sequence *sub rosa* without knowledge of his opponent's selection (Ref. Silverman). Optimal strategy here is to choose between HTT and THH in such proportions that

$$\frac{9}{8} \geq \frac{\text{HTT}}{\text{THH}} \geq \frac{8}{9}$$

which includes selecting the two sequences with equal probability.

QUANTUM COIN TOSSING

A quantum coin (a term meaningful only in a metaphorical sense) is defined by three possible states: Heads, Tails, and a mixed state of both Heads and Tails. In a game devised by D.A. Meyer (Ref.), **A** and **B** confront a closed box containing a quantum coin showing Heads. **A** has the option of turning over the box (and the coin) or leaving it as is. **B** then exercises the same option. Thence **A** is afforded another chance to flip the box (or not). **A** wins if the final configuration shows Heads. **A** is allowed to use quantum strategies; **B** is restricted to classical strategies.

In conventional game theory, the probability of the coin ending with Heads is 1/2. But here, when **A** as his first turn performs a quantum flip of the coin, he leaves it in a superposition of its two conventional states: half Heads and half Tails. Whether **B** then flips the box or not, neither action alters the coin's mixed state. **A** then performs a second quantum move that unscrambles the superposition and restores the coin to its original state of Heads and thereby wins the game.

At present, no quantum casinos have opened for business.

DIVERSE RECREATIONS

Odd Man Out

A common game among friendly competitors consists of an odd number m of players, each throwing a single coin; if, among them there is one whose coin shows an outcome different from all others, that player is declared "odd man out."

Defining p as the single-trial probability of Heads, the probability P that exactly one Heads or exactly one Tails appears among m coins is equal to

$$P = m(p^{m-1}q + pq^{m-1}) \tag{5-35}$$

For a fair coin, $p = q = 1/2$, and Eq. 5-35 simplifies to

$$P = \frac{m}{2^{m-1}}$$

With $m = 3$, the probability that one player is "odd man out" is 3/4, and with $m = 5$, that probability falls off to 5/16.

Letting $Q = 1 - P$, the probability that no player becomes "odd man out" until the nth play is $Q^{n-1}P$. And since n is a random variable that follows a geometric distribution with the parameter P, we can observe that the mean of n is equal to the mean of the geometric distribution, so that the expected duration of the game is defined by

$$\mathrm{E}(n) = \frac{1}{P} = \frac{2^{m-1}}{m}$$

which equals $1\frac{1}{3}$ plays for $m = 3$ and $3\frac{1}{5}$ plays for $m = 5$. Similarly, the variance has the form

$$\sigma^2(n) = \frac{Q}{P^2}$$

Thus, for $m = 3$, the standard deviation is 2/3 of a play, while for $m = 5$, the standard deviation is 2.65 plays.

Rotation

With three players, a variation of the "odd-man-out" theme is exemplified by the game of Rotation. Designating the players as **A**, **B**, and **C**, we match **A** versus **B**, with the winner advancing to a contest with **C**. The winner of this second match then plays the loser of the first match, and the players continue to rotate according to this procedure until one has triumphed twice in succession. If each match is a fair contest, the probabilities P_A, P_B, and P_C of **A**, **B**, and **C**, respectively, winning the game are determined by simple algebraic methods. Let p_w represent the probability of a player's winning the game immediately following the winning of a match, and p_l be the probability of ultimately winning the game following the loss of a match. Then

$$P_A = P_B = \frac{1}{2}p_w + \frac{1}{2}p_t; \quad P_C = \frac{1}{2}p_w; \quad \text{and} \quad p_w = \frac{1}{2} + \frac{1}{2}p_\ell$$

Eliminating p_w and p_ℓ,

$$P_A = P_B = 5/14 \quad \text{and} \quad P_C = 2/7$$

Player **C** obviously is handicapped by not participating in the first match.

The expected duration E(n) of the Rotation game is computable as

$$\mathrm{E}(n) = \sum_{i=1}^{\infty}(i+1)2^{-i} = 3 \text{ matches}$$

Morra

An ancient pastime of notable simplicity, Morra provides an example where the optimal strategy was unknown for centuries owing to the ignorance of elementary game theory.

Simultaneously, **A** and **B** each display one or two fingers. If both show the same number of fingers, **A** wins 2 units. If **A** shows two fingers and **B** one finger, **B** wins 1 unit. If **A** shows one finger and **B** two fingers, **B** wins 3 units. The game matrix is drawn as follows:

		A 1	**A** 2
B	1	+2	–1
B	2	–3	+2

A's optimal strategy is to display one finger with probability p and two fingers with probability $1 - p$, for a concomitant expectation of $2p - (1 - p)$ if **B** shows one finger and $-3p + 2(1 - p)$ if **B** shows two fingers. **A**'s minimax strategy equates these two expectations, so that, $p = 3/8$. The value of this game to **A** is thus $+1/8$.

Similarly, **B**'s minimax strategy is to select one finger with probability 5/8 (and two fingers with probability 3/8), securing, of course, a game value of −1/8.

Three-Fingered Morra

In this relatively recent version, each of the two players exhibits one, two, or three fingers while guessing at the number of fingers that his opponent will show. If just one player guesses correctly, he wins an amount equal to the sum of the fingers displayed by himself and his opponent. Otherwise, the trial results in a draw. Letting (i, j) be the strategy of showing i fingers while guessing j $(i, j = 1, 2, 3)$, the 9×9 payoff matrix can be evolved as illustrated in Figure 5-6.

	Player **B**								
	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
Player **A**	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
A_1 (1,1)	0	2	2	–3	0	0	–4	0	0
A_2 (1,2)	–2	0	0	0	3	3	–4	0	0
A_3 (1,3)	–3	0	0	–3	0	0	0	4	4
A_4 (2,1)	3	0	3	0	–4	0	0	–5	0
A_5 (2,2)	0	–3	0	4	0	4	0	–5	0
A_6 (2,3)	0	–3	0	0	–4	0	5	0	5
A_7 (3,1)	4	4	0	0	0	–5	0	0	–6
A_8 (3,2)	0	0	–4	5	5	0	0	0	–6
A_9 (3,3)	0	0	–4	0	0	–5	6	6	0

FIGURE 5-6 Payoff matrix for Three-Fingered Morra.

Applying a system of equations, as in Eq. 2-30, to determine the optimal strategies, we can demonstrate that there are four basic mixed strategies for each player:

$$S_A^* = S_B^* = \begin{cases} S_1: & (0, \; 0, \; 5/12, \; 0, \; 4/12, \; 0, \; 3/12, \; 0, \; 0) \\ S_2: & (0, \; 0, \; 16/37, \; 0, \; 12/37, \; 0, \; 9/37, \; 0, \; 0) \\ S_3: & (0, \; 0, \; 20/47, \; 0, \; 15/47, \; 0, \; 12/47, \; 0, \; 0) \\ S_4: & (0, \; 0, \; 25/61, \; 0, \; 20/61, \; 0, \; 16/61, \; 0, \; 0) \end{cases}$$

Actually, Three-Fingered Morra has an infinite number of optimal mixed strategies, since each convex linear combination of these four mixed strategies is also optimal. That is, for an arbitrary parameter λ, $0 < \lambda < 1$, the mixed strategy $[\lambda S_1 + (1 - \lambda)S_2]$, which is a convex linear combination of S_1 and S_2, constitutes an optimal strategy. Since the game matrix is symmetric, the value of the game is zero.

Among a set of optimal mixed strategies, there may exist one or more that are preferable in the event of an opponent's departure from optimality. In selecting such a particular mixed strategy, we assume that **B** commits an error, the consequences of which he attempts to minimize, while **A** acts to maximize his advantage. The procedure employed by **A** to determine his selection is developed as follows.

In the payoff matrix, Figure 5-6, **A**'s nine pure strategies (*i, j*) are replaced by his four optimal mixed strategies, resulting in the 4×9 matrix of Figure 5-7.

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
S_1	2/12	0	0	1/12	0	1/12	0	0	2/12
S_2	4/37	0	0	0	0	3/37	0	4/37	10/37
S_3	8/47	3/47	0	0	0	0	0	5/47	8/47
S_4	14/61	4/61	0	5/61	0	0	0	0	4/61

FIGURE 5-7 Reduced Morra payoff matrix.

Removing from this matrix **B**'s pure strategies—namely (1, 3), (2, 2), and (3, 1)—that yield the value of the game (0) against every optimal mixed strategy of **A**'s, we are left with the further reduced 4×6 matrix R:

$$R = \left\| \begin{matrix} 2/12 & 0 & 1/12 & 1/12 & 0 & 2/12 \\ 4/37 & 0 & 0 & 3/37 & 4/37 & 10/37 \\ 8/47 & 3/47 & 0 & 0 & 5/47 & 8/47 \\ 14/61 & 4/61 & 5/61 & 0 & 0 & 4/61 \end{matrix} \right\|$$

Again applying Eq. 2-30 to determine the solutions of R, we find two mixed strategies:

$$S_{\mathrm{A}}'^{*} = \begin{cases} S_1': & (120/275, \quad 0, \quad 94/275, \quad 61/275) \\ S_2': & (12/110, \quad 37/110, \quad 0, \quad 61/110) \end{cases}$$

or, in terms of **A**'s nine pure strategies, S_1' and S_2' each translates to the same solution of the original payoff matrix. Specifically,

$$S_A^* = (0,0,23/55,0,18/55,0,14/55,0,0) \qquad (5\text{-}36)$$

The value of the game to **A** with payoff matrix R is 2/55. Thus, if **A** employs the particular optimal mixed strategy of Eq. 5-36, he maximizes his minimum gain accruing from any deviation of **B**'s from an optimal strategy, winning at least 2/55 units in the event.

The Problem of Points

The second of the two classic queries that the Chevalier de Méré posed to his friend Blaise Pascal concerns the problem of points (which helped to inspire the development of probability concepts in Renaissance Europe). In its general form, this problem pertains to a game between two players who have interrupted their play at a time when **A** needs *n* points to win the stakes and **B** needs *m* points

(Ref. Siegrist). If **A**'s probability of winning a single point is p, and **B**'s single-trial probability of winning is q (p and q can be a measure of the players' relative skills), the question to be answered is how the stakes should be divided at the time of the game's interruption.

Pascal provided a solution for p and q; however, to Montmort is credited the generalized solution. **A**'s probability $P_A(n, m, p)$ of winning the game can be derived in the form

$$P_A(n,m,p) = \sum_{k=n}^{n+m-1} \binom{k-1}{n-1} p^n (1-p)^{k-n} \tag{5-37}$$

An equitable division of the stakes then specifies a ratio of $P_A(n,m,p)$ to **A** and $1 - P_A(n,m,p)$ to **B**. If $p + q < 1$, the remaining probability $(1 - p - q)$ represents a drawn contest. In either case, the stakes should be divided equitably in the same ratio.

As an example, **A** and **B** flip a fair coin ($p = q = 1/2$), with **A** winning a point for every H, and **B** for every T. The first player to gain 10 points wins the stakes. If this game is interrupted when **A** has seven points and **B** five points, Eq. 5-37 is evaluated as

$$P_A(3,5,1/2) = \sum_{k=3}^{7} \binom{k-1}{2} (1/2)^k = 99/128$$

and the stakes are divided in the ratio of 99 to 29, with **A**, of course, awarded the lion's share.

For fixed values of n and m, $P_A(n,m,p)$ increases from 0 to 1 as p increases from 0 to 1.

For the case of $n = m$ (**A** and **B** having won an equal number of points at the time of the game's interruption), $P_A(n,m,p)$ represents the probability that **A** wins the best n out of $2n - 1$ series—for example, the Baseball World Series (q.v.), $n = 4$. Also, $P_A[n,n,(1-p)] = 1 - P_A(n,n,p)$ for any n and p.

For the case of $m > n$, we can state $P_A(m,m,p) > P_A(n,n,p)$ iff $p > 1/2$.

Scissors, Paper, Rock

In a well-known children's game, **A** and **B** simultaneously display a hand in one of three configurations: two fingers in a "V" to represent "scissors" (S), palm down with fingers outstretched to represent "paper" (P), or fist clenched to represent "rock" (R). The winner is decided according to the rule: "scissors cut paper; paper covers rock; rock smashes scissors." Should both players display the same configuration, the game is drawn.[6]

[6] *Hasami, kami iwá*, better known as *Jan-Ken*, apparently originated in Japan circa 200 B.C. The Inaugural Rock-Paper-Scissors League Championship was held in Las Vegas, NV, April 2006, following months of regional qualifying tournaments.

This format defines an (uninspiring) nontransitive game with the obvious equilibrium strategy (Chapter 2) of selecting each configuration with probability 1/3. (Other strategies are susceptible to detection and exploitation by an intelligent opponent.) The player adhering to this strategy can sustain neither a long-term loss nor a long-term gain since his expected return is zero against *any* opponent's strategy.

To heighten the interest (somewhat), let S score 1 against P; let P score 2 against R; and let R score 3 against S. Each player should then adopt a mixed strategy, selecting S, P, and R with probabilities p_s, p_p, and p_r, respectively ($p_s + p_p + p_r = 1$). From the game matrix, Figure 5-8,

		B		
		S	P	R
	S	0	1	–3
A	P	–1	0	2
	R	3	–2	–0

FIGURE 5-8 Scissors-paper-rock game matrix.

A's average gain is

$$\begin{array}{ll} 3p_r - p_p & \text{when } \mathbf{B} \text{ selects scissors} \\ p_s - 2p_r & \text{when } \mathbf{B} \text{ selects paper} \\ 2p_p - 3p_s & \text{when } \mathbf{B} \text{ selects rock} \end{array}$$

While these three quantities cannot simultaneously be positive, **A** can guarantee an equitable game by setting

$$p_s = 1/3, \quad p_p = 1/2, \quad p_r = 1/6$$

Thus **A**'s expectation is zero (as is **B**'s) regardless of the opponent's strategy. Moreover, it should be noted that by exercising this zero-expectation strategy, **A** foregoes the option of exploiting any deviation by **B** from the identical strategy.

More generally, if scissors scores *a* against paper, paper *b* against rock, and rock *c* against scissors, **A** can guarantee a fair game by setting

$$p_p = \frac{c}{a+b+c} \qquad p_s = \frac{b}{a+b+c} \qquad p_r = \frac{a}{a+b+c}$$

A Three-Player Game

A, **B**, and **C** each play one game apiece against two opponents. In the event of a two-way tie for first place, the two contenders continue to play until one scores a win. In the event of a three-way tie, the match is begun a new. Let $P_w(i,j)$

and $P_t(i,j)$ represent the probabilities, respectively, of player i winning and tying player j. For a numerical example, let

$$P_w(\mathbf{A},\mathbf{B}) = 0.1 \text{ and } P_t(\mathbf{A},\mathbf{B}) = 0.9$$

$$P_w(\mathbf{A},\mathbf{C}) = 0.2 \text{ and } P_t(\mathbf{A},\mathbf{C}) = 0.8$$

$$P_w(\mathbf{B},\mathbf{C}) = 0.5 \text{ and } P_t(\mathbf{B},\mathbf{C}) = 0$$

Then the respective probabilities P_A, P_B, and P_C of winning the match are given by

$$\begin{aligned} P_A &= 0.1\times0.2\times0.5+0.1\times0.2\times0.5+0.1\times0.8\times0.5 \\ &\quad +0.1\times0.8\times0.5+0.9\times0.2\times0.5 \\ &= 0.28 \end{aligned}$$

$$P_B = 0.9\times0.8\times0.5 = 0.36$$

$$P_C = 0.9\times0.8\times0.5 = 0.36$$

Although **A** is clearly the best player, holding a winning percentage against both **B** and **C**, he has the lowest probability of winning the match—a counter-intuitive result.

Left-Over Objects

From a set of n objects 1, 2, 3, …, n, each of m players independently and hypothetically selects a single one at random. The probability of a particular object being selected by a player is $1/n$. With m independent trials, we are concerned with the binomial distribution. The expected number of players choosing a particular object is m/n, and the probability $P(0)$ that no player chooses that object is

$$P(0) = (1 - 1/n)^m = [(n - 1)/n]^m$$

The expected number of objects left over is then

$$nP(0) = n\left[\frac{n-1}{n}\right]^m = \frac{(n-1)^m}{n^{m-1}} \tag{5-38}$$

For $n = m = 10$, there will be 3.49^- objects left over. For $n = m = 50$, that number will be 18.21^-. For $n = m = 100$, the number increases to 36.64, while for $n = m = 1000$, there will be 371.5 objects unchosen. At $n = m = 10{,}000$, the number of unchosen objects rises to 3981.

Nontransitive Racing Teams

Nine runners are ranked 1, 2, …, 9 according to their order of qualifying. Three teams are then chosen: team **A** composed of runners 1, 6, 8; team **B** composed of runners 2, 4, 9; and team **C** composed of runners 3, 5, 7.

Conventionally, the winner of each race is awarded 1 point, the runner-up 2 points, and so forth. The winning team is that group scoring the least number of points.

In the two-team match-up of **A** versus **B**, runners from **A** will finish first, fourth, and fifth for a total of $1 + 4 + 5 = 10$ points; team **B** runners will finish second, third, and sixth (11 points). Matching **B** versus **C**, runners from **B** will finish first, third, and sixth (10 points); runners from **C** will finish second, fourth, and fifth (11 points). Finally, with **C** versus **A, C** runners will finish second, third and fifth (10 points), while **A** runners will finish first, fourth, and sixth (11 points). Thus

Team **A** beats team **B** beats team **C** beats team **A**

each by a score of 10 to 11.

Note that the numbers of each team form the rows or columns of the third-order magic square (albeit not in ascending sequence).

En Queue Circulaire

Let n players, numbered 1, 2, …, n, contend in cyclical order for an event with single-trial probability of success p. Thus the ith player is afforded his trial only if the previous $i - 1$ players have failed. If he too fails, his second opportunity arrives, if at all, only after another $n - 1$ failures. In general, he has an opportunity to succeed every $(i+kn)$th trial, $k = 0, 1, \ldots$.

A straightforward derivation for the ith player's probability of success, $P_{i,n}(p)$, results in

$$P_{i,n}(p) = p[q^{i-1} + q^{n+i-1} + q^{n+i} + \cdots] = \frac{pq^{i-1}}{1 - q^n} \tag{5-39}$$

where, as usual, $q = 1 - p$.

As one example, consider $n = 2$ players alternately tossing a fair coin until one obtains Heads. Then, for, $p = 1/2$, Eq. 5-39 results in

$$P_{1,2}(1/2) = \frac{1/2}{1 - (1/2)^2} = \frac{2}{3}$$

for the first player's probability of success. The second player's win probability is

$$\mathrm{P}_{2,2}(1/2) = \frac{(1/2)(1/2)}{1 - (1/2)^2} = \frac{1}{3}$$

With $n = 3$ players tossing a coin in continuing succession until Heads appears,

$$P_{1,3}(1/2) = 4/7; \quad P_{2,3}(1/2) = 2/7; \quad P_{3,3}(1/2) = 1/7$$

As the number of players becomes large, $P_{i,n}(p) \to pq^{i-1}$, and only the first few players (obviously) can anticipate a reasonable chance of success.

With $n = 2$ and, $p = 1/6$—the "Dual Russian Roulette" game—two players alternately fire a six-shot revolver, one chamber of which is loaded (the cylinder is randomly spun prior to each firing). The probability that the first player "succeeds" in suicide is

$$P_{1,2}(1/6) = \frac{1/6}{1-(5/6)^2} = \frac{6}{11}$$

And the second player's probability of firing the single bullet is

$$P_{2,2}(1/6) = 5/11$$

Note that if the cylinder is *not* spun, we have the trivial situation wherein each player "succeeds" with probability 1/2 regardless of the order of firing.

An Unfair Game

Consider the payoff matrix of Figure 5-9, which might masquerade as a fair game since the sum of what **A** can win $(1 + 4)$ equals the sum of what **B** can

		B	
		B_1	B_2
A	A_1	1	−2
	A_2	−3	4

FIGURE 5-9 Payoff matrix for an unfair game.

win $(2 + 3)$. Since 1 and 4 in one diagonal are each larger than either of the other two payoffs, this matrix offers no saddle point (Chapter 2). Consequently, **A** and **B** must employ mixed strategies.

To determine these strategies, we take the difference of the payoffs in the second row, $-3 - 4 = -7$, and the difference of the payoffs in the first row, $1 - (-2) = 3$. Thus **A**'s best strategy is to mix A_1 and A_2 in the proportion 7:3.

The column differences are $-2 - 4 = -6$ and $1 - (-3) = 4$. Thus, **B**'s best strategy is to mix B_1 and B_2 in the proportion 6:4.

The value of the game to **A** is

$$\frac{1\times 4 - (-2)(-3)}{1\times 4 - (-2) - (-3)} = -\frac{2}{9}$$

Thus **A**, using his optimal mixed strategy, averages a loss of 2/9 unit per game.

Gold and Brass Coins

A is given kn gold coins and kn brass coins, $k = 1,2, \ldots,$ to distribute as he wishes among n boxes, which are then shuffled. **A** selects one box at random and (blindly) withdraws a coin, winning if he has obtained a gold coin.

Obviously, to deposit k gold coins and k brass coins in each box offers a 0.5 probability of withdrawing a gold coin. Optimal strategy, on the other hand, consists of depositing a single gold coin in each of $n - 1$ boxes and the remaining $2kn + 1 - n$ coins in the last box. Following this procedure results in a probability of winning.

$$\frac{n-1}{n} + \frac{1}{n}\left[\frac{kn+1-n}{2kn+1-n}\right] \tag{5-40}$$

For $2kn = 100$ coins, **A**'s winning probability equals

$$\frac{n-1}{n} + \frac{1}{n}\left[\frac{51-n}{101-n}\right] = \frac{101n - 50 - n^2}{n(101-n)} \tag{5-41}$$

With two boxes, a single gold coin in one box and 99 coins in the other, **A**'s probability of withdrawing a gold coin is

$$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{49}{99} = 0.75$$

a substantial improvement over the result from distributing the coins equally between the two boxes. A fair game would require an entrance fee of 49/99 against a payoff of 1.

If the 100 coins are distributed optimally among four boxes (a single gold coin in each of three boxes and 97 coins in the remaining box), Eq. 5-41 yields

$$\frac{404 - 50 - 16}{4\cdot 97} = \frac{169}{194} = 0.87$$

For large values of k with respect to n, Expression 5-40 becomes

$$\frac{n-1}{n} + \frac{1}{2n} = \frac{2n-1}{2n}$$

and approaches unity with an increasing number of boxes.

Homeotennis, Anyone?

Competitors **A** and **B** engage in a sequence of Bernoulli trials wherein **A** has a single-trial probability p of success, and **B**'s probability of success is $q = 1 - p$. The first player to score at least m successes (points) wins the game, with the proviso that m must exceed his opponent's score by $k \geq 2$ points.

This proviso decreases the probability of success for the player whose single-trial probability of winning a point is less than 0.5. The game matrix (a "generalized tennis matrix" [Ref. Stewart]) consists of an $m \times m$ array plus a "panhandle" to accommodate the requirement of winning by more than a single point: That is, $\|a_{i,j}\|$, where $a_{i,j} = a_{i-1+j} + a_{i,j-1} + a_{m+1,m-k+1} + a_{m+2,m-k+2} + \ldots$ with a and $a_{i,j} = a_{j,i}$ for $1 \le i \le m$, $1 \le j \le m - k$. And the probability $P_{m,k}(p)$ of winning by k points in a game of m points is

$$P_{m,k}(p) = \sum_{i=0}^{m-k} a_{m,i} + \sum_{i=1}^{\infty} a_{m+i,m-k+i}$$

where both sides of the equation depend on p. The first summation represents the $m \times m$ array, and the second represents the panhandle.

For any $m \ge k$, with $k = 2$ or $k = 3$, the second expression converges nicely. For $k = 4$, it is still expressible in closed form.

With $k = 2$, the probability of winning becomes

$$P_{m,2}(p) = p^m \left[\sum_{i=0}^{m-3} \binom{m+i-1}{i} q^i + \frac{\binom{2m-3}{m-2} q^{m-2}}{1-2pq} \right] \tag{5-42}$$

And with $k = 3$,

$$P_{m,3}(p) = p^m \left[\sum_{i=0}^{m-3} \binom{m+i-1}{i} q^i + \frac{\binom{2m-3}{m-2} pq^{m-2}}{1-3pq} \right] \tag{5-43}$$

In tennis, a game winner requires at least $m = 4$ points.[7] Thus, with $p = 0.45$, for example, Eq. 5-42 yields

$$P_{4,2}(0.45) = 0.377$$

Cf. the game matrix of Figure 5-10. And the probability that the player with a single-point probability of 0.45 wins a set is then given by Eq. 5-43:

$$P_{6,2}(0.377) = 0.178$$

Cf. the set matrix of Figure 5-11.

[7]The archaic scoring system of 15, 30, 45, game originated in the late 19th century when the umpire displayed a numberless clock face with two hands, the larger one for the server. Each point scored was recorded by moving the appropriate hand one quadrant. ("45" was later simplified to "40.")

B \ A	0	1	2	3	4	5	6	7	...
0	1	1	1	1	1				
1	1	2	3	4	4				
2	1	3	6	10	10				
3	1	4	10	20	20	20			
4	1	4	10	20	40	40	40		
5				20	40	80	80	80	
6					40	80	160	160	...
7						80	160	320	...
8							160	320	...
9								320	...

FIGURE 5-10 The tennis game matrix; $m = 4$, $k = 2$.

B \ A	0	1	2	3	4	5	6	7	8	9	
0	1	1	1	1	1	1	1				
1	1	2	3	4	5	6	6				
2	1	3	6	10	15	21	21				
3	1	4	10	20	35	56	56				
4	1	5	15	35	70	126	126				
5	1	6	21	56	126	252	252	252			
6	1	6	21	56	126	252	504	504	504		
7						252	504	1008	1008	1008	
8							504	1008	2016	2016	...
9								1008	2016	4032	...
10									2016	4032	...
11										4032	...

FIGURE 5-11 The tennis set matrix; $m = 6$, $k = 2$.

Each entry *ij* (player **A** has scored *i* points to **B**'s *j* points) in these two matrices represents the number of ways that the score *i* to *j* can be reached.

For a second example, let, $p = 0.4$. Then

$$P_{4,2}(0.4) = 0.264$$

for the probability of winning a game, while the probability of winning a set decreases to

$$P_{6,2}(0.264) = 0.0339$$

To illustrate how increasing *k* further decreases the probability of success for the weaker player, consider a match wherein a game must be won by three points and a set by three games. Then

$$P_{4,3}(0.45) = 0.348$$

$$P_{6,3}(0.348) = 0.0962$$

With $k = 4$, and with all values of m,

$$P_{m,4}(p) = p^m \left\{ \begin{array}{l} \displaystyle\sum_{i=0}^{m-4} \binom{m+i-1}{i} q^i + \binom{2m-4}{m-3} pq^{m-3} \frac{\binom{2m-4}{m-3}}{m-2} \cdot \\ \displaystyle\sum_{i-2}^{\infty} [(7m-11)\mathrm{a}_{i-1} + (3m-5)\mathrm{b}_{i-1}]\, p^i q^{m+i-4} \end{array} \right\}$$

where $\mathrm{a}_i = \mathrm{a}_{i-1} + \mathrm{b}_{i-1}$; $\mathrm{b}_{i-1} = \mathrm{a}_{i-1} + (m-1)\mathrm{b}_{i-1}$; and $\mathrm{a}_1 = \mathrm{b}_1 = 1$.

With $k \geq 5$, it appears that the probability of winning does not embrace a well-behaved expression but becomes dramatically more intricate with increasing values of k.

Tennis, in reality, reveals more complexity than the format explicated here since the probability of success (winning a point) for most players is greater when serving than when receiving. Alas, partitioning the probabilities to account for this circumstance destroys convergence of the panhandle portion of the game matrix.

Duration of a Tennis Set

The expected duration of a tennis game between two equally skilled players was computed by Cooper and Kennedy (Ref.) as 6.75 points. Further, a general expression was derived for the expected game duration as a function of the probability p that one player wins a particular point. If $p = 0.6$, for example, the expected duration is only reduced to 6.48 points (somewhat surprisingly).

In its classical format, a tennis set is won by that player who wins at least six games, with the proviso that he scores two more than his opponent.[8] To determine the expected duration of a set we first lay out the set state diagram (Figure 5-12), which designates the possible transition probabilities, for players **A** and **B** having respective probabilities p and $q = 1 - p$ of winning any particular game.

Each score describes a state of the contest—as games are won or lost, we move from one state to another until reaching the set state. (Arrows pointing rightward represent **A** winning a game; those pointing leftward represent **B** winning a game.)

Expected Duration (Ref. Cooper and Kennedy)

Defining c_n as the probability that the set ends after the nth game, the expected duration of the set is expressed as

$$\sum_{n=1}^{\infty} n\mathrm{c}_n \qquad (5\text{-}44)$$

[8] Current rules (not treated here), invoke a tie-breaking system if games are even at 6-all—except for the deciding set of a 2-out-of-3- or 3-out-of-5-set match.

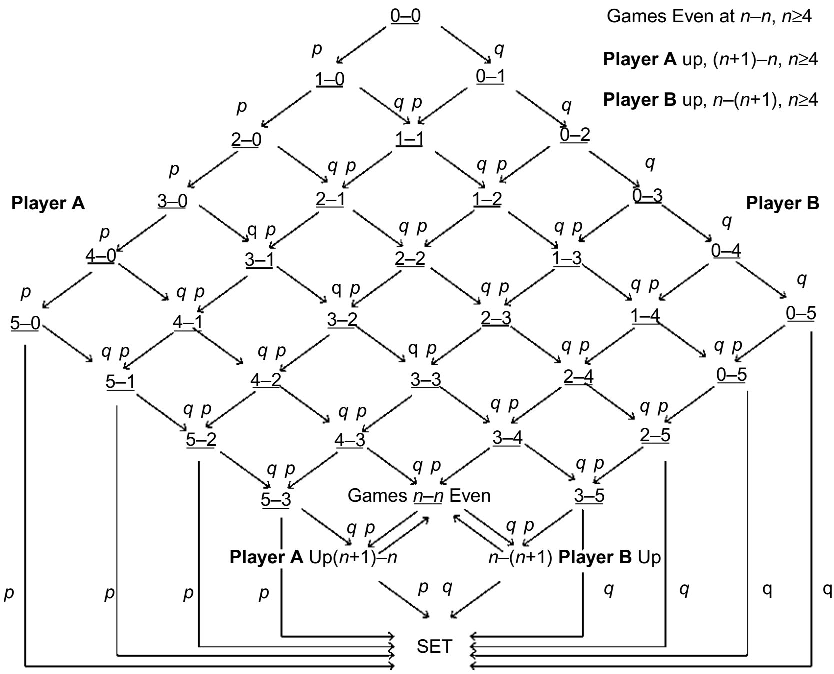

FIGURE 5-12 Set state diagram.

Since values of c_n form a binomial distribution, we can immediately determine c_n for $n = 10$—specifically,

$$c_n = \binom{n-1}{n-6}\left[p^6(1-p)^{n-6} + p^{n-6}(1-p)^6\right] \quad \text{for } 6 \leq n \leq 10$$

and $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. These values of c_n are listed in Table 5-3.

After ten games, the set will be resolved only when an even number of games has been played; thus

$$c_{2n+1} = 0 \quad \text{for } n \geq 5$$

For the set to be resolved at the 12th game, the score must have been 5-5 at the tenth game, and either **A** or **B** must win both the 11th and 12th games. Hence,

$$c_{12} = \binom{10}{5} p^5(1-p)^5\left[p^2 + (1-p)^2\right]$$

which can be generalized to

$$\begin{aligned} c_{2n} &= \binom{10}{5} p^5(1-p)^5\left[2p(1-p)\right]^{n-6}\left[p^2 + (1-p)^2\right] \\ &= 2p(1-p)c_{2n-2} \quad \text{for } n \geq 6 \end{aligned}$$

Table 5-3 Probabilities for Set Ending after *n*th Game

n	c_n
1	0
2	0
3	0
4	0
5	0
6	$p^6 + (1-p)^6$
7	$\binom{6}{1}[p^6(1-p) + p(1-p)^6]$
8	$\binom{7}{2}[p^6(1-p)^2 + p^2(1-p)^6]$
9	$\binom{8}{3}[p^6(1-p)^3 + p^3(1-p)^6]$
10	$\binom{9}{4}[p^6(1-p)^4 + p^4(1-p)^6]$
11	0
12	$\binom{10}{5}[p^5(1-p)^5][p^2 + (1-p)^2]$
13	0
For $n = 2m, m \geq 7, c_n =$	$\binom{10}{5}[p^5(1-p)^5]2^{m-6}[p^{m-4}(1-p)^{m-6} + p^{m-6}(1-p)^{m-4}]$

Finally, to evaluate Eq. 5-44 we specify the generating function

$$f(x) = \sum_{n=1}^{\infty} c_n x^n$$

which leads to

$$f(x)[1 + 2p(p-1)x^2] = \sum_{n=1}^{\infty} c_n[x^n + 2p(p-1)x^{n+2}]$$

Developing this equation (with the non-zero values of c_n):

$$f(x) = \frac{\sum_{n=5}^{35} [c_n + 2p(p-1)c_{n-2}]x^n}{1 + 2p(p-1)x^2}$$

Simplifying:

$$\begin{aligned} f(x) = \{&c_6x^6 + c_7x^7 + [c_8 + 2p(p-1)c_6]x^8 + [c_9 + 2p(p-1)c_7]x^9 \\ &+ [c_{10} + 2p(p-1)c_8]x^{10} + [2p(p-1)c_9]x^{11}\}[1 + 2p(1-p)x^2]^{-1} \end{aligned}$$

Computing the first derivative with $x = 1$, and collating all 163 terms (!):

$$\begin{aligned} f'(1) \equiv F(p) = \{&6 - 18p + 30p^2 - 18p^3 + 6p^4 + 258p^5 - 2046p^6 + 6864p^7 \\ &- 12948p^8 + 14880p^9 - 10368p^{10} + 4032p^{11} - 672p^{12}\} \\ &\bullet [1 - 4p + 8p^2 - 8p^3 + 4p^4]^{-1} \end{aligned}$$

where F(p) is defined on the interval [0,1].

Thence F(0) = F(1) = 6, and F(1/2) = $10\frac{1}{32}$. So with two equally matched players, the expected set duration is slightly more than ten games. When one player's game-winning probability is 0.6, the expected set duration is reduced to 9.60 games.

Reality Check

The probability of a player winning a game rarely remains constant and instead strongly depends on whether he serves or receives. A model closer to reality assigns a value p as *either* player's probability of winning a game as server and the complementary value $1 - p$ as *either* player's probability of winning a game as receiver.

Alas, introducing different probabilities for serving and receiving increases the "messiness" of the (already messy) equations for expected set duration by an order of magnitude. The solution is readily obtained, however, by recourse to a simulation program.[9] Table 5-4 enumerates, for this model, the expected set duration for several values of p. As p approaches 1, expected set duration reaches toward infinity—although it increases quite slowly for moderate values of p before rapidly rising to impractical heights.

Table 5-4 Expected Set Duration for Complementary Probabilities of Winning a Game

$p =$	0.5	0.55	0.6	0.75	0.90	0.99
Expected Set Duration in Games	10.03125	10.0546	10.1273	10.803	14.759	101.5

A yet more realistic model assigns noncomplementary game-winning probabilities to each of the two players, injecting yet another layer of messiness into a closed-form solution. Letting p equal the probability that one player wins his serve, and q equal the probability that the second player wins *his* serve, a simulation program q indicates that for $p = 0.6$ and $q = 0.55$, the

[9] Courtesy of Norman Wattenberger, Casino Vérité.

expected duration is 10.048. For $p = 0.65$ and $q = 0.6$, the expected duration is 10.1426.

Probability of Winning a Set

From the set state diagram, Figure 5-12, we can deduce the probabilities g(p) of winning the set in 6, 7, 8, … games. Specifically,

$$g(p) = p^6 + \binom{6}{1} p^6 (1 - p) + \binom{7}{2} p^6 (1 - p)^2 + \binom{8}{3} p^6 (1 - p)^3 + \binom{9}{4} p^6 (1 - p)^4 + \binom{10}{5} p^7 (1 - p)^5 \sum_{i=0}^{\infty} [2p(1 - p)]^i$$

Collating the powers of p,

$$g(p) \equiv G(p) = \frac{210p^6 - 888p^7 + 1545p^8 - 1370p^9 + 616p^{10} - 112p^{11}}{1 - 2p + 2p^2}$$

where G(p) is defined on the interval [0, 1]. As expected, G(0) = 0, G(1) = 1, and G(1/2) = 1/2; for values of p greater than 1/2, $G(p) > p$. Further, $G(1 - p) = 1 - G(p)$. When one player has probability $p = 0.55$ of winning each game, we have G(0.55) = 0.64 for the probability of winning the set. For $p = 0.6$, G(0.6) equals 0.77.

Baseball World Series

In the World Series of Baseball, the first team to win four games wins the series. Team **A** plays "home games" 1, 2, and, if necessary, 6, 7. Games 3, 4, and, if necessary, 5 are played on team **B**'s home field. If we assume that the probability of either team winning each game is 1/2, then the probability, $P_{n-1}(3)$, that either team has won exactly three games out of the last $n - 1$ is $2^{1-n}\binom{n-1}{3}$. Thus:

$$P_{n-1}(3) = 1/8,\ 1/4,\ 5/16,\ 5/16 \qquad \text{for } n = 4, 5, 6, 7$$

The expected number of home games for team **A** is, therefore,

$$2(1/8) + 2(1/4) + 3(5/16) + 4(5/16) = 47/16 = 2.9375$$

while the expected number of home games for team **B** is

$$2(1/8) + 3(1/4) + 3(5/16) + 3(5/16) = 46/16 = 2.875$$

Team **A** will, therefore, play home games 50.5% of the time (with equally matched teams).

If team **A** has a single-game probability of winning equal to 0.6, its probability of winning the series is

$$.6\sum_{n=3}^{6} P_n = .6\left[\binom{3}{3} .6^3 \cdot .4^0 + \binom{4}{3} .6^3 \cdot .4^1 + \binom{5}{3} .6^3 \cdot .4^2 + \binom{6}{3} .6^3 \cdot .4^3\right] = 0.710$$

With a single-game winning probability of 0.55, the probability of winning the series is 0.608.

Mathematicians wish to bet on the World Series by risking one unit on the outcome, but also wish to wager on each individual game. To accomplish this goal, consider that just prior to each game the bettor's bankroll must be one-half the sum of the two ensuing outcomes. Thus, prior to game 7, net gain in each bankroll must be 0. Prior to game 6, the net gains must be +0.5 (for three wins and two losses) or −0.5 (for two wins and three losses), etc., as shown in Table 5-5.

Table 5-5 Bets for a Win–Loss of 1 Unit

At Game	1	2	3	4	5	6	7
Result	1–0 (5/16) or 0–1 (−5/16)	2–0 (5/8), 1–1 (0), or 0–2 (−5/8)	3–0 (7/8), 2–1 (3/8), 1–2 (−3/8), or 0–3 (−7/8)	4–0 (1), 3–1 (3/4), 2–2 (0), 1–3 (−3/4), or 0–4 (−1)	4–1 (1), 3–2 (1/2), 2–3 (−1/2), 1–4 (−1)	4–2 (1), 3–3 (0), or 2–4 (−1)	4–3 (1) or 3–4 (−1)

The quantities in parentheses represent the gains (as a fraction of one unit) from a wager on that game.

The Price Is Right, Showdown Showcase

In the TV game show, "The Price Is Right," Showdown Showcase subdivision, three players, **A**, **B**, and **C**, compete in sequence by spinning a wheel with 20 increments labeled 0.05, 0.10, … 1.00 and recording the number indicated. Each player then has the option of spinning the wheel a second time and adding that number to the first. If the sum of the two numbers exceeds 1.0, that player is eliminated from the game. High score ≤1.0 wins the prize. In the event of a two-way or three-way tie, the winner is determined randomly.

Optimal strategy dictates that:

Player **A** should spin again if his first-spin score is 0.65 or less.

Player **B** should spin again if

(1) his first-spin score is less than **A**'s score (also if **A** has been eliminated *and* **B**'s first-spin score is 0.5 or less); or

(2) his first-spin score is *equal to or greater than* **A**'s score but still less than or equal to 0.5; or

(3) his first-spin score ties that of player **A** at a score less than or equal to 0.65.

Player **C** should spin again if

(1) his first-spin score is less than the higher of **A**'s and **B**'s scores; or

(2) his first-spin score is equal to the higher of **A**'s and **B**'s scores *if* that score is 0.45 or less (if the higher score is 0.50, spinning or not spinning a second time offers equal probabilities); or

(3) his first-spin score results in a three-way tie at 0.65 or less.

The probability of success for each player is composed of three components: when neither opponent is eliminated; when both opponents are eliminated; and when one opponent is eliminated. Respective winning probabilities, Pw, are enumerated as

$$Pw(\mathbf{A}) = 0.285$$

$$Pw(\mathbf{B}) = 0.340$$

$$Pw(\mathbf{C}) = 0.374$$

As is inherently obvious, the game favors **C** over **B** over **A**.

The Monty Hall "Paradox"

A contestant in the "Let's Make a Deal" TV show wins if he guesses which of three doors hides the prize. After tentatively selecting one door, he is informed that one particular door of the two remaining is incorrect and given the option of changing his guess.

Assuming the winning door to be randomly chosen by the show producers, the contestant, by accepting the option, increases his probability of success from 1/3 to 2/3, that being the probability that one of the other two doors was originally hiding the prize. (A value of 1/2 for this latter probability is maintained by many.[10])

If the contestant is afforded the option to change his guess only 50% of the time, his probability of success then improves from 1/3 to 1/2. In general, if the option to change his guess is given with probability p if he has guessed correctly and $1 - p$ if he has guessed incorrectly, he should then switch whenever $2(1 - p) > p$—that is, when $p < 2/3$. With $p = 2/3$, the contestant's decision is inconsequential, and his probability of success remains at 1/3.

Multi–Stage-Door Monty

One of N doors hides the prize. The contestant tentatively selects one door, and Monty opens a nonwinning door. The contestant is then afforded the option to stick with his original door or switch to one of the remaining N − 2 doors. Monty then opens another nonwinning door (other than the current pick), and the contestant makes another stick-or-switch decision. This process continues, the contestant faced ultimately with N − 2 decisions.

[10] The *New York Times* weighed in on this matter with a front-page article (July 21, 1991), and pursued it in a Findings column (April 8, 2008).

Correct strategy dictates that the contestant should stick at each decision point until the last opportunity, then switch. His probability of success is, accordingly, $1 - (1/N)$, since Monty eliminates one door at each stage.

With $N = 4$ doors, a first-stick-then-switch strategy offers the contestant a 0.75 probability of success.

Seven-Door Monty

Here, the contestant tentatively selects three doors out of seven, winning if the prize hides behind any of the three. Monty opens three of the remaining doors, offering the contestant the option of switching to the other unopened door. Correct strategy dictates a switch—thereby increasing the probability of success from 3/7 to 4/7.

In general, if the contestant tentatively selects N doors out of $2N + 1$, and Monty opens N non-winning doors, switching raises the contestant's probability of success from $N/(2N + 1)$ to $(N + 1)/(2N + 1)$.

Quantum Monty

In one variant of the Monty Hall game (Ref. Cheng-Feng Li et al.), the producers select a superposition of three boxes, $|0\rangle$, $|1\rangle$, and $|2\rangle$ for the placement of a quantum particle.[11] The player then chooses a particular box. A second quantum particle is introduced entangled with the first,[12] whereby the producers can make a quantum measurement on this particle as part of their strategy. If this measurement is made after a box is opened, the state of the original particle is evenly divided between the other two boxes. Thus the player has probability 1/2 of winning by either switching or sticking—as opposed to the winning probability of 2/3 gained by switching in the original game.

Another variant, contributed by Flitney and Abbott (Ref.) directly quantizes the game—with no additional quantum particles; producers and player are both afforded access to quantum strategies. Choices in this construct are represented by "qutrits" (the units of information described by a state vector in a three-level quantum-mechanical system). Both choices start in some initial state, and both strategies are operators acting on their respective qutrits. A third qutrit represents the box opened by the producers following the player's selection of a box.

Because each side has access to quantum strategies, maximal entanglement of the initial states produces the same payoffs as in the standard Monty Hall game—that is, a winning probability of 2/3 for the player when he switches boxes. If only one interest—the producers—is allowed a quantum strategy, while the other—the player—is not, the game offers zero expectation. If, conversely, the player is permitted a quantum strategy, while the producers are not, then the player wins with probability 1.

Without entanglement, this quantum game is resolved with a conventional mixed strategy.

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[11] The $|0\rangle$ symbol denotes the Dirac designation for a computational basis state.

[12] Quantum entanglement specifies the condition wherein two or more objects, though separated in space, can only be described with reference to each other. Albert Einstein famously referred to entanglement as *Spukhafte Fernwirkung* ("spooky action at a distance").

The Two-envelope Problem [First proposed by M. Kraitchik (Ref.)]

A player is presented with two envelopes, one containing x dollars and the other 2x dollars. He selects one envelope (at random) with the option, after observing its contents, of exchanging it for the second envelope. This situation defines a symmetrical game, the player's expected return, E, being

$$\mathrm{E} = (x + 2x)/2 = 3x/2$$

whether or not the envelopes are switched.

This game is often described as a paradox since, by switching envelopes, the player is offered an equal chance at double or half the amount in his envelope; thus he should always switch. Further, if he had initially selected the other envelope, he should again choose to switch—in accordance with the same "reasoning." Abbott, Davis, and Parrondo (Ref.) have shown that a switching strategy exists with a positive expectation. In particular, they adopt a strategy that dictates switching to the second envelope with a probability, $p(y)\ \varepsilon\ [0,1]$, that is a function of the amount observed in the first envelope—a concept suggested by Thomas Cover, Stanford Univ.; $p(y)$ is formulated to decrease as y increases. [A sensible desideratum renders it less likely to switch envelopes for large sums.]

Assume $p(y) = e^{-\alpha y}$ as a suitable function that decides whether or not to switch after observing y. Then the average expectation, $\bar{\mathrm{E}}$, can be written in the form

$$\begin{aligned} \bar{E} &= \int_0^\infty \left[\frac{3x}{2} + \frac{x}{2}\left(\mathrm{e}^{-\alpha x} - \mathrm{e}^{-2\alpha x}\right)\right] c\mathrm{e}^{-cx}\,\mathrm{d}x \\ &= \frac{3}{2c} + \frac{c}{2(c+\alpha)^2} - \frac{c}{2(c+2\alpha)^2} \end{aligned} \tag{5-45}$$

Differentiating $\bar{\mathrm{E}}$ with respect to α provides the maximum:

$$\frac{\mathrm{d}\bar{\mathrm{E}}}{\mathrm{d}\alpha} = -\frac{c}{(c+\alpha)^3} + \frac{2c}{(c+2\alpha)^3} = 0$$

Solving the cubic equation for $\alpha = 0.351$ and substituting this value into Eq. 5-45 yields

$$\bar{\mathrm{E}} = 1.60/c$$

Abbott, Davis, and Parrondo further show that if the arbitrary function $p(y)$ is replaced by the optimal switching function, the average return is then improved to

$$\bar{\mathrm{E}} = 1.68/c$$

A practical refinement for the Two-Envelope Problem consists of imposing an upper limit on x (the offered amount). A switching strategy for this case has been developed by McDonnell and Abbott (Ref.).

CASINO GAMES

While many ingenious games could be invented based on coin tossing or its equivalent, gambling emporia have generally ignored these possibilities. Rather, they have emphasized games that offer high odds—a large payoff against a small wager (with correspondingly low probability of winning). Roulette, slot machines, Bingo, lotteries, sweepstakes, pools, and raffles constitute the great majority of the high-payoff games; all are linked by a common characteristic: No nontrivial optimal strategy exists or has a meaning. In these games, once having selected that category of wager that maximizes the mathematical expectation (invariably negative under normal circumstances), any further refinements, such as a process to select a number for betting, is irrelevant (unless the player is guided by some eccentric subjective utility function). Thus, these games are relatively unimaginative and uninteresting since they are completely described by the most trivial aspects of gambling theory.

Roulette

Roulette remains, for many, the most glamorous of all casino games. It also stands as the world's most popular table game (although ranking third in the United States). The Roulette mechanism came together in 1796 as the hybrid of a gaming wheel invented in 1720 (with only black and white slots) and the Italian game *Biribe* that featured a layout of 36 numbers (balls were drawn at random from a sack holding 36 balls, each containing a small piece of parchment engraved with a number from 1 to 36). Introduced first into Paris, its popularity was immediate and overwhelming, quickly proving a particular favorite of the upper classes; further, it seemed immune to the manipulations of the countless sharpers then abounding in the French capital. When the Monte Carlo casino was inaugurated in 1863 (a 0 had been added in 1842 to enable a House advantage), Roulette instantly prevailed as the most frequently played game. The men who "broke the bank at Monte Carlo" almost invariably performed that feat at Roulette ("breaking the bank" referred to winning the capital posted at each table).

The American Roulette mechanism comprises a heavy rotor balanced on a well-lubricated bearing; its circumference is divided into 38 compartments separated by metal frets and numbered from 1 to 36 (colored red or black) plus 0 and 00 (colored green; in crossing from Europe to America, the wheel had acquired a 00). A stator (32 inches across on most wheels, somewhat more than its European counterpart) surrounds the rotor and is inscribed with a horizontal track that guides the roulette ball (commonly 1/2, 5/8, or 3/4 inches in diameter).

A *tourneur* launches the ball along the track in a direction counter to that of the rotor's rotation. When the ball's angular momentum drops below that necessary to hold it to the track, it spirals down across interspersed vanes and deflectors (which affect the ball's trajectory about 50% of the time), crossing onto the rotor, and eventually settling into one of the 38 compartments, thereby delineating the winning number.

The betting layout and configuration of the numbers are illustrated in Figure 5-13. This configuration is not random but represents an attempt to alternate

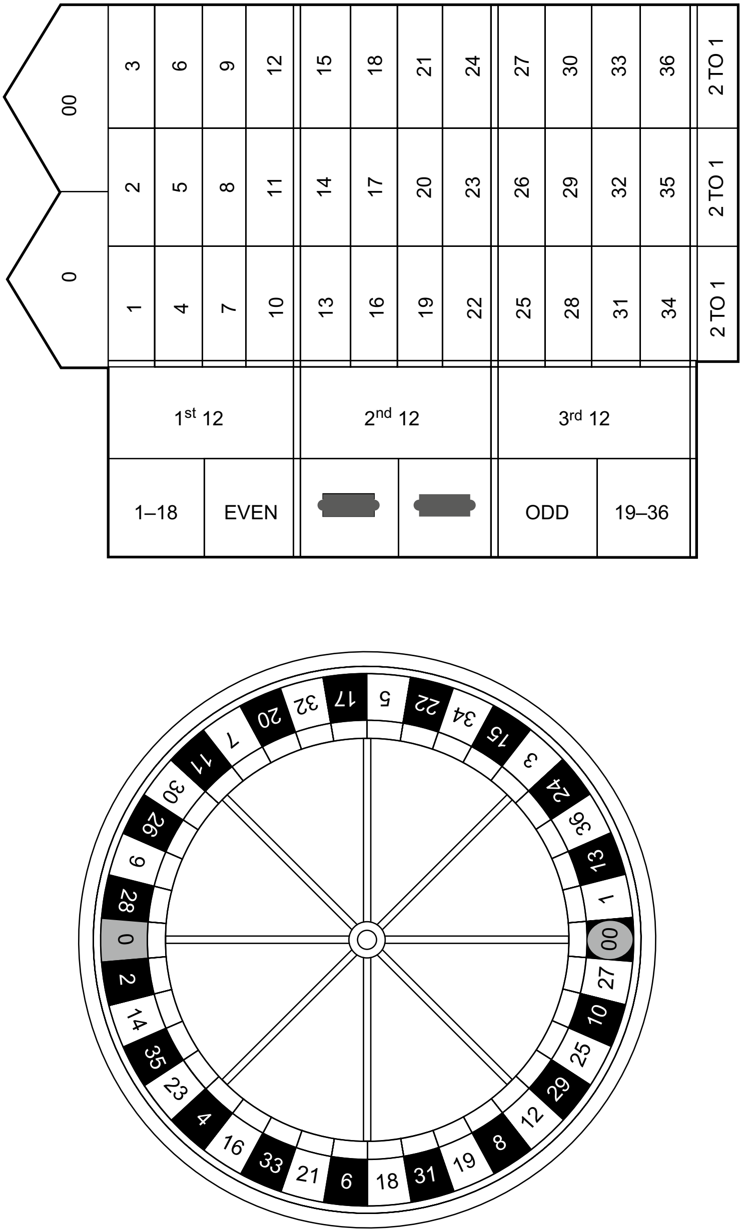

FIGURE 5-13 The American Roulette wheel and layout.

high-low and odd-even numbers as well as the red-black colors. To achieve the best possible distribution of high and low numbers, the sum of each two successive numbers of the same color must equal 37—except for 9 & 28 and 10 & 27, which are of different colors. Directly across from every odd number is the next highest even number. Also, pairs of even numbers alternate with pairs of odd numbers—except for the first pairs counterclockwise of the 0 and 00 (which are opposite each other, the 0 compartment lying between two black numbers, the 00 between two red numbers). A perfect balance is not possible since the sum of numbers 1 to 36 equals 666,[13] while the 18 odd numbers sum to 324.

European and South American wheels are fashioned with a different configuration, although each number retains the same color as its American counterpart. The European game features another concession: When the ball lands on 0, an even-money wager is not lost but held "en prise." If the succeeding spin wins for the player, his wager is released. (Some casinos follow the "La Partage (sharing) rule," returning half the wager when 0 occurs.)

Thirteen different wagers are offered by the Roulette layout, as listed in Table 5-6, with their associated payoffs. The *split* or *à cheval* is effected by placing the stake on any line separating any two numbers. To bet on three numbers, the stake is placed on the side line of the layout—only three consecutive numbers [$(n/3) - 2$, $(n/3) - 1$, and $n/3$] are permitted. For a bet on four numbers, the stake is placed on the intersection of the lines dividing four particular numbers

Table 5-6 Roulette Payoffs and Probabilities

Type of Bet	Payoff	Single-Trial Probability of Success (American wheel)	Single-Trial Probability of Success (European wheel)
Single-number (*en plein*)	35 − 1	0.4737	0.4865
Two-number, or split (*à cheval*)	17 − 1	0.4737	0.4865
Three-number, or "street" (*transversale*)	11 − 1	0.4737	0.4865
Four-number, or square (*carré*)	8 − 1	0.4737	0.4865
Five-number, or "line"	6 − 1	0.4605	
Six-number, or "line" (*sixain*)	5 − 1	0.4737	0.4865
Twelve-number, or column	2 − 1	0.4737	0.4865
Twelve-number, or dozen	2 − 1	0.4737	0.4865
Low or high (*manque*, or *passe*)	Even	0.4737	0.4931
Odd or even (*impair*, or *pair*)	Even	0.4737	0.4931
Red or black (*rouge*, or *noir*)	Even	0.4737	0.4931
Split, or a *cheval* between red/black and odd/even	2 − 1	0.4737	0.4865
Split, or a *cheval* between two columns or two dozens	1 − 2	0.4737	0.4865
Consecutive color	9 − 1	0.4783	

[13] Feared by Christian fundamentalists as "the number of the Beast" (i.e., the Devil; cf. Book of Revelations, 13:16).

according to the arrangement of the layout. The five-number bet is available only for the numbers 1, 2, 3, 0, and 00. For six numbers, the wager is placed on the intersection of the side lines and a cross line.

The method of betting on columns, dozens, high–low, odd–even, and red–black is self-evident from Figure 5-13. A bet placed on the line between red and even (or black and odd) wins only if the number at which the ball comes to rest is both red and even (this bet is not admissible in many casinos). Similarly, a split bet between two columns or two dozens is placed on the line between the columns or dozens, thereby giving the player 24 numbers out of 38 (or 37).[14]

A 14th wager, a recent innovation at some casinos, allows the player to bet that a particular color will immediately appear three times consecutively [probability $= (18/38)^3 = 0.1063$]. The payoff for this wager is 8 to 1 (4.34% take).

In American Roulette, the single-trial probability of success (defined as the probability of occurrence renormalized with respect to an even payoff) of 0.4737 (a House Take of 5.263%) holds for all classes of wagers except the five-number bet, which is inferior (House Take = 7.895%), and the consecutive color bet, which offers a reduced Take of 4.345%. For European Roulette, the House Take on the number bets is 2.703%; the even-payoff bets levy a Take of only 1.388%. For any and all bets, the game remains steadfastly unfavorable to those without recourse to extrinsic measures.

Inexplicably, Roulette has attracted more people with "systems" than any other casino game.[15] The most common approach is based on recording the numbers appearing in play in the hope of detecting a favored one or group. Some then bet on the least favored, following the "compensation principle." Others, with more reason, wager on the most favored. To encourage either practice, small boutiques in Monte Carlo sell recorded Roulette numbers (the Monte Carlo

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[14] Various contrived combinations of numbers have gained popularity at times. One example is the "red snake bet"—adjoining red numbers (1, 5, 9, 12, 13, 16, 19, 23, 27, 30, 32, 34) that "snake" along the betting layout. A bet on all 12 numbers allegedly (miraculously!) produces a positive expectation. Snake oil.

[15] An hypnotically idiot system was offered by a 1959 issue of *Bohemia*, a Cuban magazine. The system purports to exploit the fact that the third column of the betting layout has eight red and only four black numbers. One is advised to wager, consequently, one unit on black and simultaneously one unit on the third column. The "reasoning" then suggests that on the average out of 38 plays, the zero and double-zero each appear once, causing a loss of four units; red appears 18 times, of which eight are in the third column, so that the player will lose the two units ten times and win two units eight times for a net loss of four units in these 18 plays; finally, black appears 18 times, 14 times in the first and second columns, which loses one bet but wins the other for an even break, and four times in the third column, whereupon both bets win for a net gain of 12 units (the column bet pays 2 to 1) over these 18 plays. Overall, the gain is -4 on 0 and 00, -4 on the 18 red numbers, and $+12$ on the 18 black, thereby netting four units for every 38 plays. While the fallacy is evident, the gullibility of the users is beyond comment.

Equally incredible, there have been many well-known personalities who, specializing in Roulette systems, have devoted countless hours to the subject. Sir Arthur Sullivan, for one, devised 30 different systems, none of which bore fruit.

Revue Scientifique publishes each month's log of such numbers). As underscored by Theorem I of Chapter 3, no system can alter the (negative) expected gain.

The prospect of a positive expectation lies in detecting a wheel with sufficient bias. Since the numbered compartments along the circumference of the rotating cylinder are separated by thin metal partitions, or frets, it is not unlikely for one or more of these partitions to become weakened or bent, thus creating a bias (a completely flexible fret would double the expected return from 0.9474 to 1.8948). Irregular bearings, warpage, worn compartments, and other mechanical imperfections might also create a usable bias.[16]

Legion are the university students who have recorded thousands of Roulette spins, usually applying a chi-square test (Eq. 2-28) to ascertain preferred numbers. Some, doubtless, were successful; most soon encountered obstacles such as nonstationary statistics and unsympathetic casino managers who rebalanced or realigned rotors or interchanged stators and rotors.

The first problem faced by the player in search of a bias is the necessity to distinguish a random winning number from a biased winning number. No mechanism as complex as Roulette can be manufactured entirely free of defects; further, ordinary wear and tear will ultimately erode all attempts at precision. To detect and correct defective Roulette mechanisms, casinos schedule periodic maintenance that includes cleaning and oiling the ball bearings; full overhauls, however, are rarely performed more than semiannually. Rough estimates suggest that one in four or five wheels develop exploitable biases.

The wheel must be "clocked" for a sufficient number of spins n to find a winning number that has appeared m times. From the binomial distribution (Eq. 2-11) and using a confidence level of 95%, we can write

$$n_{\max}(0.95) = n/36 + 0.48\sqrt{n}$$

for the (theoretical) maximum number of occurrences of the leading number, given all numbers generated at random. If $m > n_{\max}$, we have a potential bias. (A confidence level of 95% is particularly restrictive; invoked more frequently is an 80% confidence level—which leads to $n_{\max}(0.8) = n/36 + 0.4\sqrt{n}$.)

[16]In 1947, Albert R. Hibbs and Roy Walford, then students at the California Institute of Technology, invaded the Palace and Harold's Club in Reno, Nevada, in search of biased Roulette apparatus. Using a Poisson approximation to the binomial distribution, they established confidence limits for distinguishing biased wheels. With their criterion, approximately one wheel in four exhibited an imbalance sufficient to overcome the House Take. Significant financial success was achieved at the Reno clubs and subsequently at the Golden Nugget in Las Vegas. To eliminate the labor of recording numbers, a stroboscope might be applied to detect any appreciable distortion in the frets between compartments.

Hibbs and Walford were preceded in their exploitation of biased wheels by a British engineer, Joseph Jaggers, "the man who broke the bank at Monte Carlo" (a nonexclusive appellation). Jaggers's success (in July 1875) followed an intensive frequency analysis of Roulette numbers at the Monte Carlo casino. More spectacular were the multi–million-dollar profits accumulated in 1951–1952 by a gambling syndicate at the government-owned casino in Mar del Plata, Argentina—where incompetently maintained wheels invited significant biases.

Clocking a wheel is a laborious and tedious process; in Europe and the Far East, the wheel is spun about 55 times per hour; in U.S. casinos, the rate is approximately 100 spins per hour. Thus several hours can be consumed in determining that a particular wheel is *not* a candidate for positive expectations. For those without the stamina to clock wheel after wheel after wheel, we can only recommend the modified Epaminondas procedure (see Theorem I, Chapter 3, p. 55): Bet that the previous number will repeat, then switch to that number whose cumulative total exceeds all others and continue with that choice until it is replaced by another number with a yet greater cumulative total.

An example reported by Thorp (Ref. 1984) detected a value of $m/n = 1/25$ for $n \approx 1225$ spins (i.e., 49 occurrences of the leading number); here, $n_{\max}(0.95) = 49.9$). Thus, this wheel offers an expectation of

$$E = \frac{49 \times 35 - (1225 - 49)}{1225} = 0.44$$

(with a payoff of $P = 35$ to 1).

Following the Kelly prescription (Chapter 3) of betting a fixed fraction f of the current bankroll (here, f = 0.44/35 = 0.0126), the capital growth rate, G(f), is

$$\begin{aligned} G(f) &= (m/n)\ln(1 + Pf) + [1 - (m/n)]\ln(1 - f) \\ &= 0.04\ln(1 + 35f) + 0.96\ln(1 - f) \end{aligned}$$

and after N bets, the initial bankroll will have grown by an expected $e^{NG(f)}$. In Thorp's example, the growth rate is

$$G(f) = 0.04\ln(1 + 0.44) + 0.96\ln(1 - 0.0126) = 0.00244$$

Hence, after 1000 bets, the initial bankroll will have grown by an expected $e^{2.44} = 11.47$ times.

A substantial advantage accrues when more than one number shows a bias. With two equally favorable numbers, $P = 17$, $m/n = 2/25 = 0.08$, f = 0.44/17 = 0.0259, and the capital growth rate becomes

$$G = 0.08\ln(1 + 0.44) + 0.92\ln(1 - 0.0259) = 0.00525$$

After 1000 bets, the initial bankroll will have been multiplied an expected $e^{5.25} = 190.57$ times. With three equally biased numbers, capital growth after 1000 bets equals 5202 times the initial bankroll.

An alternative approach to achieving a positive expectation relies on the very meticulousness with which casinos maintain the Roulette mechanism in striving to eliminate bias. The physics of the Roulette process suggests the means to predict that compartment where the ball comes to rest.

First, it is necessary to predict when and where the ball will fall from its orbit along the stator track. Assuming that the angular velocity of the ball depends on the number of revolutions remaining before it leaves the track, only the time of a single orbital revolution need be measured. This measurement can be implemented with an electronic clock (recording the instants when the ball passes a reference point on the stator) or with a video or motion-picture camera (with frames synchronized to a stroboscope).

Second, the time interval from the instant the ball leaves the stator track to the instant it spirals onto the rotor must be determined. This time is a constant except when the ball's path is significantly altered by the deflectors.

Third, the rotor's position and angular velocity must be measured at a fixed time, which enables the prediction of where the rotor will be when the ball arrives—and thus predicts the region where the ball comes to rest. Rotor velocity, for the duration of one play, can be assumed constant.

Thorp and Claude Shannon, in 1961, were first to program these parameters into a "wearable" computer (Ref. Thorp, 1984). To describe the number of ball revolutions left as a function of remaining time, $x(t)$, they adopted, as an initial approximation, the formula

$$x(t) = \mathrm{A}e^{\mathrm{B}t} + \mathrm{C}$$

determining that A = 10/3, B = 3/20, C = −10/3, and defining $t = 0$ as the time the ball leaves its track. Knowing T, the time for one ball revolution, the number of revolutions left, $x_0(\mathrm{T})$, is

$$x_0(\mathrm{T}) = \frac{1}{e^{3\mathrm{T}/20} - 1} - \frac{10}{3}$$

Illustratively, for $t = 0.5$ second, the ball is predicted to leave the track in

$$x_0(0.5) = \frac{1}{e^{3/40} - 1} - \frac{10}{3} = 9.507 \text{ revolutions}$$

Assuming the prediction error to be normally distributed, it can be shown that the root mean square error must be less than or equal to 16 compartments for the player to effect a positive expectation—that is, 16/38 or ~0.42 revolutions. Operational tests have indicated an expectation of ~35%.

This procedure obviously relies on the custom of allowing bets to be placed until just prior to the ball's leaving the stator track. Thus the casinos can readily thwart open computerized predictions by prohibiting bets once the *tourneur* launches the ball.

In the late 1970s, a group of University of California-Santa Cruz students, calling themselves The Eudaemons (Ref. Bass), followed in the footsteps (literally) of

Thorp and Shannon by constructing a predictive computer that fit into the sole of a shoe; ball velocity was entered by pressure from the big toe.[17] More recently (2004), a group of enterprising gamblers adapted a computer-linked laser to win £1.3 million at the Ritz casino in London.

Another method advanced by Thorp (Ref. 1969) for developing a positive expectation relies on the observation that the Roulette apparatus is occasionally aligned with a slight tilt from the horizontal. In such a configuration, the ball will not drop from that sector of the track on the "high" side. A tilt of 0.2 degrees creates a forbidden zone of about 30% of the wheel. The ball's motion along the track is described by the nonlinear differential equation for a pendulum that at first revolves completely around its point, but gradually loses momentum until it oscillates through a narrower angle.

The forbidden zone partially quantizes the angle at which the ball can leave the track and therefore quantizes the final angular position of the ball on the inner cylinder (as a function of initial conditions). This quantization is exceptionally narrow: The greater velocity (and longer path to the inner cylinder) of a ball leaving the track beyond the low point of the tilted wheel balances almost precisely the lower velocity (and shorter path) of a ball leaving at the low point. Thorp has designed and constructed a hand-held computer capable of predicting up to eight revolutions of the ball in advance of its separation from the track. He reports an expectation of +44%.

Roulette Miscellany

The number of spins n for a particular number to appear with probability 1/2 is determined by $1 - (37/38)^n = 0.5$. Thus $n \approx 26$.

The expected number of spins required to produce all 38 numbers is

$$38 \sum_{1}^{38} 1/n = 160.66$$

The probability that each number occurs once in 38 spins is given by the multinomial distribution (Eq. 2-12):

$$38!/38^{38} = 4.86^{+} \times 10^{-16}$$

while the probability that each number occurs twice in 76 spins is

$$76!/(2^{38} \cdot 38^{76}) = 5.92^{-} \times 10^{-21}$$

[17] In July, 1985, Nevada law SB-467 banned as a felony the possession or use of any electronic device to predict outcomes, compute probabilities of occurrence, analyze strategies for playing or betting, or recording the values of cards played. *Caveat aleator*.

As indicated by Eq. 5-38, 38 spins of the wheel would leave an expected 14 numbers unrepresented (more accurately, 13.79^{+}). After 76 spins, only an expected five numbers (more precisely, 5.14^{-}) would remain unrepresented. The "maturity of the chances" fallacy (see Chapter 11) would suggest an expected number close to zero.

Reddere Roulette

For favored clientele ("high rollers"), some casinos offer to refund half their potential losses after n plays (the gambler must commit to the n plays). Betting one unit on red/black (or odd/even), the player's expected loss after n plays is $-2n/38$ with a standard deviation $\sigma = \sqrt{n}$. Hence the standard score, t (see Chapter 3), is

$$\mathrm{t} = \frac{n/19}{\sqrt{n}} = \sqrt{n}/19$$

and, setting E(n) equal to t, the Reddere value is 0.276 (see Appendix Table A). Thus

$$\sqrt{n}/19 = 0.276 \quad \text{and} \quad n = (0.276 \times 19)^2 = 27.5$$

Ergo, for a sequence of 27 plays or fewer, betting red/black, the player retains a positive expectation when half his potential losses are refunded. (For a 27-play session, his expectation is $+0.018$; for a 28-play session, it falls to -0.019.) Maximal expectation (0.275^{+}) occurs for a 7-play session (approximately 1/4 the number of plays required for the casino to gain a positive expectation).

Betting on a number does not change the expectation per play ($-2/38$), but markedly increases the standard deviation to

$$\left[35^2 \times \frac{1}{38} + (-1)^2 \, \frac{37}{38}\right]^{1/2} n^{1/2} = \sqrt{33.2n}$$

Again, setting E(n) equal to t, the Reddere value is 0.276, whence

$$\frac{n/19}{\sqrt{33.2n}} = 0.276 \quad \text{and} \quad n \simeq 913$$

(Precise computation specifies $n = 930$.) Maximal expectation (8.797^{-}) occurs for a 234-play session when the player is guaranteed refund of half his potential losses.

In general, for a potential refund of r percent, the player's expectation E (for a single-play probability of winning p and a single-play probability of losing $q = 1 - p$) after n plays is given by

$$\mathrm{E} = \frac{-2n}{38} + \sum_{k=0}^{[n/36]} \binom{n}{k} (1/38)^k \, (37/38)^{n-k} \left(n - 36k\right) r/100$$

For $r = 10\%$, E maximizes at 0.241^{+} units after $n = 11$ plays. The player's expectation here is positive for $n \leq 24$ and negative for all $n \geq 25$.

If the player were relieved of his commitment to n plays (7 for an even bet, 234 for a number bet), he could achieve a considerably higher expectation. Were his potential refund of 50% payable after a single spin, for instance, his expectation would then be

$$E = (37/38)(1/2) + (1/38)35 = 0.434^{+}$$

The most liberal option permits the player, after each spin, to quit or spin again. For this game, Grosjean and Chen (Ref.) have shown that the optimal strategy dictates another spin if

$$-125{,}498 \leq \rho \leq 88{,}638$$

where ρ is the current profit. With values of ρ outside this range, the player should quit. [Note that the number of spins to achieve this profit is irrelevant (a characteristic of the Markov process).] Following this strategy results in an expectation of 13.87 for an average session of 369.6 plays—a 57.7% improvement over that from a committed 234-play session.

With a commitment to a minimum of 38 spins, the expectation suffers only a minuscule decrease to 13.86. With a (more realistic) refund of 10% of potential losses, the unconstrained option offers the gambler an expectation of 0.35, while a commitment to 11 spins yields an expected gain of 0.24. A mandated minimum of 38 spins with the potential 10% rebate produces an expectation of −0.44.

Waiting Time Between Appearances of a Particular Number

The average waiting time for a given number to appear is, of course, 38. As with the waiting time between Heads in a sequence of coin flips (pg. 107), the average waiting time between the appearances of a particular number on either side of a randomly selected point is

$$2/(1/38) - 1 = 75 \text{ spins}$$

La Boule

Another wheel game, La Boule, a light version of Roulette, still retains its popularity in some European casinos. The wheel and betting layout are illustrated in Figure 5-14. Wagers can be placed on any of the nine numbers, on *noir* (1, 3, 6, 8) or *rouge* (2, 4, 7, 9), on *impair* (1, 3, 7, 9) or *pair* (2, 4, 6, 8), and on *manque*

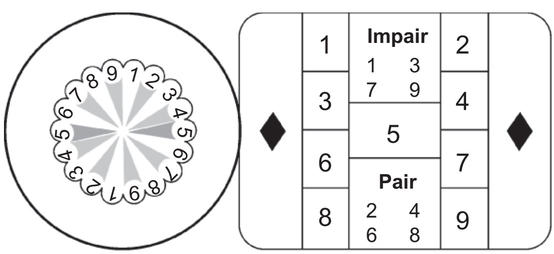

FIGURE 5-14 La Boule wheel and betting layout.

(1, 2, 3, 4) or *passe* (6, 7, 8, 9). Single-number bets are assigned a payoff of 7 to 1 (a few casinos pay only 6 to 1), compared to equitable odds of 8 to 1—for an expected return of 0.889. The other, multiple, bets offer an even payoff (compared to equitable odds of 1.25 to 1); since the number 5 is excluded,[18] each wager has four chances out of nine of winning—that is, a single-trial probability of success of 0.444 (a House Take of 11.11%).

The La Boule format can be based on numbers other than 9. For example, the game of Avions (which rated a brief appearance at Monte Carlo) is based on 17. *Pair-impair*, *manque-passe*, and *rouge-noir* pay even odds, while single-number bets offer 15 to 1. Wagers on groups of four numbers are available at odds of 3 to 1 (the number 9 is omitted from all but the single-number bets). The House Take in the game of Avions is 5.88%.

PUBLIC GAMES

Bingo

Descended from a mid-16th-century Italian parlor game, Bingo (originally "Beano") has found its niche as a fundraiser with churches and social organizations, in addition to well-attended parlors at some casinos. Its expectation varies with entrance fees and value of the prizes.

Bingo cards (Figure 5-15) are imprinted with a 5×5 array of numbers selected from the set 1 to 75 (except for the "free" center position) that are matched against a sequence of numbers generated at random (and called out by an announcer). Columns 1, 2, 4, and 5 each comprise five numbers from the sets 1 to 15, 16 to 30, 46 to 60, and 61 to 75, respectively. Column 3 (which contains the "free" space) comprises four numbers from the set 31 to 45.

B	I	N	G	O
8	26	45	51	72
9	25	34	47	67
15	22	FREE	49	75
6	27	31	60	70
10	29	37	54	38

FIGURE 5-15 Typical Bingo card.

The number of distinct Bingo cards is equal to the number of different middle columns: $\binom{15}{4} = 1365$ (since the number of cards with different rows and

[18] With reason, the 5 is sometimes referred to in France as *l'ami de la maison*.

columns is greater than this number). The number of *possible* Bingo cards is $\binom{15}{5}^4\binom{15}{4} = 1.11^4 \times 10^{17}$.

Each player's objective is to realize a colinear configuration of matching numbers—a "Bingo." (There are 12 such lines including four that contain the "free" space.)

The probability distribution for the number of calls required to complete a Bingo is shown in Table 5-7 (Ref. Agard and Shackleford).

Table 5-7 Single-Card Bingo Probabilities

Number of Calls	4	10	20	30	40	50	60
Probability	3×10^{-6}	8×10^{-4}	0.023	0.144	0.446	0.814	0.986

Typically, each player competes with scores or hundreds of others. More significant, therefore, is the distribution for the number of calls required to complete the first Bingo when n cards are contending. This distribution is shown in Table 5-8 for n = 10, 50, and 100 cards (Ref. Agard and Shackleford).

Table 5-8 Multiple-Card Bingo Probabilities

		Number of Calls				
		4	12	20	32	40
Number of Cards n	10	3×10^{-5}	0.020	0.204	0.868	0.997
	50	2×10^{-4}	0.092	0.663	1.0	1.0
	100	4×10^{-4}	0.171	0.875	1.0	1.0

Note: A simulation was run to produce these probabilities since different Bingo cards are not independent.

With 50 cards, it is virtually certain that at least one player will achieve a Bingo by the 30th call. With 100 cards, there is a probability greater than 1/2 for at least one Bingo by the 14th call. The expected number of calls before someone marks a Bingo is shown in Table 5-9.

Table 5-9 Expected Number of Calls to Produce a Bingo

Number of Cards	1	10	50	100	500	1000
Expected Number of Calls	41.4	25.5	18.3	15.9	11.6	10.1

As the number of calls increases, the Bingo is increasingly likely to be shared. With 100 cards, there are two expected winners by the 27th call; with 200 cards, there are two expected winners by the 23rd call; with 500 cards, there are two expected winners by the 17th call, three by the 20th call, and four by the 22nd call. A 1000-card game expects two simultaneous winners by the 14th call.

Bingo Subsets

The "coverall" state, as its name implies, is reached when all 24 numbers on the card have been called. A bonus is offered by certain (online) casinos if the coverall is completed within 54 calls. This probability—for a single card—is expressed by

$$\binom{54}{24}\binom{75}{24}^{-1} = 5.44\times10^{-5}$$

With 100 cards, the probability of achieving at least one coverall (in 54 calls) is

$$1-(1-5.44\times10^{-5})^{100} = 5.43\times10^{-3}$$

while the expected number of calls to complete a coverall is 63.43.

Another subset, the "four-corners" group, as *its* name implies, reaps a bonus when all four corners of the card are called. The probability of this event equals the product of the probability that in y calls, there will be x called numbers on the card and the probability that these x numbers will include the four corners:

$$\sum_{x=4}^{y} \frac{\cancel{\binom{24}{x}}\binom{51}{y-x}\binom{20}{x-4}}{\binom{75}{y}\cancel{\binom{24}{x}}}$$

The expected number of calls to cover the four corners is 60.8.

Yet other payoffs may be offered for filling in one or more of the 16 possible 2×2 subsets, or the nine possible 3×3 subsets on the 5×5 card.

A Bingo Sidelight

A little-known facet of Bingo is that, when played with numerous cards, the probability of multiple simultaneous winners in columns is greater than that in rows. There are $\binom{15}{5} = 3003$ different sets of five numbers that can appear in each column. By contrast, there are $15^5 = 759{,}375$ different rows, of which each card has five (ignoring the "free" center). Ergo, whenever there are more than 3003 cards in play, there must be two with the same set of five numbers in a column.

However, since all rows are equally likely to contain a Bingo, the first Bingo is more likely to appear in a row than in a column. For the average game, the probability of a horizontal Bingo is approximately twice that of a vertical Bingo.

Bingo Supplementary Wagers

It is herewith proposed that the Bingo player be afforded the option of wagering that his Bingo, should it occur, be along a row, column, or diagonal (no one of which is uniquely defined).

Accordingly, for an additional sum of 1/10th the price of the card, the value of the Bingo along a particular line is increased by $(N - v)/n$, where N is the total number of bets on that line, v the House Take, and n the number of simultaneously winning players with bets on that line.

A separate pool is formed for each of the three possible Bingo lines. If no winner emerges for one or more lines, the pertinent pool money, $(N - v)$, is carried over to the next drawing. Table 5-10 details a breakdown of the particular Bingo lines.[19]

Table 5-10 Row/Column/Diagonal Probabilities

Cards	Breakdown of Wins (%)				Breakdown of Combination Wins (%)			
	Rows	Columns	Diagonals	Combo	Row/ Column	Row/ Diagonal	Column/ Diagonal	All 3
1	35.9	35.8	20.6	7.6	4.1	1.6	1.62	0.28
10	34.6	32.8	28.0	4.7	1.6	2.0	0.95	0.11
50	32.4	29.4	32.4	5.8	1.3	3.5	0.87	0.11
100	31.4	27.9	33.9	6.7	1.1	4.7	0.81	0.11
200	30.1	26.6	35.2	8.2	1.0	6.3	0.73	0.11
500	27.9	25.3	35.8	11.0	0.8	9.6	0.54	0.10
1000	25.3	25.0	35.0	14.6	0.6	13.6	0.4	0.08
2000	21.6	25.4	32.7	20.3	0.4	19.6	0.24	0.06
3000	19.3	25.7	30.3	24.7	0.3	24.2	0.16	0.04
4000	17.3	26.0	28.3	28.4	0.2	28.0	0.11	0.03
5000	15.7	26.4	26.5	31.5	0.2	31.2	0.08	0.02
7500	12.8	26.3	22.8	38.1	0.1	38.0	0.03	0.01
10000	10.5	26.3	19.7	43.5	0.1	43.4	0.02	0.01

Keno

The game of Keno, brought to America by 19th-century Chinese railroad workers, has become formalized with its adoption by the Las Vegas casinos. Twenty numbers are selected at random by the House from the set of numbers 1 through 80. The player can select from 1 to 15 numbers of the set, winning if some fraction of his subset matches those included in the 20 House numbers.

[19] Courtesy of Norman Wattenberger, Casino Vérité.

In general, if the player selects n numbers, the probability $P(k)$ that $k \le n$ will be among the 20 House numbers is

$$P(k) = \frac{\binom{20}{k}\binom{80-20}{n-k}}{\binom{80}{n}} \tag{5-45}$$

When the player selects a single number, $n = k = 1$ and $P(k) = 0.25$. Equitable odds are therefore 3 to 1; the House pays 2.2 to 1 for an expected return of 0.8. For $n = 2$, both numbers are required to be among the 20 selected to qualify as a win: thus, from Eq. 5-45, with $n = k = 2$, $P(k) = 0.06$, House odds are 12 to 1, yielding an expected return of 0.782.

A more typical wager at Keno consists of selecting ten numbers, wherein the House offers a payoff if five or more of the ten are among the 20 called. Table 5-11 shows the probability of matching k numbers ($0 \le k \le 10$) for

Table 5-11 Keno Payoffs and Probabilities

k Matches	$P(k)$	House Odds	Expected Value
0	0.0458		
1	0.1796		
2	0.2952		
3	0.2674		
4	0.1473		
5	0.0514	Even	0.1029
6	0.0115	17–1	0.2066
7	0.0016	179–1	0.2900
8	1.35×10^{-4}	1299–1	0.1760
9	6.12×10^{-6}	2599–1	0.0159
10	1.12×10^{-7}	24,999–1	0.0028
			Total = 0.7942

$n = 10$ along with the associated House payoffs. The House pays the indicated return for values of k from 5 through 10, thereby proffering an expected return of 0.794 (a House Take in excess of 20%). Other bets allow the player to select any n between 0 and 15; payoffs are generally scheduled for values of k greater than about $n/2$. Overall, the House Take varies from about 20% for the smaller values of n to more than 30% for $n = 15$.

The probability of selecting 20 numbers, all of which match the 20 House numbers, is $\binom{80}{20}^{-1} = 2.83^{-} \times 10^{-19}$. There is no record of this event ever having materialized.

A Keno side bet available in Australia and the Far East is that of "Heads and Tails." The numbers from 1 to 40 are designated "Heads" and those from 41 to 80 are "Tails." A bet on Heads wins (at even odds) if a majority of the 20 House numbers falls in the 1 to 40 range—and conversely for Tails. Further, the player may wager on a 10/10 split—which pays 3 to 1.

The probability of n House numbers falling in one group and $20 - n$ in the other is

$$P(n,20-n) = \frac{\binom{40}{n}\binom{40}{20-n}}{\binom{80}{20}}$$

For $n = 10$,

$$P(10, 10) = 0.203^{+}$$

and the probability of a majority of Heads (Tails) among the 20 House numbers is

$$\sum_{6}^{9} P(n, 20 - n) = \frac{1 - P(10, 10)}{2} = 0.398^{+}$$

for an expectation of -0.204^{-}. The expectation for the 10/10 bet at odds of 3 to 1 is -0.187^{+}.

The rationale used by most players (unconsciously) in betting on Keno games (or any game with negative mathematical expectation) is that the utility of m dollars is greater than m times the utility of one dollar (perhaps the technical term "utility" should be replaced in this context by the more descriptive word "lure"). Perhaps the casinos should cater to this taste by increasing the expected value for the less probable events. For example, in the Keno game where $n = 10$, the payoff for ten matches could be increased to 1,000,000 to 1 while yet leaving a House Take of almost 10%.

The more eclectic devotee of gambling theory will tend to avoid such games as Keno, Bingo, and the like, since no significant decision-making process is involved, and their mathematical expectations remain stubbornly negative (except in those cases where a mechanical bias can be detected and exploited). Only the imposition of an urgent, subjective utility function would entice such a player to these games of pure chance.

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[20] The ancestors of the modern slot machines, called "Buffaloes," were constructed similar to wheels of fortune in an upright box. Ninety spaces were partitioned around the circumference of the wheel: 30 red, 30 black, 16 green, 8 yellow, 4 white, and 2 blue spaces. A coin was inserted into a slot representing a particular color, and the wheel spun. If the pointer stopped on a red space, and the red slot contained a coin, the payoff was 2 to 1. Blue yielded a payoff of 39 to 1 (correct odds: 44 to 1).

Slot Machines

The classic "One-armed-Bandit" was invented in 1899 by Charles Fey, a mechanic in San Francisco, California.[20] Three reels divided into ten decals with five symbols (Horseshoes, Diamonds, Spades, Hearts, and a Liberty Bell) were set in motion by pulling a lever. (The number of decals on each reel was doubled to 20 in 1910 and currently stands at 22.) A window displayed one symbol for each reel as it came to rest. Three Liberty Bells defined the jackpot, which paid 50¢ for an initial 5¢ investment. Payouts (in nickels) gushed directly from an opening at the bottom. The expected return was reportedly 0.756.

For casino owners, the "slots" have proved the most profitable game of all, accounting for more than 75% of their $33 billion annual gambling revenues in the United States.

Mechanical designs have been almost entirely replaced by computer-controlled, video-display machines—with the options of using credit cards rather than coins and pressing a button rather than pulling a lever (thereby confounding psychologists who related the action to manipulating a phallic symbol). A random number generator (RNG) with an erasable programmable read-only memory (EPROM) chip ensures that each play is independent of its predecessors and conforms to pre-set probabilities—generally an expected return ranging from 0.85 to 0.97.

For those who regard these odds as insufficiently inequitable, some casinos feature two, three, or four slot machines coupled together and operated by a single lever; the payoff is normally increased arithmetically (with the number of linked machines), while the probability of success drops exponentially. Despite the low expectations, slot machines predominate in most casinos; in Las Vegas they outnumber all other forms of gambling by more than 10 to 1.

Pools, Sweepstakes, and Lotteries

Many pools, notably those based on European football (soccer), have proved enormously popular at times. In the case of the government-sponsored English pool, the expected return is 0.5 with 30% of gross sales returned to the government and 20% to the promoters. In the United States, baseball and basketball pools frequently offer smaller expected returns (0.4 or less—sometimes referred to as "a tax on stupidity").

Similar figures apply to sweepstakes (the term originated in horse-racing: each entrant would post a fee; the winner would "sweep up the stakes"). Most well known was the quadrennial Irish Sweepstakes, a government operation launched in the Irish Free State in 1930 and disestablished in January 1986. It offered a return of about 0.4. Numbers were drawn from a large barrel and matched to one of the horses running in a major Irish or English event. Tickets were sold throughout the world, often on the black market (gambling was then generally illegal in the United States and Canada).

Both pools and sweepstakes have declined in popularity over the last several decades. A residue of informal "office pools" survives—wherein players predict the outcomes of several contests (usually football or basketball games).

Lotteries are slightly more favorable to the player. The national lotteries of England, Italy, France, and Sweden (the "penning lotteriet") offer expected returns on the order of 0.5 to 0.6. In Latin America, the return is often much less, as lotteries have traditionally been devoted to financing everything from soccer stadiums to revolutions. A peculiarly Italian institution is the Totopapa, a lottery based on the outcome of a papal election (with an expected return of ~0.5).

Lotteries were prevalent in Roman times, exploited for both public and private profit. The first known public lottery was sponsored by Augustus Caesar to raise funds for repairing the city of Rome; the first public lottery awarding money prizes, the Lotto de Firenze, was established in Florence in 1530. In the United States, lotteries and raffles were once considered indispensable to the nation's growth. The American colonies were floated on lotteries—the Virginia colonization company was organized through the permission of King James I in 1612 to conduct a revenue-raising lottery. By a 1776 resolution, the Continental Congress established a $10-million lottery "for carrying on the present most just and necessary war in defence of the lives, liberties, and property of the inhabitants of these United States." In 1963, New Hampshire became the first state to sanction lotteries since Louisiana rescinded its lottery law in 1894 following congressional action to prohibit the sending of lottery tickets through the mail. It has since been joined by 41 other states plus Washington, D.C., and the Virgin Islands.

A typical format for a state or national lottery consists of drawing $m < n$ numbers from the set 1, 2, …, n. An entry fee entitles the player to select m numbers independently, hoping to match the drawn numbers. A common game consists of $m = 6$ and $n = 49$. The probability, P_k, of $k \le m$ matches is determined from the hypergeometric distribution (Eq. 2-10),

$$P_k = \frac{\binom{6}{k}\binom{49-6}{6-k}}{\binom{49}{6}} = \begin{array}{c} 7.15\times10^{-8},\ 1.845\times10^{-5},\ 9.686\times10^{-4}, \\ 1.765\times10^{-2},\ 1.324\times10^{-1} \end{array}$$

for $k = 6, 5, 4, 3, 2$, respectively.

The two major multistate lotteries are Powerball (created in 1992 as a successor to Lotto America) and Mega Millions (a 2002 replacement for The Big Game), both of which have produced jackpots in the hundreds of millions of dollars. Neither differs conceptually from Bingo or Keno. In Powerball, five balls are selected at random from a set of white balls numbered 1 to 59. The "Powerball" number is a single ball selected at random from a second set of balls numbered 1 to 35. The player purchases a $2 lottery ticket imprinted with five numbers from the white set plus a sixth—Powerball—number. There are

$$\binom{59}{5}\binom{35}{1} = 1.75 \times 10^8 \text{ total combinations}$$

The player receives a payoff if any (or none) of his numbers match the selected white balls *and* the Powerball number, or if 5, 4, or 3 of his numbers match the selected white balls *but not* the Powerball number. The payoff schedule per $2 wager is shown in Table 5-12.

Table 5-12 Powerball and Mega Millions Payoff Schedule

	POWERBALL		MEGA MILLIONS	
	Probability	Payoff	Probability	Payoff
Match 5 + 1	5.707×10^{-9}	Jackpot	5.691×10^{-9}	Jackpot
Match 5 + 0	1.940×10^{-7}	$1,000,000	2.561×10^{-7}	$250,000
Match 4 + 1	1.541×10^{-6}	$10,000	1.451×10^{-6}	$10,000
Match 4 + 0	5.239×10^{-5}	$100	6.531×10^{-5}	$150
Match 3 + 1	8.167×10^{-5}	$100	7.266×10^{-5}	$150
Match 2 + 1	1.416×10^{-3}	$7	1.185×10^{-3}	$10
Match 3 + 0	2.777×10^{-3}	$7	3.265×10^{-3}	$7
Match 1 + 1	9.024×10^{-3}	$4	7.111×10^{-3}	$3
Match 0 + 1	1.805×10^{-2}	$4	1.337×10^{-2}	$2

The size of the jackpot begins with a $40M 30-year annuity with payments increasing by 4% annually. This sum is rolled over if no winner emerges after each semi-weekly drawing. The probability that the player receives *some* payoff is 0.0314. Expected return equals 0.4656.

The size of the jackpot varies with the number of tickets purchased. The probability that the player receives *some* payoff is 0.0273. The expected return for Powerball is 0.197 plus 6.84×10^{-9} times the jackpot. Jackpots begin at $15 million for an unappealing expected return of 0.300—before accounting for taxes and multiple winners—and are "rolled over" when no winner emerges. Jackpots occasionally exceed $300 million; at $242 million, the player has a positive expectation.

Lottery jackpots, when accepted as a lump-sum payment, are generally reduced by half of the advertised amount—or, at the winner's option, disbursed as an annuity (ranging from ten to 40 years; 29 for Powerball). The imposition of state and federal taxes often reduces the final sum by half again.

Further, because of the high number of participants in the Powerball lottery, the probability of sharing a jackpot is significant. With N tickets sold, the mean number of winners, μ, is

$$\mu = \frac{N}{1.752 \times 10^8}$$

Applying the Poisson distribution (Eq. 2-14) to determine w winners with this mean:

$$P(w) = e^{-\mu}\mu^w/w!$$

For no winners,

$$P(0) = e^{-\mu}$$

and the probability of at least one winner is then $1 - e^{-\mu}$. Thus

$$E = (1 - e^{-\mu})/\mu$$

defines the number of jackpots per winner—i.e., the expected jackpot.

For $N = 10^8$ tickets, $E = 0.719$; for $N = 2 \times 10^8$ tickets, $E = 0.545$; and with 2.245×10^8 tickets, two expected winners should share the jackpot ($E = 0.5$).

Mega Millions differs from Powerball only superficially. Five numbers are selected from the set 1, 2, …, 56 and a sixth (the Mega Ball number) from the set 1, 2, …, 46. The probability of matching all six of these numbers is one in

$$\binom{56}{5}\binom{46}{1} = 1.76 \times 10^8$$

or 5.69×10^{-9}. The probability that the player wins the jackpot or one of the lesser prizes is 0.0251. Expected return is 0.55.

Each sequence of the Mega Millions drawings (held semi-weekly) begins with a jackpot of $12M in the form of a 25-year annuity or $8M in cash. If no one selects all six winning numbers plus the Mega Ball number, the jackpot is rolled over to the next drawing.

[The largest sum offered to-date by the Mega Millions lottery: $656M (in its annuity form; $473.6M cash) at the Mar. 30, 2012 drawing. Three winners shared the humongous jackpot.]

It should be noted that although the probability of winning a lottery is fixed by its operators and is immune to any mathematical strategy, its expected return *can* be altered—by shunning numbers that are highly regarded by other players. For example, historical dates are often favored, as are small numbers. Thus, selecting numbers 32 and higher reduces the probability of sharing the grand payoff -since a substantial percentage of lottery players chooses numbers to match birthdays, wedding dates, or anniversaries (confined to the set [1, 31]). Also, "numbers in the news" (e.g., a headline reading, "44 Rescued in Mine Cave-in") will markedly increase the selection frequency of number 44 and the probability of payoff-sharing for any card-set containing that number. "Lucky" numbers include 1, 7, 13, 23, 32, 39, 42, and 48. "Unlucky" numbers (and thus more attractive) include 26, 34, 44, 45, and 46. And, in relatively rare instances, the jackpot may have grown significantly, while the number of tickets sold remained small.

California Decco

In the Decco game of the California lottery, four cards are selected by the lottery operator, one from each suit of a standard deck. The player likewise selects four cards.

For a \$1 bet, the player receives \$5000 if all four selections match [probability $1/13^4$], \$50 if three of the four selections match [probability $1^3 \cdot 12^2\binom{4}{1} = 48/13^4$], \$5 when two out of four match [probability $1^2 \cdot 12^2\binom{4}{1} = 864/13^4$], and, with a single match [probability $1 \cdot 12^3\binom{4}{1} = 6912/13^4$], he wins a ticket to a subsequent Decco game. Thus we can write for E, the expected value of the game.

$$E = 13^{-4}(5000 + 50 \cdot 48 + 5 \cdot 864 + 6912 \cdot E)$$

(Note that the expected value of the subsequent game is identical to the expected value of the current game.) Solving for E:

$$E = 0.5414$$

(an expected return of 54 cents). The probability of winning nothing is $12^4/13^4 = 0.7260$.

The "Bet on Anything" Bookmaker

More prevalent in England than in the United States are bookmakers who will quote odds on the outcome of virtually any event. Consider, for example, an election where candidate **A** is initially quoted as a 3 to 1 favorite, and candidate **B** as a 3 to 7 underdog. These probabilities sum to 1.05 (0.75 + 0.30). The excess over 1.0 is referred to as the "over-round."

If the bookmaker accepts bets in proportion to these probabilities, his profit will be 0.05/1.05 = 4.76%. If bets differ from this proportion, the bookmaker will adjust his odds to maintain an approximately constant over-round (although the over-round generally increases with the number of possible outcomes).

This procedure is used in parimutuel wagering (Chapter 9)—except that the bettor cannot take advantage of interim odds. The stock market poses a closer analogy in its quoting of bid and offer prices.

PUZZLEMENTS

The following miscellaneous games are raised as a challenge to the dedicated reader.

1. *The Jai-Alai Contest.* In a Jai-Alai game, two players from a group of n compete for a point, while the remaining $n - 2$ players form a waiting line. The loser between each contending pair takes his place at the end of the line, while the winner is then matched with the first waiting player. That player who first accumulates $n - 1$ points is declared the game winner. Letting p_{ij} represent the single-trial probability of success for the ith player

when matched against player j, what is the probability P_i that the ith player becomes the first to garner the required $n - 1$ points?

Compare P_i with $P_{i,n}$ (Eq. 5-39), the winning probability for a player competing in a queue circulaire.

2. Consider a number chosen at random from a compendium such as the Census Report. Show that the probability distribution, $P(k)$, of that number's digits, $k = 1, 2, \ldots, 9$, is equal to

$$P(k) = \log(k + 1) - \log k$$

contrary to the naïve assumption that all digits are equiprobable. For $k = 1$, $P(k) = 0.301$ (which accounts for the fact that many tables of numerical data show greater wear over the first pages).

3. *Majority wins*. Two players (or teams), **A** and **B**, of equal strength compete in a series of games. Which is greater, the probability that (1) **A** beats **B** three games out of four, or (2) five games out of eight?

   Show that (1) is more probable (1/4 compared to 7/32).

   Note that if the question were reworded to ask the probability of winning *at least* three out of four versus *at least* five out of eight, the answer would be reversed.

4. *The Schoolgirl Problem.* Posed by the 19th-century British mathematician Reverend Thomas Kirkman, this problem postulates 15 schoolgirls who walk to school each day for a week in five groups of three girls each. Determine if it is possible for no girl to walk with any other more than once.

   A generalization of this setup is known as The Fully Social Golfer Problem (Ref. Harvey).

5. *The Variable Payoff Matrix*. Each of two players is given four markers. Player **A**'s markers are labeled $+4$, $+3$, $+2$, and $+1$, while **B**'s are labeled -4, -3, -2, and -1. Beginning with **A**, the players alternately place a personal marker on any empty cell of a 3×3 matrix whose center cell is marked with an 0. When all nine cells are filled, the players then execute the game matrix that has been constructed. Compute the optimum strategies available to each player.

   For a more complex version of this game, let **A** select his four markers from the set $+1, +2, \ldots, +n$, and **B** from the set $-1, -2, \ldots, -n$. Yet another variant allows both players to choose markers from a common set.

6. *Extreme-Value Roulette*. Consider the wager—available with a conventional Roulette apparatus—that the highest outcome of the first n plays is a number that exceeds all those numbers obtained over the subsequent m plays (the 0 and 00 can be ranked low). What values of m and n and what corresponding payoff would reasonably be acceptable to the management of

a casino? Show that as the set of numbers on the wheel becomes arbitrarily large, the probability that the highest of the first n numbers exceeds the subsequent m numbers approaches $n/(n + m)$. *Note*: The general problem of exceedences lies in the domain of extreme-value statistics (Ref. Gumbel).

7. The one-child-only policy in China has been modified to permit baby-production until a boy is born. Thus each family has an expectation of two children (identical to the St. Petersburg game). Assuming that women refuse to bear more than ten babies, show that the expected number of offspring is then reduced to 1.988.

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Chapter | six

Coups and Games with Dice

A BRIEF CHRONICLE

The casting of knuckle bones for amusement, profit, or divination dates from the dawn of civilization. As bones were marked and evolved into dice, cleromancy (divination by dice) and gambling became intimately intermingled—the gods of old shaped human destiny by rolling dice and then used the same dice in games among themselves. One of the oldest mythological fables tells of Mercury playing at dice with Selene and winning from her the five days of the epact (thus totaling the 365 days of the year and harmonizing the lunar and solar calendars). In the same vein, it was reported by Herodotus that the Egyptian Pharaoh Rhamsinitus regularly rolled dice with the goddess Ceres. The entire world in the great Sanskrit epic, the Mahābhārata,[1] is treated as a game of dice in which Siva (god of a thousand names) gambles with his queen, the numbers thrown determining the fate of mankind.

Both Greeks[2] and Romans were enthusiastic dice devotees, casting the tessera (cube) and talus (elongated cube) in their games (in the Far East, the teetotum, or spinning die, was more common); the talus was occasionally numbered 1, 3, 4, 6, leaving the ends unassigned. Regal and passionate dice players in ancient Rome included Nero, Augustus, Caligula, Claudius I,[3] Domitian, and Julius Caesar. Dice served for sortilege, divination, or judicial decision, one renowned example being Caesar's tossing a die to resolve his indecision over crossing the Rubicon ("alea jacta est"). During the Middle Ages, dicing stood as a favored pastime of knights and ladies. Dicing schools and guilds of dicers

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[1] One tale from the Mahābhārata concerns the pious and righteous King Yudhishthira gambling with Sakuni, an expert with false dice. Yudhishthira lost his skirt.

[2] According to post-Homeric legend, the invention of dice is attributed to Palamedes during the Trojan War. On the other hand, Herodotus reports that the Lydians claimed that honor.

[3] Claudius (10 B.C.–A.D. 54), grandson of Mark Antony (who whiled away odd moments in Alexandria engaging in dice games), was the author of a book, *How to Win at Dice*, which, alas, has not survived.

prospered, all despite repeated interdiction. Charles V of France in 1369 forbade the play of certain games of chance, and an edict of the provost of Paris in 1397 prohibited the proletariat from playing at “tennis, bowls, dice, cards, or nine-pins on working days.”

In the late 18th century, a popular pastime for some composers entailed the creation of melodies by rolling a pair of dice to select the 16 bars of a Viennese minuet. The most famous of these compositions is the Musikalishes Würfelspiel, often (but erroneously) attributed to Mozart. (All 11 outcomes of a two-dice throw were—quite naïvely—considered equiprobable.)

For a die to be fair (all sides having equal probability of turning up), it must be isohedral (all faces exhibiting the same shape, the same relationship with the other faces and with the same center of gravity). There are 25 such isohedra (plus five infinite classes).

The Greek geometers focused on the five regular polyhedra (the Platonic solids). The tetrahedron, with four equilateral triangles as faces, does not roll well nor does it offer much variety; the hexahedron, or cube, with six square faces, appears to satisfy both the desired motion and number requirements; the octahedron, with eight faces in the form of equilateral triangles, has greater variety, but rolls a bit too smoothly; the dodecahedron, with 12 pentagons as faces, and the icosahedron,[4] with 20 equilateral triangles as faces, roll too fluidly for control in a restricted area and have more possibilities than can be comprehended by the unsophisticated mind (fittingly, these polyhedra are often used for fortune-telling). Hence, the adoption of the hexahedron as a die appears to be a natural selective process, although the octahedron could be pressed into service by some enterprising casino (cf. “Sparc,” Casino Games section). For more sophisticated gamblers, the use of irregular polyhedra might provide a provocative challenge.

The earliest dice were imprinted with consecutive numbers on opposite faces: 1 opposite 2, 3 opposite 4, and 5 opposite 6. However, as far back as the 18th dynasty of Egypt (*ca.* 1370 B.C.), the present standardized format was introduced whereby the two-partitions of 7 are ensconced on opposing faces. Under this constraint there are two topologically distinct constructions for a die. [The three two-partitions of 7 define three axes; a well-known geometric theorem states that there exist two three-dimensional coordinate systems—left-handed and right-handed—that remain distinct under rotation (and constitute mirror images of each other.)] Most American casinos employ right-handed dice—the values 1, 2, and 3 progress counterclockwise around their common corner. Only in China are left-handed dice favored.

In four-dimensional Euclidean space there are regular convex “polytopes,” such as the hypercube (eight cubes combined) and the hypertetrahedron

[4]The icosahedron offers a simple means of generating random decimal digits since its faces can be numbered (twice) from 0 to 9. Soccer balls are “truncated icosahedra,” constructed with 32 faces—20 regular hexagons and 12 regular pentagons.

(five tetrahedra combined). Polytope games are enjoyed exclusively by four-dimensional mathematicians.

The word "die" retains an obscure provenance. A possible etymology suggests that the expression *jeu de dé* is derived from *juis de dé* = *judicium dei* (judgment of God). Alternatively, *dé* may stem from *datum*, which initially signified chance—that is, that which is given or thrown.

DETECTION OF BIAS

Dice problems contributed the initial impetus to the development of probability theory. Cardano, Pascal, and Fermat enunciated the concept of equally likely outcomes for the six faces of a die; thus, by definition, the probability that any one side will face upward is 1/6 (more rigorously, one must establish the necessary and sufficient conditions for the applicability of the limiting ratio of the entire aggregate of possible future throws as the probability of any one particular throw).

Many experimenters have attempted to confirm (or dispute) this figure empirically. The most widely published dice data were compiled by the English zoologist and biometrician W.F.R. (Rafael) Weldon who (in 1894) recorded the frequencies of 5s and 6s in 26,306 throws of 12 dice.

Yet more tedious experiments were carried out by the Swiss astronomer Rudolph Wolf toward the second half of the 19th century. Wolf's feat of stamina—100,000 throws of a single die—is notable for furnishing the data that led, several decades later, to the Principle of Maximum Entropy (Ref. Jaynes). Given testable data, this procedure (more sophisticated than the χ^2 test) consists of seeking the probability distribution that maximizes information entropy, subject to the constraints of the information—that is, comparing predicted entropies to the entropies of the data.

Wolf's experiment produced 16,632 1s, 17,700 2s, 15,183 3s, 14,393 4s, 17,707 5s, and 18,385 6s. The validity of the hypothesis that the probability of each side facing up is 1/6 may be checked by the χ^2 test (Eq. 2-28):

$$\chi^2 = \sum_{i=1}^{k} \frac{(r_i - s_i)^2}{s_i}$$

where r_i is the actual number of times the ith side appears, s_i the predicted number, and k the number of possible outcomes. If $\chi^2 = 0$, exact agreement is evidenced. As χ^2 increases for a given number of trials, the probability of bias increases. Applying the χ^2 test to Wolf's data, we obtain

$$\chi^2 = \frac{12{,}474{,}769}{16{,}667} = 748.5$$

To comprehend the significance of this value, we note from Eq. 2-27 that repeating this experiment theoretically should yield a distribution of χ^2 given by

$$f(\chi^2) = \frac{(\chi^2)^{(k-3)/2} e^{-\chi^2/2}}{2^{(k-1)/2} \Gamma[(k-1)/2]} \tag{6-1}$$

with $k - 1$ degrees of freedom. (A die evidently exhibits five degrees of freedom.)

We now wish to designate a value of χ_0^2 such that the probability of χ^2 exceeding χ_0^2 represents the significance level of the experimental results. Selecting a significance level of 0.01, we can, from Eq. 6-1, find a value χ_0^2 such that

$$P[\chi^2 > \chi_0^2] = \int_{\chi_0^2}^{\infty} p_n(x)\, dx = 0.01$$

where p_n is the probability density function. It can be established[5] that $\chi_0^2 = 15.1$ appertains for the case under consideration. That is, if the die were perfect, 100,000 throws would result less than 1% of the time in a value of χ^2 greater than 15.1. Since $\chi^2 = 748.5$, we may conclude that the probability of Wolf's die being unbiased is vanishingly small. We might have suspected this fact, since all faces of the die except the 1 differ by considerably more than the standard deviation, $\sigma = 117.8$, from the expected value of 16,667.

Weldon's data recorded the event that of $12 \times 26{,}306 = 315{,}672$ possibilities, the numbers 5 or 6 appeared on the dice 106,602 times—resulting in a one-degree-of-freedom χ^2 value of 27.1 (the equivalent figure from Wolf's data is 342.5). A reasonable conclusion is that a die of poor quality (such as those manufactured in the 19th century) is highly susceptible to bias with continued use. The dice employed by the major gambling casinos are now machined to tolerances of 1/2000 inch, are of hard homogeneous material, and are rolled only a few hundred times on a soft green felt surface before being retired. Thus, the probability of finding an unintentionally biased die in a casino—and determining wherein lies the bias—is tantamount to zero.

A final question concerning biased dice relates to the number of throws of a die necessary to ascertain the existence of a bias (we assume throughout that the rolls of a die comprise a stationary statistic—though the bias is not altered with time). Although a bound to the number of throws required is given by Bernoulli's form of the law of large numbers, such a bound, applicable to any distribution, is extremely conservative. When dealing with a binomial distribution, a better estimate of the requisite number of throws to ascertain a bias is provided by de Moivre's theorem (a special case of the central limit theorem). The probability that the outcome 6 occurs between np_1 and np_2 times

[5] χ_0^2 is obtained most readily from tables of the χ^2 distribution, such as Fisher (Ref.).

in n throws of a die, where p is the estimated single-trial probability of a 6, is approximated by

$$P(np_1 \leq np \leq np_2) \simeq \frac{1}{\sqrt{2\pi}} \int_{\frac{n(p_1-p)}{\sqrt{npq}}}^{\frac{n(p_2-p)}{\sqrt{npq}}} e^{-x^2/2} dx$$

$$= \Phi\left[\frac{n(p_2-p)}{\sqrt{npq}}\right] - \Phi\left[\frac{n(p_1-p)}{\sqrt{npq}}\right] \quad (6\text{-}2)$$

where $\Phi(x)$ is the cumulative probability function (cpf). Thus, we can state with 0.95 confidence that the probability that p differs from 1/6 by not more than 0.01 is expressed by

$$P\left(-0.01 \leq p - \frac{1}{6} \leq 0.01\right) \geq 0.95$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.01\left[\frac{n}{(1/6)(5/6)}\right]^{\frac{1}{2}}}^{0.01\left[\frac{n}{(1/6)(5/6)}\right]^{\frac{1}{2}}} e^{-x^2/2} dx = 2\Phi\left(0.06\sqrt{\frac{n}{5}}\right) - 1$$

since $\Phi(-x) = 1 - \Phi(x)$. Hence, the requirement on the number of throws n is governed by the relation

$$\Phi\left(0.06\sqrt{\frac{n}{5}}\right) \geq \frac{0.95+1}{2} = 0.975$$

and, from tables of the cumulative probability function, we find that $n = 5335$ throws of the die suffice to obtain a relative frequency of $1/6 \pm 0.01$ with a confidence level of 0.95 (Bernoulli's rule suggests 27,788 throws).

Similarly, if we suspect the die of bias and assign an unknown value p to the probability of an outcome, we wish to find the number of throws n sufficient to determine p within a tolerance of 0.01 with 0.95 confidence. The Laplace-Gauss theorem states that

$$P\left[\frac{-0.01n}{\sqrt{npq}} \leq \frac{n(f_n - p)}{\sqrt{npq}} \leq \frac{0.01n}{\sqrt{npq}}\right] \simeq 2\Phi\left(\frac{0.01n}{\sqrt{npq}}\right) - 1 \geq 0.95$$

or

$$\Phi\left(\frac{0.01n}{\sqrt{npq}}\right) \geq 0.975$$

From tables of the cumulative probability function, this condition is fulfilled for

$$\frac{0.01n}{\sqrt{npq}} \geq 1.96 \qquad (6\text{-}3)$$

and, since the maximum value of pq is 1/4, Eq. 6-3 indicates that a bound occurs at $n \geq 9604$. Therefore, 9604 throws of a suspect die can test for bias to within 1% with 0.95 confidence (a value less than one-fifth that suggested by the Bernoulli rule).

As with coins in the preceding chapter, we proceed under the assumption that the dice referred to are fair and that successive throws of a die are independent. Fairness implies that the possible outcomes are equiprobable. Independence of successive events signifies that if the events $\{A_1, A_2, \ldots, A_n\}$ are independent, the probability of their joint occurrence is the product of the probabilities of their individual occurrences—that is,

$$\text{prob}\{A_1, A_2, \cdots, A_n\} = \text{prob}\{A_1\} \cdot \text{prob}\{A_2\} \cdots \text{prob}\{A_n\}$$

Finally, it follows that successive dice throws constitute an ergodic process: n throws of one die are equivalent to one throw of n dice.

DIVERS DICE PROBABILITIES

Dice problems generally succumb to straightforward analysis. A few simple formulas encompass virtually all forms of dice play.

De Méré's Problem

This 17th-century "gambler's conundrum" (cf. pp. 2 and 131), which contributed to the inception of probability theory, is readily solved by consideration of simple repetitions. If a chance event has a constant probability p of occurrence on any given trial, then the number of trials n required for its first occurrence is a discrete random variable that can assume any of the infinitely many positive integral values $n = 1, 2, 3, \ldots$. The distribution of n is the geometric distribution (cf. Eq. 2-13)

$$f(n) = q^{n-1}p$$

and the cumulative distribution function $F(n)$ is

$$F(n) = \sum_{t=1}^{n} f(t) = 1 - q^n$$

Therefore the probability P_n that the event occurs in n trials is expressed by

$$P_n = F(n) = 1 - (1 - p)^n \qquad (6\text{-}4)$$

De Méré had posed the question of the probability of obtaining at least one 6 in one roll of four dice as compared to the probability of obtaining at least one double-6 in 24 rolls of two dice. His "reason" argued that the two probabilities should be identical, but his experience insisted that the former dice game offered a positive and the latter a negative expected gain.

The single-trial probability of rolling a 6 with one die is defined as $p = 1/6$, and that of throwing a double-6 with two dice is $p = (1/6)^2 = 1/36$. Thus the probability P_4 of obtaining at least one 6 in one roll of four dice (or four rolls of one die) is, from Eq. 6-4,

$$P_4 = 1 - (1 - 1/6)^4 = 0.5177$$

and the probability P_{24} of obtaining at least one double-6 in 24 rolls of two dice has the value

$$P_{24} = 1 - (1 - 1/36)^{24} = 0.4914$$

De Méré also inquired as to the number of rolls necessary to ensure a probability greater than 0.5 of achieving the 6 or double-6, respectively. Solving Eq. 6-4 for n,

$$n = \frac{\log(1 - P_n)}{\log(1 - p)}$$

For $p = 1/6$ and $P_n > 1/2$,

$$n > \frac{\log 2}{\log 6 - \log 5} = 3.81$$

Hence, four rolls of a die are required to obtain a probability greater than 1/2 of rolling a 6. For $p = 1/36$ and $P_n > 1/2$,

$$n > \frac{\log 2}{\log 36 - \log 35} = 24.61 \tag{6-5}$$

Therefore, 25 is the smallest number of rolls of two dice that offers a probability greater than 1/2 of achieving a double-6.

Newton–Pepys Problem

Another contribution to early probability theory was triggered by an exchange of letters between Samuel Pepys and Isaac Newton. Pepys, the 17th-century English diarist, questioned which was most likely:

P_{1+} = probability of at least one 6 in a fling of six dyse,
P_{2+} = probability of at least two 6s in a fling of twelve dyse, or
P_{3+} = probability of at least three 6s in a fling of eighteen dyse.

Newton responded that the odds were "in favour of ye Expectation of [ye first chance]."

From the vantage point of more than three centuries, we can express the probability of an outcome of n or more 6s in a cast of $6n$ dice as

$$P_{n+} = \sum_{j=n}^{6n} \binom{6n}{j} (1/6)^j \, (5/6)^{6n-j}$$

Therefore,

$$P_{1+} = 0.665, \quad P_{2+} = 0.619, \quad P_{3+} = 0.597,$$

and P_{1+} is the most likely event. Newton calculated only the probabilities of P_{1+} and P_{2+}, then contented himself by noting that "in ye third case, ye value will be found still less." He also neglected to mention that $P_{n+} \rightarrow 0.5$ as n increases.[6]

Mean versus Median for Two Dice

P_{2+} provides an illustration of the difference between mean and median (Ref. Griffin; Chapter 2).

The number n computed in Eq. 6-5 represents the *median* of the distribution of the number of throws necessary to obtain a pair of 6s—that is, half the values of the distribution are less than or equal to 24, and half are greater or equal to 25.

In a proposed game, **A** throws the two dice until (6,6) appears—at throw n_1. **B** pays **A** 30 units initially, and, subsequently, **A** pays **B** n_1 units. The game *would seem* favorable to **A**, who will win 55.85% of the time. However, the expected number of throws, $E(n_1)$, or *mean*, for (6,6) to appear is

$$E(n_1) = p \bullet 1 + (1 - p)[1 + E(n_1)] = 1/p$$

where p is the single-trial probability of throwing (6,6). Ergo, $E(n_1) = 36$, and the value of the game is $36 - 30 = 6$ to **B**, though **B** will win only 42.93% of the time (1.22% of the time $n_1 = 30$, and a tie will result). (**A**'s gain is limited to 29 units, whereas **B**'s gain is theoretically unlimited.)

Note that, for standard dice, the mean is greater than the median (since the probability distribution is skewed rightward).

Probability of a Particular Outcome

Analogous reasoning to that leading to the formula for simple repetition will conclude that the probability of achieving a particular outcome exactly m times in n throws is the binomial expression

$$P_n^m = \binom{n}{m} p^m q^{n-m} \tag{6-6}$$

[6] Newton was not particularly attracted to probability theory; this dice problem marks his sole venture into ye field.

Thus, the probability P_4^1 of exactly one 6 in four rolls of a die is computed to be

$$P_4^1 = \binom{4}{1}(1/6)(5/6)^3 = 0.386$$

and for the probability P_{24}^1 of obtaining exactly one double-6 in 24 rolls of two dice,

$$P_{24}^1 = \binom{24}{1}(1/36)(35/36)^{23} = 0.349$$

Similarly, if a chance event has constant single-trial probability p, and n denotes the number of trials up to and including the rth success, the distribution of n has the form

$$P_n^r = \binom{n-1}{r-1} p^r q^{n-r}, \quad n = r, r+1, \ldots \tag{6-7}$$

which is recognizable as the negative binomial distribution,[7] aka Pascal's distribution.

By extension from Eq. 6-7 we can also determine the probability P_n that, in n throws of a die, a particular outcome occurs between $n/6 - t$ and $n/6 + t$ times:

$$P_n(n/6 - t \le x \le n/6 + t) = \sum_{x=n/6-t}^{n/6+t} \binom{n}{x} (1/6)^x (5/6)^{n-x}$$

Applying the normal approximation to the binomial distribution,

$$P_n(n/6 - t \le x \le n/6 + t) = \Phi\left(\frac{n/6 + t - n/6 + 1/2}{\sqrt{n(1/6)(5/6)}}\right) - \Phi\left(\frac{n/6 - t - n/6 - 1/2}{\sqrt{n(1/6)(5/6)}}\right)$$

akin to Eq. 6-2. With $n = 6000$ and $t = 50$ as an example, tables of the cpf result in

$$P_{6000}(950 \le n/6 \le 1050) = 2\Phi(1.7494) - 1 = 0.919$$

A perfect die, then, will produce a particular outcome 1000 ± 50 times 91.9% of the time over 6000 trials.

Expected Sum of Single-Die Outcomes

A die is thrown repeatedly until a 6 appears. We inquire as to the expected sum of all outcomes of the die prior to the first 6.

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[7] For an interesting application of the negative binomial distribution, see Epstein and Welch (Ref.).

Let x equal the sum at any given throw. The expected sum, E(x), can then be written as

$$\begin{aligned} \mathrm{E}(x) &= (1/6)[1 + \mathrm{E}(x)] + (1/6)[2 + \mathrm{E}(x)] + (1/6)[3 + \mathrm{E}(x)] \\ &\quad + (1/6)[4 + \mathrm{E}(x)] + (1/6)[5 + \mathrm{E}(x)] \\ &= (1/6)[16 + 5\mathrm{E}(x)] \end{aligned}$$

whence $\mathrm{E}(x) = 15$.

Probability of a Particular Sum

A general formula, useful for the solution of many dice games, provides the probability $P(s)$ of obtaining a given sum s of the outcomes of n dice (or n throws of one die). Of the 6^n different configurations of n dice, the number of configurations whereby the n outcomes sum to a number s is equivalent to the number of solutions of the equation

$$a_1 + a_2 + \cdots + a_n = s \tag{6-8}$$

where each of the a_i's are integers between 1 and 6. A theorem of the discipline of number theory states that the number of solutions of Eq. 6-8 for any n corresponds to the coefficient of x^s in the expansion of the polynomial

$$(x + x^2 + x^3 + x^4 + x^6)^n = \left[\frac{x(1 - x^6)}{1 - x}\right]^n$$

Applying the binomial theorem,

$$x^n(1 - x^6)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x^{n+6j} \tag{6-9}$$

and

$$(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{n + k - 1}{n - 1}x^k \tag{6-10}$$

Multiplying the two series of Eqs. 6-9 and 6-10 and dividing by 6^n, the total number of configurations, we obtain

$$P(s) = \mathrm{coeff}(x^s) = 6^{-n}\sum_{j=0}^{[(s-n)/6]}(-1)^j\binom{n}{j}\binom{s - 6j - 1}{n - 1} \tag{6-11}$$

for the probability of a total of s points on n dice, where $[(s - n)/6]$ is the largest integer contained in $(s - n)/6$. Two-dice probabilities are displayed in Table 6-1.

Table 6-1 Two-Dice Probabilities

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

The most common outcome for n dice is $7n/2$ for n even, and $(7n \pm 1)/2$ for n odd. Thus 7, 10 & 11, 14, 17 & 18, 21, … are the most common outcomes for $n = 2, 3, 4, 5, \ldots$, with respective probabilities 1/6, 1/8, 73/648, 65/648, 361/3888, … .

The game of Razzle-Dazzle is played (in Havana) with eight dice. Pertinent is the probability that a throw (of all eight) totals of 30 or higher is obtained. Eq. 6-11 indicates that

$$P(s \geq 30) = \sum_{s=30}^{48} P(s) = 0.0748 + 0.0677 + 0.0588 + \cdots = 0.380$$

Probability of All Possible Outcomes

We can also inquire as to the probability that one throw of n dice ($n \geq 6$) will produce each of the outcomes, 1, 2, …, 6 at least once. By extending the theorem of total probability (Eq. 2-2), we can state, as in Chapter 3, that the probability of m events, $A_1, A_2, \ldots, A_m$, occurring in a throw of $m \leq n$ dice (or in n throws of a single dice) is given by

$$P(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \cdots \qquad (6\text{-}12)$$

For the $m = 6$ events possible from the throw of a die, Eq. 6-12 leads to

$$P_n = \sum_{k=0}^{5} (-1)^k \binom{6}{k} \left(1 - \frac{k}{6}\right)^n$$

for the probability that all outcomes of a die are represented in a throw of n dice (or n throws of a single die).

Numerically, for $n = 8$, $P_8 = 0.114$. With ten dice, $P_{10} = 0.272$, and, with the minimum of six dice, P_6 has the value 0.0154 (equivalent to $6!/6^6$). Thirteen dice (the median number) are required to obtain a probability greater than 1/2 for the occurrence of all six outcomes ($P_{13} = 0.514$).

Occurrence of All Possible Outcomes

The expected number of throws for all outcomes to appear at least once is

$$6\sum_{1}^{6} 1/k = 14.7$$

which represents the (arithmetic) mean.

The mode, the most likely number of throws following which all outcomes are represented, is determined by defining P(n, m) as the probability that, out of $n \geq 6$ throws, there are m distinct outcomes, $n \geq m \geq 1$. Then (Ref. Nick's Mathematical Puzzles),

$$P(n, m) = \frac{m}{6} P(n-1, m) + \frac{7-m}{6} P(n-1, m-1)$$

For $m = 6$,

$$P(n, 6) = P(n-1, 6) + (1/6)\, P(n-1, 5)$$

$$P(n, 1) = 6^{1-n} \quad \text{and} \quad P(i, i) = i!\binom{6}{i} 6^{-i}, \;\; 2 \leq i \leq 6$$

From these probabilities the mode is calculated as 11, since

$$\max[P(n, 6) - P(n-1, 6)] = 0.317 - 0.233 = 0.084 \text{ for } n = 11$$

Further, the minimum number of times a die must be thrown for at least an even chance that all outcomes occur is 13 [$P(12, 6) + 0.438$; $P(13, 6) = 0.514$].

Occurrence of All Possible Outcomes - Two Dice

A pair of dice is rolled repeatedly until each outcome (2 through 12) has occurred at least once. We inquire as to the expected number of rolls N for this event to take place.

The problem is solved most proficiently by embedding it in a Poisson process (Ref. Ross):

$$\mathrm{N} = \int_0^\infty \left(1 - \prod_{j=1}^{m}\left(1 - e^{-p_j t}\right)\right) dt$$

where there are 11 events with probability p_j, $j = 1, 2, \ldots, 11$ (as per the two-dice probabilities displayed in Table 6-1). Carrying through the indicated operations,

$$\mathrm{N} = \frac{7.69^{+} \times 10^{11}}{1.25^{+} \times 10^{10}} = 61.217$$

Potpourri

1. In $6q$ throws of a die, the probability that each of the six numbers will appear *exactly* q times is immediately determined from the multinomial distribution (Eq. 2-12):

$$P_{6q} = \frac{(6q)!}{(q!)^6 6^{6q}}$$

With 6 throws ($q = 1$), $P_6 = 0.0154^+$ (as per Eq. 6-12 with six dice). With 12 throws ($q = 2$), $P_{12} = 0.00344^-$. With 18 throws ($q = 3$), $P_{18} = 1.359 \times 10^{-3}$. And with 24 throws ($q = 4$), $P_{24} = 8.879 \times 10^{-4}$.

Applying Stirling's formula for larger values of q,

$$P_{6q} \cong \sqrt{6(2\pi)^{-5}}\, q^{-5/2} = 0.025 q^{-5/2}$$

2. With n dice, the expected value, E_n, of the highest outcome $k \le 6$ is expressed by

$$E_n = 6 - \sum_{1}^{5} (k/6)^n$$

For $n = 6$ dice, $E_n = 5.560^+$; for $n = 12$, $E_n = 5.880^-$. Obviously, as n increases, $E_n \to 6$.

3. Let $N(n, m)$ denote the number of ways of throwing n dice, each with m faces. We can readily derive this number as

$$N(n, m) = \binom{n + m - 1}{n}$$

For two conventional dice, $N(2, 6) = 21$; and for three conventional dice, $N(3, 6) = 56$.

4. Eq. 6-11 can be generalized to specify the probability of obtaining the sum s from n dice, each with m faces:

$$P(s, n, m) = m^{-n} \sum_{j=0}^{[(s-n)/m]} (-1)^j \binom{n}{j} \binom{s - mj - 1}{n - 1}$$

5. We wish to determine the expected number of throws of n dice for the appearance of two consecutive all-1s.

Let X represent the state that the last throw of the n dice did *not* show all-1s for the second consecutive time (and is also the state at the first throw). Let Y represent the state that the last throw of the n dice *did* show all-1s for the second consecutive time. After the first throw there is probability $1 - (1/6)^n$ that state X still pertains and probability $(1/6)^n$ that state Y pertains. Then the expected values of X and Y can be written as

$$E(X) = 1 + [1 - (1/6)^n]\, E(X) + (1/6)^n\, E(Y)$$
$$E(Y) = 1 + [1 - (1/6)^n]\, E(X)$$

Solving for E(X):

$$E(X) = 6^n(6^n + 1)$$

Thus with a single die, the expected number of throws to obtain two consecutive 1s is $6 \bullet 7 = 42$. With two dice, the expected number of throws to obtain consecutive "snake eyes" is $36 \bullet 37 = 1332$; and, for $n = 3$, $E(X) = 216 \bullet 217 = 46{,}872$.

6. With a k-sided die, numbered 1, 2, …, k, the probability P_2 of obtaining a particular number twice in m rolls is

$$P_2 = 1 - [(k-1)/k]^m - (m/k)[(k-1)/k]^{m-1}$$

For six rolls of a conventional die, $P_2 = 1 - (5/6)^6 - (5/6)^5 = 0.263^+$, and with 12 rolls, $P_2 = 0.619^-$.

STRUCTURED DICE GAMES

First or Second or Third or Time Up

In one of the oldest formal dice games, a single die is thrown until a particular number appears (or, in alternative versions, until it appears twice or three times). If p is the single-trial probability of that event, and n is the number of throws until it occurs for the rth time, Eq. 6-7 applies (the parameter $n - r$ obeys the negative binomial distribution), and with $p = 1/6$ and $r = 1$, yields

$$P_n^1 = 5^{n-1}6^{-n} \tag{6-13}$$

To obtain the probability, $P_n(r)$, that the particular outcome occurs for the rth time *by* the nth throw of the die, we sum P_n^r from r to n:

$$P_n(r) = \sum_{i=r}^{n} \binom{i-1}{i-r} p^r q^{i-r} \tag{6-14}$$

For $r = 2$, this event occurs by the seventh throw with a probability greater than 0.5 (specifically, 0.536). For $r = 3$, Eq. 6-14 indicates a probability greater than 0.5 (specifically, 0.513) that the third occurrence of this outcome appears by the 16th throw.

Nicolas Bernoulli's Game—Expected Number of Throws to Achieve a Particular Outcome

A nearly identical problem posed by Nicolas Bernoulli can be considered the forerunner of the St. Petersburg game. Bernoulli inquired as to the expected value of n in a game consisting of rolling a die until a 6 appears for the first time at the nth throw, with a payoff of n units.

Evidently, the largest value of P_n^1 in Eq. 6-13 occurs for $n = 1$. Therefore, the most probable number of throws to obtain a 6 is 1. The expected number is expressed by

$$\mathrm{E}(n) = \frac{1}{6}\sum_{n=1}^{\infty} n(5/6)^{n-1} \tag{6-15}$$

Using the identity $1 + 2x + 3x^2 + \cdots = (1 - x)^{-2}$, Eq. 6-15 becomes

$$\mathrm{E}(n) = \frac{1}{6}[1 - (5/6)]^{-2} = 6 \tag{6-16}$$

and, as we would have anticipated, six throws are expected for obtaining one particular outcome of the die. This value should be clearly distinguished from the number of throws (four) required to obtain a 6 with probability greater than 1/2.

The specific example of Eq. 6-16 can be generalized by the following theorem.

> In any series of trials where the probability of occurrence of a particular event *X* is constantly *p*, the expected number of trials E(*X*) to achieve that event is the reciprocal of the probability of its occurrence.

We can prove this statement by noting that if p is constant, the probability of X first occurring at the ith trial follows the geometric distribution (Eq. 2-13)

$$X(i) = p(1 - p)^{i-1}$$

The expected number of trials is then defined by

$$\mathrm{E}(X) = \sum_{i=1}^{\infty} ip(1 - p)^{i-1} = p + \sum_{i=2}^{\infty} ip(1 - p)^{i-1} \tag{6-17}$$

With the substitution $j = i - 1$, Eq. 6-17 is rewritten in the form

$$\mathrm{E}(X) = p + (1 - p)\sum_{j=1}^{\infty}(j + 1)p(1 - p)^{j-1}$$

Therefore

$$\mathrm{E}(X) = p + (1 - p)[\mathrm{E}(X) + 1]$$

and

$$\mathrm{E}(X) = 1/p \tag{6-18}$$

As an example, since the probability of rolling the sum of 9 with two dice is 4/36, the expected number of rolls of two dice to achieve a 9 is nine. (Eq. 6-18 also specifies the expected length of a sequence of throws of a biased coin until the first occurrence of Heads [single-trial probability = p].)

James Bernoulli's Game

To illustrate one of the pitfalls inherent in skewed distributions, consider the following dice game initially propounded by James Bernoulli. Let a solitary player perform one throw of a single die, securing the result i ($i = 1, 2, \ldots, 6$). He then selects i dice from a set of six and throws them once (or throws the single die i times), obtaining the sum s_i. If $s_i > 12$, he receives one unit payment; if $s_i < 12$, he loses one unit; and if $s_i = 12$, the game is declared a tie. The total number of outcomes is 6×6^6; enumerating those that contribute a sum greater than, equal to, or less than 12:

$$P(s_i > 12) = \frac{130{,}914}{6\times 6^6} = 0.468$$

$$P(s_i = 12) = \frac{13{,}482}{6\times 6^6} = 0.048$$

$$P(s_i < 12) = \frac{135{,}540}{6\times 6^6} = 0.484$$

Comparing these values, it is apparent that the player is burdened with a negative mathematical expectation (specifically, $0.468(1) - 0.484(-1) = -0.016$). Yet, the expected sum obtained from throwing i dice according to the game rules is

$$\mathrm{E}(s_i) = [\mathrm{E}(i)]^2 = (3.5)^2 = 12.25$$

for which the player receives a unit payment. *Were* the distribution about 12.25 symmetrical rather than skewed, the game would be favorable to the player.

A Maximization Game

A dice game popular among military personnel consists of throwing five dice simultaneously, declaring one or more to be "frozen," then repeating the process with the remaining dice until all five are frozen (in five or fewer throws). Objective: to maximize the sum of the points on the five frozen dice. By obtaining a total of s points, a player receives $s - 24$ units from each of the other players ($24 - s$ units are paid to each other player if $s < 24$). We wish to determine an optimal strategy and its corresponding mathematical expectation.

The expected sum arising from a throw of one die is, as per Eq. 2-16,

$$\mathrm{E}(s_1) = (1/6)(1 + 2 + 3 + 4 + 5 + 6) = 3\tfrac{1}{2}$$

With two dice plus the option of rethrowing one, the mathematical expectation of the sum of the pips on the two dice is given by

$$\begin{aligned}
E(s_2) &= (1/36)\Big[12 + 2\cdot 11 + 3\cdot 10 + 3\cdot 2\left(9\tfrac{1}{2}\right) + 2\cdot 9 + 3\cdot 2\left(8\tfrac{1}{2}\right) \\
&\quad + 8 + 3\cdot 2\left(7\tfrac{1}{2}\right) + 5\left(6\tfrac{1}{2}\right) + 3\left(5\tfrac{1}{2}\right) + 4\cdot\tfrac{1}{2}\Big] \\
&= 8.236
\end{aligned}$$

With three dice plus the option of rethrowing two, one of which may be thrown for the third time, the expectation of the sum can be calculated as

$$E(s_3) = 13.425$$

Similarly, throwing four dice with rethrow options according to the stated rule produces an expectation of the sum of

$$E(s_4) = 18.844$$

and, with five dice, according to the game rules, the mathematical expectation of the sum has the value

$$E(s_5) = 24.442$$

Optimal strategy is specified by these expectations. Thus, the option of rethrowing one die should be exercised if its outcome is 3 or less, since the expectation of the rethrow is higher (3.5). With the option of rethrowing two dice, combinations totaling 9 or less should be (at least partially) rethrown. For example, 4-4 is rethrown, and 5-4 declared frozen; with the combination 5-3, only the 3 should be rethrown, resulting in an expectation of $5 + 3\frac{1}{2} = 8.5$ (rethrowing the 5 and the 3 results, in the lower expectation of $E(s_2) = 8.236$). With the option of rethrowing three dice, all combinations totaling 15 or less should be examined to determine whether rethrowing one, two, or three dice maximizes the expectation. As examples, with 5-5-3 we freeze the two 5s and rethrow the 3 [$10 + E(s_1)$ is the maximum value], and with 5-4-4 we rethrow all three dice [$E(s_3)$ is maximum].

Following the first roll of the game, we have the option of rethrowing four dice. Here we examine all combinations totaling 21 or less to determine whether rethrowing one, two, three, or four dice maximizes the expectation. Illustratively, with 5-5-5-3, we would rethrow all four dice [resulting in $E(s_4)$], and with 6-5-4-4 we would rethrow the 5-4-4 [resulting in $6 + E(s_3)$]. Clearly, a 6 is never rethrown; 5 and 4 are rethrown or not, depending on the number of dice remaining with rethrow options; and 3, 2, and 1 are always rethrown whenever permitted by the rules. The expected gain E(G) of the game is the sum of the products of each possible payoff by its corresponding probability, $P(s)$:

$$\mathrm{E}(G) = \sum_{s=5}^{30} (s - 24)P(s) \tag{6-19}$$

where s can range from 5 (all 1s on the five dice) to 30 (all 6s). The values of $P(s)$ are computed by delineating the various combinations of dice that total each value of s; the results are presented in Table 6-2. Substituting the appropriate numerical values into Eq. 6-19, we obtain

$$\mathrm{E}(G) = 0.438$$

The game is thus highly favorable to the player rolling the dice since 25 is the most probable sum achieved by optimal play. Prevailing opinion that the game is nearly equitable likely rests on the assumption of nonoptimal strategies.

Table 6-2 Maximization Game Probabilities

s	$P(s)$	s	$P(s)$	s	$P(s)$	s	$P(s)$
30	0.0109	25	0.1410	20	0.0397	15	0.0015
29	0.0383	24	0.1291	19	0.0250	14	0.0005
28	0.0809	23	0.1076	18	0.0145	13	0.0001
27	0.1193	22	0.0817	17	0.0076	12	—
26	0.1401	21	0.0588	16	0.0036	11	—

All Sixes

A game similar in its mechanics, although not in its computations, consists of rolling n (usually five) dice, attempting to achieve the outcome 6 on all n. Following each throw of the dice, those showing a 6 are set aside and the remainder rethrown; this process is then continued indefinitely until all n dice exhibit a 6. We wish to determine the expected number of throws $\mathrm{E}_n(N)$ to obtain all 6s.

The classical method of solving this problem proceeds by computing $P_n(N)$, the probability of achieving the goal with n dice by the Nth throw, and then summing the product $NP_n(N)$ over all values of N and n (from 1 to ∞)—thus prescribing a large number of tedious calculations. Fortunately, there exists a clever insight leading to a direct solution of this problem.

The probability of throwing at least one 6 in a roll of n dice is $1 - (5/6)^n$; hence, from Eq. 6-18, the expected number of rolls to obtain at least one 6 is $6^n/(6^n - 5^n)$. If the throw that achieves the first 6 contains no other 6s, the probability of which is $n5^{n-1}/(6^n - 5^n)$, the problem reduces to one with $n - 1$

dice. If the throw achieving the first 6 contains exactly one additional 6, the probability of which is

$$\frac{n(n-1)5^{n-2}}{2(6^n - 5^n)}$$

then the problem reduces to one with $n - 2$ dice. Proceeding by induction, we are subsequently led to the expression

$$E_n(N) = \frac{6^n + \sum_{k=0}^{n-1} 5^k \binom{n}{k} E_k(N)}{6^n - 5^n} \tag{6-20}$$

where $E_0(N) = 0$. For $n = 5$ dice, Eq. 6-20 is evaluated as

$$E_5(N) = 13.024$$

Thus, slightly more than 13 throws are expected to obtain all 6s on five dice with repeated throws of those dice exhibiting non-6s.

Turn the Die

A simple strategic dice game suggested by Martin Gardner involves a single die that is initially thrown by **A** and then rotated a quarter-turn by **B** and **A** as alternating moves, recording a running sum of the numbers facing up. The object of the game is to maneuver the cumulative sum of the up-faces exactly to a specified goal.

Strategy for either player dictates the attempt to force the running sum to a value equal to the residue of the goal, mod 9, or to prevent the opponent from achieving that value. For example, if the goal is set at 31 (4, mod 9), and **A** throws a 4, he strives to include the totals 4-13-22-31 in the running series. Thus, if **B** plays 6 or 5, **A** counters with 3 or 4, respectively (totaling 13); if **B** plays 1 or 2, **A** shows 4 or 3, respectively, which totals 9 and prevents **B** from reaching 13 (the 4 is not available). In general, letting r be the residue of the goal, mod 9, one should play to $r - 5$, leaving the 5 on the top or bottom of the die, and to $r - 3$, $r - 4$, or $r + 1$, leaving the 4 unavailable.

Except for $r = 0$, a win for **A** exists for at least one roll of the die. Table 6-3 indicates the winning rolls for each residue of the goal, mod 9. Clearly, if the first player is afforded the option, he chooses a goal with residue 7, mod 9, thereby obtaining a fair game.

Table 6-3 Winning Numbers for Turn-the-Die Game

r	0	1	2	3	4	5	6	7	8
Winning first roll of die		1,5	2,3	3,4	4	5	3,6	2,3,4	4

Roller Coaster Dice

A game found on the Internet and in a few unpretentious casinos, Roller Coaster Dice pits a single player against the House. To begin, following a unit wager, the player rolls two conventional dice and then predicts whether a second roll will produce a sum greater or less than the first. If correct, he repeats this procedure for a third, a fourth, and a fifth roll (if he continues to predict successfully).

Obviously for any sum ≤ 6, he will predict a higher number for the next roll; and, for any sum ≥ 8, he will predict a lower number. For an initial outcome of 7, his two choices are equivalent.

A single incorrect prediction loses the wager. With four correct choices, the player may end the game and gain a sum from the House equal to three times his wager. He also has the option of proceeding under the same rules—risking his stake of 4 units. Two more consecutive correct predictions (a total of six) will increase his net gain to 6 units. He may then end the game or opt to continue—whence yet two more correct predictions (for a total of eight) will raise his net gain to 12 units. Finally, he may either end the game at this stage or continue, striving for yet two more correct predictions (for a total of ten) and a net gain of 29 units.

Appropriate arithmetic shows the player to have a probability of 0.2324 of achieving four correct predictions—for an expected return of $4(0.2324) - 1 = -0.0704$.

If the fifth dice roll has shown 4 through 10, he should end the game. With a 2 or 12 (3 or 11), he has a probability of 891/1296 or (825/1296), respectively, of reaching the sixth stage—and should therefore continue. His expected return is thereby increased by

$$0.2324 \cdot \frac{6}{36}\left\{\frac{2}{36}\left[3\left(\frac{891}{1296}\right) - 4\left(\frac{405}{1296}\right)\right] + \frac{4}{36}\left[3\left(\frac{825}{1296}\right) - 4\left(\frac{471}{1296}\right)\right]\right\} = 0.0037$$

After six correct predictions, the player should end the game if the (seventh) roll shows a 5 through 9. With a 2 or 12, 3 or 11, 4 or 10, he should continue, similar calculations indicating an additional expected return of 0.0026. After eight correct predictions, he should stop if the (ninth) roll shows a 6 through 8. With a 2 or 12, 3 or 11, 4 or 10, 5 or 9, he should continue—accruing a further expected return of 0.00608.

Thus, with optimal strategy, the game entails a value of $-0.0704 + 0.0037 + 0.0026 + 0.0061 = -0.0580$.

La Queue Circulaire Aux Dés

Each of six players in a ring rolls a die in sequence; the first to roll a 6 wins. With a non-6, the die is passed on to the next player, *seriatim*; the first player regains a turn if no 6 has been rolled by his five competitors, and so forth.

Eq. 5-39 indicates that the first player's probability of success, $P_{1,6}$, is

$$P_{1,6} = \frac{1}{6}\left[(5/6)^0 + \frac{(5/6)^6}{1-(5/6)^6}\right] = 0.2506$$

Similarly, for players 2 to 6,

$$P_{2,6} = 0.2088;\; P_{3,6} = 0.1740;\; P_{4,6} = 0.1450;\; P_{5,6} = 0.1204;\; P_{6,6} = 0.1006$$

With a large number of competitors, the first player's probability of success approaches 1/6, since it is unlikely he will have further opportunities with the die. A player in 25th position along a ring of 50 persons has a probability of success of 0.013, and a player midway around a 100-person ring faces a dismal probability success of 0.00014.

A Parrondo-Type Dice Game

Although Parrondo's principle (Chapter 4) is illustrated with biased coins, it can also be applied to a game with fair dice (or a fair coin) (Ref. Astumian). Consider the five-square Astumian diagram of Figure 4-3 with a checker at the START position. We then define two games as follows.

Game I: The player rolls two dice, moving the checker to the RIGHT square if the dice total 11, to the LEFT square for a total of 2, 4, or 12. With any other total, the dice are rolled again. With the checker positioned at LEFT or RIGHT, a roll of 7 or 11 moves it one square rightward; a roll of 2, 3, or 12 moves it one square leftward. When the checker reaches WIN or LOSE, the game is over.

The probability of reaching WIN non-stop is (2/7)(8/12) = 4/21, and of reaching LOSE non-stop is (5/7)(4/12) = 5/21. Ergo, *Game I* is losing.

Game II: Here the player follows a prescription reversing the *Game I* rules, moving right from START with a roll of 7 or 11, and left from START with a roll of 2, 3, or 12. With the checker at LEFT or RIGHT, he moves rightward with 11, and leftward with 2, 4, or 12.

His probability of reaching WIN non-stop is therefore (8/12)(2/7) = 4/21, and of reaching LOSE non-stop is (4/12)(5/7) = 5/21. Thus *Game II* is also losing.

Since both games continue after the dice mandate a non-move, the probability of ending with WIN is 4/9, and of ending with LOSE is 5/9, whence both games entail an expectation of $-1/9$.

Now, in accordance with the Parrondo principle, we mix the two games in random fashion, flipping a fair coin prior to each move: with Heads, *Game I* is played; with Tails, *Game II*. Then the probability of reaching WIN is the product of the average number of rightward moves: $[(2 + 8)/2] \cdot [(8 + 2)/2] = 25$. Similarly, the probability of reaching LOSE is the product of the average number of leftward moves: $[(5 + 4)/2] \cdot [(4 + 5)/2] = 20.25$.

The combined game, therefore, offers an expectation of $+0.105$.

The Racing Game

The general racing game is defined by two players, **A** and **B**, alternately throwing m dice, the winner being that player whose cumulative total reaches or exceeds T points.

The mean number of points gained by throwing the m dice is $m7/2$ with a standard deviation of

$$\sigma = \sqrt{\mathrm{M}_2(X) - (7m/2)^2} = \sqrt{35m/12}$$

where $\mathrm{M}_2(X)$ is the second moment (the binomial distribution pertains).

Let t_a and t_b represent the number of points short of the goal T for **A** and **B**, respectively. After n throws, $n = 2(t_a + t_b)/7m$, and the standard deviation in terms of the number of throws becomes $\sigma = \sqrt{5(t_a + t_b)/6}$ (independent of m).

The advantage of having the next throw is equal to 1/2 the average number of points to be gained: $t_a - t_b + 7m/4$.

We wish to determine the probability $P(x)$ that the player's opponent fails to gain this number of points in $2(t_a - t_b)/7m$ throws. Accordingly, we set

$$x = \frac{t_a - t_b + 7m/4}{\sqrt{5(t_a + t_b)/6}}$$

and find $P(x)$ from the tables of standard normal distribution. More directly, for small x, we can apply the approximation

$$P(x) \approx 0.5 + x/\sqrt{2\pi} \qquad (6\text{-}21)$$

To illustrate this general formula, consider $t_a = t_b = t$, as is the case at the beginning of the game or at some stage where the two players are tied. For a single die ($m = 1$) and a goal of T = 100 points, the probability P_t that the player who next throws will win the game is (Eq. 6-21):

$$P_{25} = 0.5 + 7/(20\sqrt{10\pi/3}) \approx 0.61 \quad \text{for } t = 25 \text{ points short of the goal}$$

$$P_{50} \approx 0.58 \quad \text{for } t = 50$$

$$P_{100} \approx 0.55 \quad \text{for the player who throws first at the start}$$

With $m = 2$ dice and a goal of 100 points,

$$P_{25} \approx 0.72$$

$$P_{50} \approx 0.69$$

$$P_{100} \approx 0.61 \quad \text{for the player who throws first at the start reflecting the advantage of first move.}$$

And for a goal of 200 points,

$$P_{25} = 0.65$$

$$P_{50} = 0.63$$

$$P_{100} = 0.58$$

$$P_{200} = 0.55 \quad \text{for the player who throws first at the start—still a significant advantage}$$

The racing game can be rendered more equitable by awarding the second player a handicap of $7m/2$ points.

Of Piglets, Pigs, and Hogs

Jeopardy race games define a class of probabilistic and strategic contests wherein a player's fundamental decision is (1) whether to jeopardize previous gains by continuing to play for potentially greater gains; or (2) surrender his turn voluntarily and thereby safeguard his gains accumulated to date. In general, the objective is to be first to reach a specified goal.

In the jeopardy game Piglet, a player rolls a single die; the sum of the points obtained at each roll is added to the sum achieved on previous rolls. The player continues to roll the die until either the outcome 1 appears or he voluntarily ends his turn—whereby his accumulated sum is designated a "plateau." In subsequent turns, the appearance of a 1 reverts his sum to the previous plateau. Objective: to reach a cumulative total of 100.

Optimal strategy is a function of the previous plateau, the current score, the opponents' plateaus, and the number of opponents. To simplify, consider the solitaire form of piglet (Ref. Humphrey, 1979, 1980) with the objective of reaching 100 in as few turns as possible. Let k equal the optimal number of accumulated points at which to declare a plateau. Then, given k points, the expected number at the next roll is

$$\frac{1}{6}[0 + (k + 2) + (k + 3) + (k + 4) + (k + 5) + (k + 6)] = \frac{5k + 20}{6}$$

Therefore, optimal strategy dictates ending a turn whenever

$$\frac{5k + 20}{6} < k \quad \text{or} \quad k > 20$$

that is, whenever the player has accumulated a total more than 20 points above his previous plateau (at $k = 20$, the expected gain from continuing is identical to that from stopping).

One Piglet variation scores the squares of the outcome appearing on the die. In this case, the expected number at the next roll is

$$\frac{1}{6}[0 + (k + 4) + (k + 9) + (k + 16) + (k + 25) + (k + 36)] = \frac{5k + 90}{6}$$

and optimal strategy declares a plateau whenever $k > 90$. An ultimate goal of 400 or 500 is more consonant with this game.

With one or more opponents, it can be difficult to quantify an optimal strategy. For example, against an opponent holding a wide lead in points, it is advisable to continue a turn for lower values of k. The same argument pertains when competing against a large number of opponents who may not be adhering to the optimal policy.

Pig differs from Piglet in its use of two dice—a difference that engenders immensely greater complexity. Here, the outcome 1 on either die again reverts the player's score to the previous plateau; in addition, a 1 on both dice reduces the score to zero.

As with Piglet, we consider a simplified solitaire Pig whose objective is to reach 24 points. In this elementary case, the player pays one unit for each turn at the dice—that is, until a 1 appears on either die or a plateau is declared (here, "snake eyes" adds no further penalty).

We wish to determine the strategy that minimizes the expected number of units paid (plus some fixed payoff for reaching 24).

Let (a, b) represent the position of the player with current score a and plateau b. To prove that the optimal strategy for this game is never to declare a plateau, it is sufficient to examine this strategy for the case of $a = 23$ and $b = 0$.

Let V be the expected additional cost under a strategy of declaring a plateau at (23, 0) and paying one unit. Two events can subsequently occur: The player can throw a 1 on either die, remaining at (23, 23), or he can obtain an outcome without a 1 on either die, in which case the game is concluded. The probability of this latter event is 25/36. Thus in the (23, 23) case, the expected cost is $1/(25/36) = 36/25$. As a consequence we can write for V,

$$V = 1 + (11/36) \cdot (36/25) = 36/25 \tag{6-22}$$

Now, if we let W be the expected number of units paid in the (0, 0) case under a strategy of never declaring a plateau, and V_1 be the expected cost at (23, 0) under this strategy, we have

$$V_1 = (11/36)(1 + W) \tag{6-23}$$

To prove that the optimal strategy is never to declare a plateau, we must show that $V_1 < V$—i.e., from Eqs. 6-22 and 6-23, that

$$W < \frac{36^2}{11 \times 25} - 1 = 3.71$$

The quantity W is the expected number of times a 1 is thrown on either die before achieving a continuous run (without 1s) adding up to 24 or greater. Further, $W = 1/P$, where P is the probability of accumulating a score of 24 or more without the appearance of a 1 (Eq. 6-18).

Enumerating all possible combinations, we can calculate the probabilities, p_i, of reaching 24 with i rolls of two dice without a 1 appearing (i can assume the values 2, 3, 4, 5, 6). Numerically, $p_2 = 7.7 \times 10^{-4}$, $p_3 = 0.186$, $p_4 = 0.100$, $p_5 = 2.5 \times 10^{-3}$, and $p_6 = 3.3 \times 10^{-6}$.

Thus, $P = 0.289$, and $W = 3.465$, which is less than the critical value of 3.71. Hence, no plateau should be declared in this game.

If this solitaire version of Pig is played with a goal of 100, it appears that the first plateau should be declared at about the level of 25.

The unpruned version of Pig has been analyzed by Neller and Presser (Ref.), exploiting the didactic technique of value iteration.

Of several Pig mutations extant, Hog (Ref. Neller and Presser) can claim the most interesting facets. Players are restricted to one roll per turn, but with as many dice as desired. If no 1s occur, the sum of the dice is scored. With one or more 1s, no points are scored for that turn. As customary, the first player to reach 100 points wins. (Hog is equivalent to a version of Piglet where the player must commit beforehand to a specific number of rolls in his turn.)

Let $P(n, k)$ be the probability that rolling n dice results in a turn score of k points, $k \geq 0$ and $0 < n \leq n_m$, where n_m is the number of dice beyond which the optimal policy remains unchanged. Here, $n_m \geq 26$. Feldman and Morgan (Ref.) have shown that the probability of winning, $P_{i,j}$, is expressible as

$$P_{i,j} = \max_{c<n<n_m} \sum_{k=0}^{6n} P(n,k)(1 - P_{j,i+k})$$

where i is the player's score, and j the opponent's, $0 \leq i,j < 100$. This probability can be computed with dynamic programming. Results indicate an optimal strategy akin to that of Pig. And, as with Pig, the first player wins with probability ~0.53. The maximum expected score per turn (8.04) occurs with either five or six dice—and that number should be rolled on the first turn.

Other variants include Piggy Sevens (throwing a 7 with the two dice loses the turn), Big Pig (the player's turn continues after throwing two 1s), and Hog Wild, a variant devised by the author wherein the sum of the two numbers rolled (2 through 12) is multiplied by their difference (0 through 5), with the product added to the player's score.

Yahtzee®

Sometimes described as "poker with dice," Yahtzee[8] entails five standard dice and from two to several players. Each player, in turn, is allowed a maximum of three throws of the dice with the option of "fixing" any non-zero subset of the outcomes of each throw and then rethrowing the remaining dice (if any). The configuration showing after the final throw constitutes the player's score for that turn—see Table 6-4. Further, an additional 35 points are awarded if the player's total score from the first six categories exceeds 62. Additional yahtzees after the first are rewarded with 100 points each.

Table 6-4 Yahtzee Scoring

Outcome	Point Score
Aces (1s)	Sum of 1s
Twos	Sum of 2s
Threes	Sum of 3s
Fours	Sum of 4s
Fives	Sum of 5s
Sixes	Sum of 6s
Three of a Kind	Sum of all 5 dice
Four of a Kind	Sum of all 5 dice
Full House	25
Small Straight (four consecutive numbers)	30
Straight	40
Yahtzee (five of a kind)	50
Chance (any outcome)	Sum of all 5 dice

Each player's objective is to maximize his score.

At the end of his turn, the player applies his configuration to one of the 13 scoring categories. Each category may be selected only once (if the player's configuration does not fit any unused category, he receives a 0 for that turn). Thus, a game consists of 13 turns.

The maximum score achievable is 375 (without Yahtzee bonuses); the expected score, employing optimal decisions, is 255.

Phil Woodward (Ref.) has evaluated all 1.27×10^9 possible outcomes of the game to determine the optimal playing strategies—detailed in Appendix Table B (Ref. Verhoeff and Scheffers.). To illustrate the use of this table, consider the combination 3 3 4 5 6 obtained with the first throw. Optimal strategy dictates fixing 3 3 and rethrowing the remaining three dice. If this configuration occurs after the second throw, optimal strategy calls for fixing 3 4 5 and rethrowing the other two dice. If this configuration occurs after the third throw, it should be scored as 30 in the Small Straight (SS) category.

Applying this optimal strategy yields an expected final score of 254.59 (with a standard deviation of 59.61).

[8] A corruption of "yacht at sea."

Of the 13 categories, Large Straight (LS) proves to be the greatest contributor to the final score, followed by Small Straight (SS) and Full House (FH). Each contribution is enumerated in Table 6-5.

Table 6-5 Yahtzee Categories

Category	Expectation
Large Straight	32.71
Small Straight	39.46
Full House	22.59
Chance (CH)	22.01
Three-of-a-Kind	21.66
Sixes	19.19
Yahtzee (YZ)	16.87
Fives	15.69
Four-of-a-Kind	13.10
Fours	12.16
Threes	8.57
Twos	5.28
Aces	1.88

Poker Dice

Analysis of dice games and determination of an optimal strategy are conceptually trivial in most instances. Pig is a notable exception. Liars Dice (where the application of judicious prevarication is the essential ingredient) and Poker Dice (equivalent to conventional Poker played with an infinite number of decks) are two other examples.

Poker Dice (each die admits of six outcomes: Nine, Ten, Jack, Queen, King, and Ace) invokes the theory of partitions. The occupancy problem posed is that of five dice occupying six cells; we therefore consider a partition of the six cells into six subpopulations (0 through 5). A throw of five dice can yield one of seven results whose respective probabilities are obtained by enumerating the partitions for each case:

			Number of Dice Re-Rolled	**Probability of Improvement**
P (no two alike)	$= 6! \cdot 6^{-5}$	$= 0.0926$	4	319/324
P (one pair)	$= \left(\frac{6!}{2!3!} \cdot \frac{5!}{2!}\right) 6^{-5}$	$= 0.4630$	3	13/18
P (two pairs)	$= \left(\frac{6!}{3!2!} \cdot \frac{5!}{2!2!}\right) 6^{-5}$	$= 0.2315$	1	1/3
P (three alike)	$= \left(\frac{6!}{2!3!} \cdot \frac{5!}{3!}\right) 6^{-5}$	$= 0.1543$	2	4/9
P (Full House)	$= \left(\frac{6!}{4!} \cdot \frac{5!}{3!2!}\right) 6^{-5}$	$= 0.0386$	0	–

P (four alike)	$=\left(\frac{6!}{4!}\bullet\frac{5!}{4!}\right)6^{-5}$	$= 0.0193$	1	1/6
P (five alike)	$=\left(\frac{6!}{5!}\bullet\frac{5!}{5!}\right)6^{-5}$	$= 0.0008$	0	–

In the standard version of Poker dice, each player has the option of retaining one or more dice after the first throw and rethrowing the others. Probabilities of improving the "poker hand" are listed in the rightmost column.

NONTRANSITIVE DICE

Consider two sets of three dice marked as follows:

I:	A: (1, 1, 4, 4, 4, 4)	**II:**	A: (1, 4, 4, 4, 4, 4)
	B: (3, 3, 3, 3, 3, 3)		B: (2, 2, 2, 5, 5, 5)
	C: (2, 2, 2, 2, 5, 5)		C: (3, 3, 3, 3, 3, 6)

In a two-person game, the first player selects either A, B, or C. The second player then selects C against A, A against B, and B against C. Each player casts his die, with the higher number winning the stake.

With set **I**, C wins against A with probability 5/9; A wins against B, and B wins against C, each with probability 2/3. If the first player selects A, B, or C equiprobably, the value of the game is 7/27 to the second player.

With set **II**, A wins against C with probability 25/36; C wins against B, and B wins against A, each with probability 7/12. If the first player selects A, B, or C equiprobably, the value of the game is 13/54 to the second player.

These two sets constitute the only nontransitive sets wherein no die is marked with a face value higher than 6. Without this constraint, any number of nontransitive sets can be devised. Three such examples are generated by the magic square:

	D:	E:	F:
A:	6	1	8
B:	7	5	3
C:	2	9	4

[Rows, columns, and diagonals have the same sum (15).] The three numbers of each row and column define a die with each number marked on two faces.

III:	A: (1, 1, 6, 6, 8, 8)	**IV:**	D: (2, 2, 6, 6, 7, 7)
	B: (3, 3, 5, 5, 7, 7)		E: (1, 1, 5, 5, 9, 9)
	C: (2, 2, 4, 4, 9, 9)		F: (3, 3, 4, 4, 8, 8)

For these two sets, A(D) wins over B(E), B(E) over C(F), and C(F) over A(D), each with probability 5/9. The value of either game is 1/9 to the second player.

A nontransitive set with the same numbers as set **III** is obtained by interchanging 3 with 4 and 6 with 7.

The same game format, but with each player rolling his selected die twice—and scoring the cumulative total—retains the nontransitivity but reverses the dominance for the cases of sets **I** and **II**.

Specifically, for set **I**, A wins against C with probability 0.613; C wins against B, and B wins against A, each with probability 5/9. The value of the game, with the first player selecting A, B, or C at random, equals 0.150 to the second player.

For set **II**, C wins over A with probability 0.518; A wins over B, and B wins over C, each with probability 0.590. With the first player selecting A, B, or C equiprobably, the value of the game to the second player is 0.132.

Set **III** loses its nontransitivity in this game format (each die cast twice)!

With set **IV** and two rolls of each die, D wins over F with probability 4/9; F wins over E, and E wins over D, each with probability 42/79. (The probability of a tie—the first player choosing D, E, or F equiprobably—is 8/243.) The value of this game is $149/1659 = 0.0898$ to the second player.

With each person casting a selected die three times or more and then comparing cumulative totals, the game does not exhibit nontransitivity.

R.P. Savage (Ref.) has constructed a nontransitive set of three dice using the numbers 1, 2, …, 18:

$$\text{A: } (18,9,8,7,6,5) \qquad \text{B: } (17,16,15,4,3,2) \qquad \text{C: } (14,13,12,11,10,1)$$

Here, A wins over B, and B wins over C, both with probability 7/12; C wins over A with probability 25/36.

Schwenk's dice (Ref. Hobbs) constitute yet another three-member nontransitive set:

$$\text{A: } (1,1,1,13,13,13) \qquad \text{B: } (0,3,3,12,12,12) \qquad \text{C: } (2,2,2,11,11,14)$$

For a single roll of these dice, matched pairwise, A wins over B; B wins over C; and C wins over A, each with probability 7/12. However, when each die is rolled twice, preferences are reversed: C^2 now wins over B^2 with probability 25/48, while B^2 wins over A^2, and A^2 wins over C^2, each with probability 77/144.

Curiously, rolling each die three times returns the order of preference to $B^3 > C^3$ (with probability 0.5067), $C^3 > A^3$, and $A^3 > B^3$ (each with probability 0.5028). (For this game, a significant probability of tying exists.)

Bradley Efron[9] has devised sets of four dice with nontransitive properties:

I: A: (0, 0, 4, 4, 4, 4)	**II:** A: (2, 3, 3, 9, 10, 11)	**III:** A: (1, 2, 3, 9, 10, 11)
B: (3, 3, 3, 3, 3, 3)	B: (0, 1, 7, 8, 8, 8)	B: (0, 1, 7, 8, 8, 9)
C: (2, 2, 2, 2, 6, 6)	C: (5, 5, 6, 6, 6, 6)	C: (5, 5, 6, 6, 7, 7)
D: (1, 1, 1, 5, 5, 5)	D: (4, 4, 4, 4, 12, 12)	D: (3, 4, 4, 5, 11, 12)

In a two-person game, each player selects A, B, C, or D, and then compares the number rolled with their respective dice, the higher number winning the stake.

With sets **I** or **II**, the second player wins with A against B, B against C, C against D, and D against A, each with probability 2/3. The value of the game is always 1/3 to the second player.

With set **III**, the same selection strategy results in a win probability of 11/18 for the second player, for whom the game offers a value of 5/18. (The probability of a tie equals 1/18.)

Another set of four nontransitive dice, notable for using each number 1 through 24 just once, has been devised by Columbia University physicist Shirley Quimby (Ref.):

A: (3, 4, 5, 20, 21, 22)	C: (10, 11, 12, 13, 14, 15)
B: (1, 2, 16, 17, 18, 19)	D: (6, 7, 8, 9, 23, 24)

Here, $A > B > C > D > A$. Whichever set the first player selects, the second player then selects the superior set, winning with an average probability of 2/3. (It has been proved that 2/3 is the greatest possible advantage achievable with four dice.)

With three sets of numbers, the maximum advantage is 0.618; however, the sets must contain more than six numbers (thus precluding conventional dice). With more than four sets, the possible advantage approaches a limit of 3/4.

Oskar van Deventer, a Dutch puzzle-maker, devised a three-person game with a nontransitive set of seven dice:

A: (2, 2, 14, 14, 17, 17)	E: (1, 1, 12, 12, 20, 20)
B: (7, 7, 10, 10, 16, 16)	F: (6, 6, 8, 8, 19, 19)
C: (5, 5, 13, 13, 15, 15)	G: (4, 4, 11, 11, 18, 18)
D: (3, 3, 9, 9, 21, 21)	

The first and second players each select a (mutually exclusive) die from one of the sets A, B, …, G. The third player then wins with a strategy illus-

[9] A Stanford University statistician.

trated by a Fano[10] plane—defined as a two-dimensional finite projective plane of order 2—an incidence system wherein a set of seven points is partitioned into a family of seven "blocks" in such a way that any two points determine a block with three points, and each point is contained in three different blocks.

The diagram of Figure 6-1 shows the Fano plane with the six points labeled F, A, E, C, B, and G, moving clockwise around the triangle beginning with

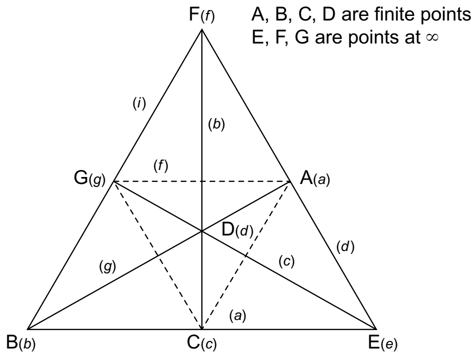

FIGURE 6-1 The Fano plane.

F at the top vertex. Set A dominates the three pairs along line *a* (B-C, B-E, and C-E), set B dominates the three pairs along line *b*, etc., as illustrated in Table 6-6.

Table 6-6 Seven-Dice Dominance Pattern

Dice Set	Dominates
A	B-C, B-E, and C-E
B	C-D, C-F, and D-F
C	D-E, D-G, and E-G
D	A-E, A-F, and E-F
E	B-F, B-G, and F-G
F	A-C, A-G, and C-G
G	A-B, A-D, and B-D

The examples of nontransitivity presented here have been limited to six-sided dice. Sets of *n*-sided dice that exhibit nontransitivity can also be contrived—see Finkelstein and Thorp (Ref.).

[10] Gino Fano, 1871–1952, the preeminent precursor of David Hilbert in the field of projective geometry.

SICHERMAN'S DICE

The probability distribution of the outcomes 2 through 12 from two conventional dice (Table 6-1) can be duplicated by a second pair with faces not restricted to numbers 1 through 6.[11]

The two-dice polynomial can be represented by

$$\sum_{n=2}^{7}(n-1)x^n + \sum_{n=8}^{12}(13-n)x^n$$

which can be factorized into the form $f(x)g(x)$, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 6$. The two factorizations are

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \qquad (6\text{-}24)$$

and

$$(x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8) \qquad (6\text{-}25)$$

where each exponent represents the face of a die, and each coefficient indicates the number of times that face appears on the die.

The expression of 6-24 delineates the two conventional dice, while the two dice represented by that of 6-25 are

$$(1, 2, 2, 3, 3, 4) \text{ and } (1, 3, 4, 5, 6, 8)$$

By definition, these are the only two pairs of dice with the standard probability distribution; it is further apparent that n pairs of Sicherman dice have the same probability distribution as $2n$ conventional dice.[12] For n odd, pairs of Sicherman dice plus a conventional die exhibit the same distribution as n conventional dice.[13]

CASINO GAMES

Craps

Of the various dice games enlisted by the gaming casinos, the most prevalent, by far, is Craps. Its direct ancestor, the game of Hazard, was reputedly (accord-

[11] Noted by Col. George Sicherman of Buffalo, NY.

[12] The probability of throwing a double number with a pair of Sicherman dice is 1/9—compared to 1/6 with traditional dice. In games where double numbers are significant (such as Backgammon and Monopoly), the substitution of Sicherman's dice would alter basic strategies.

[13] As an off-beat alternative, a pair of dice, one conventional and the other marked (0, 0, 0, 6, 6, 6), will produce a uniform distribution of outcomes 1 through 12 (the probability of each total equals 3/36).

ing to William of Tyre, *d.* 1190) invented by 12th-century English Crusaders during the siege of an Arabian castle. Montmort established formal rules for Hazard (originally: "Hazart"), and it subsequently became very popular in early 19th-century England and France. In the British clubs the casts of 2, 3, or 12 were referred to as "Crabs." Presumably, the French adaptation involved a Gallic mispronunciation, which contributed the word "Craps," and since the game immigrated to America with the French colony of New Orleans, its currency became firmly established.[14] Adopted by American blacks (the phrase "African dominoes" is often a synonym for dice), Craps spread out from New Orleans across the country. A rather disreputable aura surrounded the game during the latter half of the 19th century, owing to its manipulation by professional "sharpers" frequenting the steamboats and Pullman cars of the era.[15] However, its widespread popularity among the soldiers of World War I gained it a veneer of social respectability. It was subsequently enshrined in the salons of Monte Carlo and on the green baize table-tops of Las Vegas; today the cabalistic signs of the Craps layout can be seen in every major casino of the world.

Craps is played with two conventional dice. The types of wagers are displayed in the betting layout shown in Figure 6-2 on the next page. The basic bet is the "pass line," which wins unconditionally when the player initially throws a 7 or 11 ("naturals") and loses unconditionally when the initial outcome is a 2, 3, or 12 (referred to as "craps"). The remaining outcomes—4, 5, 6, 8, 9, 10—are each known as a "point." When a point is set, the player continues to roll the dice; the pass-line bettor then wins if that point is rolled again before the appearance of a 7 and loses if the 7 appears first. The latter event, known as a "seven-out," ends the player's tenure with the dice.

The probabilities of the outcomes 2 through 12 of a throw of two dice are obtainable directly from Eq. 6-11 and are tabulated in Table 6-1. Thence the probability of throwing a 7 is 6/36, of throwing an 11 is 2/36, and of throwing a point is 24/36. Table 6-7 shows the probabilities that a point is repeated

Table 6-7 Probabilities of Making a Point

Point	4	5	6	8	9	10
Probability of point recurring before a 7	1/3	2/5	5/11	5/11	2/5	1/3

[14] Another plausible legend credits Count Bernard Mandeville Marigny with introducing the game to New Orleans. Marigny was a Creole and as such was known as Johnny Crapaud (French for "toad," the term was originally applied to Frenchmen by Nostradamus in allusion to the fleur-de-lis pattern of the French national standard that, as altered by Charles VI in 1365, resembled three toadlike flowers; Guillim, in *Display of Heraldrie*, 1611, refers to the device as "three toads, erect, saltant"—the modern pejorative is "frog"). Thus the game was referred to as Crapaud's or, finally, Craps.

[15] Misspotted dice (called "tops") were often switched in. With the faces of each die marked 1, 1, 3, 3, 5, 5, it becomes impossible to roll a 7 with two dice.

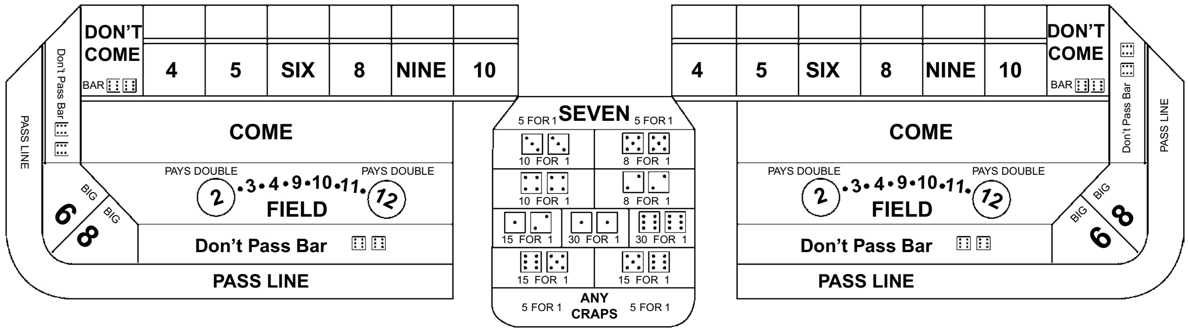

FIGURE 6-2 Craps layout.

before the occurrence of a 7. Thus, the probability of making a pass is

$$\begin{aligned} P &= p(7) + p(11) + p(\text{point + point preceding 7}) \\ &= 6/36 + 2/36 + (24/36)[(1/4)(1/3) + (1/3)(2/5) + (5/12)(5/11)] \\ &= 244/495 = 0.49\overline{29} \end{aligned} \tag{6-26}$$

for an expected gain of $-0.0\overline{14}$

In casino Craps, the opposite of the pass-line bet is the "don't pass, bar twelve" bet—meaning that a throw of 12 on the first roll of a sequence is deemed a tie. Hence its probability of success is

$$P = 1 - p(\text{pass}) - p(12)(1 - P) = 949/1925 = 0.49\overline{298701}$$

for an expected gain of $-0.01\overline{402597}$, slightly more favorable than that offered by a pass-line bet.[16]

The "come" and "don't come" bets are identical to pass and don't pass, respectively, and may be initiated at any time rather than exclusively on the first roll of a sequence.

A "field" bet wagers that the next outcome of the dice is a 2, 3, 4, 9, 10, 11, or 12, with the 2 and 12 each paying 2 to 1—for a winning probability of 0.472 and an expectation of $-1/18$ (some casinos offer 3 to 1 for a 12, increasing the expectation to $-1/36$).

Betting the "big 6" or "big 8" constitutes wagering that that outcome will appear before a 7; the probability of this event is (5/36)/[(5/36) + (6/36)] = 5/11 . Betting a 7, 11, 2, 12, and 3 are self-evident wagers that the next roll will produce that particular outcome. A "hard-way" bet defines the wager that a specified even-numbered outcome (4, 6, 8, or 10) will occur with the same number of pips showing on each die (sometimes referred to as a double number) before the same outcome with different pips on each die (the "easy way") or the outcome 7 occurs.

Finally, an "odds" bet is permitted following the establishment of a point during a pass, don't pass, come, or don't come wager (events that occur 2/3 of the time). A sum up to 100 times the original bet (depending on individual casino rules) wagers that the point will reappear before the outcome 7. By itself, "taking the odds" constitutes a fair bet and thereby nudges the player's negative expectation per expected unit bet closer to zero.

Some casinos permit different amounts to be wagered for the odds bet on the 4&10, 5&9, and 6&8 points. In general, if the odds bet allows up to x times odds on 4&10, y times odds on 5&9, and z times odds on 6&8, the player's expectation is

$$(-7/495)/[1 + (3x + 4y + 5z)/18]$$

---

[16]The tie stemming from a come-out 12 can be evaluated as either a possible resolution of the bet or as a delay in its eventual resolution. We advocate the latter perspective. With the former criterion, a factor of 35/36 is introduced, and a lesser negative expectation of $-0.1\overline{36}$ is specified [Online Ref. (2)].

As opposed to taking the odds, the contrary play, combined with a don't pass (or don't come) bet, wagers that a 7 appears before the point and is referred to as "laying the odds." The player's expectation for this bet is

$$-3/220(1+\omega)$$

where the player is permitted ω times the odds.

A tabulation of the various wagers for Craps is given in Table 6-8. The overall expectations from combining the pass line (or come) bet with taking the odds and from combining the don't pass (or don't come) bet with laying the odds are shown in Table 6-9.

Table 6-8 Probabilities for Craps

Bet	Payoff Odds	Winning Probability	Expected Number of Rolls per Bet	Expected Gain
Pass line/Come	1–1	0.493^-	3.38	-0.0141^-
Don't pass/Don't come, bar 12	1–1	0.493^-	3.38	−0.0143
Field	1–1 plus 2–1 on 2&12	0.472^-	1	-0.0556^-
Big six/Big eight	1–1	0.455^-	3.27	−0.0909
Seven	4–1[a]	0.417^-	1	−0.1667
Hard way six/Eight	9–1	0.455^-	3.27	−0.0909
Hard way four/Ten	7–1	0.444^+	4	−0.1111
Any craps	7–1	0.444^+	1	−0.1111
Two/Twelve	29–1	0.417^-	1	−0.1667
Three/Eleven	14–1	0.417^-	1	−0.1667
Take odds (not an independent wager)	2–1 on 4&10 3–2 on 5&9 6–5 on 6&8	0.500 0.500 0.500	4 3.6 3.27	0.000 0.000 0.000
Lay odds (not an independent wager)	1–2 on 4&10 2–3 on 5&9 5–6 on 6&8	0.500 0.500 0.500	4 3.6 3.27	0.000 0.000 0.000

[a]In many casinos, the inferiority of this bet (and others similar) is camouflaged by stating the odds as 5 *for* 1, since the original stake is returned with the winning sum. The intent of this wording is not altogether above suspicion.

As is evident, Craps shooters are accorded no significant strategy other than selection of the wager with the least unfavorable odds. Because of its rapid pace, however, the game has attracted a plethora of systems and has developed a small anthology of esoteric machinations, anthropomorphic analogies, and mumbled incantations ("Baby needs a new pair of shoes"). As with Roulette, dice possess "neither conscience nor memory"[17] and are democratic to a fault.

[17] Joseph Bertrand, 18th-century French mathematician.

Table 6-9 "Odds" Expectations

Amount of "Odds" Bet (Times)	Expectation from "Taking the Odds"	Expectation from "Laying the Odds"
1	−0.0085	−0.0068
2	−0.0061	−0.0045
5	−0.0033	−0.0023
10	−0.0018	−0.0012
100	−0.0002	−0.0001

Of those determined experimenters who have tested the probabilities of Craps, perhaps the greatest stamina was demonstrated by B.H. Brown (Ref.), who conducted a series of 9900 Craps games, obtaining 4871 successes and 5029 failures compared with the theoretically expected values of 4880 successes and 5020 failures, a discrepancy well within one standard deviation. Although the major gambling casinos do not maintain statistical records, one astounding event *has* been recorded: On May 23, 2009, at the Borgata Casino in Atlantic City, Patricia Demauro did not seven-out her Craps hand until the 154th roll. The probability for a particular such run is 1.79×10^{-10} (Ref. Ethier and Hoppe).

Proposition (Side) Bets

Associated with Craps are a number of so-called proposition bets. These wagers generally entail the probability of a specified outcome occurring in a specified number of throws or before another particular outcome occurs. Six of the more common proposition bets:

1. The probability $P_{5\text{-}9}$ that the outcomes 5 and 9 *both* appear before a 7. *Either* 5 *or* 9 will occur before a 7 with probability (8/36)/(6/36 + 8/36) = 8/14. The probability that the *other* outcome then occurs before a 7 is (4/36)/(6/36 + 4/36) = 4/10. Therefore,

$$P_{5\text{-}9} = (8/14)(4/10) = 8/35 = 0.2286.$$

   A payoff of 3 to 1 results in a House Take of 8.57%.

2. The probability $P_{4\text{-}6}$ that the outcomes 4 and 6 *both* appear before a 7. Thorp (Ref.) has shown that

$$P_{4\text{-}6} = 1 - P(7)\left\{\sum_{k_1} \frac{1}{P[\mathrm{E}_{k_1} \cup \mathrm{E}(7)]} + \cdots\right\}$$

where $P(7)$ is the probability of the event E(7) occurring. $E_{k_1} \cup E(7)$, the union of preceding events and the event E(7), means that at least one of the events E_{k_1} and E(7) occurs. Numerically,

$$P_{4\text{-}6} = 1 - (6/36)(36/9 + 36/11 - 36/14) = 50/231 = 0.2165^{-}$$

[The unexpected complexity of determining $P_{4\text{-}6}$ lies in the fact that $P(4)$ and $P(6)$ are not equal.] Here, a 3-to-1 payoff leaves a House Take of 13.42%.

3. Thorp has also computed the probability P_{6p} that all six points will appear before a 7 as 0.0622^{-}, and the probability that *all* numbers other than 7 will appear before a 7 as 0.0053.

4. The probability P_{12} of rolling 12 before two consecutive rolls of 7 (Ref. Griffin). Player **A** wagers on the former outcome, **B** on the latter. We can write immediately

$$P_{12} = 1/36 + (29/36)P_{12} + (1/6)(1/36) + (1/6)(29/36)P_{12}$$

which represents the probability that the first roll is a 12 (first term) plus the probability that the first roll is neither a 12 nor a 7 (second term) plus the probability that the first roll is a 7 and the second roll a 12 (third term) plus the probability that the first roll is a 7 and the second roll is neither a 7 nor a 12 (fourth term). Solving for P_{12}, we have

$$P_{12} = 7/13 = 0.538$$

for the probability that 12 appears first. Hence the game offers a positive expectation of 1/13 to **A**. (The House, of course, accepts only bets that two consecutive 7s appear first.)

Fallacious reasoning argues that the two outcomes are equiprobable since 12 occurs with probability 1/36 and two successive 7s with probability (1/6)(1/6). And the game *is* equitable *if* it ends immediately upon the occurrence of a second (consecutive) 7 regardless of the number of 12s that precede it.

With a continuous sequence of dice rolls (rather than beginning anew as each bet is resolved), **A**'s probability of success decreases slightly—since the previous roll was either a non-7 (which returns the game to its starting point) or a 7 (whence **A**'s probability of a win is 6/13). Then, for the continuous sequence,

$$P_{12} = (5/6)(7/13) + (1/6)(6/13) = 41/78 = 0.526^{-}$$

While **A**'s probability of winning the initial wager is 7/13, his probability of winning the nth wager is $(1 + 1/13^n)/2$, maintaining the game as a positive but diminishing expectation for **A**.

Curiously, if we inquire as to **A** and **B** winning specifically on the nth roll of the dice, we find these probabilities to be equal. **A**'s probability of a win remains (as always) 1/36; for **B** to win, the nth roll *and* the $n-1$st roll must be 7s—that is, $(6/36)(6/36) = 1/36$. Hence the game is equitable for any wager *after the first*. Further, if the game is defined for exactly n rolls, **A**'s expectation is $1/36 = 0.02\overline{7}$, a value that holds for all n.

If, on the other hand, the game is defined for exactly *n wagers*, **A**'s expectation on the nth wager is $1/13^n$. **A**'s total expectation is then $\sum_1^n 1/13^k$. For n large, this summation equals $(1/13)/(1 - 1/13) = 1/12$.

5. The probability $P_{6\text{-}8}$ of rolling 6 *and* 8 before 7 occurs twice. Note that the probability that a 6 *or* 8 precedes a 7 is 10/16; the probability that the remaining 8 or 6 precedes a 7 is 5/11; the probability that a 7 precedes the remaining 6 or 8 is 6/11; and the probability that a 7 precedes both 6 and 8 is 6/16. Therefore,

$$P_{6\text{-}8} = \frac{10}{16}\left[\frac{5}{11} + \frac{6}{11}\left(\frac{5}{11}\right)\right] + \frac{6}{16}\left(\frac{10}{16}\right)\left(\frac{5}{11}\right) = 0.5456^{-}$$

6. A more interesting variation on a proposition bet concerns **A** and **B** alternately throwing two dice: **A** begins the game and wins if he throws a 6 before **B** throws a 7. **B**'s successful outcome has probability 6/36 compared to 5/36 for that of **A**; however, since **A** throws first, the game is more equitable than would first appear. Specifically, we can show that if the game is limited to n throws, **A**'s probability p_n of winning is

$$P_n = \frac{30}{61}\left[1 - \left(\frac{155}{216}\right)^{(n+\delta)/2}\right] \quad \text{for } \delta = 0 \text{ or } 1 \text{ as } n \text{ is even or odd}$$

Similarly, **B** has probability q_n of winning, expressed by

$$q_n = \frac{31}{61}\left[1 - \left(\frac{155}{216}\right)^{(n-\delta)/2}\right]$$

and the probability r_n of a tie (no winner after n throws) is

$$r_n = \left(\frac{155}{216}\right)^{(n-\delta)/2}$$

As n increases indefinitely, the probability of a tie declines to zero, and p_n and q_n approach 30/61 and 31/61, respectively. Hence, the long-term game favors **B** with an expectation of 0.0164.

Duration of a Craps Hand

It is customary in casino Craps for the shooter to continue rolling the dice until he loses his wager by sevening out. We ask the expected (mean) number of rolls, E, for a shooter's turn—that is, before he "sevens out" after establishing a point. Then

$$\mathrm{E} = \bar{N}_p \bar{N}_s$$

where $\bar{N}_p$ is the average number of rolls per pass-line decision, and $\bar{N}_s$ is the average number of pass-line decisions per seven out. For $\bar{N}_p$ we can write

$$\begin{aligned}\bar{N}_p &= \frac{12}{36}(1) + \frac{3}{36}(1 + 36/9) + \frac{4}{36}(1 + 36/10) + \frac{5}{36}(1 + 36/11) \\ &\quad + \frac{5}{36}(1 + 36/11) + \frac{4}{36}(1 + 36/10) + \frac{3}{36}(1 + 36/9) \\ &= 557/165 = 3.3\overline{75}\end{aligned}$$

where the first term represents the probabilities of 7, 11, 2, 3, or 12 on the first roll, and the remaining terms represent the probabilities of points 4, 5, 6, 8, 9, and 10, respectively, multiplied by the expected number of rolls to obtain that point or roll a 7. The total probability of a seven out for the six points, $1/\bar{N}_s$, can be calculated, analogously to Eq. 6-26, as

$$1/\bar{N}_s = \frac{24}{36}\left(\frac{1}{4} \bullet \frac{2}{3} + \frac{1}{3} \bullet \frac{3}{5} + \frac{5}{12} \bullet \frac{6}{11}\right) = \frac{196}{495} = 0.3\overline{95}$$

Therefore,

$$\mathrm{E} = 3.3\overline{75}(1/0.3\overline{95}) = 8.53$$

So the shooter can expect to hold the dice for slightly more than $8\frac{1}{2}$ rolls.

A related question asks the median number of rolls—i.e., the number for which the shooter has a probability $\geq 1/2$ of sevening out. We can compute this number from the theorem of complete probability (Eq. 2-6), where A_i are the eight possible states faced by the craps shooter:

$$\begin{aligned}\mathrm{A}_1 &= \text{initial roll ("coming out")} \\ \mathrm{A}_2\text{–}\mathrm{A}_7 &= \text{points } 4,\ 5,\ 6,\ 8,\ 9,\ 10\text{, respectively} \\ \mathrm{A}_8 &= \text{seven out}\end{aligned}$$

State A_8 represents the probability P_n that the shooter will seven out on or before the nth roll. Donald Catlin (Ref., 2002) has computed values of A_8, determining that for $n = 5$, $P_5 = 0.424^-$; and for $n = 6$, $P_6 = 0.503^-$.

Thus the probability that the shooter will have sevened out *by* the sixth roll is slightly greater than 1/2. The probability of the shooter sevening out *precisely on* the sixth roll is the difference in these two values: 0.079^-.

(Some casinos pay a bonus to the shooter who makes all six points before sevening out. The probability of this feat is $1.62^+ \times 10^{-4}$.)

Average Number of Points Made by a Shooter

Given that a point has been established, the probability p that the shooter repeats that point before the appearance of a 7 is

$$\begin{aligned} p &= \sum_{i=4}^{6} \text{Prob(pt.} = \text{i or } 14 - \text{i}) \cdot \text{Prob(rolling i or } 14 - \text{i)} \\ &= (6/24)(3/9) + (8/24)(4/10) + (10/24)(5/11) = 0.406 \end{aligned}$$

The expected number of times E that an event of probability p would occur before failure equals p/(1–p). Ergo

$$E = .406/(1-.406) = 0.684^-$$

Card Craps

Some casinos in locales where gambling with dice is prohibited (e.g., California) reclaim the game of Craps by using certain subsets of the conventional deck of cards (Ref. Online-3). Typically, a deck of four suits with six cards of each suit, A, 2, ..., 6, pinch-hits for each die. Five such decks usually constitute the playing pack.

Two cards are dealt openly from the pack. Their sum is treated as the sum of two dice to resolve bets on a standard Craps layout. Probabilities of the outcomes 2 through 12 are displayed in Table 6-10.

Table 6-10 Card Craps Probabilities

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probabilities	19	40	59	80	99	120	99	80	59	40	19

Grinding these figures through Eq.6-26, we determine the probability of winning the pass-line bet as

$$\begin{aligned} P &= 120/714 + 40/714 + (476/714)[(59/233)(59/179) \\ &\quad + (80)/(233)(2/5) + (99)/(233)(33/73)] \\ &= 0.4993^+ \end{aligned}$$

for an expected gain of -0.001322^-, constituting a game perilously close to even.

Similarly, we can calculate the probability of winning the Don’t-Pass bet as

$$P = 0.487^+$$

for an expected gain of -0.0259^-

The difference between these probabilities and those of a conventional card game reflects the effect of card removal. This effect can be eliminated by using a separate pack for each of the two cards dealt or by adding a 7th card, a "doubler," to each A-through-6 group; the "doubler" indicates that the previous card drawn is repeated (if the "doubler" is drawn first, it is returned to the pack).

Card Craps—particularly the pass-line wager—invites the use of a counting system. The Hi-Lo system (pg. 279) can be applied to obtain a substantially positive expectation.

Reddere Craps

Analogous to their practice for Reddere Roulette (Chapter 5), major casinos cater to their more affluent clientele by offering a refund of from 10 to 50% of potential losses after n plays. With the pass-line bet, the player's expected loss is then

$$\frac{(251-244)n}{495} = \frac{7n}{495}$$

As with any even wager, the standard deviation for n plays is $\sqrt{n}$. Standardizing, $t = 7\sqrt{n}/495$, and, setting E(n) equal to t, the Reddere integral (Appendix Table A) has a value of 0.276. Thus,

$$7\sqrt{n}/495 \sim 0.276 \quad \text{and} \quad n \sim 381$$

(precise computation yields $n = 382$). Thus for a sequence of 382 plays or fewer at the pass line, the player retains a positive expectation when half his potential losses are refunded. His greatest expectation occurs at $n = 93$ (approximately 1/4 the number of plays necessary for the casino to gain a positive expectation) for an expected profit of 0.960.

Other wagers at Reddere Craps with a potential 50% refund after n plays are detailed in Table 6-11.

Table 6-11 Reddere Craps with 50% Rebate

Bet	Number of Plays or Fewer for Player's Positive Expectation
Any 7	9
Any craps	39
2 or 12	65
3 or 11	32
Big 6 or 8	9
Hardway 6 and 8	76
Hardway 4 and 10	39
Field	27

In general, for a potential refund of r percent, the player's expectation E (for a single-play probability of winning p and a single-trial probability of losing $q = 1 - p$) after n plays is given by

$$\mathrm{E} = n(p - q) + \sum_{k=0}^{[n/2]} \binom{n}{k} p^k / q^{n-k} (n - 2k) r/100$$

For $r = 10$, E is positive for nine or fewer plays. The maximum expectation of 0.036^{+} occurs at $n = 1$, with a close-to-maximum of 0.035^{-} at $n = 3$.

5-Count

A distant cousin to the Epaminondas system (Chapters 3 and 5), the 5-count was introduced in 1980 by a gambler identified only as "The Captain." Designed to uncover any bias—deliberate or not—in the probability distribution of the two-dice outcomes, it kicks in when the shooter first rolls a point. His subsequent three rolls define the 2-count, 3-count, and 4-count (regardless of their outcomes if the shooter has not sevened out). The fifth roll becomes the 5-count *only* if its outcome is a point; otherwise, the count remains at 4 (assuming a 7 has not been rolled). At this stage, the bettor is advised to wager on the come option.

Simulations run by Donald Catlin (Ref.) indicate that an SRR (Seven-to-Rolls Ratio) of 1 to 7 results in a positive return of 0.18%. With an SRR reduced to 1 to 8, the bettor on pass or come is offerred a 1.15% advantage.

Certain Craps aficionados claim the ability to control the dice sufficiently to reduce the SRR significantly below its unbiased value of 1 to 6. Wagering on 6 or 8, say, the bettor will have an expectation of

$$\mathrm{E} = [(7/\mathrm{SRR}) - 43]/[(6/\mathrm{SRR}) + 30]$$

which is positive for SRR $< 7/43$ (Catlin, 2003). An SRR of 1 to 7 offers an expectation of $+0.083$, while an SRR reduced to 1 to 8 confers an advantage of $+0.167$ for this wager.

Craps Variations

Four dice are thrown in this variant, with the highest and lowest numbers discarded and the other two employed as in a conventional game of Craps. The probability distribution is displayed in Table 6-12. The mean equals 7, and the probability of winning is

Table 6-12 Craps with Two Middle Dice

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probability ×1296	21	54	111	156	201	210	201	156	111	54	21

$$P_w = \frac{1}{1296}\left[210 + 54 + \frac{2 \cdot 111^2}{111 + 210} + \frac{2 \cdot 156^2}{156 + 210} + \frac{2 \cdot 201^2}{201 + 210}\right] = 0.517^+$$

As a casino game, if the outcome 11 is classified as a draw rather than a player win, then $P_w = 0.497^-$, an acceptable profit for the House.

In another four-dice variant the highest and lowest numbers are used and the other two discarded; the probability distribution in this case (again with a mean of 7) ensues as shown in Table 6-13. The probability of winning (retaining the standard rules of Craps) is 0.529^+.

Table 6-13 Craps with High–Low Dice

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probability ×1296	1	14	51	124	245	426	245	124	51	14	1

When the four dice are used for a game wherein **A** bets on the sum of the highest and lowest numbers, and **B** bets on the sum of the two middle numbers, the result is zero expectation. The probability of a tie is 0.1426^+, while both **A** and **B** share a winning probability of 0.4287^-.

Sparc: Craps with Octahedral Dice

The player wins if his opening throw of two octahedral dice (8 sides numbered 1 to 8) shows 9 (the most probable outcome with probability 1/8), 4, 14, 15, or 16 (total probability 17/64). He loses if his opening throw is either 2 or 3 (probability 3/64). All other totals establish points, winning if repeated before a 9 is thrown, losing if a 9 intervenes.

The probability of making a pass (defined similar to conventional Craps) is calculated analogously:

$$P = \frac{17}{64} + \frac{44}{64}\left[\frac{2}{11} \cdot \frac{1}{3} + \frac{5}{22} \cdot \frac{5}{13} + \frac{3}{11} \cdot \frac{3}{7} + \frac{7}{22} \cdot \frac{7}{15}\right] = 0.4915$$

which qualifies Sparc as a casino game. (Side bets generally entail a slightly smaller House edge. For example, Big 8 and Big 10 pay even odds, with an expected gain of $-1/15$.)

In an alternative Sparc format, the player wins with an initial throw of 9, 14, 15, or 16 (probability 7/32). Totals of 2, 3, or 4 (probability 3/32) are not registered, whence the player throws again. His probability of making a pass now equals 0.4906. The appeal of this variant stems from the fact that the player cannot lose on his first throw.

Chuck-A-Luck

Originally known as "Sweat-Cloth," Chuck-A-Luck has long been played illegally in British pubs under the name Crown and Anchor—the six sides of the dice are inscribed Clubs, Diamonds, Hearts, Spades, Crown, and Anchor. The game was exported to the United States about 1800. Three dice are agitated inside a double-ended, rotatable cage with an hourglass cross section (the "Bird Cage"). A player may wager upon any of the outcomes 1 through 6. If one (and only one) die exhibits that outcome ($p = 75/216$), the player wins at even odds; if two dice exhibit that outcome ($p = 15/216$), the payoff is 2 to 1; if all three dice show the player's choice ($p = 1/216$), the payoff is 3 to 1; otherwise (that is, if the specified outcome appears on none of the three dice), the player loses. Elementary calculations indicate a probability of success of 0.461. An equitable game would offer odds of 1 to 1, 2 to 1, and 20 to 1, or of 1 to 1, 3 to 1, and 5 to 1 for the specified outcome occurring once, twice, or three times on the three dice.

Barbotte, Passe-Dix, Counter Dice, and Twenty-Six

Hundreds, if not thousands, of dice games have evolved since the cavemen first cast their knucklebones. In Barbotte, a Canadian version of Craps, the player wins if the two dice produce 3-3, 5-5, 6-6, or 6-5 and loses with the outcomes 1-1, 2-2, 4-4, and 2-1. The remaining values are not registered. In this form, Barbotte is a fair game.

Passe-dix (past ten) is an ancient game that fell into disuse until, in the 17th century, it attracted the attention of Galileo. Three dice are thrown, and the player can wager on *manque* (an outcome of 10 or lower) or *passe* (an outcome greater than 10)—whence the name. Passe-dix is also a fair game.

Counter Dice is played occasionally in taverns. The player selects a number 1 through 6 and then rolls ten dice from a cup ten times in all. If the player's number occurs 20 times, he is paid at odds of 2 to 1; if the number appears 25 times, the payoff is 4 to 1; and if the number occurs 30 or more times, the payoff is 20 to 1. (The expected frequency for each number is, of course, $16\frac{2}{3}/100$.)

A version of Counter Dice, well known in Chicago bars, is "Twenty-six." Here, a player selects a particular outcome of a die as his point; he then rolls ten dice 13 times, recording the total number of occurrences of his point (expected number equals $130/6 = 21\frac{2}{3}$). There is no standard payoff; however, most establishments pay 4 to 1 if the point occurs 26 times, with increasing odds (usually to about 8 to 1) for 33 or more occurrences. The probability of obtaining 26 particular outcomes in 13 throws of ten dice is given directly by Eq. 6-6:

$$P_{130}^{26} = \binom{130}{26}(1/6)^{26}(5/6)^{104} \simeq 5.33 \times 10^{-2}$$

The probability of obtaining 26 or more occurrences of the point is, similar to Eq. 6-2,

$$P_{130}(26 \le m \le 130) \simeq \Phi\left(\frac{130 - (130/6) + (1/2)}{\sqrt{130(1/6)(5/6)}}\right) - \Phi\left(\frac{26 - (130/6) - (1/2)}{\sqrt{130(1/6)(5/6)}}\right)$$
$$= 1.0000 - 0.8164 = 0.1836$$

and the probability of achieving 33 or more occurrences of the point is

$$P_{130}(33 \le m \le 130) \simeq 1.0000 - \Phi\left(\frac{33 - (130/6) + (1/2)}{\sqrt{130(1/6)(5/6)}}\right)$$
$$= 1.0000 - 0.9845 = 0.0155$$

Other dice games that have attained a fleeting popularity in gambling casinos include Cusek (a type of Dice-Roulette in Macao), In-and-In, and Carabino.

BACKGAMMON

A two-person game with complete information and probabilistic moves, Backgammon constitutes one of the oldest board games known, originating along the eastern borders of the Persian empire in the third millennium B.C. It is played throughout the world, with numerous tournaments and regular world championship matches (as well as in the International Computer Olympiad). There are likely more professional Backgammon players than professional Chess players.

Computer Programs

With 30 checkers and 26 possible locations (including the bar and off-the-board), it is evident that the number of possible configurations of the checkers ($>10^{20}$) far exceeds the memory capabilities of any currently realizable computer. Further, the game tree has branching ratios of several hundred per "ply" (one player's turn)—far too many to be amenable to the heuristic search procedures effective in Chess and Checkers.

The first practical Backgammon-playing computer, the BKG 9.8,[18] functioned by evaluating board positions throughout the course of the game. The program applies techniques of fuzzy logic to smooth over the transitions between phase changes. In 1979, the BKG 9.8 became the first program to defeat a world champion. Ten years later, Backgammon-playing software, most notably TD-Gammon

[18] Programmed by Hans Berliner, professor of computer science, Carnegie Mellon University, on a PDP-10 in the early 1970s.

(for Temporal Difference learning [Ref. Tesauro]), achieved even greater success using a neural network that trains itself to be a heuristic evaluation function by playing against itself and learning from the outcome.

TD-Gammon incorporates an algorithm to learn complex numerical functions. At each time step, the algorithm is applied to update the network's "weights" (real-valued parameters for each of the network's connections), a procedure that has enabled the program to defeat Backgammon players of world-class caliber.

Subsequent software programs—such as *BGBlitz* (2005 World Backgammon Champion), *Jellyfish*, and *Snowie*—refined the neural network approach yet further and are superior to the best human players.

Neural networks, it may be noted, are better suited for Backgammon—rather than the alpha-beta search and pruning techniques that have proved more efficient for Chess.

DICE DIVERTISSEMENTS

Fanciers of dice idiosyncrasies may wish to grapple with the following problems.

1. Removing the restriction that the pips on opposite faces of a die must total 7, show that the number of distinct dice totals 30.

2. *Pair O'dice Lost*. Prove that no weighting of two conventional dice is possible such that the outcomes of their sums—2 through 12—occur with equal probability (i.e., 1/11).

   Prove that the sums of two dice, P and Q, weighted as follows:

$$P_1 = P_6 = 1/2; \quad P_2 = P_3 = P_4 = P_5 = 0$$

$$Q_1 = Q_6 = 4/33; \quad Q_2 = Q_3 = Q_4 = Q_5 = 25/132$$

   to yield the following distribution:

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probability	2/33	25/264	25/264	25/264	25/264	4/33	25/264	25/264	25/264	25/264	2/33

   represent the "closest to equality" that can be implemented—that is, with minimal deviations from 1/11.

3. An ancient Chinese dice game entails each of two players casting six dice. Two specific outcomes are noted: any single pair of numbers and any two pairs. Compute their respective probabilities.

   This wording admits an ambiguity. If we define one pair as *not excluding* additional but different pairs or three-of-a-kind or four-of-a-kind with the remaining four dice, show that the probability of rolling a pair is 0.7399. And if "two pairs" is similarly defined as not excluding a third pair with the remaining two dice, show that this probability is 0.3445.

Compute the probabilities of casting specifically one pair and specifically two pairs (excluding any combinations of the remaining dice that improve the "values" of the six dice cast).

Compute the odds of rolling *at least* one pair and at *least two* pairs without exclusions for the remaining dice.

4. In a popular children's game, each of two players casts six dice (or a single die six times). The outcomes are summed, but with the restriction that any number may be used only once. The sum, therefore, can extend from 1 to 21.

   Compute the probability that each player obtains the same sum (a tie game).

5. In an infinite sequence of dice throws, prove that the probability of the six outcomes ever occurring with equal frequencies is 0.022.

6. Determine the order of preference for $n \geq 4$ rolls of Schwenk's dice.

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Chapter | seven

The Play of the Cards

ORIGINS AND SPECIES

The invention of playing cards has been generally credited to the Chinese during the latter years of the Tang Dynasty (~900 A.D.). Chinese records attest that the first modern pack was shuffled in the reign of Sèun-ho,[1] about A.D. 1120, and that cards became commonplace by the reign of Kaou-tsung, who ascended the Dragon throne in 1131; however, card games *per se* date back more than two millenia, preceding the invention of paper.[2]

Card playing was introduced into Europe through the Crusades (the Arabs endured lengthy sieges with cards and other gambling paraphernalia) and gained currency in Western society during the 14th century. In 1423, the Franciscan friar St. Bernardino of Siena preached a celebrated sermon against cards (*Contra Alearum Ludos*) at Bologna, attributing their invention to the devil. Despite such ecclesiastic interdiction, Johannes Gutenberg printed playing cards the same year as his famous Bible (1440). The pack struck by Gutenberg's press consisted of Tarot cards, from which the modern deck is derived. Initially there were 78 cards: 22 "atouts" (literally: "above all," later known as "triumph" or "trump") including a Joker (*le Fou*) and four denominations of 14 cards each (ten numbered ranks plus a King, Queen, Cavalier or Knight, and a Valet or Knave). The four denominations, or suits, represented the four major divisions of medieval society: Swords symbolized the nobility; Coins, the merchant class; Batons or Clubs, the peasants; and Cups or Chalices, the Church. These suits are still reflected in the contemporary playing cards of Italy, Spain, Andorra, and Portugal.

About 1500, the French dropped the 22 atouts and eliminated the Knight, leaving a deck of $4 \times 13 = 52$ cards. They also transformed Swords, Coins, Batons, and

[1] According to the Chinese dictionary, Ching-tsze-tung, compiled by Eul-koung and first published in 1678.

[2] The word "card" stems from the Greek word for paper, $\chi\acute{\alpha}\rho\tau\eta$s. Similarly, the Hebrew term for playing, card, *qlaf*, is equivalent to "parchment."

Cups into Piques (soldiers' pikes), Carreaux (building tiles, lozenge- or diamond-shaped), Trèfles (trefoils or clover leaves), and Coeurs (hearts). In 16th-century Spain, Swords were Espados; Coins were square pieces of currency, Dineros; Batons, Clubs, or Cudgels were Bastos; and Cups or Chalices were Copas. Hence, we have adopted the French symbols, translated the names of two (Hearts and Diamonds) into English and applied to the other two English translations of Spanish names for different symbols. Such is the illogic of language.

Despite a number of coincidences, no firm connection has been found between the structure of the deck and the calendar. It seems palpable to link the four suits with the seasons and the 13 ranks with the lunar months (corresponding to one card per week of the year). Further, if we assign a weight of 13 to the Kings, 12 to the Queens, 11 to the Jacks, and so forth, the 52-card pack totals 364; adding 1 for the Joker equals the number of days in a (nonbissextile) year. Such is the treachery of numerology.

Initially, the court cards[3] were designed as portraits of actual personages, and stylistic vestiges of the individuals remain today. The four original kings (invested in 14th-century Europe) were Charlemagne (Hearts), the biblical David (Spades), Julius Caesar (Diamonds), and Alexander the Great (Clubs). Distaff royalty included Helen of Troy as the Queen of Hearts, Pallas Athena as the Queen of Spades (the only queen to be armed), and the biblical Rachel as the Queen of Diamonds. Regal ladies passingly honored as card monarchs were Joan of Arc, Elizabeth I, Elizabeth of York (wife of Henry VII), Roxane, Judith, Hecuba, Florimel, and Fausta, among many. Famous Jacks were frequently renowned warriors, such as Hogier La Danois, one of Charlemagne's lieutenants (Spades); Etienne de Vignoles, a soldier under Charles VII of France (Hearts); Roland, Charlemagne's nephew (Diamonds); and the gallant knight Sir Lancelot (Clubs).

Many cultures have evolved decks partitioned in other ways. The Hindus employ a deck with ten denominations, representing the ten incarnations of Vishnu. The Italians use, generally, a 40-card deck, and the Spanish use a 48-card deck. The French play games with both the 52-card pack and the 32-card Piquet pack (omitting twos through sixes). Clearly, the number of cards comprising a deck is arbitrary and could be selected to satisfy the needs of a particular game. However, we shall confine ourselves, unless otherwise specified, the 52-card pack partitioned into 4 denominations and 13 ranks.

RANDOMNESS AND SHUFFLING

Bias

The question of the "fairness" of a deck of cards is distinct from questions of bias relating to coins, dice, or roulette. Intuitively, a deck of cards is fair if each

[3]A corruption of *coat* cards, so called because they bear the representation of a clothed or *coated* figure (not because the King, Queen, and Knave may be considered as members of a royal court).

outcome is enumerated and is represented identically as all others. Experimenters have conducted tests to establish the veracity of this intuition. Most notably, the statistician C.V.L. Charlier performed 10,000 drawings (with replacement) from a conventional deck, recording 4933 black cards and 5067 red cards for a deviation of 67 from the expected value. The standard deviation is

$$\sigma = \sqrt{10{,}000\times\tfrac{1}{2}\times\tfrac{1}{2}} = 50$$

assuming red and black cards to be equiprobable. A result with a 1.34σ deviation is hardly unusual.

Cut and Shuffle Operations

Rather than bias, the appropriate parameter for cards is randomness. Since the ordering of a deck is neither random nor nonrandom in itself (every possible ordering is equiprobable *a priori*), we must adopt a dynamic concept of randomness. That is, the question of randomness is meaningful only when the ordering is being changed by some process such as shuffling. We consider here five types of processes that transform one ordering of a deck into another.

A *simple cut* c translates the top card of the deck to the bottom; thus the power c^i of this operation separates the deck after the ith card and is merely a phase shift transforming the order $1, 2, 3, \ldots, i, j, k, \ldots, 2n$ into the order $j, k, \ldots, 2n, 1, 2, 3, \ldots, i$.

The *perfect shuffle* (or "out shuffle") *operation* s on a $2n$-card deck separates the top n cards from the bottom n cards and precisely interleaves them, the top and bottom cards remaining unchanged in position. Functionally,

$$\begin{aligned} &s(1,\ 2,\ 3,\ \ldots, n-1, n, n+1, \ldots, 2n-1, 2n) \\ &\quad = (1, n+1,\ 2,\ n+2,\ \ldots\ , n, 2n) \end{aligned}$$

From this equation it follows that if the card in position a_0 is moved to position a_1 by the perfect shuffle s, then

$$a_1 \equiv 2a_0 - 1,\ \bmod\ (2n-1)$$

noting that $a_1 = 1$ for $a_0 = 1$, and $a_1 = 2n$ for $a_0 = 2n$. After k iterations of s, the card originally in position a_0 has moved to position a_k. Thus,

$$a_k \equiv 2^k a_0 - (2^k - 1),\ \bmod\ (2n-1)$$

The *modified perfect shuffle* (or "in shuffle") s' constitutes the same operation, but with the $(n+1)$st card moving to the top and the nth card to the bottom.

Defining this operation:

$$\begin{aligned} &s'(1,\ 2, 3,\ \ldots, n-1, n, n+1, \ldots, 2n-1, 2n) \\ &\quad = (n+1, 1, n+2,\ 2,\ \ldots,\ 2n, n) \end{aligned}$$

and if the card in position b_0 is moved to position b_1 by the modified perfect shuffle s',

$$b_1 \equiv 2b_0, \ \bmod (2n + 1)$$

In general, after k iterations of s',

$$b_k \equiv 2^k b_0, \quad \bmod (2n + 1)$$

One mildly interesting characteristic of the modified perfect shuffle is that, for the standard deck, each shuffle interchanges cards in the 18th and 35th position. Also, the 52-card deck reverses its order after 26 such shuffles.

A curious phenomenon arising from the mixing of successive operations s and s' (Out and In shuffles) was discovered by Alex Elmsley (Ref. Gardner) who found that card c_1 would be moved to position c_x by a sequence of shuffles that spells out x–1 in binary form. For example, to move the card in first position to the 15th position, perform three In shuffles followed by an Out shuffle: IIIO, the binary representation of 14. Further, reversing this sequence of shuffles reverses the card movement. Thus an Out shuffle followed by three In shuffles (OIII) moves card c_{15} to position c_1.

The *amateur shuffle operation* s_a divides the deck approximately into two groups of n cards and interleaves them singly or in clusters of 2, 3, or 4 according to some probability distribution. An amateur shuffle operation, wherein the probability of the ith card in the final sequence originating in either half of the deck is 1/2, provides 2^{52} possible sequences. Hence, the maximum amount of information associated with this shuffling process is $\log_2 2^{52} = 52$ bits. With a known probability distribution constraining the s_a operator, the information required to specify s_a is evidently less than 52 bits.

Finally, we define the *random shuffle* s_r as an operation equivalent to having each card selected in turn by a blindfolded inebriate. Since the random shuffle renders equiprobable all possible orderings of the 52 cards, it is associated with the greatest amount of information. Specifically, $\log_2 52! = 225.7$ bits of information are required to describe this operation.

(A *perfect shuffle* is completely deterministic and therefore is not accompanied by an information parameter.)

Randomness and Cycling

It can be shown that to randomize the deck, the amateur shuffle operation must be repeated *at least* five times. Typical forms of s_a dictate 20 to 30 replications to achieve randomization.

If the perfect shuffle operation s is iterated f times on a $2n$-card deck, where f is the exponent of 2, mod $(2n - 1)$, the deck is restored to its original ordering. Specifically, eight perfect shuffles cycle the 52-card pack to its original ordering,[4]

[4] It is worth noting that two perfect shuffles of a "fresh" deck (arranged customarily by suits in ascending ranks) followed by simple cuts order the cards for a deal of four perfect hands at bridge (all 13 cards of a suit comprising each hand). This fact may account partially for the astounding number of perfect hands reported each year.

since $2^8 = 1$, mod 51. Table 7-1 tabulates the values of f for decks up to $2n = 60$ cards.

Table 7-1 Deck Cycles with Perfect Shuffles

$2n$	f	$2n$	f	$2n$	f
2	1	22	6	42	20
4	2	24	11	44	14
6	4	26	20	46	12
8	3	28	18	48	23
10	6	30	28	50	21
12	10	32	5	52	8
14	12	34	10	54	52
16	4	36	12	56	20
18	8	38	36	58	18
20	18	40	12	60	58

For the modified perfect shuffle operation s', 52 iterations are necessary to cycle the deck, since 2 is primitive, mod 53—that is, $f' = 52$ is the lowest solution to $2^{f'} = 1$, mod 53. Thus, f and f' are the orders of s and s', respectively. The operation c has order $2n$, since c is a primitive cyclic permutation on $2n$ cards. The operations c and s together (cut and shuffle) generate the entire symmetric group S_{2n} of permutations on the $2n$ cards (c and s' also generate an entire symmetric group of permutations). The order of the product operation cs assumes one of two values. If the deck size is such that $2n - 1$ is a power p^m of a prime p, where n is any positive integer, and 2 is primitive, mod p^m—that is, $f = \phi(p^m) = p^{m-1}(p - 1)$, where ϕ is Euler's function—then the order of cs, $O(cs)$, is

$$O(cs) = (f + 1)\,\phi(p^{m-1})$$

If, on the other hand, either $2n - 1$ has two or more distinct prime factors or, although $2n - 1$ is a power of a prime, 2 is not primitive, mod$(2n - 1)$, then the order of cs is given by

$$O(cs) = f(f + 1)$$

as is the case for the conventional deck ($n = 26$).

A proof of the theorem that the operations c and s (or c and s') applied to a deck of $2n$ cards generate the entire symmetric group S_{2n} of permutations on the $2n$ cards has been published by S.W. Golomb (Ref.). This theorem expresses the equivalence between the operations $c^i s^k$ and s_r—that is, the perfect shuffle and single-cut operations sufficiently iterated can produce complete randomness.

For decks of $2n - 1$ cards, we adjust our definition of the perfect shuffle $\bar{s}$ as an operation that separates the bottom n cards from the top $n - 1$ cards and precisely interleaves the two sets, the bottom card remaining invariant, and the nth card emerging on the top. The operation $\bar{s}$ has order f (the exponent of 2, mod $[2n - 1]$), and the simple-cut operation $\bar{c}$ is definable as with an even-numbered deck and has order $2n - 1$. A theorem developed by Golomb states that the operations $\bar{c}$ and $\bar{s}$ applied iteratively to a deck of $2n - 1$ cards generate a subgroup of the symmetric group S_{2n-1} of order

$$O(\overline{cs}) = (2n - 1)f \leq (2n - 1)(2n - 2)$$

This subgroup is proper for $n > 2$. Therefore the operation $c^i s^k$ can never be made equivalent to the random shuffle operation s_r, since not all permutations of odd-sized decks are obtained by the cutting and perfect shuffling operations.

Characteristics of the Amateur Shuffle

Of more practical concern is the amateur shuffle operation s_a. Experiments undertaken by the author indicate that neophyte shufflers tend to interleave cards in clusters at the beginning and end of each shuffle; intermediate positions are generally alternated singly or by twos. More proficient shufflers appear to overcome the clustering tendency, which is caused by the first and final cards of the new sequence adhering to the fingers. Indeed, postgraduate shufflers, such as dealers in Las Vegas casinos, exhibit a uniform type of card interleaving with quite small variances. The nature of the s_a operator was investigated by recording the sound waves produced by shuffling a deck of cards (a flat-response microphone and sharp-skirted high-pass filters were used with a fast-response sonograph). Duration of a beginner's shuffle is about 700 milliseconds, while virtuoso shufflers average about 1/2 second for a shuffle.

It was found that highly expert shufflers create sequences with single-card interlacings approximately eight times more frequently than two-card interlacings; a group of three cards appears less than once per shuffle. Thus, to an excellent approximation, the geometric distribution (Eq. 2-13)

$$P(\eta) = (8/9)\,(1/9)^{\eta-1}, \quad \eta = 1,2,3,\ldots \tag{7-1}$$

represents the probability that a sequence consists of η cards. It is evident from this equation that a large measure of orderliness is preserved for a small number of shuffles. This fact suggests the feasibility of predicting the position of a card following the s_a operation. However, the problem of prediction is subsequent to the question of the correlation between successive card sequences.

Correlation of Card Sequences; Markov Chains; Entropy

The shuffling operation evidently is not a function of the order of the deck on which it operates nor of the past history of the deck. Hence, successive operations correspond to independent trials with fixed transition probabilities. The sequence of cards is therefore said to form a *Markov chain*, and the matrix of transition probabilities from one of the 52! states of the deck to another is *doubly stochastic* (both the row sums and the column sums are unity).

As a measure of the relationship between the initial state S_j and the state S_k that results after n shuffles, the concept of entropy logically suggests itself (connected, as it is, to Markov processes). In statistical mechanics, Boltzmann's hypothesis, which relates it to probability. Therein, p_i is defined as the probability of a system being in cell (or state) i of its phase space. The entropy H of the set of probabilities $p_1, p_2, \ldots, p_n$ is

$$H = -K \sum_i p_i \log p_i$$

Entropy, then, is a measure of uncertainty or of disorder; the greater the entropy, the less the preservation of order. (The constant K simply specifies the unit of measure; it is expedient to set $K = 1$.)

Applying the entropy concept to the phenomenon of card shuffling, we define

$$H_{j,k} = -\sum p_{j,k}^{(n)} \log p_{j,k}^{(n)}$$

as the desired measure. Here, $p_{j,k}^{(n)}$ denotes the probability of finding the deck in state S_k at step $t + n$, given that at step t it was in state S_j.

To compute numerical values of entropy for a conventional deck imposes a Herculean burden of arithmetical labor. The transition probabilities form a $52! \times 52!$ matrix ($52! \simeq 10^{68}$); evaluations of $H_{j,k}$ are thus far beyond the reach of any practical computer. Fortunately, since the ultimate goal is to develop a useful prediction function, there exists a pragmatic simplification that dispenses with excessive computational chores.

Prediction

We can state the problem of prediction as follows: Given the deck of cards in state S_j, and, following n shuffles, given the first m cards of the new sequence represented by state S_k, what predictions can be made concerning the $(m + 1)$st card? Note that since we are applying the prediction to the goal of increasing the probability of success at some game, we wish a probability distribution of this $(m + 1)$st card; that is, we wish to obtain a maximum-expectancy decision rather than a maximum-likelihood prediction. Rigorously, we should now examine all those states reachable from state S_j after n shuffles whose sequences begin with the m cards specified. The corresponding transition probabilities for each allowable state are then tabulated according to the possible cards that can occur in the $(m + 1)$st position—and the probability distribution of the $(m + 1)$st card is obtained. However, we now invoke our pragmatic observation that

> given the original sequence and the first m cards of the final sequence, the $(m + 1)$st card must be one of 2^n cards, n shuffles having transformed one sequence into the other.

To apply this observation practically, we consider only those transition processes comprising one, two, or three shuffles; for n equal to six or more shuffles, the statement is virtually useless with a conventional deck. First, we must specify for the shuffling process a probability distribution that is applicable to a finite-length deck (Eq. 7-1 implies an infinite deck). From experience, an expert shuffling operation interleaves alternate packets of cards where each packet consists of one, two, or three cards (with amateur shuffling the packets may comprise four or more cards).

Accordingly, we let r_1, r_2, and r_3 represent the probabilities that a packet selected at random is composed of one, two, or three cards, respectively (where $r_1 + r_2 + r_3 = 1$), Or, equivalently, we can define p_1 as the probability that a packet ends with the first card. Given that the packet does not end with the first card, the (conditional) probability that it ends with the second card is designated by p_2. And, given that the packet does not end with the second card, the (conditional) probability that it ends with the third card is $p_3 = 1$. Thus,

$$r_1 = p_1, \quad r_2 = (1 - p_1)p_2, \quad r_3 = (1 - p_1)(1 - p_2)$$

If three consecutive cards of the new sequence are observed, one of four patterns must occur: AAA, AAB, ABA, or ABB, where A represents a card from one section of the divided deck and B represents a card from the other section. The probability of occurrence of each of these four patterns is readily calculated as

$$P(\text{AAA}) = \frac{r_3}{R} \qquad P(\text{ABA}) = \frac{r_1}{R}$$

$$P(\text{AAB}) = \frac{1 - r_1}{R} \qquad P(\text{ABB}) = \frac{1 - r_1}{R}$$

where $R = r_1 + 2r_2 + 3r_3$.

We can now compute the transition probabilities between packets after observing one of the four allowable patterns. Let $P_t(X)$ be the probability of a transition, given the event X, and let $Q_t(X)$ be the probability of no transition. Then it is elementary to demonstrate that

$$P_t(\text{AAA}) = 1 \qquad Q_t(\text{AAA}) = 0$$

$$P_t(\text{AAB}) = p_1 = r_1 \qquad Q_t(\text{AAB}) = 1 - p_1 = 1 - r_1$$

$$P_t(\mathrm{ABA}) = p_1 = r_1 \qquad Q_t(\mathrm{ABA}) = 1 - p_1 = 1 - r_1$$

$$P_t(\mathrm{ABB}) = p_2 = \frac{r_2}{1 - r_1} \qquad Q_t(\mathrm{ABB}) = 1 - p_2 = \frac{r_3}{1 - r_1}$$

For a second shuffle, consider the A cards to be divided into C and E cards and the B cards to be divided into D and F cards, depending on the deck partitioning. The first test is to apply these equations for transition probabilities between members of the CD set and those of the EF set. Second, we apply the same equations to determine the conditional transition probabilities for the C set versus the D set and the E set versus the F set. The next card in sequence within each set occurs with the probability accorded that set. This procedure can evidently be generalized to three or more shuffles; for n shuffles, there are 2^n sets of cards.

A numerical example will clarify these concepts. Observations of professional card dealers indicate a shuffling operation described by

$$P(\eta) = \begin{cases} (8/9)(1/9)^{\eta-1} & \text{for } \eta = 1,2 \\ (1/9)^{\eta-1} & \text{for } \eta = 3 \\ 0 & \text{for } \eta \geq 4 \end{cases} \tag{7-2}$$

(which approximates Eq. 7-1). In terms of packet lengths, $r_1 = 8/9$, $r_2 = 8/81$, and $r_3 = 1/81$. Thus, assuming an equipartition of the deck, the probability of successive cards of the post-shuffle sequence alternating between members of the opposing halves of the original sequence is 8/9; at any point in the new sequence, the probability of obtaining two consecutive members of the original sequence is 8/81; and the probability of obtaining, at any point, three consecutive members of the original sequence is 1/81.

With this probability distribution for the shuffling process, the prediction function for the $(m + 1)$st card is readily obtained. Consider the initial state S_j defining the sequence of cards $c_1, c_2, \ldots, c_{52}$. A single shuffle is performed according to Eq. 7-2 (with equipartition of the deck and the first card remaining invariant). The first m cards forming the new state are then composed of α cards of one half and $m - \alpha$ cards of the other half. Thus, the $(m + 1)$st card is either the $(\alpha + 1)$st card of the first half (the A set) or the $(m - \alpha + 1)$st card of the second half (the B set), with respective probabilities corresponding to Eq. 7-2 as a function of the $(m - 2)$nd $(m - 1)$st, and mth cards. As an example, let $m = 10$, and let the first ten cards of the new sequence following one shuffle be $c_1, c_2, c_{27}, c_3, c_{28}, c_4, c_{29}, c_5, c_{30}, c_{31}$. The 11th card must be either c_{32} or c_6 with probabilities 1/9 and 8/9, respectively, as seen from the equations for $Q_t(\mathrm{ABB})$ and $P_t(\mathrm{ABB})$.

Continuing the example: After two shuffles (we assume that results of the first shuffle are not disclosed) with the first ten cards of the new state forming the sequence $c_1, c_{27}, c_{13}, c_2, c_{39}, c_{40}, c_{28}, c_{14}, c_{15}, c_3$, the 11th card must be c_4,

c_{29}, c_{16}, or c_{41}. Let the A set initially encompass cards $c_1, c_2, \ldots, c_{26}$, and the B set cards $c_{27}, c_{28}, \ldots, c_{52}$. Then, following two shuffles, the classification of the first ten cards of the new sequence is

$$\text{C set: } c_1, c_2, c_3 \quad \text{E set: } c_{13}, c_{14}, c_{15}$$

$$\text{D set: } c_{27}, c_{28} \quad \text{F set: } c_{39}, c_{40}$$

The first test, as described above, is to distinguish between the CD and EF sets. Referring to the new sequence, the last three cards form an AAB pattern; then, from the equations for $P_t(\text{AAB})$ and $Q_t(\text{AAB})$,

$$P\{\text{EF}\} = r_1 = 8/9 \quad \text{and} \quad P\{\text{CD}\} = 1 - r_1 = 1/9 \tag{7-3}$$

Equations 7-3 represent the probabilities that the eleventh card is a member of the EF or CD set, respectively. Now, given that the eleventh card belongs to the EF set, we test for E versus F by noting that the last three cards of the EF set (c_{40}, c_{14}, c_{15}) form an ABB pattern. Hence, applying the equations for $P_t(\text{ABB})$ and $Q_t(\text{ABB})$,

$$P\{\text{F}|\text{EF}\} = \frac{r_2}{1 - r_1} = 8/9 \quad \text{and} \quad P\{\text{E}|\text{EF}\} = \frac{r_3}{1 - r} = 1/9 \tag{7-4}$$

Similarly, in testing for C versus D, we note that the last three cards of the CD set (c_2, c_{28}, c_3) form an ABA pattern. Thence, from the equations for $P_t(\text{ABA})$ and $Q_t(\text{ABA})$,

$$P\{\text{D}|\text{CD}\} = r_1 = 8/9 \quad \text{and} \quad P\{\text{C}|\text{CD}\} = 1 - r_1 = 1/9 \tag{7-5}$$

The probability that the 11th card is c_4, c_{29}, c_{11}, or c_{41} is then determined by Eqs. 7-3 through 7-5 in the following manner:

$$P\{c_4\} = P\{\text{C}|\text{CD}\} \cdot P\{\text{CD}\} = (1 - r_1)^2 = \frac{1}{81}$$

$$P\{c_{29}\} = P\{\text{D}|\text{CD}\} \cdot P\{\text{CD}\} = r_1(1 - r_1) = \frac{8}{81}$$

$$P\{c_{16}\} = P\{\text{E}|\text{EF}\} \cdot P\{\text{EF}\} = \frac{r_1 r_3}{1 - r_1} = \frac{8}{81}$$

$$P\{c_{41}\} = P\{\mathrm{F}|\mathrm{EF}\} \bullet P\{\mathrm{EF}\} = \frac{r_1 r_2}{1 - r_1} = \frac{64}{81}$$

Continuation of this procedure for three shuffles is straightforward, albeit hampered by arithmetical drudgery. Practical applications of card prediction, even with our pragmatic theorem, invite the aid of a modest computer.

Monge's Shuffle

There are other methods of card shuffling apart from the alternate interleaving of card packets.[5] Each method might be associated with one or more convenient theorems for card prediction, depending on the regularities involved in the shuffling procedure. As one example, consider a pack of cards arranged in the order $c_1, c_2, c_3, \ldots$; we shuffle this pack by placing c_2 above c_1, c_3 under c_1, c_4 above c_2, c_5 under c_3, etc.—an operation known as Monge's shuffle.[6] Then, illustratively, if the pack contains $6n - 2$ cards, the $2n$th card always retains its original position. With 22 cards ($n = 4$) shuffled repeatedly by this process, the eighth card never changes place; the fifth and 16th cards oscillate, exchanging positions at each shuffle; and the third, 13th, and 18th cards circulate in an independent cycle, regaining their original positions every third shuffle. Cyclic periods of the full deck are generally longer with Monge's method as compared with the perfect shuffle (interleaving alternate cards).

There exists a theorem stating that the cyclic period ρ resulting from Monge's shuffle on a $2n$-card deck is equal to the smallest root of the congruence

$$2^\rho \equiv 1, \operatorname{mod}(4n + 1)$$

and if this congruence possesses no solutions, ρ equals the smallest root of the congruence:

$$2^\rho \equiv 1, \operatorname{mod}(4n + 1)$$

With 52 cards ($n = 26$), Monge's shuffle operations cycle the deck after 12 shuffles ($2^{12} \equiv 1$, mod 105), compared to eight for the perfect shuffle. For a 16-card deck, the original order returns after five Monge shuffles, compared to four perfect shuffles needed for cycling.

Monge's shuffle, as well as other shuffling operations considered here, are examples of binary shuffling, wherein the deck is divided into two parts that are then interleaved. It is quite feasible, of course, to divide the deck into three

[5] For more esoteric studies of shuffling phenomena, the reader is referred to the works of Henri Poincaré and Jacques Hadamard on probability chains.

[6] Analyzed by Casper Monge, Comte de Péluse.

or more parts and then perform a shuffling operation (Ref. Medvedoff and Morrison); however, in practical terms, such shuffles are inconvenient.

CARD PROBABILITIES

Sampling without Replacement

Combinations and permutations of cards generally involve the hypergeometric distribution (Eq. 2-10)—as opposed to dice or coin combinations, which are governed by the binomial distribution. Card statistics, then, involve the principle of sampling without replacement. An elementary illustration of this principle, which demonstrates the considerable difference from sampling with replacement, is that of drawing two cards at random from a complete deck. The probability of both cards being Aces (say) is $(4/52) \cdot (3/51) = 0.0045$, whereas if the first card is replaced before drawing the second, the probability of obtaining two Aces is $(4/52)^2 = 0.0059$, a 31% change.

Another example supplies greater emphasis: A single red card is removed from the deck. Then 13 cards are drawn and found to be the same color; the (conditional) probability that they are black is

$$\frac{26!/13!}{26!/13! + 25!/12!} = \frac{2}{3}$$

a greater difference than might be expected from the corresponding probability of 1/2 for sampling with replacement. Of course, if the population size (52, initially) is large compared with the sample size (4, for the example of the card ranks), the binomial and hypergeometric distributions produce approximately the same results. The striking differences occur as cards are withdrawn from the deck. If we consider the number of cards drawn without replacement until a Spade appears (at the mth draw), then

$$P(m) = \frac{13\binom{39}{m-1}}{(53-m)\binom{52}{m-1}}, \quad m = 1, 2, \ldots, 40$$

constitutes the probability distribution over m. Clearly, $P(m)$ becomes quite small as more cards are drawn without a Spade appearing. For $m = 11$, $P(m)$ has decreased to 0.0124 from its value of 0.25 at $m = 1$. For $m = 40$, $P(40) = 1.575 \times 10^{-12}$.

Similarly, the number of cards drawn without replacement until an Ace appears is

$$P(m) = \frac{4\binom{48}{m-1}}{(53-m)\binom{52}{m-1}}, \quad m = 1, 2, \ldots, 49$$

For $m = 11$, $P(11)$ is 0.0394 as compared with $P(1) = 1/13$. For $m = 49$, $P(49) = 3.694 \times 10^{-6}$.

An example demanding careful distinction between *a priori* and *a posteriori* probabilities in sampling without replacement concerns a deck of cards from which one card has been removed. Two cards are then drawn at random and found to be Spades. We wish to determine the probability that the missing card is also a Spade. *A priori*, this probability is 1/4. The joint probability that the missing card and the two drawn cards are Spades is

$$\frac{1}{4} \times \frac{12}{51} \times \frac{11}{50} = \frac{11}{850}$$

while the joint probability that the missing card is a non-Spade and two Spades are drawn is

$$\frac{3}{4} \times \frac{13}{51} \times \frac{12}{50} = \frac{39}{850}$$

Hence, applying Bayes theorem, Eq. 2-4, to determine the *a posteriori* probability of the missing card being a Spade, we obtain

$$P(\text{Spade}|\text{two Spades drawn}) = \frac{P(\text{Spade})\,P(\text{two Spades drawn}|\text{Spade})}{P(\text{two Spades drawn})}$$

$$= \frac{P_i}{P_1 + P_2} = \frac{11/850}{11/850 + 39/850} = \frac{11}{50}$$

Expected Number of Cards to Achieve Success

An important consideration in card probabilities is the expected number of cards drawn without replacement to achieve a particular success; we could ask, for example, the expected number of cards drawn to uncover the first Ace, or the first Spade, etc. The solution of this type of problem is obtained from the discrete analog to the nonparametric technique used in order statistics. An equivalent to the density function of ordered observations can be derived directly. With a deck composed of a kinds of cards and b of each kind (a total of $n = ab$), the probability $P_i(r)$, $i < a$, that none of the i kinds is represented in a random sample of r cards is given by

$$P_i(r) = \frac{\binom{n-ib}{r}}{\binom{n}{r}} = \frac{(n-r)_{ib}}{(n)_{ib}}, \quad r = 1,2,\ldots,n-ib+1$$

Now, if we define $p_i(r)$ as the probability of failure through the first $r - 1$ cards times the probability of success on the rth card, we have

$$p_i(r) = \frac{ibP_i(r-1)}{n-r+1} = \frac{ib(n-r)_{ib-1}}{(n)_{ib}}, \quad \begin{cases} r = 1, 2, \ldots, n - ib + 1 \\ i < a \end{cases} \tag{7-6}$$

For $i = a$, $p_a(1) = 1$.

The expected number of cards to achieve the event specified by the probability $p_i(r)$ is defined by

$$E_i(r) = \sum_{n=1}^{n-b+1} rp_i(r) = \sum \frac{rib(n-r)_{ib-1}}{(n)_{ib}} = \frac{n+1}{ib+1} \tag{7-7}$$

with the expression for $p_i(r)$ substituted from Eq. 7-6. By extension, the expected number of cards $E_i^m(r)$ to uncover the mth number of one of i groups of b cards is

$$E_i^m(r) = \frac{n+1}{ib+1}m, \quad m \leq ib \tag{7-8}$$

[If the cards are drawn with replacement, $E_i^m(r)$ is expressed by nm/ib.]

We can now answer the questions posed. The expected number of cards to uncover the first Ace is, from Eq. 7-7 with $n = 52$, $i = 1$, and $b = 4$,

$$E_1(r) = 53/5 = 10.6$$

The expected number of cards until the appearance of the first Spade is also given by Eq. 7-7 with $i = 1$ and $b = 13$:

$$E_1(r) = 53/14 = 3.786$$

And, according to Eq. 7-8, it is expected that

$$E_1^{13}(r) = 49.214$$

cards are required to expose all Spades in the deck. It also follows that the expected number of cards between successive Spades remains unchanged. That is, if **A** deals from the pack until a Spade appears, then passes the remainder of the deck to **B**, who deals until a second Spade appears and then passes the remainder of the deck to **C**, who deals cards until the appearance of the third Spade, the expectations of the number of cards dealt by **A, B**, and **C** are identical.

From Eq. 7-7, we can also answer the following question: What is the expected number of cards drawn without replacement until at least one card of each suit is exposed? With four suits, S, H, D, and C, we can write this expectation in the form

$$\begin{aligned}\mathrm{E}(\mathrm{S} \cup \mathrm{H} \cup \mathrm{D} \cup \mathrm{C}) = {} & \mathrm{E}(\mathrm{S}) + \mathrm{E}(\mathrm{H}) + \mathrm{E}(\mathrm{D}) + \mathrm{E}(\mathrm{C}) - \mathrm{E}(\mathrm{S} \cap \mathrm{H}) \\ & - \mathrm{E}(\mathrm{S} \cap \mathrm{D}) - \cdots - \mathrm{E}(\mathrm{D} \cap \mathrm{C}) \\ & + \mathrm{E}(\mathrm{S} \cap \mathrm{H} \cap \mathrm{D}) + \mathrm{E}(\mathrm{S} \cap \mathrm{H} \cap \mathrm{C}) + \cdots \\ & + \mathrm{E}(\mathrm{H} \cap \mathrm{D} \cap \mathrm{C}) - \cdots \\ & - \mathrm{E}(\mathrm{S} \cap \mathrm{H} \cap \mathrm{D} \cap \mathrm{C})\end{aligned}$$

After appropriate substitutions,

$$\mathrm{E}(\mathrm{S} \cup \mathrm{H} \cup \mathrm{D} \cup \mathrm{C}) = (n+1)\left[\frac{\binom{a}{1}}{b+1} - \frac{\binom{a}{2}}{2b+1} + \frac{\binom{a}{3}}{3b+1} - \cdots \pm \frac{\binom{a}{a}}{2b+1}\right]$$

For the conventional deck ($a = 4$, $b = 13$),

$$\mathrm{E}(\mathrm{S} \cup \mathrm{H} \cup \mathrm{D} \cup \mathrm{C}) = 53\left(\frac{4}{14} - \frac{6}{27} + \frac{4}{40} - \frac{1}{53}\right) = 7.665 \qquad (7\text{-}9)$$

This figure suggests the following game: The House deals out eight cards, winning one unit from the Player if all four suits are represented; otherwise the Player is paid two units. The probability of this event is expressed by

$$P = (\mathrm{C}_1 + \mathrm{C}_2 + \mathrm{C}_3 + \mathrm{C}_4 + \mathrm{C}_5)\binom{52}{8}^{-1}$$

where $\mathrm{C}_1 = \binom{13}{2}^4$; $\mathrm{C}_2 = 12\binom{13}{3}\binom{13}{2}^2\binom{13}{1}$; $\mathrm{C}_3 = 6\binom{13}{3}^2\binom{13}{1}^2$;

$$\mathrm{C}_4 = 12\binom{13}{4}\binom{13}{2}\binom{13}{1}^2; \quad \mathrm{C}_5 = 4\binom{13}{5}\binom{13}{1}^3$$

Thus $P = 0.685$, and the House Take is $3P - 2 = 5.63\%$.

Interestingly, replacement of the cards creates a problem easier to solve. Drawing until each suit is exposed at least once represents four exclusively ordered events. The probability of obtaining the first suit at the first card is $p_1 = 1$. The probability that the second card is a suit other than the first is $p_2 = 3/4$. Subsequently, $p_3 = 1/2$ and $p_4 = 1/4$ by the same reasoning. Thus, by applying Eq. 6-18, we have

$$\mathrm{E}_1(r) = 1, \quad \mathrm{E}_2(r) = 4/3, \quad \mathrm{E}_3(r) = 2, \quad \mathrm{E}_4(r) = 4$$

The sum of these expectations produces

$$E(S \cup H \cup D \cup C) = 8\tfrac{1}{3}$$

which, as might well be anticipated, is greater than the value obtained by sampling without replacement (Eq. 7-9).

Combinations

The number of combinations of n objects taken r at a time is given by Eq. 2-10 as $\binom{n}{r}$. Thus, the number of distinct Poker hands totals

$$\binom{52}{5} = 2{,}598{,}960 \tag{7-10}$$

and the number of different hands possible at cribbage is $\binom{52}{6} = 20{,}358{,}520$.

An extension of this combinatorial formula considers the probability P that, in a random sample of size r selected without replacement from a population of n elements, all of N given elements are contained in the sample. It is straightforward to prove that

$$P = \frac{\binom{n-N}{r-N}}{\binom{n}{r}}$$

Illustratively, the probability that a Poker hand contains all four Aces is

$$P = \frac{\binom{52-4}{5-4}}{\binom{52}{5}} = 1.847 \times 10^{-5}$$

Similar combinatorial formulae can answer most of the elementary probability questions that arise in games of cards.

Dividing the Deck

A conventional deck of cards is divided, at random, into two parts—that is, the dividing point is randomly distributed and, therefore, is equally likely to be in the left half as in the right half of the deck. Thus, on average it will be at one-half of that half or one-fourth of the deck length, the deck being divided into segments of 39 and 13 cards.[7]

[7] In practice, cutting the deck, although a random process, is not uniformly distributed since the cutter usually aims for a point about midway.

To determine the average ratio of the smaller segment to the larger, let the dividing point be in the right half of the deck. Then $(1 - x)/x$ is the ratio, and, since x is evenly distributed from 1/2 to 1, the average value is

$$2\int_{1/2}^{1} \frac{(1-x)}{x}\,dx = 2\log_e 2 - 1 \simeq 0.386$$

rather than the intuitive 1/3. Thus there will be 14.48 cards rather than 13 in the smaller segment.

It can be shown that when the deck is randomly divided into three parts, the average lengths of the segments form the proportions 1/9, 5/18, and 11/18. (For a 54-card deck [two Jokers added], the three segments comprise 6, 15, and 33 cards.)

Positions of the Aces

Cards from a standard deck are turned over one by one. We ask the most likely positions for the two black Aces.

The probability that the first black Ace turns up in position p_1 ($1 \le p_1 \le 51$) is

$$\frac{2(52 - p_1)}{52 \cdot 51}$$

which has its maximum value at $p_1 = 1$; therefore the most likely position for the first black Ace is on top. By symmetry, the probability that the second Ace turns up in position p_2 ($2 \le p_2 \le 52$) is

$$\frac{2(p_2 - 1)}{52 \cdot 51}$$

which has its maximum value at $p_2 = 52$; thus the most likely position for the second black Ace is on the bottom.

If we inquire as to the probability of the first of the four Aces being in position p_1', we can derive the expression

$$\frac{4(52 - p_1')(51 - p_1')(50 - p_1')}{52 \cdot 51 \cdot 50 \cdot 49} \qquad 1 \le p_1' \le 49$$

which has its maximum value (1/13) at $p_1' = 1$. The probability that an Ace appears in the first eight cards is close to 1/2 (more precisely, 0.4986).

Higher Card

Two cards are drawn successively from a standard deck. We are interested in the probability that the second card is higher in rank than the first card.

Since $P(\text{higher}) + P(\text{lower}) + P(\text{same}) = 1$; since $P(\text{higher}) = P(\text{lower})$; and since $P(\text{same}) = 3/51$, we can immediately write

$$P(\text{higher}) = \frac{(1 - 3/51)}{2} = \frac{24}{51} = 0.47$$

Sens Dessus-Dessous (Ref. Nick's Mathematical Puzzles, No. 106)

A standard deck is thrown into a high-velocity wind tunnel—so that each card, independently and equiprobably, lands face up or face down. The numerical rankings of the face-up (or face-down) cards are summed, with Ace = 1 through King = 13. We ask the probability P that this sum is divisible by 13.

First, we establish the generating function

$$g(x) = \prod_{n=1}^{13} (1 + x^n)^4 = \sum_{k=0}^{364} a_k x^k$$

Each exponent in $g(x)$ represents a total value, and each corresponding coefficient a_k represents the number of ways of obtaining that value. Thus, we can write for P:

$$P = \frac{1}{2^{52}}\left[a_0 + \sum_{k=13}^{364} a_k\right]$$

since there are 2^{52} configurations of the deck.

To express this summation, let t be a complex primitive root of 1. Then $t^{13} = 1$, and $\sum_{k=0}^{12} t^k = 0$, which leads to

$$P = \frac{1}{13} + \frac{12}{13} 2^{-48}$$

That P differs from 1/13 is counterintuitive. That it differs by such a minute amount (in the 18th decimal place) is a further surprise.

MATCHING PROBLEMS (RENCONTRES)

In the simplest form of card matching, a deck of n distinct cards numbered 1, 2, …, n is randomized and dealt out. If the number of a card coincides with its position in the deck, that circumstance is designated a match. We inquire as to the probability of one or more matches occurring.[8]

Let P_n equal the number of permutations without a match. Then

$$P_n = (n - 1)(P_{n-1} + P_{n-2}); \quad P_2 = 1; \quad P_3 = 2$$

[8] First proposed and solved by Montmort, 1708.

Since the total number of permutations is $n!$, the probability of no matches is $P_0 = P_n/n!$ Table 7-2 shows that P_0 converges rapidly for even relatively small decks.

Table 7-2 Probability of No Matches versus Deck Size

n	2	3	4	5	6	7	8
P_0	0.5	0.333	0.375	0.367	0.368	0.368	0.368

The probability of at least one match, therefore, equals 0.632^+. For $n = 52$, $P_0 = 0.368^-$. As n increases, $P_0 \to 1/e = 0.36788^-$ (a value—surprisingly—virtually independent of n).

The probability P_m of *exactly* m matches among n events (e.g., a deck of n cards) was calculated by Montmort. Specifically,

$$P_m = \sum_{i=0}^{n-m} (-1)^i \binom{m+i}{m} \frac{1}{(m+i)!} \tag{7-11}$$

As an example, for three matches with decks of any length greater than 6, $P_3 = 0.0613^+$. Note that $P_{n-1} = 0$ since exactly $n - 1$ matches are not possible, and $P_n = 1/n!$ for the case where both decks are in the same order.

Equation 7-11 can further be applied to evaluate the validity of "psychic" perceptions (see Chapter 11).

A matching problem credited to Levene[9] consists of shuffling together the Hearts and Spades of a conventional deck. The 26 cards are then divided into two piles of 13 and matched against each other; a match is recorded if the ith of Spades and the ith of Hearts occur at the same position in the two piles. We wish to determine the probability of exactly r matches. The probability distribution of r can be derived as

$$p(r) = \binom{n}{r} \frac{i}{2^r [N - (1/2)]_r} \sum_{j=0}^{N-r} \frac{(N-r)_j (-1)^j}{[N - r - (1/2)]_j 2^j j!}$$

where N is the number of cards in each pile. For $N = 13$, the probability of no matches is $p(0) = 0.5948$. For $r = 1$, $p(1) = 0.3088$; also $p(2) = 0.0804$, and $p(3) = 0.0140$. As $N \to \infty$, the number of matches assumes a Poisson distribution, and $p(r)$ approaches the values 0.6065, 0.3033, 0.0758, and 0.0126 for $r = 0$, 1, 2, 3, respectively.

Oddly, Levene's problem assumes a simpler form (suggested by Kullback[10]) if the compositions of the two piles are determined by random, independent draw-

[9] Howard Levene, Department of Mathematics, Statistics, and Genetics, Columbia University, New York.

[10] Solomon Kullback, 1903–1999, cryptologist, Chief Scientist, National Security Agency.

ings from two populations. Let the first pile be composed by drawing cards from a population containing a_i cards of the ith kind, $i = 1, 2, \ldots, k$, where p_i is the constant probability that a card of the ith kind is drawn. Similarly, let the second pile be formed by drawing cards from a second population containing b_i cards of the ith kind, $i = 1, 2, \ldots, k$, and let p_i' be the constant probability of drawing a card of the ith kind from this population. Each pair of cards in the same position in the two piles constitutes an independent event; thus the probability p of a match is simply

$$p = \sum_{i=1}^{k} p_i p_i'$$

The probability of r matches therefore forms a binomial distribution (Eq. 2-11):

$$p(r) = \binom{N}{r} p^r (1-p)^{N-r}$$

As N and k both increase indefinitely, $p(r)$ becomes Poisson-distributed.

A more difficult matching problem, not readily handled by conventional means, is the matching of two n-card decks, each of which comprises a suits and b cards in each suit. A match is now defined as the coincidence of two cards of the same rank without regard to suit (we could, obviously, interchange rank and suit and inquire as to matching by suits).

This problem is best attacked through the application of *rook and hit polynomials*. The former is defined as the generating function of the number of ways of placing non-mutually-attacking rooks on a chessboard. Thus to formulate the rook polynomial $R(x)$, we envision the equivalent n x n chessboard as containing n rooks, no two in the same row or column; i rooks on the negatively sloped diagonal correspond to i matches. The rook polynomial is then expressible as

$$[R(x)]^n = (1 + x)^n$$

Further, we define a *hit* as the occurrence of an element of a permutation in a forbidden position. Thus the *hit polynomial* $N_n(t)$ engenders the permutations by enumerating the number of hits. That is,

$$N_n(t) = \sum N_i^i$$

where N_i is the number of permutations of n distinct elements, i of which are in restricted positions. Clearly, the rook polynomial determines the hit polynomial, and vice versa.

To determine the rook polynomial for $a = 4$ and $b = 13$, consider a 4×4 chessboard. There are $4^2 = 16$ ways of placing a single rook on such a board; two mutually nonattacking rooks can be placed in $(4^2 \times 3^2)/2! = 72$ ways; three nonattacking rooks can be placed on this board in $(4^2 \times 3^2 \times 2^2)/3! = 96$ ways; and four rooks allow $(4^2 \times 3^2 \times 2^2 \times 1^2)/4! = 24$ ways. It is clearly impossible to place five or more nonattacking rooks on a 4×4 chessboard. The rook polynomial is therefore written as

$$R(x) = 1 + 16x + 72x^2 + 96x^3 + 24x^4$$

We are specifically concerned with the polynomial $[R(x)]^{13}$ arising from the 13 cards in each suit. The corresponding hit polynomial assumes the form

$$N_n(t) = 52! + 13 \cdot 16 \cdot 51!\,(t-1) + \cdots + (24)^{13}(t-1)^{52}$$

and the quantity $N_n(0)/n!$ is the *probability* of no matches. The probabilities of exactly i matches are displayed in Table 7-3. Hence the probability of at least one match is $1 - P(0) = 0.984$, somewhat higher than might be expected intuitively.

Table 7-3 Probability of *i* Matches

Number of matches	Probability (*P*)	Number of matches	Probability (*P*)
0	0.0162	6	0.1058
1	0.0689	7	0.0586
2	0.1442	⋮	
3	0.1982	51	0
4	0.2013	52	1.087×10^{-50}
5	0.1611		

We can also calculate the expected number of matches as

$$\mathrm{E}(i) = \sum_{i=0}^{52} iP(i) = 4$$

which is also the most probable number. The standard deviation is $8/\sqrt{17}$.

Rook polynomials are not restricted to square chessboards; rectangular, triangular, and trapezoidal chessboards have also been called on to solve problems in combinatorics.

Rank Matching

Cards are dealt out one at a time from a standard deck. We inquire as to the expected number of cards, E, to realize a match—that is, two cards of the same rank. This expectation[11] can be expressed as

$$\mathrm{E} = \sum_{k=0}^{13} \frac{4^k \binom{13}{k}}{\binom{52}{k}} = 5.70 \tag{7-12}$$

[11] Contributed by Stewart Ethier.

The probability P(n) that a match occurs for the first time with the nth card, $2 \le n \le 14$, is given by

$$P(n) = \frac{12!\,3(n-1)\,4^{n-2}(52-n)!}{51!\,(14-n)!}$$

For $n = 2, 3, 4, 5$, Eq. 7-12 yields

$$P(2) = 0.0588; \quad P(3) = 0.1094; \quad P(4) = 0.1521; \quad P(5) = 0.1690$$

for a cumulative total of 0.4893. Thus the wager that a match (of ranks) will occur by the fifth card dealt has an expected gain of -0.0214.

Rank Color Matching

Here, we define n distinct pairs of same-colored cards: $n/2$ Red pairs and $n/2$ Black pairs. With the cards exposed one at a time, the expected number to realize a matched pair is

$$E = 2^{2n}/\binom{2m}{m}$$

For the standard deck, $n = 26$, and

$$E = 2^{52}/\binom{52}{26} = 9.081$$

The probability that the first rank-and-color match occurs at the exposure of the nth card, $2 \le n \le 27$, is expressed by

$$P(n) = \frac{26!\,2^{n-1}(52-n)!\,(n-1)}{52!\,(27-n)!} \qquad (7\text{-}13)$$

Evaluating Eq. 7-13 for values of n between 2 and 9:

$P(2) = 0.0196$	$P(3) = 0.0392$	$P(4) = 0.0576$	$P(5) = 0.0728$
$P(6) = 0.0852$	$P(7) = 0.0934$	$P(8) = 0.0968$	$P(9) = 0.0956$

for a cumulative total of 0.5102. Wagering that a match occurs *by* the ninth card dealt therefore entails a positive expectation of 0.0204.

Suit Matching

Dealing cards one by one from a standard deck, the probability of obtaining a suit match on the nth card, $2 \le n \le 5$, can be written as

$$P(n) = \frac{4!\,12(n-1)\,13^{n-1}(52-n)!}{52!\,(5-n)!}$$

which yields

$$P(2) = 0.2353; \qquad P(3) = 0.3671; \qquad P(4) = 3921; \qquad P(5) = 0.1055$$

With a payoff of 3 to 1 for achieving a match at the second card dealt, the bettor on that event is faced with an expected gain of -0.0588.

Color Matching

Considerably more complex is a particular problem of color matching. From a deck of $4n$ cards, $2n$ red and $2n$ black, cards are dealt out in pairs. If a pair is matched by color, it is placed in Pile 1. Pairs of mismatched colors are retained, shuffled, and then dealt out again in twos, this time with color-matched pairs deposited in Pile 2. This process is continued until the deck is depleted. We wish to determine E(n), the expected number of piles resulting (including piles containing zero cards).

For any n, there are $\binom{2n}{n}$ ways of pair-matching on the first round. Noting that for every matched red pair there must be a matched black pair (since a mismatched pair consists of one red and one black card), there will be $2k$ pairs, $2n \geq k \geq 0$, discarded in each round. After the first round there will be $2n - 2k$ pairs remaining—having discarded $2k$ pairs in $\binom{2n}{2k}\binom{2k}{k}2^{2(n-k)}$ ways ($2k$ out of $2n$ pairs are same-colored, and k out of these $2k$ pairs are red, leaving $2^{2(n-k)}$ possible ways for the remaining unmatched $2n - 2k$ pairs). The number of pairs retained is therefore $2n - 2k$, and the expected number of further rounds required to discard them is E($n - k$). We can then write

$$\binom{4n}{2n}E(n) = \binom{2n}{n} + \sum_{k=0}^{n-1}\binom{2n}{2k}\binom{2k}{k}2^{2(n-k)}[1 + E(n-k)]$$

The $k = 0$ component can be expressed distinctly, reconstituting this equation to

$$E(n) = \frac{\binom{2n}{n} + 2^{2n} + \sum_{k=1}^{n-1}\binom{2n}{2k}\binom{2k}{k}2^{2(n-k)}[1 + E(n-k)]}{\binom{4n}{2n} - 2^{2n}} \tag{7-14}$$

For $n = 1$ (a deck of 2 red and 2 black cards), we can immediately write

$$E(1) = 2/6 + (4/6)[E(1) + 1] = 3$$

Then, from Eq. 7-14,

$$E(2) = 6/70 + (16/70)[E(2) + 1] + (48/70)[E(1) + 1] = 107/27 = 3.963$$

$$E(3) = (20/924) + (64/924)[E(3) + 1] + (480/924)[E(2) + 1]$$
$$+ (360/924)[E(1) + 1] = 8789/1935 = 4.452$$

$$E(4) = 7457/1505 = 4.955$$

For the conventional deck, E(13) = 6.653. With increasing n, E(n) slowly edges toward infinity.

The probability P(n) that the deck will be depleted on or before the nth deal is tabulated as follows:[12]

$P(1) = 2.10\times10^{-8}$	$P(4) = 0.184$	$P(7) = 0.718$
$P(2) = 6.8\times10^{-4}$	$P(5) = 0.397$	$P(8) = 0.811$
$P(3) = 0.0339$	$P(6) = 0.582$	$P(9) = 0.874$

Thus, for example, a wager (at even odds) that the deck will be depleted by the sixth deal entails an expected gain of +0.164.

SELECTED SIDELIGHTS

Highest Rank in a Sample

It is informative to consider a game wherein the cards of a pack are exposed one by one until all four suits are represented, with the first-drawn card of each suit set aside. Of the four cards so collected, we inquire as to the probability P_m that m is the highest rank (Ace = 1, King = 13). Clearly, the probability P that the highest rank does not exceed m is $P = P_1 + P_2 + \cdots + P_m$, which is equivalent to $(m/13)^4$, since the probability that each suit contributes a rank m or less is $m/13$. Therefore, replacing m by $m - 1$, we can write

$$P - P_m = P_1 + P_2 + \cdots + P_{m-1} = \left(\frac{m-1}{13}\right)^4$$

Combining the expressions for P and $P - P_m$,

$$P_m = \frac{m^4 - (m-1)^4}{13^4}$$

which is monotonically increasing with m. For $m = 1$ (Ace), $P_1 = 3.5 \times 10^{-5}$; for Queen and King, $P_{12} = 0.211$ and $P_{13} = 0.274$.

Information: The Exposed Ace

It is enlightening to consider an example that illustrates the effect of information on card probabilities. Suppose that in a random sample of five cards, one is known to be an Ace. Then the probability P that the remaining four cards comprise at least one additional Ace is

$$P = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{4}{1}\binom{48}{4} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}} = 0.122$$

[12]Simulation courtesy Norman Wattenberger, Casino Vérité.

Now suppose that of the five-card sample, one is known to be the Ace of Spades. The probability P' that the remaining four cards include at least one more Ace is then determined to be

$$P' = \frac{\binom{3}{1}\binom{48}{3} + \binom{3}{2}\binom{48}{2} + \binom{3}{3}\binom{48}{1}}{\binom{51}{4}} = 0.221$$

In this instance the knowledge of the specific Ace has increased the probability of one or more additional Aces in the sample.

Knowledge of color-only results in an intermediate value of the probability for additional Aces. Specifically, the probability P'' of the four unknown cards comprising one or more Aces, given the first card to be a red Ace, is

$$P'' = \frac{\binom{2}{2}\binom{2}{0}\binom{48}{3} + \binom{2}{1}\binom{2}{1}\binom{48}{3} + 2\binom{2}{2}\binom{2}{1}\binom{48}{2} + \binom{2}{2}\binom{2}{2}\binom{48}{1}}{\binom{2}{1}\binom{48}{4} + \binom{2}{2}\binom{2}{0}\binom{48}{3} + \binom{2}{1}\binom{2}{1}\binom{48}{3} + 2\binom{2}{2}\binom{2}{1}\binom{48}{2} + \binom{2}{2}\binom{2}{2}\binom{48}{1}} = 0.190$$

This probability is closer to the specified-Ace value than to the value for the case of an unspecified Ace. In order to avoid semantic confusion, each case should be considered as a distinct dichotomy—the particular condition (red Ace, for example) is either specified or not; alternations between color and suit produce equal probabilities for all cases.

INFORMAL CARD GAMES

In addition to the conventional card games that have evolved throughout the centuries, many rudimentary games can be invented spontaneously to illustrate certain principles.

First Ace Wins

Consider the simple game wherein first **A** and then **B** alternately draw cards from a deck, the winner being that player who first draws an Ace. **A** wins the game immediately with probability

$$p_1 = a/n$$

where a is the number of aces and n the total number of cards. **A** also wins on his second try if the first two cards are non-Aces and the third is an Ace. The probability p_2 of this occurrence, by the theorem of compound probabilities, is expressed in the form

$$p_2 = \frac{a(n-a)(n-a-1)}{n(n-1)(n-2)}$$

Similarly, the probability p_3 of **A** winning on his third try (the first four cards drawn being non-Aces and the fifth an Ace) is

$$p_3 = \frac{a(n-a)(n-a-1)(n-a-2)(n-a-3)}{n(n-1)(n-2)(n-3)(n-4)}$$

And the probability P_A that **A** wins the game is the sum $p_1 + p_2 + p_3 + \cdots + p_{(n-a)/2}$:

$$P_A = \frac{a}{n}\left[1 + \sum_{i=1}^{(n-a)/2} \frac{(n-a)_{2i}}{(n-1)_{2i}}\right]$$

Analogously, **B**'s probability P_B of winning the game is

$$P_B = \frac{a}{n} \sum_{i=0}^{[(n-a)/2]-1} \frac{(n-a)_{2i+1}}{(n-1)_{2i+1}}$$

For $a = 4$ and $n = 52$, we have, numerically,

$$P_A = 0.52 \quad \text{and} \quad P_B = 0.48$$

Thus, **A** gains a 4% advantage by being first to draw.

With three players **A, B**, and **C** competing in that order to draw the first Ace, we can derive the expressions

$$P_A = \frac{a}{n}\left[1 + \sum_{i=1}^{(n-a)/3} \frac{(n-a)_{3i}}{(n-1)_{3i}}\right]$$

$$P_B = \frac{a}{n} \sum_{i=0}^{[(n-a)/3]-1} \frac{(n-a)_{3i+1}}{(n-1)_{3i+1}}$$

$$P_C = \frac{a}{n} \sum_{i=0}^{[(n-a)/3]-1} \frac{(n-a)_{3i+2}}{(n-1)_{3i+2}}$$

For $a = 4$ and $n = 52$, these equations are evaluated numerically as

$$P_A = 0.360 \quad P_B = 0.333 \quad P_C = 0.307$$

Obviously, as the number of players increases, the greater becomes the advantage of drawing first.

Polish Bank

Several players contend against the banker who deals three cards, face up, to each player and a single card, face up, to himself—from a deck of 32 cards with four suits ranked 7 to Ace. A player wins—at 2 to 1 odds—only if one of his cards is of a higher rank than the banker's card and in the same suit. A hand of three 7s or of 7, 8, 9 in the same suit is declared *non avenu* and replaced with another three cards.

Let n_1, $0 \le n_1 \le 21$, denote the number of cards the player can beat, and n_2 the number of different hands that can beat n_1 cards. Then eliminating the *non-avenu* hands, the total number of possible games, N_t, is

$$N_t = 29\,\Sigma n_2 = 29 \times 4940 = 143{,}260$$

while the number of games, N_p, that the player can win is

$$N_p = \Sigma n_1 n_2 = 42{,}540$$

Therefore, the winning probability P_w for the player is

$$P_w = N_p / N_t = 0.297$$

Thus, with 2-to-1 odds, the player's expectation is -0.109.

Stein's Game[13]

From a deck of cards numbered from 1 to n, cards are turned over in random sequence. Whenever the rank of the turned-over card exceeds the highest rank of those cards preceding it, the player receives X units from the House, which charges an entry fee for the game.

The probability that the player wins on a particular card is $1/n$. Therefore, his expectation is simply

$$X(1 + 1/2 + 1/3 + \cdots + 1/n)$$

With $n = 10$, an entry fee of $2.93X$ units would constitute a fair game. With $n = 52$, a fair game would require an entry fee of $4.54X$, and for $n = 100$, the fee rises only to $5.187X$. For large n, the player's expectation asymptotically approaches

$$X(\ln n + \gamma)$$

where $\gamma = 0.577 \ldots$ is Euler's constant.

Odds and Evens

Invented by Richard Cowan (Ref. 1), Odds and Evens is a two-person game played with n cards numbered 1 through n. **A** and **B** each receives $(n - 2)/2$ cards. Simultaneously, each selects and turns over one of his cards. **A** wins if the sum of the two exposed cards is odd, **B** wins if the sum is even.

Table 7-4 shows the payoff matrix for $n = 8$, along with the card played (odd or even) and the probabilities for each three-card hand.

[13] Suggested by E.P. Stein.

Table 7-4 Payoffs and Probabilities for Odds and Evens

			B		
		EEE	EEO	EOO	OOO
	Probability	0	0	3/70	2/70
		E	E O	E O	O
	EEE E			0 1	1
	Probability	0	9/70	18/70	3/70
	E		0 1	0 1	1
A	EEO O		1 0	1 0	0
	Probability	3/70	18/70	9/70	0
	E	0	0 1	0 1	
	EOO O	1	1 0	1 0	
	Probability	2/70	3/70	0	0
	OOO O	1	1 0		

In this instance, **A**'s optimal strategy is to play E (or O) with a probability inversely proportional to the ratio of Es (or Os) in his hand. Thus, from hand EOO, **A** should play E with probability 2/3 and O with probability 1/3. **A**'s probability of winning with this strategy is readily computed as 18/35 (game value of +1/35).

For $n = 10, 12, 14$ cards, **A** retains winning probabilities of 11/21, 39/77, 229/429, respectively. (From hand E000, as another example, **A** should play E with probability 3/4 and 0 with probability 1/4.)

With the number of cards dealt to each player fixed at three, **A**'s winning probabilities for $n = 10$, 12, and 14 cards are 85/168, 83/165, and 287/572, respectively. As $n \to \infty$, the winning probability $\to 1/2$.

Further analysis of Odds and Evens is available in Cowan (Ref., 2007).

Red and Black

From a deck of R red and B black cards, where $R \geq B \geq 0$, player **A** at each step guesses the color of the top card—which is then exposed and discarded (Ref. Knopfmacher und Prodinger), To maximize the number of correct guesses, he chooses that color corresponding to the majority of cards remaining in the deck.

Let $p(R,B;k)$ be the probability of guessing k cards correctly:

$$p(R,B;k) = \binom{R+B}{R}^{-1} \left[\binom{R+B}{k} - \binom{R+B}{k+1}\right] \quad R \leq k \leq R+B$$

And the expected number of correct guesses E(R, B) is expressed by

$$E(R,B) = R + \binom{R+B}{R}^{-1} \sum_{k=0}^{B-1} \binom{R+B}{k}$$

For a standard (symmetric) deck, R = B = 26, and

$$E(26,26) = 26 + \binom{52}{26}^{-1} \sum_{k=0}^{25} \binom{52}{k} = 30.04^{+}$$

A second player, **B**, using a random strategy, has a probability of guessing k cards correctly given by

$$q(\mathrm{R};k) = \frac{1}{4\mathrm{R}} \binom{2\mathrm{R}}{k} \quad 0 \le k \le 2\mathrm{R}$$

with, of course, an expectation equal to R. The probability P_A that **A** wins in competition against **B**'s random strategy is

$$P_A = \sum_{0 \le k < \mathrm{R}} q(\mathrm{R};k) + \sum_{\mathrm{R} \le k < j \le 2\mathrm{R}} q(\mathrm{R};k) p(\mathrm{R};j)$$

which can be expressed as

$$P_A = \frac{1}{2}\left[1 - \binom{2\mathrm{R}}{\mathrm{R}} 4\mathrm{R}^{-\mathrm{R}}\right] + \binom{4\mathrm{R}}{2\mathrm{R}-1}\left[2\binom{2\mathrm{R}}{\mathrm{R}} 4^{\mathrm{R}}\right]^{-1}$$

For R = 26, $P_A = 0.793$. And the game has a value to **A** of 0.586.

More Red and Black

Two cards are selected at random from a deck of R red and B black cards. The probability P_s that the two cards show the same color is

$$P_s = \frac{\mathrm{R}^2 + \mathrm{B}^2 - (\mathrm{R} + \mathrm{B})}{(\mathrm{R} + \mathrm{B})(\mathrm{R} + \mathrm{B} - 1)} \tag{7-15}$$

For B = R, Eq. 7-15 reduces to

$$\frac{\mathrm{B} - 1}{2\mathrm{B} - 1}$$

which, for a conventional deck (B = 26), equals 25/51 = 0.490.

Somewhat curiously, if B = R − 1, the same result ensues—owing to the fact that we are dealing, in this case, with a deck comprising an odd number of cards.

The Kruskal Count

Devised by Martin Kruskal (while at Rutgers University), this mathematical "trick" consists of a subject who selects a card according to a prescribed counting procedure and a "magician" who then *identifies the selected card without having observed the procedure*. Underlying the mathematics is a principle related to coupling models for Markov chains.

A deck of N cards is shuffled and spread out face up. The subject then chooses—privately—an integer j, $1 \leq j \leq 10$, and counts from one end of the deck to the jth card, referred to as the first key card. The rank of this card—with Aces equal to 1, each face card assigned the value 5, and each other card contributing its numerical value—defines the number to be counted to the second key card, whose rank, in turn, specifies the number to be counted to the third key card, and so forth. The final key card before exhausting the deck is designated the chosen one and is noted by the subject (and revealed by the magician).

It can be proved that, for *most* orderings of the cards, *all the numbers 1 through 10 will lead to the same chosen card*. (Setting face card values at 5 rather than 10 increases the number of key cards in an average chain and thereby increases the success rate by ~16%.)

The proof relies on the *ensemble success probability* averaged over the set of all possible permutations of the deck. Although this procedure is a non-Markovian stochastic process, it can be approximated by a geometric distribution.

Lagarius, Rains, and Vanderbei (Ref.) have determined that the optimal first key card is the first card of the deck (i.e., $j = 1$, which also increases slightly the expected number of key cards in the chain and affords a concomitant increase in the probability of success; ergo, the magician is best advised to select $j = 1$ as *his* first key card), and the expected key card size is 70/13. Thus we have a geometric distribution G_P defined by

$$g_k = (1 - p)^{k-1}, \quad 0 < p < 1$$

and let $G_N(p)$ define the deck distribution induced on a deck of N cards.

Further, let $P[t > N]$ denote the probability that the subject and the magician have no key card in common (i.e., the probability of failure, P_F). For $G_N(p)$ with initial geometric value distributions, G_p, we have

$$P[t > N] \equiv P_F = p^N (2 - p)^N$$

Choosing $j = 1$ reduces the failure probability by $1/(2 - p)$ to

$$\frac{p^N (2 - p)^N}{2 - p} = p^N (2 - p)^{N-1}$$

For $p = 1 - 1/(70/13) = 57/70$ and $N = 52$,

$$P_F = \left(\frac{57}{70}\right)^{52} \bullet \left(\frac{83}{70}\right)^{51} = 0.136$$

Thus the probability that the magician's final key card will match the subject's chosen card is $1 - 0.136 = 0.864$.

Using two decks, $N = 104$, and the probability of matching equals 0.978. As the deck size increases, the probability that both final key cards are identical approaches one.

In the Kruskal count, any two Markov chains, as defined here, tend to become coupled, after which they lead, of course, to the same chosen card.

Two-Card Hi-Lo

Two cards, marked Hi and Lo, are randomly dispensed, one to each of two players. **A** looks at his card and may either pass or bet. If he passes, he pays **B** an amount $a > 0$. If he bets, **B**, without looking, has the option of (1) passing, whereby he pays a units to **A**, or (2) calling, whereby **A** or **B** receives an amount $b > a$ from his opponent according to whether **A**'s card is Hi or Lo. It is readily seen that if **A**'s card is Hi, he bets with unity probability, whereas if he draws the Lo card, he should bet with probability P_1:

$$P_1 = \frac{b-a}{b+a}$$

B's strategy is to call with probability P_2:

$$P_2 = 1 - P_1 = \frac{2a}{b+a}$$

and the value γ of the game (to **A**) is

$$\gamma = aP_1 = \frac{a(b-a)}{b+a}$$

FORMAL CARD GAMES

Literally thousands of formalized card games have evolved or have been invented over the past few centuries. The majority of them have vanished into a well-deserved oblivion. Of the remainder, only a very few entail greater complexity than a pure or simple mixed strategy. Many games possess only a rudimentary structure, but have strategies involving selection among an astronomical number of alternatives. Bridge and Poker are generally nominated as the two card games demanding the highest level of skill—particularly the former, which is recognized by international competition.

A partial enumeration of the more popular card games might include Bézique, with its variations of Pinochle, Rubicon Bézique, and Chinese Bézique; Hearts, with its variations of Black Lady, Cancellation Hearts, Omnibus Hearts, Polignac, and Slobberhannes; the diverse forms of Poker; Conquian (a corruption of the Spanish *con quien*?), the direct ancestor of all Rummy games, which bred the Canasta family (including Samba, Bolivia, and Oklahoma) and the Gin Rummy family (including Seven-Card Rummy, Kings and Queens, Five-Hundred Rummy, Persian Rummy, Contract Rummy, Continental Rummy, Brelan—a 15th-century French rummy game—and Panguingue or "Pan"); All Fours, with its derivatives of Seven Up, Auction Pitch, and Cinch; Skat (which has inspired a World Congress of Skat players) with its variations of Schafkopf, Doppelkopf, Calabrasella, and Six Bid Solo; Klaberjass or Kalabriasz (among many spellings); Cribbage, invented by Sir John Suckling (1609–1642), who based it on the contemporarily popular game of Noddy; the Cassino family; the Whist family, with its offshoots of Black Maria, Mort, Cayenne, Boston, Preference, and Vint; Whist's successors: Auction Bridge, Contract Bridge, Brint, Calypso, and Towie (originated by J. Leonard Replogle), *inter alia*.

Further the many Solitaire games, including Klondike, Canfield, Spider, Forty Thieves, Golf, and the varieties of Patience; Bassette (a predecessor of Faro devised by Pietro Cellini at the end of the 16th century); Primero (Cardano's favorite game); Primero's descendants: Brag and Loo; Gilet, a French version of Primero; Michigan; the varieties of Euchre; Handicap; Muggins; Gleek; Ecarté; Belote (a relative of this French game, Jo-Jotte, was created by Ely Culbertson in 1937); Monte and its many variations; Svoyi Kozin; Durachki; Tresillo; Quadrille (or Cuartillo); Ombre; "66"; Tablanette; Hasenpfeffer; Spoil Five (the national game of Ireland); Manilla; Quinze; Angel–Beast; and innumerable others. The procedures for these games can be found in rule books for card games, such as those listed at the end of this chapter.

Recently, a number of card games have been devised with the intent of embedding more sophisticated analytic techniques than those required by games surviving the evolutionary filter. For example, the game of "Eleusis," invented by Robert Abbott, involves principles of pattern recognition similar to those discussed (Chapter 5) for coin-matching phenomena. Each of $n - 1$ players attempts to analyze a card-acceptance rule established by the nth player, thus permitting the successful analyzer to contribute the most cards from his hand in accordance with that rule. The appeal of Eleusis and its siblings has been confined, thus far, to an eclectic minority of game fanciers.

Poker

Of the considerable multitude of card games, likely the most widespread among Americans is that of Poker. Derived from the ancient Persian pursuit called *Ās Nâs* or *Dsands*, the game was played with 20 cards[14] during its popularity in 18th-century Paris. A variation known as *Poque* arrived in the United States with the French colonists of New Orleans, whence arose the word "Poker" from the American mispronunciation of *Poque*. An English version of the game is Brag, and another French offshoot is named Ambigu. The full 52-card deck was adapted around 1830, although the form of the game remained essentially that of straight Poker. Draw Poker was not introduced until the early 1860s during the Civil War.

Embodying psychological elements as well as being a game of imperfect information, Poker is not amenable to the deep-search techniques that have proved successful for Chess, Checkers, *et similia*. Further, developing an optimal strategy lies beyond the scope of current game theory owing to the large number

[14] The deck consisted of four suits, each of five ranks: Lion, King, Lady, Soldier, and Dancing Girl.

of (often ill-defined) tactical alternatives that present themselves. For these reasons plus, Poker may be described as something of a black art.

Parallels between Poker and political/economic/military concerns are frequently drawn. Chance, deceit, and cost-effectiveness form some of the basic elements of Poker. It is axiomatic that the best hand may not win the pot—strong players with weak hands can deploy bluffing techniques to finagle pots from weak players with strong hands. Bluffing, in particular, qualifies Poker as a useful model for nations with opposing aims and ideals.

Elementary combinatorial formulae enable us to calculate the probabilities of all subdivisions of the 2,598,960 five-card Poker hands (Eq. 7-10). Ten categories are defined, and within each category the winning hand is that composed of the highest rank or suit. In ascending order of value:

$$P(\text{high card only}) = \left[\binom{13}{5}4^5 - \binom{4}{1}\binom{13}{5} - (10\bullet 4^5 - 10\bullet 4)\right]D^{-1} = 0.5012$$

where $D = \binom{52}{5} = 2{,}598{,}960$

$$P(\text{one pair}) = \binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 D^{-1} = 0.4226$$

$$P(\text{two pairs}) = \binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} D^{-1} = 0.0475$$

$$P(\text{three of a kind}) = \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 D^{-1} = 0.0211$$

$$P(\text{straight})^{15} = \left[10\binom{4}{1}^5 - 10\binom{4}{1}\right]D^{-1} = 0.00392$$

$$P(\text{flush}) = \left[\binom{4}{1}\binom{13}{5} - 10\binom{4}{1}\right]D^{-1} = 0.00199$$

$$P(\text{full house}) = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} D^{-1} = 0.00144$$

$$P(\text{four of a kind}) = \binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1} D^{-1} = 2.40\times 10^{-4}$$

$$P(\text{straight-flushes})\ (\text{common}) = 9\binom{4}{1} D^{-1} = 1.385\times 10^{-5}$$

$$P(\textit{Royal flush}) = \binom{4}{1} D^{-1} = 1.539\times 10^{-6}$$

These results are summarized in Table 7-5. The relative values of the hands are, logically, ordered inversely to their probabilities.

[15] For straights and straight flushes, the Ace is permitted to rank either high or low.

Table 7-5 Poker-Hand Probabilities

Hand	Probability
High card	0.5012
One pair	0.4226
Two pairs	0.0475
Three of a kind	0.0211
Straight[a]	0.00392
Flush[a]	0.00199
Full house	0.00144
Four of a kind	2.40×10^{-4}
Straight-flush (common)	01.385×10^{-5}
Royal flush	1.539×10^{-6}
Four flush	0.043
Four straight (open)	0.035
Four straight (middle)	0.123
Four straight (end)	0.0087
Four straight-flush	1.23×10^{-4}

[a]Excluding straight-flushes.

Each of the figures in this table is strictly applicable only to independent hands. As an illustration of the change in probability associated with a second, dependent hand, consider the probability of obtaining a straight, given another hand composed of A, 2, 3, 4, 5. For this case, $P = 137/32{,}637 = 0.00420$ compared with 0.00392 for the probability of a straight in a singular hand—a $7^{+}\%$ increase.

In diverse forms of Poker, the player is permitted to replace one, two, or three cards in his hand with an equivalent number drawn from the remainder of the deck. The probabilities of achieving certain improvements are enumerated in Table 7-6.

It does not follow that optimal play can be derived directly from the probabilities of Tables 7-5 and 7-6. There exist strategies such as "bluff" or "inverse bluff" that, in some circumstances, might create an advantage to drawing two cards rather than three to a pair, as an example, although the latter drawing effects an improvement of 0.287 compared with 0.260.

The next sensible step in Poker analysis is that which provides the probable ranking of all possible hands, before and after the draw, as a function of the number of players contending (and of the number of cards each has drawn). Knowing the wager risked on each hand, it is then theoretically possible to calculate the expectation and devise an optimum strategy. However, because of the formidable computational labor attendant to the large number of hands, such a description is rendered inaccessible to precise mathematical treatment.

For games that permit increasing the initial bet (ante) during the course of playing out a hand—Poker being the prime example—*element of risk* offers a better measure than House Edge to evaluate comparative bets. Element of

Table 7-6 Draw Possibilities

Original Hand	Cards Drawn	Improved Hand	Probability
One pair	3	Two pairs	0.160
	3	Three of a kind	0.114
	3	Full house	0.0102
	3	Four of a kind	0.0028
	3	Any improvement	0.287
	2	Two pairs	0.172
	2	Three of a kind	0.078
	2	Full house	0.0083
	2	Four of a kind	0.0009
	2	Any improvement	0.260
Two pairs	1	Full house	0.085
Three of a kind	2	Full house	0.061
	2	Four of a kind	0.043
	2	Any improvement	0.104
	1	Full house	0.064
	1	Four of a kind	0.021
	1	Any improvement	0.085
Four straight (open)	1	Straight	0.170
Four straight (gap)	1	Straight	0.085
Four flush	1	Flush	0.191
Three flush	2	Flush	0.042
Two flush	3	Flush	0.0102
Four straight-flush (open)	1	Straight-flush	0.043
	1	Any straight or flush	0.319
Four straight-flush (gap)	1	Straight-flush	0.021
	1	Any straight or flush	0.256

risk is defined as the ratio of expected gain to the total sum wagered on a particular hand.

Countless varieties of Poker have evolved over the past two centuries, and more will undoubtedly be invented. Some, such as Stud Poker (five cards per hand, one hidden and four exposed successively, no draw) or Seven-Card Stud,[16] are tight games, almost purely strategic. Others, such as those with certain cards or groups of cards declared "wild" (allowed to assume any value desired), tend to increase the variance and decrease the strategic content of the basic game.

[16] Featured by "The Thanatopsis Literary and Inside Straight Club," Seven-Card Stud was extremely popular but little understood during the 1920s. It was also celebrated at Charles Steinmetz's famed Poker gathering, "The Society for the Equalization and Distribution of Wealth."

Table 7-7 lists the probabilities for the $\binom{52}{7} = 133{,}784{,}560$ possible seven-card hands.

Table 7-7 Seven-Card Hand Probabilities

Hand	Probability
High card	0.174
One pair	0.438
Two pair	0.235
Three of a kind	0.0483
Straight	0.0462
Flush	0.0303
Full house	0.0260
Four of a kind	0.00168
Straight-flush	0.000311

Texas Hold'em

Currently the most popular and most strategically complex Poker variant, Texas Hold'em,[17] holds a virtual monopoly in high-stakes poker tournaments and dominates as the main event in the World Series of Poker. Touted by its advocates as "the brightest star in the Poker universe," it has largely displaced its predecessors in casinos, Poker parlors, and informal home gatherings.

Terminology peculiar to Texas Hold'em is provided in numerous works on the subject (e.g., Ref. Yao).

Of the $\binom{52}{2} = 1326$ possible two-card combinations, there are three distinct groupings of starting hands: $\binom{13}{1}\binom{4}{2} = 78$ pairs, $\binom{13}{2}\binom{4}{1} = 312$ same-suited cards, and $\binom{13}{2}\binom{4}{1}\binom{3}{1} = 936$ unsuited cards. The value of the two *hole* cards—and the concomitant strategy—can vary markedly with the player's position (with respect to the "designated dealer") and over subsequent developments—such as the composition of the five community cards, the betting structure of each round, pot size at each round, the number of opponents and their actions, implied strategies, and inferred characteristics, *inter alia*. Ergo, the game is notably path-dependent, making it difficult to specify *a priori* the probability of a particular hand leading to a winning combination (the best poker hand drawn from the two *hole* cards and the five community cards) in the showdown.

As a *general rule*, however, proficient Hold'em players advocate folding with *most* unsuited cards and *some* same-suited cards as soon as the rules permit—i.e., in the *preflop* round except for the mandatory blind bets—and as

[17] The game was born a century ago in Robstown, Texas.

a function of position and the number of opponents remaining. The following two-card hands, under the qualifications stated, are recommended as playable [Refs. Shackleford (1); Galfond]:

In early position:

Pairs: 7 and higher	
Suited Ace and T-K	Unsuited Ace and T-K
K and T-Q	K and J, Q
Q and T-J	
T and 9	

There are 170 such hands (12.8% of the total number).

In later positions, additional hands may be engaged in play—particularly if the pot has not been raised.

Additional hands:

Pairs: 5, 6	
Suited Ace and 6-9	Unsuited K and T
K and 8, 9	Q and T, J
Q and 8, 9	J and T
J and 8, 9	
T and 8	
9 and 8	

There are 168 such hands (12.7% of the total number). Thus, a conservative player will enter the flop stage less than 25.5% of the time. In last (or next-to-last) position—again with no pot raise and with fewer opponents remaining—the player might hazard continuing in the game with a pair of 4s; Suited A,5, K,8, J,7, T,7; and Unsuited K,9, Q,9, J,9, T,9—for a further 5.3% participation (a positive return dependent on a favorable flop).

Regardless of these considerations, *any* deterministic strategy is, *ipso facto*, ill advised; an intelligent opponent could then readily devise an effective counter-strategy. Indeed, a corollary to this statement warrants that it may prove advantageous at times to play incorrectly to advertise a false pattern.

For each player's hand, the three-card flop entails $\binom{50}{3} = 19{,}600$ possible combinations. Independent of the hand, the probabilities for the flops are readily calculated as follows:

	Probability
No pair	0.828
One pair	0.169
Three of the same suit	0.0518
Three in sequence	0.0347
Three of a particular suit	0.0129
Three of a kind	0.00235

Computer simulations indicate that the best hand after the flop wins the pot with a probability greater than 0.7.

To estimate the probability of winning after each card of the flop has been dealt, a parameter widely applied is the number of "outs" (defined as the number of cards remaining in the deck that will improve the hand to best in the game, referred to as "the nut hand"). Since the opponents' cards are unknown, it may be impossible to compute the precise number of outs.[18] It should be noted that the unseen cards do not exhibit a uniform density, but are somewhat more likely to be skewed toward low-ranking values since those hands still contending are more likely to contain high-valued cards (this effect is minimal, however, except when all—or most—players remain). Further, clues to those cards may be revealed in the opponents' betting (or nonbetting) actions and in their "tells" (indiscreet body or facial movements) or behavior patterns. Also feigned clues may be signaled to mislead opponents. As with most Poker strategies, counting outs is an amalgam of art and science.

Texas Hold'em Computers

The history of Hold'em computer programs extends back more than three decades (Ref. Findler). The most notable—and successful—programs have issued from the Computer Poker Research Group (CPRG) at the University of Alberta, Canada:

LOKI—a probabilistic formula-based program.

POKI—incorporates opponent-modeling to adjust its hand evaluations. Based on enumeration, simulation techniques, and expert knowledge, POKI, designed for a ten-player game, performs well against weak opposition.

PsOPTI (for pseudo-optimal Poker program)—a range of programs that apply Nash equilibrium game-theoretic methods. However, computation of a true Nash equilibrium solution for Hold'em lies beyond the foreseeable future.

Polaris—a composite Artificial Intelligence program consisting of ten different bots, each with a distinct playing style, and a number of other fixed strategies. Development of Polaris began in 1991. Matched against human experts in July 2007 (the first Man–Machine Poker Championship), the program was narrowly defeated. For the rematch, July 2008, a component of learning was added, enabling Polaris to identify which common Hold'em strategy its opponents were using and to switch its own strategy in counteraction. Further, the model size was enlarged—thus more closely approximating a realistic game. Polaris gained the victory with three wins, two losses, and a tie.[19]

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[18] Online computer programs are available to estimate the value of any particular hand against a range of opponent hands.

[19] Polaris is credited with showing a perfect poker-face.

Despite the admirable success of Polaris, considerable further development is required before Poker programs master the game well enough to surpass all human players.

Although more ingenious and challenging card games may be proposed by gamblers or mathematicians, it is improbable that any game will so inspire game theorists and analysts or will achieve the public success and addiction as that gained by the pastime of Poker.

Poker Models

Simplified Poker models are frequently called upon to illustrate certain fundamental principles since they present the right degree of challenge—neither impossibly difficult as many real games nor trivial as most games of pure chance. To date, these models have shed little light on the actual game—although certain analogies to business, war, and politics have proved rewarding.

For example, Rhode Island Hold'em, a model for Texas Hold'em, reduces each hand and the flop to a single card. Its underlying structure differs markedly from the actual game. Poker models have been analyzed by Borel (Ref.), von Neumann, H.W. Kuhn, Bellman and Blackwell, and Goldman and Stone, among others.

Red Dog

A realistic model, Red Dog (or High Card Deal) is played with the standard deck. Each of N players receives a hand of five cards (or four as N exceeds 8). Following an ante of 1 unit, each player in turn may either fold or bet an amount ranging from 1 unit to the size of the pot that his hand contains a card of greater value *in the same suit* than one to be drawn from the remainder of the deck.

An analysis of this game, results in a strategy dictating that the player should wager the maximum stake (the entire pot) when his hand falls in the region

$$\sum_{i=1}^{4} x_i \geq 2 \tag{7.16}$$

where x_i is the relative ranking in the ith suit of the highest card in that suit (Ace is ranked high). A minimum bet should be risked when the hand is described by

$$2 \geq \sum_{i=1}^{4} x_i \geq 1$$

and the folding region is defined by

$$\sum_{i=1}^{4} x_i \leq 1$$

As an illustration, consider a hand consisting of the Ace of Spades ($x_1 = 13/13$), the Queen and Three of Hearts ($x_2 = 11/13$ for the Queen), the Four of Diamonds ($x_3 = 3/13$), and the Nine of Clubs $x_4 = 8/13$. (Ace is ranked high.) Thus

$$\sum_{i=1}^{4} x_i = \frac{13 + 11 + 3 + 8}{13} = \frac{35}{13}$$

which, being greater than 2, the value specified by Eq. 7-16, advises that we should bet the maximum that a card in this hand can exceed, in its suit, the rank of any card exposed from the remainder of the deck.

One-Card Poker (Ref. Beasley)

Player **A** draws a card randomly from a conventional deck (concealed from player **B**) and wagers either 1 unit or 5 units that it is a face card (J, Q, K). **B** then concedes or doubles the wager. If he concedes, he pays **A** the amount of the wager. If he doubles, the card is turned over, and the wager resolved (with either 2 or 10 units at stake from each of the players).

An initial impression might suggest that the game retains a negative expectation for **A**, since he draws a face card with probability 3/13. However, applying the proper mixed strategy proves the contrary. Specifically:

- **A** bets 5 units whenever he draws a face card; when drawing an Ace through 10, he bets 5 units with probability 1/10 and 1 unit otherwise—that is, he bluffs randomly with probability 1/10. (This strategy is simple to implement by wagering the higher sum whenever the drawn card is J, Q, K, or A.)
- **B** responds by always doubling a bet of 1 unit, with a tentative bet of 5 units, he doubles at random with probability 7/15 and concedes with probability 8/15.

The game therefore has a value to **A** of $(4/13)5 + (9/13)(-2) = 2/13$.

Note that if **A** bluffs with a probability less than 1/10, **B** can then gain a positive expectation by conceding all bets of 5; if **A** bluffs with probability greater than 1/10, **B** gains by doubling all bets of 5.

Similarly, if **B** doubles bets of 5 with probability less than 1/3, **A** gains by bluffing on every card; if **B** doubles with probability greater than 1/3, **A** gains by never bluffing.

More One-Card Poker

A and **B** are each dealt one card face down. **A** may keep his card or exchange it for **B**'s card. **B**'s options are to retain his card (be it the original one or the one given him in exchange) or trade it for the next card in the deck. The higher-ranking card wins the stake. Ties are "no bets" [Ref. Shackleford (1)].

A's optimal strategy is to switch with a 7 or lower. **B** will then maximize his expectation by trading (for the next card in the deck) with an 8 or lower. The game is favorable to **A** with an expectation of 0.082.

Steve Shaefer (Ref.) has proposed a variation wherein **B** wins the ties unless **A** and **B** have just exchanged cards of equal rank, in which case he must draw a new card (or flip a coin) to break the tie. **A**'s optimal pure strategy is, as before, to switch with 7 or less. Then, if **A** hasn't switched, **B** will again draw to 8 or lower—resulting in a positive expectation for **B** of $3/13^3$. However, if **A** hazards a mixed strategy, trading 8s with probability p_1, he compels **B** also to adopt a mixed strategy p_2 for drawing to a 9—so that **B**'s expectation becomes

$$13^3 y = p_2(-7 + 12p_1) + (1 - p_2)(3 - 2p_1)$$

For each player to maximize his expectation against his opponent's optimal strategy, the two partial derivatives, $\partial y/\partial p_1$ and $\partial y/\partial p_2$, must be zero:

$$0 = 13^3 \partial y/\partial p_1 = 14p_2 - 2$$

$$0 = 13^3 \partial y/\partial p_2 = 14p_1 - 10$$

Thus

$$p_1 = 5/7; \quad p_2 = 1/7$$

So **A** should trade an 8 with probability 5/7. Then, when **A** does not trade, **B** should draw to a 9 with probability 1/7. The game is slightly favorable to **B** with an expectation of $(11/7)/13^3 = 0.000715$.

Le Her

An 18th-century card game whose solution was first sought by N. Bernoulli and Montmort, Le Her is played between two persons, Dealer and Receiver, with the standard 52-card deck. Dealer dispenses one card to Receiver and one to himself. If Receiver is dissatisfied with his card, he may exchange it for Dealer's—unless Dealer has a King, which he is permitted to retain (*le droit de refuser*). If Dealer is then dissatisfied with his original card or that obtained in exchange from Receiver, he may replace it with a third card selected from the deck; however, if the new (third) card is a King, the replacement is canceled, and Dealer must retain his undesired card. The object of the game is to conclude with the higher-valued card (Ace equals 1). In the event of a tie, Dealer wins.

Three moves comprise the game: a chance move, which selects the first two cards; a personal move by Receiver; and a personal move by Dealer. Since cards of 13 ranks can be either held or exchanged, there are 2^{13} strategies available to both contenders. It is straightforward to demonstrate that two strategies dominate all others. Obviously, Receiver will retain cards with a value of 8 or greater, and he will exchange cards of value 6 or lower. His decision relates to rank 7. As a consequence, Dealer's decision relates to rank 8. The payoff matrix of Le Her is therefore a 2×2 matrix; its elements are displayed in Figure 7-1.

Dealer's Strategies	Receiver's Strategies: Hold 7 and over	Receiver's Strategies: Change 7 and under
Hold 8 and over	$\frac{16{,}182}{33{,}150}$	$\frac{16{,}122}{33{,}150}$
Change 8 and under	$\frac{16{,}146}{33{,}150}$	$\frac{16{,}182}{33{,}150}$

FIGURE 7-1 Le Her payoff matrix.

Applying the minimax principle, we can determine that optimal strategy for Dealer dictates mixing his two strategies in the ratio of 3:5. Receiver's optimal strategy is (5/8, 3/8). Probability of success for Dealer is 0.487 and therefore 0.513 for Receiver, an expectation (for Dealer) of −0.026. Although it may appear that the advantage lies with Dealer since he wins the ties, the reverse is true owing to the greater ability of Receiver to influence the game's outcome.

CASINO GAMES

Despite the plethora of card games, only a few have flourished in the major gambling emporia. Trente et quarante, Baccarat, Poker in specialized settings, and Blackjack comprise the most popular of the contemporary card games available to the gambling public. [Blackjack offers sufficient depth and interest to warrant a chapter (8) of its own.] Faro, once the dominant card game in 19th-century Western America (virtually every saloon in the Old West featured at least one Faro Bank), has now all but vanished from the scene.[20]

Trente et Quarante

Trente et Quarante is a game of pure chance found in virtually all French casinos (there are more than 160 state-licensed gambling establishments in the Gallic republic), although rarely offered in the United States. It is played with six decks of cards shuffled together, forming a total pack of 312 cards. Suits are of no significance; each card is assigned a value equal to its rank (Ace = 1), with Jacks, Queens, and Kings set equal to 10. Dealer representing the House lays out two rows of cards, the first row corresponding to Noir and the second to Rouge, until the cumulative total of the card values in each row reaches or exceeds 31. Thus, Noir and Rouge are each associated with a number that ranges between 31 and 40. Players may place wagers according to the diagram of Figure 7-2.

[20] Whatever its fate in the world's casinos, Faro will be remembered from Pushkin's classic short story, "The Queen of Spades."

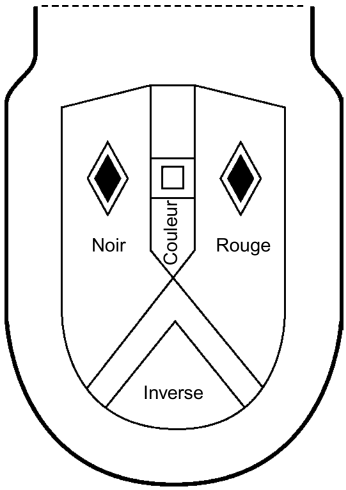

FIGURE 7-2 The Trente et Quarante layout.

A bet on Rouge wins if the cumulative total of the second row is lower (closer to 31) than that of the first row (Noir). If the two totals are equal, the bet is a tie unless the equality exists at 31, in which case the House pockets one-half of all stakes. Similarly, a wager on Noir wins if the first-row total is the lesser. The payoff for a winning bet is at even odds. Couleur is determined by the first card dealt. If Rouge (Noir) wins and the first card is red (black), Couleur wins; otherwise, Inverse wins. A player betting on Couleur or Inverse also ties if the two totals are equal at 32 through 40 and loses one-half the stake if the two totals are 31.

Probabilities for the totals 31 through 40 were first computed by Denis Poisson and subsequently by Joseph Bertrand with the assumption of sampling with replacement (equivalent to an infinite deck); the concomitant error is extremely small with a pack of six conventional decks. Table 7-8 summarizes

Table 7-8 Trente et Quarante Probabilities

Total	Probability
31	0.148
32	0.138
33	0.128
34	0.117
35	0.106
36	0.095
37	0.084
38	0.072
39	0.061
40	0.052

the pertinent probabilities—from which it is readily calculated that the game of Trente et Quarante offers the player an expected return of 0.989 (a House Take of 1.1%) since the probability of a tie at 31 equals $(0.148)^2 = 0.0219$. Other ties (at 32 through 40) occur with a frequency of 8.783%. A reduction to a House Take of 0.9% is available through an "insurance" bet, permitting the player to wager an additional 1% stake (quantized to the higher 1-euro level) that cancels the loss in the event of a tie at 31; the insurance bet is garnered by the House in all instances except ties at 32 through 40.

Baccarat

Introduced into France from Italy where it originated *ca*. 1490, Baccarat was devised by a gambler named Felix Falguiere, who based it on the old Etruscan ritualism of the "Nine Gods."[21] According to legend, 26 centuries ago in "The Temple of Gold Hair" in Etruscan Rome, the Nine Gods prayed standing on their toes to a golden-tressed virgin who cast a *novem dare* (nine-sided die) at their feet. If her throw was 8 or 9, she was crowned a priestess. If she threw a 6 or 7, she was disqualified from further religious office and her vestal status summarily transmuted. And if her cast was 5 or under, she swam resolutely out to sea. Baccarat was designed with similar partitions (albeit less dramatic payoffs) of the numbers, modulo 10.

Three accepted versions of the game have evolved: Baccarat Banque (wherein the casino as the Banker opposes the Player), Baccarat Chemin de Fer[22] (a Player assumes the role of Banker), and Punto Banco (the casino banks the game, and no strategic choice is involved). Chemin de Fer became the dominant version in the United States in 1920; more recently, it has been supplanted by Punto Banco, a game of lesser appeal. [Conceptualized in the 1480s with a Tarot deck, the modern construction ("punto" stems from the Spanish for "player") was installed at George Raft's Capri Casino in pre-Castro Havana and soon found its way to Las Vegas.]

Six decks (in a shoe) are conventionally employed in Punto Banco; thus, most probability calculations can be performed under the assumption of independence (sampling with replacement) with small error. Each card from Ace through Nine is assigned the face value of its rank; Tens, Jacks, Queens, and Kings contribute zero.

Banker and Player are initially dealt two cards each, which are turned face up. The two cards are summed mod 10. If either hand totals 8 or 9, no further cards are drawn. With a total of 0 to 5, Player must draw an additional card; with a total of 6 or 7, he must stand. Banker then draws or stands according to the schedule presented in Table 7-9.

[21] Macaulay, *Horatius at the Bridge*: "Lars Porsena of Clusium/By the Nine Gods he swore/…"
[22] James Bond's favorite casino game, it was featured in several films before being replaced by Texas Hold'em (in the 2006 adaptation of CASINO ROYALE).

Table 7-9 Punto Banco Rules for Drawing

Player Draws	Banker Draws with Total	Banker Stands with Total
No card	0–5	6 or 7
2 or 3	0–4	5–7
4 or 5	0–5	6 or 7
6 or 7	0–6	7
8	0–2	3–7
A, 9, T	0–3	4–7

The hand with the higher total wins. Equal totals for both hands (ties) are "no bets." Any participant at the Baccarat table may wager on either Banker or Player, with a payoff of 1 to 1 for the latter and 0.95 to 1 for the former (a 5% House commission). A bet on Player entails an expectation of -0.0124^{-}; and a bet on Banker an expectation of -0.0106 (after commission). A bet on ties (Égalité) pays 8 to 1, for an expectation of -0.144^{-}. (Some casinos pay 9 to 1 on a tie bet—increasing the expectation to -0.0484^{+}.)

In Chemin de Fer, Player's sole decision is whether to stand or draw on 5, his optimal strategy being (1/2, 1/2). Banker's fixed response is as follows:

Stand on 6 when Player stands on 5
Stand on 4 when Player draws a 1
Draw on 5 when Player draws a 4
Draw on 3 when Player draws a 9

Under this rule, the value of the game is -0.0137 for a bet on Player, and -0.0116 for a bet on Banker (after the 5% commission).

A 4-5-6 side bet is sometimes available—i.e., a wager that the final number of cards in the two hands totals 4, 5, or 6, as illustrated in Table 7-10.

Table 7-10 Baccarat Side Bet

Total Cards	Probability	Payoff	Expectation
4	0.379	3 to 2	−0.053
5	0.303	2 to 1	−0.089
6	0.318	2 to 1	−0.047

Another side bet pays 9 to 1 that Banker's (or Player's) two cards will total 9 (or 8). Knowledge of the remaining 9s and non-9s in the eight decks thus enables occasional positive-expectation wagers. Thorp has calculated that for a deck composition of n cards remaining, t of which are 9s, the probability $P_{n,t}$ of Banker receiving a two-card total of 9 is

$$P_{n,t} = \frac{2(n-t)(32n+351t-32)}{1149n(n-1)}$$

The side bet yields a positive expectation whenever $P_{n,t} > 0.1$—i.e., whenever there is at least one or more 9s among 12 remaining cards, two or more 9s among 26 remaining cards, or three or more 9s among 40 remaining cards.

However, for the principal bets at Baccarat, *no positive expectation can be derived even with an exact knowledge of the remaining cards*. The effect on the expectation from the removal of one card of a given rank (Ref. Thorp, 1984) can be shown to be about one-ninth of the equivalent effect in Blackjack (Ref. Griffin).

Reddere Baccarat

With a 10% rebate on losses, the Punto Banco player has a positive expectation, 0.0035, for coups of 13 or fewer. At 14 coups, the expectation drops to −0.0087; at 3 coups, it reaches a maximum of 0.0417.

CARD CONUNDRA

1. Consider a game with m winning and n losing cards, $m + n$ players, and a banker who pays out and takes in the wagers. Each player is dealt a single card, receiving a units from the banker if it is one of the m winners, and paying b units to the banker if it is one of the n losers. If the amount won is equal to the amount lost—i.e., $ma = nb$—and if m and n are relatively prime, show that the probability of the banker always having sufficient funds to pay the winners is $1/(m + n)$. Show that the probability of a bankrupt banker at any particular point in the game is also $1/(m + n)$.
2. Cards are dealt from a deck of R red and B black cards until the first red card appears. Show that the expected number of black cards preceding the first red card is B/(R + 1). For a conventional deck, this expectation is 26/27.

   Let B = R + 1 (one more black card). Then the expected number of black cards preceding the first red card is 1 for any deck of finite length.
3. *A Three-Person Money Game*. Three players simultaneously expose either a \$1, \$5, \$10, or \$20 bill. If all three expose the same denomination, the game is a draw. If the three bills are all different, that player with the middle value wins the money shown by the other two. If two of the bills are equal, that player with the different denomination wins from the other two. Determine the optimal (mixed) strategy. *Note*: For a simpler version, the problem can be formulated with \$1 and \$5 bills only, deleting the middle-man-wins possibility.
4. *Hi-Lo Forehead Poker*. Three players ante one unit apiece, and each receives a random integer $\leq N$. Without looking at the integer, it is exposed on the

forehead for the other two players to observe. Simultaneously, each player announces for either "Hi" or "Lo." The pot is split between the two winners of Hi and Lo. If all declare for Hi (Lo), a single winner is determined by the highest (lowest) integer. Compute the optimal strategy.

A modification of this game permits the players to draw lots to determine the order of announcing for Hi or Lo.

5. A card is drawn from a standard deck and then replaced. Denoting by n the smallest number of drawings that produces all 13 ranks, show that the expected value of n is

$$\mathrm{E}(n) = 13\sum_{m=1}^{13}\frac{1}{m} = 41.78$$

6. A casino built along the imaginary axis, *The House of Cards*, offers a Passe-dix game (p. 214) where the Card-Craps pack is used in lieu of the three dice. A payoff of 7 to 1 is tendered for a wager on the number 10. Is this casino a viable operation?

[*The House of Cards* also features a Roulette game played with a deck composed of A through 9 of the four suits plus two Jokers.]

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Chapter | eight

Blackjack

MEMORABILIA

The exact origin of Blackjack is rather murky; it was not invented by a single person, but apparently evolved from related card games in the 19th-century and gained currency during World War I. Often referred to as vingt-et-un, it exhibits structural similarities to Baccarat, "7½," Quinze, and also to Trente et Quarante. However, the connection with these games remains historically tenuous.

Transplanted from France to America, it was initially spurned. To heighten interest, casinos awarded special bonuses, one of which was a 10-to-1 payment if the player's hand comprised the Ace of Spades and either black Jack. While the bonus generally has been discontinued, the name remains—although "21" would be a more logical appellation.

Blackjack does not constitute a game in the usual game-theoretic sense, since one player's line of play is fixed *a priori*—the dealer cannot exercise independent judgment or intelligence (we might refer to it as a "one-person game"). Its special appeal lies in its property of being the only form of public gambling where the knowledgeable player is able to obtain a positive mathematical expectation. Unlike Craps and Roulette, Blackjack does possess a memory (the interdependence of the cards) and a conscience (inferior play will inevitably be penalized) and is not democratic (the mental agility and retentiveness of the player are significant factors). These facts have been realized in a qualitative sense by many professional gamblers for nearly a century. In the 1950s, numerical analyses of the Blackjack strategy were performed concurrently by engineers and mathematicians at various engineering firms across the United States. Quantitative results through computer programming were reported by the author in lectures at UCLA, Princeton, and other universities in 1955. The first published strategy (Ref. Baldwin et al.) appeared in 1956. Allan Wilson of San Diego State University programmed a simulation of several hundred thousand hands of Blackjack in 1957, obtaining the most accurate results to that date.

Then, in 1962, Edward O. Thorp (Ref. 1962) published *Beat the Dealer*, a work instantly extolled as the bible of Blackjack votaries. Here, for the first time, was a comprehensive mathematical treatment of the full range of Blackjack strategies and their underlying probabilities plus the first quantitative synthesis of card-counting systems.

Blackjack has since become the favored pastime in the Nevada gambling emporia and in the resort casinos of the Caribbean area. It is featured as well in casinos throughout South America, Australia, and the Far East. In Europe, the game sometimes rates a less-than-first-class status (although it is featured at the Casino Municipale di Venezia under a sign announcing "Gioco del Blak Jak o 21").

RULES AND FORMAT

There are variations in rules as played in different cities, in different casinos of the same city, and even in a single casino from time to time, at the whim of a gaming manager. In general, however, these various rules are in accord on the principal points and differ only in minutiae. Here we adopt a set of rules that, to the greatest extent possible, reflects the degree of commonality that exists; for the fine-structured regulations, we adhere to those prevailing along the "strip" of Las Vegas.

Specifically, we postulate that the dealer stands on soft 17 (S17), the player is permitted to double down after splitting pairs (DAS), may split pairs three times (SPL3) though Aces but once (SPL1), and is offered the options of Early Surrender (ES) and Late Surrender (LS).[1]

An illustration of the Blackjack layout is shown in Figure 8-1. From one to seven players compete against the dealer representing the House. A single,

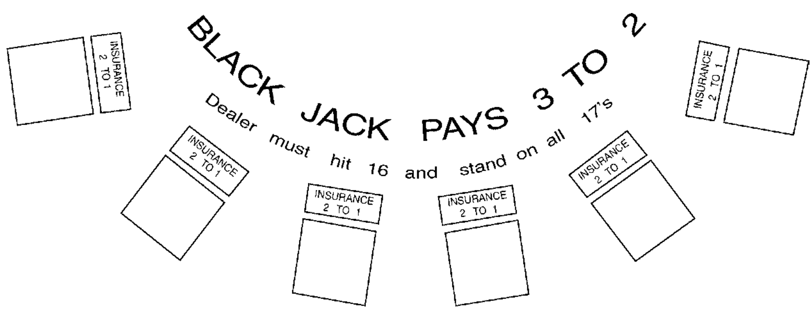

FIGURE 8-1 The Blackjack layout.

[1] This concept was first displayed publicly in 1958 by the Continental casino in Manila and is currently in force in other Far East gambling houses such as the Casino de Macao as well as in Las Vegas.

conventional deck of cards was used exclusively until the advent of card-counting drove the casinos to develop countermeasures. Now, 4-, 6-, and 8-deck packs predominate (and are dealt from a shoe). A \$5 minimum bet and a \$3000 maximum are customary but not universal.

ASSUMPTIONS

In the mathematics leading to optimal Blackjack strategies, our premise—as the only viable choice—is that of a strictly objective utility function. The mathematical expectation of an event that has probability of success p and probability of failure q is defined as $p - q$. If, in addition, the event has probability r of remaining unresolved (a tie), only the objective utility function permits the retention of $p - q$ as defining the mathematical expectation. Specifically, in the game of Blackjack it is frequently necessary to select between one event, characterized by (p', q'), and another, characterized by (p, q, r); for example, standing with a total of 16 or less (which excludes the possibility of a tie) must be compared with the action of drawing a card; the latter strategy includes the possibility of a tie. According to the objective utility function, time is irrelevant, and significant probabilities are calculated on a per-play basis; thus ties are resolved by sharing the probability r equally between probabilities p and q. That is,

$$p' = p' + r/2 \quad \text{and} \quad q' = q + r/2$$

so that the game (p, q, r) still exhibits a mathematical expectation of $p - q$.

A different utility function might emphasize time (i.e., the number of plays to achieve a given goal) but consider the variance to be irrelevant (perhaps because of an immense initial fortune). In such an instance, a tie would be resolved by renormalizing the probabilities p and q. Thus,

$$p' = \frac{p}{1-r} \quad \text{and} \quad q' = \frac{q}{1-r}$$

Other subjective utility functions could produce yet other values of p' and q'.

OPTIMAL STRATEGIES

Basic Strategy

With the preceding considerations, expectations by standing or hitting (drawing a card) in each of the 550 initial situations are calculated (the player's hand comprises one of 55 two-card combinations, the dealer's up-card one of ten). The greater expectation in each case dictates optimal strategy—here adapted for a multi-deck game. The same procedure is applied for the options of doubling down, splitting, Insurance, and Surrender. These data are then processed to produce the optimal basic strategy displayed in Table 8-1 (Ref. Schlesinger, p. 487). (Where it is significant, the composition-dependent strategy [C-D] is given. Otherwise, the total-dependent strategy [T-D] based on a generic total suffices.)

Table 8-1 Basic Strategy—Multi-Deck, Dealer Stands on Soft 17, Doubling after Split Permitted

Player's Hand	Dealer's Up-Card									
	2	3	4	5	6	7	8	9	T	A
5–7	x	x	x	x	x	x	x	x	x	x[a]
8	x	x	x	x	x	x	x	x	x	x
9	x	D	D	D	D	x	x	x	x	x
10	D	D	D	D	D	D	D	D	x	x
11	D	D	D	D	D	D	D	D	D	x
12	x	x				x	x	x	x	x[a]
13						x	x	x	x	x[a]
14						x	x	x	x[a]	x[a]
15						x	x	x	x[a,b,c]	x[a]
16						x	x	x[a,b]	x[a,b]	x[a,b]
17										[a]
18–21										x[a]
A–2	x	x	x	D	D	x	x	x	x	x
A–3	x	x	x	D	D	x	x	x	x	x
A–4	x	x	D	D	D	x	x	x	x	x
A–5	x	x	D	D	D	x	x	x	x	x
A–6	x	D	D	D	D	x	x	x	x	x
A–7		D	D	D	D			x	x	x
A–8,9,T										
A–A	S	S	S	S	S	S	S	S	S	S
2–2	S	S	S	S	S	S	x	x	x	x
3–3	S	S	S	S	S	S	x	x	x	x[a]
4–4	x	x	x	S	S	x	x	x	x	x
5–5	D	D	D	D	D	D	D	D	x	x
6–6	S	S	S	S	S	S	x	x	x	x[a]
7–7	S	S	S	S	S	S	x	x	x[a]	x[a]
8–8	S	S	S	S	S	S	S	S	S[a]	S[a]
9–9	S	S	S	S	S		S	S		
T–T										

x=hit
D=double down
S=split

[a]Early Surrender (when available)
[b]Late Surrender (when available)
[c]Late Surrender 8-7 only with 7 or more decks
Stand where no other action is indicated

In the column listing the player's hands, those entries not referring to two specific cards are generic totals—which may represent totals of two or more cards or totals resulting from a pair-split and subsequent draw.

Consolidating the properly weighted individual expectations arising from this strategy yields an overall mathematical expectation of -0.00419. (The virtually extinct single-deck game is susceptible to an expectation of $+0.0017$ and thereby constitutes the only positive-expectation game offered by the public gaming casinos.)

In addition to the exact strategy for the player's initial hand, we might compute optimal strategy for some of the remaining 1953 hands[2] containing three or more cards, particularly for those hands where the decision is composition-dependent (see Appendix Table D, for example).

The multi-deck game inherently entails lowered mathematical expectations, decreasing with increased number of decks. The advantage of doubling down is reduced with multiple decks, as are the benefits of standing on totals of 12 to 16 against the dealer's 2 to 6 (owing to the lesser effects of removing lower-valued cards). For a specific example, consider the player with two non-Ts versus a non-T dealer up-card. In this instance, the single-deck probability of drawing a T, or of the dealer having a T as his hole card, is $16/49 = 0.3265^{+}$; with eight decks, that probability drops to $128/413 = 0.3099^{+}$; in the infinite-deck game, it drops again to $16/52 = 0.3077^{-}$. Further, with two or more decks, the advantage of applying a composition-dependent strategy over a total-dependent strategy becomes minuscule. Changes in basic strategy for one and two decks are indicated in Table 8-2.

Table 8-2 Changes from Multi-Deck Strategy

Single deck	Double down on 6-2 vs. 5 and 6; 11 vs. A; 2-2 vs. 4 Split 3-3 and 7-7 vs. 8; 4-4 vs. 4; 6-6 vs. 7; 9-9 vs. A Late Surrender 7-7 vs. T and A
Two decks	Double down on 9 vs. 2; 11 vs. A Hit 16 vs. 9 Stand on A-7 vs. A Split 6-6 vs. 7; 7-7 vs. 8

Surrender

Surrender is relinquishing the hand at the cost of one-half the amount wagered. Basic strategy dictates Early Surrender (prior to the dealer checking his hole card) for 16 vs. 9; 14, 15, 16, 7-7, 8-8 vs. T; and 5, 6, 7, 12-17, 3-3, 6-6,

[2] For hands comprising 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 cards, there are, respectively, 55, 179, 336, 434, 426, 319, 175, 68, 15, and 1 possible combinations (for a total of 2008). Of the 179 three-card combinations, 66 constitute hard totals from 12 to 16. Each dealer's hand (single-deck game), composed of from two to nine cards, is one of 48,532 possible combinations.

7-7, 8-8 vs. A. This strategy adds an expectation of 0.007 to the player's game value—for an overall expectation of +0.0087.

Late Surrender (LS)—subsequent to the dealer checking his hole card—is less advantageous to the player, entailing an added expectation of 0.00023. Optimal strategy: surrender 16 vs. 9; T-5, 9-6, 9-7, and T-6 vs. T; 9-7 and T-6 vs. A; and 8-7 vs. T with seven or more decks.

Some casinos, mostly in the Far East, permit Surrender after doubling down. Optimal strategy with this option: surrender hands totaling 12 through 16 vs. 8, 9, T, A; further, Surrender 17 vs. A. The added expectation accruing from this option is 0.005.

Table 8-3 compares the overall expectations from variously sized decks under the rules stated above (doubling allowed on two cards and after splitting; dealer stands on soft 17). (Advantages accruing from the Surrender rule and from bonuses are computed separately.)

Table 8-3 Overall Expectations versus Deck Size

1 deck C-D	1 deck T-D	2 decks	4 decks	6 decks	8 decks	∞ deck
0.0017	0.0014	−0.0019	−0.0035	−0.0041	−0.0043	−0.0051

Bonuses

Premiums awarded for "special" hands vary erratically among casinos. The more common ones are listed in Table 8-4 along with their expectations to the player. "Charlies" are 5-, 6-, or 7-card combinations that total 21 or less.

Table 8-4 Various Bonuses

Bonus	Expected Value
Five-card Charlie, pays 1 to 1	0.0146
Six card Charlie, pays 1 to 1	0.0016
Seven-card Charlie, pays 1 to 1	0.0001
Five-card 21, pays 2 to 1[a]	0.0020
Six-card 21, pays 2 to 1	0.0010
Three 7s, pays 2 to 1	0.0003
Three 7s, pays 3 to 1	0.0005
6-7-8, pays 2 to 1	0.0003
6-7-8, same suit, pays 2 to 1[b]	0.0001

[a]Alter basic strategy to hit four-card totals of 14 vs. 2; 13 vs. 2, 3, 4, and 6; 12 vs. 4, 5, and 6; soft 19 vs. T and A; and soft 18 vs. 2, 3, and 4.

[b]Alter basic strategy to hit 6-7 vs. 2.

Insurance and Side Bets

"Insurance" (wagering at 2 to 1 that the dealer's hole-card with an A showing is a T)[3] and side bets (e.g., Over/Under, Royal Match) constitute negative-expectation bets. The latter permit a separate stake independent of ongoing play. Only through the application of card counting can these options prove of value. (Insurance changes to a positive-expectation bet whenever the number of Ts remaining in the deck comprise more than one-third of all remaining cards.)

CARD COUNTING

Because of fluctuations in the composition of the deck as it is depleted over successive (and *dependent*) trials, it is intuitively apparent that altering the magnitude of the wager or the strategic decisions or both in accordance with the fluctuations should prove advantageous to the player.

At the onset of each round, there is a probability distribution F_c that describes the player's expectation, given knowledge of c cards. A theorem, proven by Thorp and Walden (Ref.), states that as c increases, F_c spreads out; therefore, increasing the bet size for positive expectations and betting the minimum for negative expectations improve the player's overall expectation, $E(F_c)$—a phenomenon that Thorp and Walden term the "spectrum of opportunity." It should be noted that the F_c are dependent—when a deck evolves into a favorable (or unfavorable) composition, it tends to retain that structure (at least for the short run).

Card-counting systems that track fluctuations in deck composition can yield player expectations in excess of $+0.02$.

The foundation of card counting rests on the quantitative *effect on the player's expectation of removing individual cards*—referred to as the EOR (Ref. Griffin)—from the deck as the hands are dealt out. Altering the composition of the deck obviously alters the probabilities that underlie the six options (hit, stand, double down, split pairs, Insurance, Surrender).

Table 8-5 details the EOR (Ref. Schlesinger, p. 522) for each rank (under our postulated ground rules but without Surrender). It is immediately apparent

Table 8-5 Betting EOR values (in %)

Rank	2	3	4	5	6	7	8	9	T	A	Sum of Squares
EOR	0.381	0.434	0.568	0.727	0.412	0.282	−0.003	−0.173	−0.512	−0.579	2.842

[3] Appropriation of the word "insurance" to describe this wager must be classed as a stroke of genius. The word implies that high-total hands, such as those comprised of two Ts, should be insured to protect their value, whereas low-total hands are not worthy of being insured. Such "reasoning," contrary to an objective appraisal, leads the susceptible player into increasing the "House Take."

that the presence of high-valued cards remaining in the deck favors the player, while low-valued cards favor the dealer.

Point-count systems assign to each rank a value that approximates the relative EOR for that rank. This value is generally expressed by an integer to accommodate the limitations of human memory. As each card is dealt, the appropriate integer is added in to maintain a *running count* that reflects the deck's composition. To apply the running count, it must be normalized to a *true count* (TC) by dividing by the number of decks (or fraction of a deck) remaining. (TC need only be estimated roughly since the criterion is whether or not it exceeds a particular integer.) Ergo, TC is approximately proportional to the player's expectation (albeit for both exceptionally low and high TCs, the change in expectation declines from linearity).

The desiderata for an effective point-count system are (1) simplicity, preferably characterized by a single parameter, so that it can be quickly computed and deployed; (2) accuracy, so that its measure reflects the current state of the deck; and (3) equilibrium, so that its values for the full pack sum to zero. With equilibrium, the *direction* of the pack's deviation from its initial state is given by the algebraic sign of the running count regardless of the number of cards remaining.

Two measures are applied to assess the worth of a point-count system: *betting correlation* and *playing efficiency*.

Betting correlation (BC) is defined as the correlation between the point values of the counting system and the best linear estimates of deck expectation—equivalent to the sum of the product of the respective value assigned to each rank and the concomitant payoff for that rank divided by the square root of the sum of squares of the point-count values (SSPC) and the sum of squares of the EOR values (SSEOR). That is,

$$\mathrm{BC} = \frac{\rho}{\sqrt{\mathrm{SSPC} \cdot \mathrm{SSEOR}}} \tag{8-1}$$

where ρ is the *inner product* of the point-count values and the EOR values.

The strategic (or playing) efficiency of a card-counting system is defined as the quotient of the expected gain it engenders and the (hypothetical) expected gain achievable with complete knowledge of the remaining cards; Griffin (Ref.) has shown that the total expected game E(n) resulting from appropriate departures from basic strategy with n cards remaining in the deck is approximated by

$$\mathrm{E}(n) \sim \frac{\sigma_n}{\sqrt{2\pi}} e^{-\mu^2/2\sigma_n^2} - \frac{\mu}{\sqrt{2\pi}} \int_{\mu/\sigma_n}^{\infty} e^{-x^2/2} dx \tag{8-2}$$

where μ is the full-deck expectation from basic strategy and σ_n is the variance for the mean of an n-card subset of the deck:

$$\sigma_n^2 = \frac{\sigma^2}{n}\left(\frac{51-n}{50}\right),$$

σ being the variance of the single-card payoffs. It should be noted from Eq. 8-2 that E(n) increases as n decreases and also as σ increases and as μ decreases. To derive the similar expression for a point-count system, it is assumed that the average payoff for the system and the average payoff for the n-card subset of the deck are bivariate normally distributed with correlation coefficient γ. Then the expected gain $F(\gamma)$ for the point-count system is approximated by

$$\mathrm{F}(\gamma) = \frac{\gamma\sigma_c}{\sqrt{2\pi}} e^{-\mu_c^2/2r^2\sigma_c^2} - \frac{\mu_c}{\sqrt{2\pi}} \int_{\mu_c/\gamma\sigma_c}^{\infty} e^{-x^2/2} dx$$

where μ_c and σ_c are the expectation and variance resulting from the point-count application. [$F(\gamma)$ is identical to E(n) as expressed in Eq. 8-2 with $\gamma\sigma_c$ replacing σ_n.] Strategic efficiency is thus given by $F(\gamma)$/E(n) and increases with decreasing n (as the deck is depleted). For values of $\gamma < 1$ and $\mu_c \neq 0$, efficiency is less than γ; for $\mu_c = 0$, efficiency $= \gamma$.

Some of the more practical and efficient point-count systems applicable to the multi-deck game are listed in Table 8-6. To describe the relative strength of these systems, a particular measure, SCORE (Standard Comparison of Risk and Expectation),[4] can claim the greatest expedience. SCORE provides a rating number—the expected hourly return in dollars—that embodies an initial bankroll ($10,000), a specific risk of ruin [13.5%, the value derived from the Kelly criterion without bet sizing (pp. 61–62)], a representative bet spread (1 to 12), the number of decks in play (6), a typical deck penetration (through 5 of the 6 decks), and the number of rounds dealt per hour (100); for completeness, it also stipulates four players at the Blackjack table. Although any of these seven parameters can be changed, with a concomitant change in the SCORE rating (a modified SCORE), the ordinality of the point-count-system rankings remains unchanged.

Table 8-6 presents the SCORE with these parameters for eight-count-systems under our postulated rules (plus Late Surrender). The highest SCORE is not, *per se*, the sole gauge of the "best" system. Ease of application, for example, would, for most players, favor a first-level system over higher-level counts.

The Hi-Lo system provides an optimal combination of simplicity and effectiveness and is employed more extensively than other systems. Devised

[4] Devised by Don Schlesinger (Ref.).

Table 8-6 Point-Count Systems

System	Point Values										Strategic Efficiency	Betting Correlation	SCORE
	A	2	3	4	5	6	7	8	9	10			
Hi-Lo	−1	1	1	1	1	1	0	0	0	−1	0.511	0.966	49.91
K-O	−1	1	1	1	1	1	1	0	0	−1	0.553	0.976	47.86
Einstein Hi-Opt	0	0	1	1	1	1	0	0	0	−1	0.609	0.879	43.97
Uston +/−	−1	0	1	1	1	1	1	0	0	−1	0.551	0.948	46.45
Second-level Zen	−1	1	1	2	2	2	1	0	0	−2	0.627	0.962	47.71
Revere Point Count	−2	1	2	2	2	2	1	0	0	−2	0.554	0.981	52.11
Zen (Unbalanced)	−1	1	2	2	2	2	1	0	0	−2	0.616	0.964	49.83
Third-level Wong Halves	−1	½	1	1	1½	1	½	0	−½	−1	0.565	0.992	53.80

in 1963 by Harvey Dubner, it was also developed by Julian Braun as the Braun plus-minus system. Eq. 8-1 computes its betting coefficient as

$$\mathrm{BC} = \frac{5.150}{\sqrt{10 \cdot 2.842}}$$

since $\rho = 1(0.381 + 0.434 + 0.568 + 0.727 + 0.412) - 4(-0.512) - 1(-0.579) = 5.150$. Playing efficiency for the Hi-Lo system is 0.511.

For the amount wagered to be proportional to the player's (favorable) expectation, the prescribed bet β as a function of the TC is specified by

$$\beta = \tfrac{1}{2}[1 + \mathrm{TC} + \mathrm{TC}^2/4], \qquad \mathrm{TC} \geq 1 \tag{8-3}$$

where $[X]$ signifies the largest half-integer contained in X. The quadratic term arises from the contribution of the Insurance wager. Table 8-7 lists the Hi-Lo bet spread in accordance with Eq. 8-3 for the multi-deck game.

Table 8-7 Bet Spread for the Hi-Lo Point Count

TC	1	2	3	4	5	6
β	1	2	3	4½	6	8
Prob. of Occurrence	0.098	0.059	0.040	0.028	0.020	0.010

Unbalanced systems render conversion to TCs unnecessary; their Ace values consequently cannot be 0, but instead are a function of the number of decks in the game (e.g., K-O, Zen). Systems beyond the second level intrude upon the desideratum of simplicity and demand exceptional feats of mental agility.

Appendix Tables C(A) through C(J), originally computed by Griffin (Ref.) and refined by Schlesinger (Ref., pp. 506–515) and "Zenfighter," detail the full-deck(s) EORs for Player's hand versus Dealer's up-card. Each entry, in percent, represents the difference in probabilistic gain between drawing a card and standing—or between doubling and either hitting or standing, or between splitting and either hitting or standing. The quantities m_2, m_6, and m_8 represent the full-pack favorability factor for 2, 6, and 8 decks, respectively.

Illustratively, consider the example of T-6 versus T. Appendix Table C(I) indicates that the full-pack favorability for two decks (m_2) is -0.19. The three EORs—for two Ts and a 6—are 2(1.12) and 1.64, which sum to 3.88. For n decks with k cards removed, this figure must be multiplied by $51/(52n - k)$, which for $n = 2$ and $k = 3$, equals 51/101. Thus (51/101)(3.88) = 1.96 (the adjusted EOR). Adding this number to the full-pack favorability results in $+1.77$. Therefore, correct strategy dictates drawing a card—as it does for this combination with six and eight decks.

If, as a counter example, Player's 16 were composed of T-5-A (again vs. Dealer's T), the four EORs are 2(1.12), -2.57, and -0.50, which total -0.83. For two decks with four cards removed, we adjust this figure by 51/100, whence $(51/100)(-0.83) = -0.42$, and optimal strategy calls for standing. A more detailed list of Player's 16 vs. T is computed in Appendix Table D.

The Insurance wager can also be resolved by EOR calculations—for each rank A through 9, the EOR is 1.81, while Ts engender an Insurance EOR of -4.072. The full-pack favorability is, of course, $-1/13$ (for any number of decks). With seven non-Ts and one T dealt out of two decks, the eight EORs sum to $7(1.81) - 4.072 = 8.60$. Multiplying by 51/96 equals 5.30, and optimal strategy calls for the player to accept the Insurance wager. (In this instance, less mental effort is required to divide the number of non-Ts by the number of Ts exposed.)

In addition to determining the optimal wager, point-count systems indicate advantageous changes in strategy—i.e., in hitting/standing, pair-splitting/doubling/Insurance/Surrendering decisions. For low (or negative) True Counts, with bets of 1 or 2 units, deviations from basic strategy offer the only significant means to further potential gain. For high TCs, with bets in the range of 12 to 16 units, bet variations are considerably more significant than strategy variations.

Table 8-8 (Ref. Wong) displays the Hi-Lo TC strategy numbers. Player should stand (or split or double down) when the TC is equal to or greater than the number given; he should hit (and not split or double down) for a TC less than the number given.

The preponderant majority of the card combinations relating to these numbers (116 in all) occurs infrequently and with a modest concomitant gain.

Table 8-8 Hi-Lo True Count Strategy Numbers (Multi-deck)

Player's Hand	Dealer's Up-Card								
	2	3	4	5	6	7	8		T	A
10	−8	−9	−10			−6	−4		4	4
11						−9	−6		−4	1
12	3	2	0	−1	0					
13	0	−1	−3	−4	−4					
14	−3	−4	−6	−7	−7	10	10	8	4	10
15	−5	−6	−7	−9	−9	10				
16	−8	−10				9	7	5	0	8
17										−6
2-2	−3	−5	−7	−9			5			
3-3	0	−4	−7	−9			4			
4-4										
5-5										
6-6	−1	−4	−6	−8	−10					
7-7	−9									
8-8									8[a]	
9-9	−2	−3	−5	−6	−6	3	−8	−9		3
T-T		8	6	5	4					
A-A						−9	−8	−7	−8	−3
A-2		7	3	0	−1					
A-3		7	1	−1	−4					
A-4		7	0	−4	−9					
A-5		4	−2	−6						
A-6	1	−3	−7	−10						
A-7	0	−2	−6	−8	−10					
A-8	8	5	3	1	1					
A-9	10	8	6	5	4					

[a]For this combination, Player splits only if the True Count is *less than* 8.

Courtesy of Stanford Wong.

A more expeditious procedure, suggested by Don Schlesinger (Chapter 5 in Ref.), focuses on 18 card-combinations (the "Illustrious 18") that can claim some 80 to 85% of the gain arising from strategy changes. Further, 90% of this gain is concentrated in 12 card combinations—the "top 12" that are listed in Table 8-9. It should be noted that the Insurance wager alone accounts for 30% of the gain from all 116 strategy changes (and for some 28% of the gain from the Illustrious 18).

A similar economy can be instituted with regard to the Surrender option. Table 8-10 (Ref. Wong) displays the Hi-Lo TC strategy numbers for Early Surrender (S17, DAS).

Limiting the options to four hands forfeits less than 10% of the possible gains. These four are listed in Table 8-11. An additional gain of 0.00121 is thereby added to the basic-strategy surrender value of 0.00067—for a total gain of 0.00188.

Table 8-9 The Top 12 Hi-Lo Strategy Changes

Hand	Prob. of Occurrence
Insurance	0.077
16 vs. T	0.035
15 vs. T	0.035
T-T vs. 5	0.0073
T-T vs. 6	0.0073
T vs. T	0.0115
T vs. 3	0.0075
12 vs. 2	0.0075
11 vs. A	0.0025
9 vs. 2	0.0028
T vs. A	0.0020
9 vs. 7	0.0028

Table 8-10 Hi-Lo TC Numbers for Early Surrender

Player's Hand	Dealer's Up-Card			
	8	9	T	A
4				2
5				0
6				−1
7				−2
8				9
12			8	−4
13			3	−7
14		6	0	−9
15	7	2	−2	x
T-6, 9-7	4	0	−5	x
8-8		7	−2	x
17		9	5	x

x = Always surrender if TC equals or exceeds the tabulated number.

Courtesy of Stanford Wong.

Table 8-11 Hi-Lo Surrender TCs

Hand	Prob. of Occurrence	TC
14 vs. 10	0.0202	3
15 vs. 10	0.0202	0
15 vs. 9	0.0055	2
15 vs. A	0.0038	1

Courtesy of Stanford Wong.

Late Surrender is more commonly available and entails fewer decision points—as shown in Table 8-12.

Table 8-12 Hi-Lo TC Numbers for Late Surrender

Player's Hand	Dealer's Up-Card			
	8	9	T	A
13			8	
14		6	3	6
15	7	2	0	2
T-6, 9-7	4	0	−2	−1
8-8		7	0	

Courtesy of Stanford Wong.

SHUFFLE TRACKING (Ref. Hall)

Virtually every casino skimps on the amount of shuffling necessary to produce a randomly ordered pack of cards (Ref. Hannum) since thorough shuffling delays the game more than the casino is willing to tolerate and since only a minute fraction of Blackjack players are sufficiently astute to exploit a nonrandom pack.

Generally, with multi-deck packs, cards are divided into four piles with shuffling between groups of two. Thus a subset of cards has zero probability of ending up in a certain portion of the shuffled pile. At the conclusion of a round (when approximately 1/4 of the pack remains in the shoe) and before the shuffle ensues, the clump of unplayed cards is typically broken into two and "plugged" into two different locations in the discard pile. This clump obviously reflects the negative of the final running count.

Most shuffles are effected between the two quarters that comprise the top half of the played cards and between the two quarters that comprise the bottom half. Whenever two clusters of cards are shuffled together, the estimated count of the combined cluster is, of course, the sum of the counts in each cluster.

It has been calculated that shuffle-tracking yields an overall accuracy in all regions of the multi-deck pack greater than the accuracy of TCs halfway through the pack. On average, depending on the nature of the shuffle, shuffle-tracking offers an increased expectation on the order of that obtained from first-level card-counting systems. In those casinos that repeatedly follow an identical shuffle procedure, it is of value to develop a "shuffle profile" and approach the Blackjack tables "prearmed."

Ace Prediction

Tracking and predicting the location of one or more Aces is both less complex and more expedient than a general shuffle-tracking system. A procedure first outlined by Thorp and Walden (Ref.) and practicalized by David McDowell (Ref.) consists of noting the "key card"—defined as that card immediately preceding the "target Ace" after the players' hands have been "scooped up" by the dealer (after paying the winning bets) and returned to the discard pile. This process is prescribed so as to preserve the original order of the cards, thus enabling reconstruction of the hands from the discard tray should a dispute later arise.

Empirical data indicate that, in a one-riffle shuffle, two initially adjoining cards will most likely—probability 0.69—be separated by one intervening card in the next round dealt; with probability 0.18, the two adjoining cards will remain adjoining. In a two-riffle shuffle, the most likely event is for three cards to intervene between two neighboring cards—probability 0.38. One- and two-card separations occur with probabilities 0.13 and 0.18, respectively.

When the first card of a player's hand is an Ace, his expected gain from that hand, averaged over all Dealer up-cards, is 0.51. However, this figure is achievable only with a 100%-accurate prediction. Given a two-riffle shuffle, a more realistic prediction value is 38%—resulting in an expected gain of $0.38 \times 0.51 = 0.194$. Ergo, by applying this Ace-prediction function and by increasing his bet-size accordingly, the player's overall expectation is increased from -0.00419 to $+0.1898$.

SIDE BETS

Usually side bets entail limits on allowable wagers and are resolved independent of the ongoing play.

Over/Under

Here, the player may wager that his initial two cards total either over 13 or under 13. Aces are counted as 1s (even when paired with a T). A total of exactly 13 loses for either bet.

The player's multi-deck expectation is -0.0656 for the Over bet and -0.1006 for the Under bet (averaged over all the dealer's up-cards). Two-card totals greater and less than 13 occur with probabilities 0.4672 and 0.4497, respectively—leaving 0.0831 for the probability of an exact total of 13.

Jake Smallwood (Ref.) has devised a "crush count" to exploit this option: A and 2 are each counted $+2$; 3 is counted $+1$; and 9 and T are counted -1. An average expectation of $+0.024$ results from betting Over with card counts ≥ 3 and Under with card counts ≤ -4.6.

Royal Match, Single Deck

The player is paid 3 to 1 if his initial cards are of the same suit and 10 to 1 for a same-suited King and Queen (the Royal Match). There are $4\binom{13}{2} = 312$ same-suited pairs including the four K-Q matches. Hence the player's expectation for this wager is -0.038^{-}. With multiple decks, the expectation is yet lower.

Arnold Snyder (Ref.) has proposed a facile system to identify those deck compositions that produce a positive expectation: Whenever m = seven cards of a suit have been dealt out, while at least one of the other three suits has lost no cards, the bet offers a positive expectation of 0.01 to 0.03 (depending on the number of K-Q pairs remaining and on the distribution of the two other suits). If one suit has been depleted by m = eight cards with no cards dealt out from at least one other suit, the player's expectation ranges from 0.03 to 0.10. With m = nine cards dealt out from one suit and zero cards from a second suit, the player's expectation extends from about 0.06 to 0.20.

The probabilities of these events are expressed by

$$P(m) = \sum_{a=b}^{m-1}\sum_{b=0}^{m-1} \frac{\binom{n}{m}\binom{n}{0}\binom{n}{a}\binom{n}{b}}{\binom{4n}{n}} \bullet \delta \quad \delta = \begin{cases} 4 \text{ if } a = b = 0 \\ 12 \text{ if } a = b \neq 0 \\ 24 \text{ if } a \neq b \end{cases}$$

where $n = m + 0 + a + b$. [a and b represent the number of cards depleted from the third and fourth suits.] For $m = 7$, 8, and 9, $P(m) = 1.7 \times 10^{-3}$, 5.0×10^{-4}, and 9.0×10^{-6}, respectively. Associated expected values are 0.0211, 0.0439, and 0.0868.

BLACKJACK VARIATIONS

Numerous other blackjack variants have aspired to popularity; most have attracted only transient interest. The few exceptions follow.

Full Disclosure

As an illustration of the efficacy of the knowledge of undisclosed cards, we consider the game of Blackjack played under the rule that both Dealer cards are exposed. The optimal basic strategy then changes markedly from that applicable under the conventional blackjack format. Applying this strategy, displayed in Appendix Table E, results in a mathematical expectation of +0.099. Appendix Table F details the individual expectations for each of Dealer's 55 possible hands.

As a practical casino game, the House would levy a 10% charge on all winning bets.

Double Exposure

This game, under the name "Zweikartenspiel," was initially proposed by the author in the first edition of this work. It has since, with minor modifications, been installed in several gambling emporia.

Dealer's cards are dealt face-up. In compensation, Dealer wins ties except for Player natural versus Dealer natural—which Player wins for an even payoff. Basic strategy for the multi-deck game (four or more decks are standard) is displayed in Appendix Table G (Ref. Wong, Tables 72 and 73)—with Dealer standing on soft 17, resplitting of pairs not permitted, and doubling allowed only on the first two cards. More frequent splitting and doubling (an obvious consequence of Dealer's visible stiff hands) serves to counterbalance the loss of ties.

Player expectation under this format equals −0.002. Were resplitting allowed, an additional positive expectation of 0.0030 would ensue. Allowing doubling down after splitting adds another 0.002—for an overall expectation of +0.003.

The EORs in Double Exposure, shown in Table 8-13, indicate an incremental unit of TC of about 0.007 compared to about 0.005 for standard Blackjack. As these values imply, this game sustains considerably greater fluctuations. Elimination of the 3-to-2 Blackjack payoff accounts for the lower EOR for the Ace.

Table 8-13 Double Exposure EORs (%)

Rank	2	3	4	5	6	7	8	9	10	A
EOR	0.43	0.60	0.85	1.11	0.59	0.32	−0.15	−0.51	−0.78	0.11

The Hi-Lo card-counting system is equally well suited to Double Exposure as it is to conventional Blackjack. Appendix Table H displays the corresponding TC strategy numbers based on the six-deck game (Ref. Wong, 1994, Tables 75 and 76). Player should stand (or double or split) at a TC equal to or greater than the number given for the specified card-combination. For Insurance, the strategy number is 3.0—although greater accuracy is derived from using the usual ratio of non-Ts to Ts (<2 to 1 indicating a positive expectation).

Blackarat

In this variation (suggested by the author), Player receives two cards whose ranks are added modulo 10 to define their value. Dealer receives two cards, one face-up, the other face-down. Dealer's total is also computed modulo 10. Dealer has no drawing options.

Player may exercise the option of drawing one (and only one) additional card. Player may also split any pair (at the cost of another bet) and play each hand separately.

A hand of 9-9 immediately receives a bonus, after which play continues as prescribed.

Dealer's and Player's totals are compared (after Dealer's hole card is turned over); higher total wins the wager—with the exception that Dealer's hand of 0-0 (two ten-valued cards) automatically wins. Player may buy Insurance against this possibility as in conventional Blackjack.

Ties (pushes) are "no bet."

Player's optimal hit/stand/split-pair strategies and the consequent mathematical expectations are indicated in Appendix Tables I and J for games with a single deck and with six decks, each paying 5 to 1 for the 9-9 bonus. In the former case, Player's expectation is -0.00478. In the latter case, the expectation improves slightly to -0.00437. Changes in expectation for different payoff bonuses are listed separately in Appendix Table K.

3-Way Blackjack (suggested by the author)

Initially, Player places his wager and is also afforded the option of a Blackjack side bet at an amount up to one-fifth the original wager. Player is then dealt three cards (the side bet pays off if any two of them can form a Blackjack). Dealer receives two cards, one face-up, one face-down.

Certain three-of-a-kind hands are awarded an instant pay bonus—as detailed in Table 8-14.

Table 8-14 3-Way Blackjack Bonuses

	Probability	Payoff	Return (%)
Jumpin' Jackpot (3 suited Jacks)	0.000016	100 to 1	0.1596
Jumbled Jacks (3 unsuited Jacks)	0.000388	5 to 1	0.1939
Busting triplets (3 Aces, Kings, Queens, 10s, or 9s)	0.002019	1 to 1	0.2019
	Total bonus return		0.5554
	Base House edge		1.96
	Total House edge		1.41

Should the sum of Player's three cards exceed 26 with Aces counted as 11, Player busts and loses his main wager. However, if Player has made the side bet, and if his hand contains a Blackjack, that bet is paid at 6 to 1 even if the hand busts.

When the sum of Player's three cards equals 26 or less, Player discards one of the three and continues the play, as in conventional Blackjack, with a two-card hand. [The usual options for splitting, doubling, Insurance, and Surrender apply.]

Optimal strategy for discarding from the three-card hand is displayed in Table 8-15. The hand can be partitioned into two or three distinct two-card hands (or a single pair from three-of-a-kind). Each two-card total for a particular Dealer's up-card is specified by a numerical rating in this table. That holding showing the highest rating should be retained for the playing phase, and the third card discarded.

Against Player's optimal strategy and with the bonuses as specified, the House advantage is 1.41%.

Spanish 21

Pip tens are removed from the pack (usually six or eight decks), leaving 48 cards per deck (the "Spanish deck"). In partial compensation, certain favorable rules are offered: (1) Player's Blackjack and Player's 21 *always* win; (2) five-card

Table 8-15 3-Way Blackjack Decision Matrix

Player's Hand	Dealer's Up-Card									
	2	3	4	5	6	7	8	9	T	A
5	10	10	9	8	7	8	11	12	12	12
6	9	9	7	6	6	6	8	9	10	10
7	12	12	11	9	9	11	10	11	11	9
8	13	15	13	14	12	17	16	14	14	15
9	21	21	22	22	22	20	21	23	21	23
10	26	28	28	28	27	22	26	26	26	25
11	29	29	29	29	29	27	27	27	29	27
12	1	1	0	0	0	5	7	8	8	7
13	0	0	0	0	0	3	5	6	5	6
14	0	0	0	0	0	2	4	3	4	5
15	0	0	0	0	0	1	1	1	1	3
16	0	0	0	0	0	0	0	0	0	0
17	7	7	6	5	5	10	2	4	6	2
18	22	22	21	20	19	23	22	16	19	17
19	27	28	25	25	26	29	29	29	28	29
A-2	20	20	19	18	18	19	20	24	24	24
A-3	19	17	17	16	15	16	19	22	23	22
A-4	17	16	16	17	17	14	18	19	20	20
A-5	14	14	15	15	16	13	14	17	16	19
A-6	18	19	20	21	21	15	15	18	17	16
A-7	23	23	23	23	23	25	23	21	22	21
A-8	28	26	26	26	25	30	30	30	27	30
A-9	32	32	32	32	30	32	31	32	32	31
A-A	30	30	30	30	32	28	28	28	30	28
2-2	11	11	12	12	14	12	12	13	13	13
3-3	8	8	10	11	11	9	9	10	9	11
4-4	16	13	14	13	13	18	17	15	15	14
5-5	25	27	27	27	28	26	25	25	25	26
6-6	4	5	5	7	8	4	6	7	7	8
7-7	6	6	8	10	10	7	3	2	3	4
8-8	15	18	18	19	20	21	13	5	2	1
9-9	24	24	24	24	24	24	24	20	18	18
T-T	31	31	31	31	31	31	32	31	31	32

Courtesy of Norman Wattenberger, Casino Vérité.

21 pays 3 to 2, six-card 21 pays 2 to 1, seven-card 21 pays 3 to 1; (3) 6-7-8 or 7-7-7 of mixed suits, same suit, or Spades pays 3 to 2, 2 to 1, and 3 to 1, respectively; and (4) Player may Surrender after doubling, forfeiting his original bet; this option is referred to as "double-down rescue."

Basic strategy (Ref. Shackleford, Spanish 21) differs appreciably from that of conventional Blackjack. Player's expectation for Spanish 21 (Dealer S17) is −0.0040. Potential returns, however, are insufficient to warrant investment of additional memorization.

Double Attack Blackjack

Eight Spanish decks are deployed in this game. Dealer's up-card is the fist card dwelt. Each player is then afforded the option to increase his wager (an amount up to the original bet). Basic strategy is detailed in Ref. Shackleford, Double Attack Blackjack. Player's expectation equals –0.0062

Triple Attack Blackjack

Also played with eight Spanish decks, this game entails a particularly elaborate set of rules. After an initial wager, Player is dealt a single card face-up; he may then increase his wager by an amount equal to his original bet. Dealer's first card is then dealt face-up, after which Player may again increase his wager by an amount equal to his original bet whether or not he has exercised the first option. Dealer continues with a second card to Player and to himself, the latter card face-down. If Dealer's up-card is an Ace, he now offers Insurance, the wager for which may not exceed one-half of Player's combined wagers to-date. Insurance pays 5 to 1 for a suited blackjack and 2 to 1 for all other blackjacks.

Player may now request additional cards. With a point total of 21—or with six cards totalling 21 or less (the "Six-card Charlie")—he automatically wins all wagers regardless of Dealer's point total.

In addition, if Dealer's point total equals 22, while Player has not busted, then all wagers are ruled as ties.

Player may double down on any number of cards (including split hands). Aces may be split and re-split (receiving a single card on each Ace).

With optimal strategy (Ref. Shackleford, Triple-Attack Blackjack), the game offers an expected value of –0.0118.

Pontoon

In Pontoon, the Australian variant of Spanish 21 (played with four to eight Spanish decks), Dealer does not take a hole card until conventional play concludes. Payoffs are similar to those of Spanish 21, as is basic strategy (Ref. Walker). Player's expectation: –0.0042

No Bust 21

Up to eight decks are deployed for this game, with two Jokers added to each deck. A hand of two Jokers defines a "natural" that pays 2 to 1 (a natural for both Player and Dealer constitutes a tie). A Joker paired with any other card is counted as 21 (Ref. Shackleford, No Bust 21—Rules).

As per the eponymous feature of the game, Player does not automatically lose if his total exceeds 21; in that instance, he ties if Dealer exceeds 21 by a greater amount. Player loses if his total is greater than Dealer's total when both exceed 21.

Basic strategy is mostly identical to that of conventional Blackjack, with minor differences that can be ignored with minimal penalties. Overall expectation for No Bust 21 is −0.0189 (for the six- or eight-deck game).

Blackjack Switch

In this variant Player places bets of equal size on each of two hands (dealt face-up) and is permitted to interchange the second cards dealt to each hand. Dealer total of 22 ties any Player total of 21 or less (but loses to a Player Blackjack). Conventional basic strategy accommodates this game with few exceptions. Player's expectation equals -0.0018 with six decks and -0.0016 with eight decks (Ref. Shackleford, Blackjack Switch).

Grayjack

Lastly, we have contrived a simplified version of Blackjack, termed "Grayjack," Here, the deck is comprised of but 13 cards: 1 Ace, 2 Twos, 2 Threes, 2 Fours, 2 Fives, and 4 Sixes. Each card is accorded its face value except for the Ace, which is valued as 1 or 7, at Player's discretion. Two cards are dealt to Player and Dealer, one of Dealer's cards being face-down. The object of the game for Player is to obtain a numerical total greater than Dealer's and less than or equal to 13. A sum exceeding 13 automatically loses. Doubling down with any original two-card hand is permitted, as is splitting of any initial pair. Grayjack is defined by the Ace and a Six, and, when dealt to Player, is paid 3 to 2 (when Dealer's up-card is the Ace, the option of "Insurance" is offered against the hole-card's being a Six with a payoff of 2 to 1). Dealer is constrained to draw with totals of ten and less and to stand with 11, 12, and 13. The Ace must be valued as 7 if thereby Dealer's hand totals 11 through 13, but not 14 or above. Player may draw or stand at each instant (with totals of 13 or less) at his discretion.

GRAYJACK weighs in with a positive expectation (~7½%), which mitigates its suitability as a casino game.

Optimal strategy and game expectation are readily computed owing to the reduced size of the deck. These calculations are left as an exercise for the interested reader.

Snackjack (suggested by Stewart Ethier, Ref.)

The deck is composed of eight cards: two Aces, two deuces, and four treys. Player is initially dealt one of the six possible two-card hands and ultimately may have to resolve one of 33 decision points. Aces count as 1 or 4. A hard total >7 busts. For an untied Natural (A,3), Player is paid at 3 to 2. Dealer, initially dealt one card down and one up. Must draw to 5 or less and stand on (hard or soft) 6. Split Aces receive one card only. Treys may not be re-split. A mimic-the-dealer strategy suggests a positive expected value of approximately 9.5% with basic strategy (for which 33 decision points must be resolved).

POOLED RESOURCES

In Blackjack, only a modest advantage attends the pooling of resources among the m players opposing the dealer. The win–lose correlation coefficient between any two players in the same game is about 0.5; that is, discounting ties, two players will incur the same result (win or lose) 75% of the time and opposite results 25% of the time. Thus, the effective bankroll Z_{eff}, for m cooperating players, is approximately (cf. Eq. 3-12)

$$z_{eff} < \binom{m+1}{2}^{-1/2} \sum_{i=1}^{m} z_i$$

where Z_i is the initial fortune of the ith player. For seven players, each with an initial bankroll of 1000, the effective bankroll is

$$z_{eff} < \binom{8}{2}^{-1/2} 7000 = 1323$$

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Chapter | nine

Statistical Logic and Statistical Games

STRATEGIC SELECTION

There are certain phenomena, such as weather and ocean wave movements, whose statistics are pseudostationary—in the sense that they would be statistically predictable if we possessed sufficient microscopic information regarding their nature. Since such data may never be practically available, they are treated as nonstationary. On the other hand, games whose outcomes depend to a large extent on human, animal, or psychological factors are truly nonstationary in nature. Statistics of horse-racing events, for example, change with improved training methods and breeding controls. Stock market activity, as another example, motivates different people to react differently, and the same person may well react differently to the same stimulus applied at different times. Eighty years ago, if J.P. Morgan advised, "Buy!," almost everyone bought (while J.P. Morgan likely sold short). At present, no financier can influence investment behavior to the same extent.

Many of those professionally concerned with more serious games such as warfare, both military and economic, have long been accustomed to regard problems of strategy, where nonstationary statistics are involved, as too enigmatic to be subjected to intellectual analysis. Such problems were considered the exclusive province of men who had crystallized long experience into a set of heuristic rules of play. Mathematical models were viewed merely as exercises in fantasy; the usual approach was that precarious system of trial and error referred to by its adherents as pragmatism and by its opponents as willful bungling. Then, the rise in respectability of such disciplines as Operations Research instituted an incisive, analytic approach to such previously arcane domains as war strategy and weapons development. One consequence of the newer approach, on the national level, is a posture of greater political sophistication.

Yet, efforts to reduce human or animal behavior to precise mathematical formulae can prove less than convincing. We usually strive to determine a "best" decision constrained by the limitations of a practical model and its underlying assumptions.

The choice of a "probable best" course of action from two or more alternatives based on statistical inference as to their relative merits defines a process of strategic selection by weighted statistical logic. There are four basic steps leading to an ultimate decision:

1. Recognition and evaluation of the individual factors pertinent to the final decision.
2. Expression of the alternate courses of action in terms of the weighting assigned each of the individual pertinent factors.
3. Derivation of a composite ranking that represents the overall relative merit of each alternate course of action.
4. Identification and designation of the highest ranking alternative as the optimal course of action.

The key to this approach is the judicious choice and manipulation of the pertinent factors affecting the selective decision. In attempting to predict the outcome of a football contest, we might list such factors as weather, field conditions, home team, and the number of disabled players. To the extent that we include irrelevant parameters, our prediction rests on shaky foundations.

Each ingredient in our dialectical stew is examined for its likely level of contribution. The space of permissible outcomes is then described in a probabilistic sense. If our assessment of the relative importance of each factor is erroneous, our final ranking becomes questionable. Finally, our optimal prediction falls impotent in the face of a blocked kick, a fumble, or some other unforeseen catastrophic departure from planned behavior.

THE STOCK MARKET

A complex game comprising weighted subjective judgments and nonstationary statistics is that tendered by the stock market[1] (or commodity markets). Constituting a gigantic decision-making phenomenon, the stock market—with its codified folklore and certified respectability—poses a broad appeal to the gambler (euphemistically referred to as a "speculator"). Like horserace handicappers, stock market analysts frequently attempt to predict the future on the basis of past performances, and, like horse racing, the market has successfully withstood all nostrums for financial success; the pot of gold exists, but the labyrinthine path to its den has not yet been charted.

Rigorous technical analysis of the stock market encounters a vertiginous array of factors, some real, some imagined, to which security prices respond with varying sensitivities. Treatment of the apparently whimsical fluctuations of the stock quotations as truly nonstationary processes requires a model of such complexity that its practical value is likely to be limited. An additional

[1] So-called from the custom of English brokers of recording transactions by duplicating notches in two wooden stocks (the customer retained one stock and the broker the other; a matching of the notches presumably ensured honesty).

complication, not encompassed by most stock-market models, arises from the manifestation of the market as a non-zero-sum game.

The theory of games has shown that consideration of non-zero-sum games immediately injects questions of cooperation, bargaining, and other types of player interaction before and during the course of the game. Also, considerations of subjective utility functions arise—for non-zero-sum games we are generally concerned with equilibrium point-preserving vectors of utility functions. Even when treated as a zero-sum game, stock-market speculations imply utility functions that are *non-strategy preserving* (by definition, if a player's strategy is independent of his fortune at the time of his play, his utility function is *strategy preserving*; it can be shown that such utility functions must be either linear or exponential).

A further peculiarity of the stock market arises from the time duration of a complete play (a buy-and-sell transaction, in either order). What constitutes a profit over an extended period of time is complicated by the time-varying purchasing power of money, taxation rates, availability of merchandise, inventions, war, government changes, political upheavals, and shifts in mass psychology influencing the mores of investors. Reflecting an amalgam of economic, monetary, and psychological factors, the stock market represents possibly the most subtly intricate game invented by man.

Descriptive Models

Tradition has it that the New York Stock Exchange (the "Big Board")[2] was established on Wall Street in 1792 under the sheltering limbs of a buttonwood tree. By 1815 it had moved indoors and was offering 24 issues for trade. Over 216 years it has grown into the world's largest exchange with more than 2800 companies listed and a global capitalization of $22.6 trillion, five times the size of its nearest competitor (the Tokyo Stock Exchange). In 2005, the NYSE merged with the fully electronic exchange, Archipelago Holdings; accordingly, more than 50% of all orders are, at present, delivered to the floor electronically, a percentage that promises to increase.

In 1965, the NYSE Composite Index was created as a measure of the change in aggregate value of all common stocks traded on the Exchange (in contrast to the 30 stocks encompassed by the Dow Jones Industrial Average). Revised in 2003, the Index is weighted using free-float market capitalization and calculated on the basis of both price and total return (and is announced every half-hour). The four years from January 2003 to January 2007 witnessed a rise of 81% in the Index (from 5000 to 9069).

Throughout the more than two centuries of open stock auctions, multitudinous models have been proposed as descriptive of the behavior of security prices. Classically, efforts are directed toward relating the values of securities with industrial production levels or other measures of economic health. Yet industrial stocks and industrial production have moved together but 51% of the

[2] In addition to the New York exchange, there are ten registered exchanges in the United States plus three Canadian exchanges.

time since mid-19th century (36% of the time both have risen, while 15% of the time both have fallen; rising industrial production has seen a falling market 25% of the time, and stocks have risen despite falling production 24% of the time). The industrial cycle itself is exhibiting markedly changing characteristics. "boom" and "bust" periods are becoming less frequent and violent under the constraints of government controls. From 1854 through 1933, the United States experienced 21 economic depressions, an average of one every 45 months; between 1933 and 1965, six depressions resulted in an average of one every 64 months. From 1986 through 2006, economists have identified only five such cycles. (Depressions, now bowdlerized to "recessions," are officially acknowledged by the National Bureau of Economic Research—NBER.)

Of those models that deal directly with security prices—short-circuiting economic and monetary analysis—the great majority are of an interpretive nature. The Dow theory and its host of relatives typify those models based on crude measurements with such ambiguous formulation as to require interpreters, most of whom disagree among themselves. We can only marvel at the contradictory conclusions that have evolved from a single collection of data.

Other models are predicated upon assumed (or imagined) correlations between stock price fluctuations and various fluctuating phenomena in and out of the world. Sunspot activity, astrological events, ionic content of the air, and Jovian radio signals are but some of the supposed correlated events; one enterprising individual ran, for a time, a fairly successful investment service based on his "readings" of comic strips in the *New York Sun*. Not surprising to the statistician, good correlations can be found over discretely chosen limited data samples *in the past*. Their failure to survive the transition from past to future can surprise only the irrepressible optimist.

Although the first serious study of a stock market (Vienna) was conducted prior to 1871 by Carl Menger (as a background for his theory of prices and the concept of marginal utility), it was not until 1900 that a competent mathematician began a scientific and empirical investigation into the nature of security prices. In that year, Louis Bachelier (Ref.), in his doctoral dissertation, developed an elaborate mathematical study of market prices that launched the theory of stochastic processes, established the probability law for diffusion in the presence of an absorbing barrier, and foreshadowed many other areas of interest to probability theorists.

Through the first half of the 20th century and beyond, several investigations into the ability of publicized forecasters to predict stock prices found that no success beyond that of chance could be attributed to their prognostications.

Subsequently, mathematicians and economists have combined efforts to understand the principles of trading securities. The University of Chicago, in 1960, established a Center for Research in Security Prices to investigate the behavior of stocks, individually and collectively. Monthly closing prices of all common stocks listed on the New York Exchange from 1926 through 1960 were recorded. It was determined that a "random" investor (the particular security and the buying and selling moments were selected at random) acquired a median gain of 10% per annum on his capital.

Virtually all academic investigators agree on one point: To a good approximation, price changes in speculative markets, such as commodities and securities, behave like independent, identically distributed random variables with finite variances. It then follows, according to the central limit theorem, that price changes over a small interval of time are normally distributed. (Actually, the log-normal distribution provides a better fit.) The model thus depicted is referred to by statisticians as a random walk and by physicists as Brownian motion. It asserts that *a history of past prices alone is of no substantial value in the prediction of future prices*. No gambling rule exists that can produce a profit (assuming truly independent price increments—see Theorem I, Chapter 3). Except for appreciation from earnings retention, the conditional expectation of tomorrow's price, given today's price, is today's price.

The random-walk model is consistent with the postulate of stock exchanges as "perfect" markets. That is, if a substantial group of investors believed that prices were too low, they would enter buying bids, thereby forcing prices higher, and vice versa, quickly forcing prices into conformity with the "intrinsic value" of the securities.

Numerous studies have applied the technique of spectral analysis to time series of stock prices. A spectrum of a time series represents completely the autocorrelation function for any stationary stochastic process with finite variance and also determines the best linear predictor of that time series. This technique is resorted to with reservation in connection with economic data, since the basic phenomenon is nonstationary. However, if the underlying structure of the time series is not a fast-changing function of time, spectral analysis can prove useful. In virtually every case, the first differences of various stock aggregates exhibit a spectrum quite flat over the entire frequency range, thus providing additional support to the random-walk thesis.

Analyses of security prices are characterized by two distinct approaches. Fundamental analysis relies upon evaluation of economic and financial data to ascertain the intrinsic value of a corporate security—generally measured by the price-to-earnings ratio. It can be particularly difficult, however, to estimate earnings more than a year or two into the future.

Technical analysis, on the other hand, is concerned solely with market price trends and patterns—price histories are examined in search of patterns that portend future prices. The "technician" or "chartist" then speaks in terms of "head and shoulders," "double tops," "support levels," "resistance points," and the like, referring to structures delineated by a chart of price versus time. (In practice, most investment counselors combine fundamental and technical principles in appraising securities.) The random-walk model renders unavailing all technical analysis. Historical price patterns are merely statistical fossils (Ref. Thorp and Kassouf)—though their advocates, like astrologers, alchemists, psychics, and inventors of perpetual motion machines, refuse to give up the ghost.

Markets may be defined as strongly or weakly efficient. In the former category, all participants are afforded equality of opportunity but with average expectations. Only to the extent that a market deviates from strong efficiency can a participant exceed the average return on investment. (In a semistrongly efficient market, a favored few have gained access to "inside" information and can profit thereby. Government regulations militate against this type of market.)

In a weakly efficient market, pertinent information is made available to the investing public after a time lag (termed "relaxation time," it describes the period for new information to be absorbed into the marketplace; relaxation time has grown ever shorter with greater and more rapid diffusion of knowledge). Essentially, all major stock exchanges constitute weakly efficient markets.

Burton P. Fabricand (Ref.) has demonstrated that strategic selection of stocks in a weakly efficient market can yield positive expectations. His suggested procedure arises from the fact that roughly 15% of listed companies will report quarterly earnings (RE) that exceed prior projections (PE) by 10% or more (while 7% of earnings reports will exceed projections by 20%). (Such projections are available from Value Line Investment Survey; actual earnings are promptly reported in the *Wall Street Journal*.) Fabricand's strategy entails purchase of stocks (*immediately upon the issuance of earnings reports*) with RE to PE ratios >1.1 and holding them so long *and only so long* as the ratio exceeds 1.1.

Clearly, the success of this procedure depends on the persistence of the weakly efficient nature of the stock market. To date, there is no evidence to suggest that this persistence has diminished; indeed, it seems likely that it is ingrained in the DNA of an auction market.

Sergei Maslov and Yi-Cheng Zhang (Ref.) have shown that processes similar to the Parrondo principle (Chapter 4) can be applied to the stock market to increase the value of a portfolio despite a long-term decline in each of its individual stocks. Specifically, the entire portfolio must be sold frequently (the "readjustment interval"), with the proceeds immediately reinvested in the same proportions as in the original portfolio. The net effect of such frequent rebalancing is to apply the gains from stocks temporarily performing better than average (and have therefore increased their relative proportions in the portfolio) to buy a higher percentage of those stocks that are temporarily performing below average. It has been demonstrated that the investor following this (counterintuitive) strategy improves his return over that of the investor who follows a static buy-and-sell strategy (Ref. Marsili, Maslov, and Zhang).

An interesting offshoot of the Maslov game is the "toy model" (impractical but possibly insightful) that exploits Parrondo's principle by combining a stable stock (A), unlikely to engender significant gains or losses, with a highly volatile stock (B), the value of which equiprobably doubles or halves each day. An opposite strategy, proposed by D.G. Leunberger (Ref.) and termed *volatility pumping,* consists of selling both stocks each day and rebuying them in equal monetary proportions, thus rebalancing the portfolio. The result, quite remarkably, produces an exponential growth of the portfolio. Figure 9-1 (Ref. Abbott) illustrates a typical profile obtained by exercising the Parrondo principle on the two stocks, A and B. By the one-hundredth day, the initial portfolio has increased in value 10,000-fold.

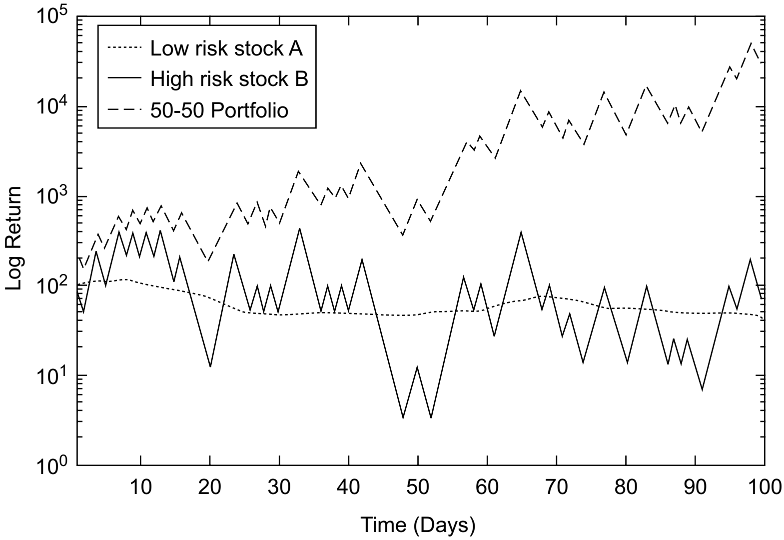

FIGURE 9-1 Volatility pumping a low-risk stock with a high-risk stock.

The principle can be equally adapted to two volatile stocks, each defined as stock B. For this case, the sell-and-rebalance procedure will have increased the value of the portfolio 1,000,000-fold after 100 days.

Transactions costs, for volatility pumping and for the Maslov strategy, would severely crimp the theoretical benefits.

Warrant Hedging

The practice of warrant hedging (simultaneously buying a security and shorting its associated warrant) has been recognized for several decades. Its effectiveness, its anatomy, and its operation, however, were not analyzed until the 1960s, when Edward O. Thorp and Sheen T. Kassouf (Ref.) performed a thorough autopsy.

A warrant is defined as an option to buy a share of common stock at a fixed price—referred to as the exercise price. Warrants themselves are bought and sold in essentially the same manner as common securities although, unlike stocks, they are (usually) issued with an expiration date, beyond which the warrant retains no value. Adroit hedging of the warrant against the stock largely cancels the risk attendant to each separately while still offering a substantial return on investment if the warrant is overpriced relative to the stock.

The specifics of a warrant-hedging transaction involve shorting m warrants at selling price w and exercise price e for every share of stock bought at price s (m is known as the mix). If, at or before the expiration date of the warrant, the stock is selling at price S and the warrant at price W, the percentage profit return R from concluding the transaction is

$$R = \frac{S - s + m(w - W)}{mw + s\mu} \qquad (9\text{-}1)$$

where μ is the margin required for stock purchase and $mw + s\mu \equiv I$ is the initial investment capital.

As its expiration date approaches, the warrant will have a price approaching zero if the stock price is at or below the exercise price; if the stock price exceeds the exercise price, the warrant will sell for about the difference between the stock and exercise prices. That is,

$$W \sim \begin{cases} S - e, & S > e \\ 0, & S \leq e \end{cases}$$

Equation 9-1 can then be written in the form

$$R \sim \begin{cases} \dfrac{(1-m)S + m(w+e) - s}{I}, & S > e \\[2ex] \dfrac{S + mw - s}{I}, & S \leq e \end{cases} \tag{9-2}$$

R remains positive throughout the range

$$(s - mw) < S < \left[\frac{s - m(w+e)}{1-m}\right]$$

and assumes its maximum value when $S = e$ (disregarding commissions). According to the random-walk model, the mean value of S is s; Eq. 9-2 for this case shows a profit of $R = mw/I$.

[Conventionally, it was assumed that the mean value of S is $Se^{k(t^*-t)}$ for some $k > 0$, where $S \equiv S(t^*)$ and $s \equiv S(t)$, provides a fairly good fit. If k is set equal to r_0, the riskless rate for the period (t, t^*), then the Black-Scholes model[3] follows by integration.]

Thorp and Kassouf restrict the hedging system to those warrants whose expiration date is prior to four years from the transaction date and whose stock is selling at less than 120% of the exercise price. Further, they advance two "commonsensical rules" relating warrant prices to stock prices:

1. The price of a warrant should be less than the price of its associated common stock.
2. The price of a warrant plus its exercise price should be greater than the price of the stock.

For each transaction they select a mix that provides a conservative balance between potential risk and potential profit. Often the mix is chosen so that the zone of potential profit is distributed logarithmically equal above and below the stock price. Applying these guidelines, they report consistent profits of about 25% per year on invested capital.

As a numerical illustration, consider a stock selling at 20 with a warrant selling at 5 and an exercise price of 25. A mix of 2 provides equal logarithmic safety against stock fluctuations. Then, if we buy 100 shares of the stock and short 200 warrants,

[3] A partial differential equation that describes the price of an option over time.

the invested capital with 50% margin is $I = 200(5) + 100(20)(0.5) = \$2000$. The safety zone ranges throughout $10 < S < 40$ with its maximum value at $S = 25$, whence (from Eq. 9-2)

$$R = \frac{1500}{2000} = 75\%.$$

Examples of warrants suitable for hedging appear with decreasing frequency as the hedging system has become increasingly exercised in the marketplace. A different market with further profit potential was initiated in April 1973 when the Chicago Board of Options Exchange (CBOE) permitted public buying and selling of call options on selected stocks. Since a call can be treated identically to a warrant, a system of option-hedging (selling calls while buying the associated stock) is mathematically equivalent to warrant-hedging. The classic construct in this field is the Black–Scholes option model, which demonstrates that hedging options short and common stock long and continuously adjusting the mix lead to an essentially risk-free rate of profit return.

HORSE RACING

Of the various games with formalized wagering, perhaps none so embodies the concept of subjective evaluation as horse racing, the "Sport of Kings." Competitive racing can be traced back some 6500 years to the nomadic tribesmen of Central Asia. Wagering on the outcome of such races apparently originated among the Hittites in the second millennium B.C.; archaeologists have uncovered a lengthy treatise on the breeding and training of horses written for a Hittite king about 1500 B.C. The earliest full-length account of a chariot race appears in Book xxiii of the *Iliad*. Competitive rules for prescribed distances were instituted for the Thirty-third Olympiad of Greece, *ca.* 624 B.C. The Romans added handicapping and the concept of "betting against the House" for their chariot races. Thence, racing declined with the onset of the Middle Ages and was not renewed until early in the 17th century when James I sponsored its establishment at Epsom and Newmarket; his grandson Charles II was such an avid racing addict that he became known as the "father of the British turf." In mid-18th century, the Jockey Club was formed as the governing board of racing in Great Britain. And, in 1894, the United States followed suit with the formation of the American Jockey Club, the "combined Congress and Supreme Court of American racing."

Three major types of horse racing have held sway in the United States: Thoroughbred (by far the most popular), Standardbred, and Quarter-Horse racing (operating on 20-some major and hundreds of smaller tracks). By definition, a Thoroughbred is lineally descended from one of three Near East progenitors (the Byerly Turk, the Darley Arabian, and the Godolphin Barb) bred to English mares about the turn of the 18th century. Standardbred horses are descended from an English Thoroughbred named Messenger, brought to the United States in 1788. Quarter-horses (a mixture of Arabian, Spanish, and English stock with 11 different bloodlines) are bred for compact bodies appropriate to great speed at short distances (from 220 to 870 yards); they are also featured at horse shows and rodeo events. (The name derives from the practice of early 17th-century Virginia settlers to race them through quarter-mile clearings in a forest.)

Steeplechases (a remnant from the "sport" of fox hunting), more popular in the British Isles than elsewhere, are run over a 2- to 4-mile course that interposes obstacles such as hedges, stone walls, timber rails, and water jumps. England's Grand National, held annually since 1839 at Aintree, is arguably the world's most famous horserace.

Competition is generally staged among 6 to 12 horses and occurs over certain standard distances measured counterclockwise[4] about an oval track: six furlongs, seven furlongs, one mile, $1\frac{1}{8}$ miles, $1\frac{1}{4}$ miles, and $1\frac{1}{2}$ miles. Races are also categorized according to prize, sex (three classes are recognized: stallions, mares, and geldings), age of the contending horses (all equine birthdays are celebrated on January 1 by decree of the Thoroughbred Racing Association), and the horse "rating" or quality. The general divisions are Handicap and Stakes races, Claiming and Optional Claiming races, Allowance and Classified Allowance races, and Maiden races.

For any and all data regarding the history, breeding, or racing of horses, the library on the grounds of the Keeneland (Kentucky) racetrack provides the most comprehensive source. Devoted exclusively to hippology, its volumes of racing literature and turf records date from the 18th century.

Predicting the outcome of a horse race is an activity that exerts a continuing appeal despite its complexity. A strategic selection by weighted statistical logic may entail numerous elements, many subjective in nature. An enumeration of the pertinent factors might include post position, track condition, weather, weight carried, previous performances, appearance or condition of the horse, earnings, jockey, owner, trainer, class of race, equipment changes, *inter alia*. No few of these factors comprise statistical phenomena of a nonstationary nature. For this reason, data samples gathered over a few seasons of racing at several tracks offer little predictive value, although copious records of past races have been sifted in the search for significant patterns. Any finite data sample, if analyzed persistently, will divulge patterns of regularity. A simple prediction, however, cannot sensibly be constructed on the shifting sands of nonstationary processes.

Pari-Mutuel Betting

Horse racing survives to this day as a major professional sport solely because of its venue for legalized gambling. The pari-mutuel betting system is based on the consensus of subjective probabilities assigned to the contending horses by the aggregation of bettors (thus constituting a weakly efficient market within a compressed time scale) and specifies payoff odds on each horse inversely proportional to the amount of money wagered on that horse. Invented in 1865 by a Parisian parfumeur, Pierre Oller, it essentially ensures a fixed profit to the

[4] English, Australian, and Japanese horses, among others, often race in clockwise fashion.

track operator independent of the winning horses (pari-mutuel wagering is also applied for Greyhound racing and Jai Alai contests). In 1933, the first completely electronic totalizator, automatically computing odds and issuing tickets, was introduced at Arlington Park, Chicago, and subsequently installed at all major racing establishments.[5]

Generally (depending on the country or the state within the United States) track operators pocket from 15 to 25% of the total wagered on each race and redistribute the remainder among those bettors selecting the winning horses. Additionally, a 2.5% loss is sustained by each winner owing to the "breakage" rule—the payoff is quantized to the lower 10-cent level (based on the conventional minimum wager of $2.00). Standard bets include Win, Place (the horse to finish first or second), Show (the horse to finish first, second, or third), Across the Board (a minimum $6.00 wager divided equally among Win, Place, and Show), Daily Double (two horses in different races, usually the first and second, both to win), Quinella (two horses in the same race to finish first and second, without regard to order), Exacta or Perfecta (the first two finishers in exact order), Trifecta (the first three finishers in exact order), Superfecta (the first four finishers in exact order), Pick 3 (winning horses in three consecutive races), and Pick 6 (winners in six consecutive races). These last few bets obviously entail extravagant odds.

The legal minimum return on a $2.00 winning ticket is normally $2.10; when the totalizator indicates a smaller payoff (after deducting the House Take), the situation is termed a "minus pool," and the difference to $2.10 is borne by the track operators.

To describe mathematically the pari-mutuel model, we postulate n bettors $B_1, B_2, \ldots, B_n$ concerned with a race involving m horses $H_1, H_2, \ldots, H_m$. Each better B_i applies his weighted statistical logic to what he deems to be the pertinent factors and thereby achieves an estimate of the relative merits of each horse H_j expressed in quantitative terms—i.e., a subjective probability distribution over the m horses. Specifically, we have an $n \times m$ matrix $\{p_{ij}\}$, where p_{ij} designates the probability, in the judgment of B_i, that H_j will win the race. A sum $b_i > 0$ is then wagered by B_i in a manner that maximizes his subjective mathematical expectation. That is, B_i observes the pari-mutuel probabilities π_1, $\pi_2, \ldots, \pi_m$, as indicated by the track tote board, that horses $H_1, H_2, \ldots, H_m$, respectively, might win the race; he then follows a strategy of distributing the amount b_i among those horses H_j for which the ratio p_{ij}/π_j is a maximum.

We assume that the sum b_i is small with respect to the total amount wagered by the n bettors on the race and therefore does not appreciably influence the pari-mutuel probabilities. We further assume that each column of the matrix $\{p_{ij}\}$ contains at least one entry; otherwise, if the jth column consists of all zeros, no bettor has selected horse H_j, and it can theoretically be eliminated from consideration.

[5]Totalizators are normally limited to the issuance of 12 different tickets. With 13 or more horses going to the post, two or more are coupled together as a betting unit, either because they share common ownership or because they are labeled as "field horses" (a group defined by their long odds).

The pari-mutuel system is described by three conditions. First, if β_{ij} is the sum wagered by B_i on H_j, we have

$$\sum_{j=1}^{m} \beta_{ij} = b_i \tag{9-3}$$

Second, the pari-mutuel format imposes the relation

$$\sum_{i=1}^{n} \beta_{ij} = k\pi_j \tag{9-4}$$

where k is the constant of proportionality relating the amount bet on each horse to its pari-mutuel probability. Third, each B_i bets so as to maximize his subjective expectation E_i—that is, when

$$\mathrm{E}_i = \frac{p_{ij}}{\pi_j} > 1 \tag{9-5}$$

and he bets only on horses for which the inequality holds. Nonnegative numbers π_j and B_{ij} that satisfy Eqs. 9-3, 9-4, and 9-5 are termed *equilibrium probabilities* and *equilibrium bets*, respectively. Their existence is readily proved by means of fixed-point theorems (a clever, elementary proof has been given by Eisenberg and Gale [Ref.]).

Pari-mutuel probabilities for the Win position are determined by the proportionality constant

$$k = (1 - K)\sum_{i=1}^{n} b_i$$

where K is the House Take, so that

$$\pi_j = \frac{\sum_{i=1}^{n} \beta_{ij}}{(1 - K)\sum_{i=1}^{n} b_i} \tag{9-6}$$

The odds on each horse are quoted as $(1 - \pi_j)/\pi_j$ to 1.

Pari-mutuel probabilities for Place are slightly more complicated. The total amount wagered for Place is decreased by the House Take and then divided into two halves, one half being distributed among bettors on the winning horse and the other among bettors on the placing horse. Similarly, pari-mutuel probabilities for Show are computed by dividing the total amount bet for Show less the House Take into three equal parts, which are distributed among bettors on the winning, placing, and showing horses. Thus, the payoff for a horse finishing Place or Show is a function of the other two horses sharing the mutuel pool. That payoff will be higher, the greater the odds on these two horses.

"Real" Probabilities for a Race

There are horse fanciers who profess the ability to determine the objective odds of each horse in a race. Such "dopesters," can, presumably, assess the "real" probability ρ_j (as opposed to the subjective, pari-mutuel probability π_j) that horse H_j will finish first in a particular gymkhana. It is of interest to derive optimal betting strategy, given the distribution of real probabilities for the contenders.

Let the total capital bet on the jth horse, $k\pi_j$ (Eq. 9-4), be partitioned into the sum s_j wagered by the subjective "crowd" and the amount t_j contributed by our knowledgeable dopester. The dopester's profit $F(t_1, t_2, \ldots, t_m)$ from an investment spread over the m horses is defined by

$$F(t_1, t_2, \ldots, t_m) = (1-K)\left[\sum_{j=1}^{m}(s_j + t_j)\right]\sum_{j=1}^{m}\frac{\rho_j t_j}{s_j + t_j} - \sum_{j=1}^{m} t_j$$

It is desired to select a value of $t_j \geq 0$ so as to maximize $F(t_1, t_2, \ldots, t_m)$, where the maximal F possesses a positive magnitude.

Rufus Isaacs (Ref.), in his original statement of the problem, showed that F has a positive maximum if

$$\max_{1\leq j\leq m}\frac{\rho_j}{s_j} > \frac{1}{(1-K)\sum_i s_i}$$

Consequently, for a solution $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m)$, we can write

$$\frac{\partial F}{\partial t_i} = (1-K)\sum_{j=1}^{m}\frac{\rho_j \bar{t}_j}{s_j + \bar{t}_j} + (1-K)\frac{\rho_i s_i}{(s_i + \bar{t}_i)^2}\sum_{j=1}^{m}(s_j + \bar{t}_j) - 1 = 0$$

Then the quantity $\rho_i s_i/(s_i + \bar{t}_i)^2$ exhibits the same value for all i such that $\bar{t}_i > 0$; defining this value by $1/\lambda^2$, we have

$$\bar{t}_i = \lambda\sqrt{\rho_i s_i} - s_i$$

A maximal form of F never occurs for all $\bar{t}_i > 0$ (that is, the dopester never bets on all the horses). Rather, there may exist some number of horses k whose real expectation is greater than the subjective expectation realized from the actual capital wagered on them by the crowd. Then

$$\bar{t}_1 = \bar{t}_2 \cdots = \bar{t}_{k-1} = 0$$
$$\bar{t}_k > 0, \quad \bar{t}_{k+1} > 0, \ldots, \bar{t}_m > 0$$

where $\bar{t}_i$ is uniquely determined by k:

$$\bar{t}_i = \lambda_k \sqrt{\rho_i s_i} - s_i, \quad \text{for } i = k, k+1, \ldots, m \tag{9-7}$$

and

$$\lambda_k^2 = (1 - K)\sum_{j=1}^{k-1} s_j \left[1 - (1 - K)\sum_{j=k}^{m} \rho_j\right]^{-1} \tag{9-8}$$

As a numerical example, consider the horse-race data in Table 9-1. Horse H_7 offers the soundest investment, since it returns 75 times the wager less the

Table 9-1 Hypothetical Horse Race

Horse	"Real" Probability of Winning	Amount Wagered by the "Crowd"
H_1	0.40	$35,000
H_2	0.18	10,000
H_3	0.12	9,000
H_4	0.10	8,000
H_5	0.10	7,000
H_6	0.05	5,000
H_7	0.05	1,000
		$75,000

House Take, yet possesses a win probability of 0.05. However, if the statistically minded dopester plunges heavily on H_7, the pari-mutuel reaction will alter the odds appreciably.[6] To maximize his expected profit, the dopester should examine

[6] An illustration of the sensitivity of the pari-mutuel system is supplied by the so-called builder play, a betting coup whereby some members of a gang obstruct the mutuel windows at a track, placing small wagers on unlikely winners, while confederates are placing large wagers with off-track bookmakers on the more probable winners. Notable builder plays include the 1964 coup at the Dagenham (East London) Greyhound Stadium and a 1932 feat staged at the Agua Caliente (Mexico) racetrack. In the latter instance, odds on the winning horse (Linden Tree) were increased from a logical 7 to 10 to almost 10 to 1. The Dagenham coup was better organized, allowing only a single on-track ticket to be sold on the winning combination (Buckwheat and Handsome Lass), yielding pari-mutuel payoff odds of 9872 to 1.

the subset (possibly null) of horses whose real expectation is greater than their subjective expectation. The subset for the race illustrated here is composed of horses H_2, H_5, and H_7. Thus, from Eq. 9-8 with $K = 0.15$ (15% House Take),

$$\lambda_k = \left[\frac{0.85(35{,}000 + 9000 + 8000 + 5000)}{1 - 0.85(0.18 + 0.10 + 0.05)}\right]^{1/2} = 259.5$$

And, from Eq. 9-7, the optimal wager on horse H_2 is computed as

$$\bar{t}_2 = 259.5\sqrt{0.18 \times 10{,}000} - 10{,}000 = \$1018.98$$

No wager should be placed on horse H_5, since the House take of 15% leads to a negative value of $\bar{t}_5$:

$$\bar{t}_5 = 259.5\sqrt{0.05 \times 7000} - 7000 < 0$$

Horse H_7 should receive a bet of magnitude

$$\bar{t}_7 = 259.5\sqrt{0.05 \times 1000} - 1000 = \$837.68$$

Herein, we have avoided pursuing the conceptual connotations of real probabilities as applied to horse races. However, it is intuitively apparent that any individual who can assess the probability of a given horse winning a race more accurately than the ensemble of other bettors can apply this profit maximization method.

The Equus Computer

Currently one of the most successful approaches to horse-race forecasting is the genetically programmed system EDDIE[7] (Ref. Tsang, Butler, and Li). With this methodology (inspired by Darwinian evolution theory), a candidate solution is represented by a genetic decision tree (GDT) whose basic elements consist of *rules* and *forecast values* that correspond to the functions (relevant questions) and terminals (proposed actions) in genetic programming (GP). EDDIE applies GP to search the space of decision trees and to channel available "expert" knowledge into forecasting and the generation of pertinent rules.

Each function in a GDT consists of attributes (e.g., finishing positions in the horse's previous races, weight carried in previous and forthcoming races, whether stepping up or down in class, etc.) and operators (various logical and relational operations that compare attributes). Processing these input data, EDDIE generates decision trees that rank the horses by ability.

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[7] Evolutionary Dynamic Data Investment Evaluator.

As the functions are satisfied for each horse, the process is repeated—so as to evolve by way of natural selection a decision tree—and progressively moves down the tree until a terminal forecast value—an integer—is reached. This integer defines a confidence rating describing how well the specified horse is expected to perform in its impending race. That horse accruing the highest confidence rating is presumed the most probable winner.

EDDIE was put to the test with a database of 180 handicap races in the UK. A spectacular 88.2% return-on-investment resulted, an outcome that far outperformed other systems evaluating the same data.

This procedure is obviously analogous to financial forecasting, and genetic programs have also been adapted to financial markets with moderate if not dramatic success.

Psychological Betting Systems

Because of the inherently subjective nature of the pari-mutuel betting format, it may be feasible to devise methods of horse selection based solely on the psychological behavior of the bettors. For example, one "system" consists of wagering on the favorite (shortest odds) horse in the last race of a day's program. While favorite horses are not more prone to win the last race than any other, the payoff is frequently greater than it would be in earlier races since there apparently exists a tendency to place a larger percentage of the total wagers on the longer-odds horses. This tendency possibly arises from the losers' attempts to regain their losses and the winners' inclinations to leave prior to the last race.

Another system advises wagering on the favorite horse if the previous two or more races have been won by favorites, and conversely. The rationale in this instance suggests that many bettors are susceptible to a "maturity of the chances" doctrine, and therefore tend to avoid wagering on a favorite horse when preceding races have produced winning favorites; thus the favorite offers higher odds than would arise from independent considerations. Both systems have been tested by examination of a limited data sample. The results indicate an increase in expectation of approximately 15% over a random selection of horses. Presumably, as bettors gain additional mathematical sophistication, this figure would decrease.

More cyclopedic means exist for utilizing the relatively invariant psychological behavior of bettors as reflected by the pari-mutuel probabilities. A logical method involves a statistical classification of races according to the subjective probability distribution of each horse's winning chances and a comparison of the mathematical expectation with previous data. Each race can be distinguished by a distribution of the probabilities π_j, as specified by Eq. 9-6. One of several parameters might be chosen as a measure of this distribution; for simplicity, we select the natural and convenient measure of entropy. Thus, we categorize all races according to the entropy H of that race:

$$H = -\sum_{j=1}^{m} \pi_j \log \pi_j$$

A maximum entropy race is one wherein all horses are accorded identical pari-mutuel probabilities (that is, identical sums are wagered on each horse). A uniform distribution of π_j over the interval 0 to 1 constitutes a minimum-entropy race. By examining records of past races, we can obtain a frequency distribution $f(H)$ (actually a histogram) of pari-mutuel entropies.

For every race, we can then compute the frequency distribution $g(\pi_j)$ of the pari-mutuel probabilities and therefore the distribution

$$F(\mathrm{E}) = \pi_j\left[\frac{g(\pi_j)}{\pi_j} - 1\right]$$

of expectations. Performing this computation over all possible entropies, we can construct a surface of expectation density as a function of entropy. To determine whether to bet on a particular race and on which horses, we first calculate the entropy of the race just prior to its running. For a particular entropy we have a pari-mutuel expectation density profile $F_1(\mathrm{E})$ available from past data. A typical example is illustrated in Figure 9-2. The m pari-mutuel probabilities associated with the m horses are compared with this profile; if one or more values of π_j leads to a positive expectation, a wager on the corresponding horse(s) is advised.

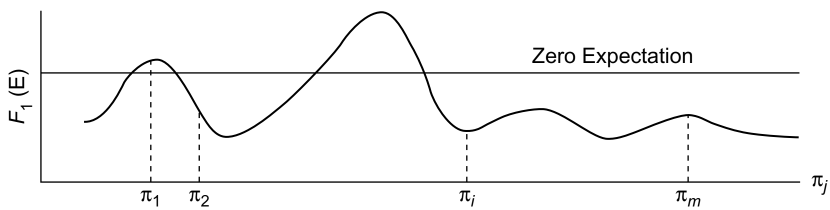

FIGURE 9-2 Expectation density profile of a particular entropy.

Compilation of data over a limited period of time at several tracks has revealed that a positive expectation usually occurs in races of relatively high entropy. For these situations, horses with odds of 6 to 1, 7 to 1, and 8 to 1 have more often exhibited a positive payoff probability.

Other than lotteries or football pools, horse racing constitutes one of the poorest forms of betting available to the average gambler, owing to the 15+% House Take plus breakage. Overcoming the highly negative mathematical expectation of horse racing poses virtually insurmountable obstacles in terms of acquiring the requisite knowledge, constructing an appropriate model, and performing the attendant computations. Records of the published handicappers indicate a lack of success for all methods based solely on subjective evaluations.

Horseplay

We inquire as to the number of ways, $S(n, k)$, that n horses can cross the finishing line, allowing for all possible ties (Ref. *qbyte.org*). Here, k is the number of different blocks that describe the finishing order.

We can immediately write the recurrence relation

$$S(n+1, k) = S(n, k-1) + kS(n, k), \qquad n \geq k$$

with initial conditions

$$S(n, 1) = S(n, n) = 1$$

Thus the total number of ways, H_n, that n horses can finish is given by

$$H_n = \sum_{k=1}^{n} S(n, k)k!$$

Table 9-2 lists the values of H_n for $n \leq 10$.

Table 9-2 Number of Possible Finishes in an n-Horse Race

n	1	2	3	4	5	6	7	8	9	10
H_n	1	3	13	75	541	4683	47,293	545,835	7,087,261	35,719,563

DUELS AND TRUELS

Dueling games, a subset of those games whose key element is timing, constitute another category with interesting ramifications; such games were first conceived and analyzed by Blackwell and Girschick (Ref.). Duels can be either *noisy* or *silent*. In the latter case, a player is unaware of the moment his opponent has fired. Duels can also be characterized by the number of bullets each player has at his disposal as well as his firing rate. Further, the bullets may exist only in a probabilistic sense.

In-Place, Noisy Duel, Unlimited Ammunition

One of the simplest forms of dueling constrains the two combatants **A** and **B** to remain in place, firing in turn with prescribed accuracies a and b, respectively. Firing continues indefinitely until only one duelist survives (Ref. Knuth).

Let $f(a, b)$ denote the probability that **A** wins the duel, given that **A** fires first. Evidently,

$$f(a, b) = a + (1-a)[1 - f(b, a)]$$

since **A** wins if his first shot is successful or if his opponent fails. Thus we have the linear equation

$$\begin{aligned} f(a, b) &= a + (1-a)\big(1 - \{b + (1-b)[1 - f(a, b)]\}\big) \\ &= \frac{a}{a + b - ab} \end{aligned}$$

And, of course, **A**'s probability of winning increases as a increases and decreases as b increases. For equal accuracies, $a = b$, and $f(a, a) = 1/(2 - a)$; this game is equitable for very low values of a and a certain win for **A** for values of a approaching 1 (since **A** fires first).

The probability that **B** wins the duel, given that **A** fires first, is

$$1 - f(a, b) = \frac{b(1 - a)}{a + b - ab}$$

When **B** fires first, **A**'s probability of winning is

$$f(b, a) = \frac{a(1 - b)}{a + b - ab}$$

while **B**'s probability of winning, given that **B** fires first, is

$$1 - f(b, a) = \frac{b}{a + b - ab}$$

A Variant

In a somewhat more complex variant of the in-place duel, **A** may select any firing accuracy P. **B**'s accuracy is 0.5, and he fires first with that same probability P.

If **A** selects 1.0, the game is equitable since **B** is given the first shot; however, **A** can improve his prospects. To complete **A**'s optimal value for P, we can express **B**'s probability of winning as

$$P_{B_1} = \frac{1}{2} + \frac{1}{2}(1 - P)P_{B_1} = 1/(1 + P) \quad \text{when } \mathbf{B} \text{ fires first (prob. } P\text{)}$$

and

$$P_{B_2} = (1 - P)P_{B_1} = (1 - P)/(1 + P) \quad \text{when } \mathbf{B} \text{ fires second [prob. } (1 - P)\text{]}$$

Thus **B**'s overall probability of winning, P_B, is

$$P_B = PP_{B_1} + (1 - P)P_{B_2} = (1 - P + P^2)/(1 + P)$$

The derivative of P_B with respect to P determines the minimum value of P_B. To wit,

$$\frac{dP_B}{dP} = \frac{P^2 + 2P - 2}{(1 + P)^2} = 0$$

which yields

$$P = \sqrt{3} - 1 = 0.732^+ \quad \text{and} \quad P_B = 2\sqrt{3} - 3 = 0.464^+$$

Thus **A** maximizes his probability of winning at $1 - 0.464^{-} = 0.536^{-}$ by selecting an accuracy of 0.732^{+}. Note that **A** thereby gains an additional 7.2% over the strategy of selecting $P = 1.0$.

More generally, if **B**'s firing accuracy is q, we have

$$P_{B_1} = q + (1-q)^2 P_{B_1} = \frac{-q}{(1-q)(1-P)-1} \quad \text{when } \mathbf{B} \text{ fires first,}$$

$$P_{B_2} = \frac{-q(1-P)}{(1-q)(1-P)-1} \quad \text{when } \mathbf{B} \text{ fires second,}$$

and

$$P_B = \frac{-qP - q(1-P)^2}{(1-q)(1-P)-1}$$

Setting the derivative of P_B with respect to P equal to zero and solving for P, we have

$$P = \frac{-q + \sqrt{q^2 - q + 1}}{1-q}$$

as **A**'s optimal firing accuracy. If, $q = 0$, **A** selects $P = 1$ (since allowing **B** to fire first does no harm). If, $q = 1$, $P = 1/2$, and the game is equitable.

Noisy Duel, Single Bullet Each

Here, **A** and **B** face each other at a distance X_0 at time t_0 and approach each other at a linear rate until $X = 0$ at time t_f. Each player possesses a single bullet and an accuracy $p(t)$; that is, $p_a(t)$ and $p_b(t)$ are the probabilities that **A** and **B**, respectively, will succeed when firing at time t. It is presumed that $p(t)$ is monotonically increasing to a value of unity at t_f. Thus, if one player fires at his opponent at time $t < t_f$ and misses, the opponent's obvious best course is then to await time t_f, thereby securing a sure kill. In general, if **A** fires at time t_a, $t_0 \leq t_a \leq t_f$, and **B** fires at time t_b, $t_0 \leq t_b \leq t_f$, the payoff to **A**, $f(T)$, can be written in the form

$$f(T) = \begin{cases} p_a(T) - [1 - p_a(T)] = 2p_a(T) - 1, & \text{if } T = t_a < t_b \\ -p_b(T) + [1 - p_b(T)] = 1 - 2p_b(T), & \text{if } T = t_b < t_a \\ p_a(T) - p_b(T), & \text{if } T = t_a = t_b \end{cases} \qquad (9\text{-}9)$$

where $T = \min(t_a, t_b)$. It is straightforward to show that the three forms of Eq. 9-9 take on equal values when

$$p_a(T) + p_b(T) = 1 \qquad (9\text{-}10)$$

Thus, Eq. 9-10 describes a minimax solution to the duel, and both players should fire simultaneously at this value of t (it is assumed that the value of killing an opponent is unity, while the value of being killed is -1). The value γ of the game is given by

$$\gamma = p_a(T) - p_b(T)$$

If the two duelists are equally skilled, they should fire when their accuracies reach 0.5; the value of such a game is, of course, zero.

Noisy Duel, Many Bullets, Probabilistic Bullets

If the two duelists possess equal accuracy functions $p(t)$ but different numbers of bullets, their optimal strategies are immediately apparent. Specifically, if **A** is equipped with $b_a(t)$ bullets, and **B** with $b_b(t) \le b_a(t)$ bullets, an optimal strategy for **A** consists of firing a bullet when

$$p(t) = [b_a(t) + b_b(t)]^{-1} \tag{9-11}$$

If **B** is accorded fewer bullets, he should shoot only after **A**'s first firing time and then only if **A** has not fired. The value γ of this game is expressed by

$$\gamma = \frac{b_a(t_0) - b_b(t_0)}{b_a(t_0) + b_b(t_0)} \tag{9-12}$$

where t_0 designates the starting time of the duel.

As an example, if **A** is initially furnished three bullets and **B** two bullets, then **A** should pull the trigger once when the accuracy function reaches the value $p(t) = 1/5$, as indicated by Eq. 9-11. If **A** misses, then both **A** and **B** fire one shot when $p(t) = 1/4$, and if both miss, the last bullets should be discharged simultaneously when $p(t) = 1/2$. Value of the game, from Eq. 9-12, is 1/5 to **A**.

Blackwell and Girschick (Ref.) considered the case of two duelists with equal accuracy, where each has only a probability of possessing a bullet—P_1 for **A** and $P_2 < P_1$ for **B**. Here **A** should fire with the density

$$f(T) = \frac{1 - P_2}{2P_2[(2T/t_f) - 1]^{3/2}}, \quad \frac{(1 + P_2^2)t_f}{(1 + P_2)^2} \le T \le t_f$$

a fraction Γ of the time and should fire at time t_f the remaining fraction $1 - \Gamma$ of the time, where

$$\Gamma = \frac{P_2(1 + P_1)}{P_1(1 + P_2)}$$

Evidently the moment at which **A** fires is a function only of his opponent's probability of possessing a bullet, while the percentage of the time that he fires at that moment is a function of both P_1 and P_2. To obtain the optimal strategy for the case where $P_2 > P_1$, we simply interchange P_1 and P_2. The value of the game to **A** is

$$\frac{P_1 - P_2}{1 + \min(P_1, P_2)}$$

Alternate forms of dueling include silent duels (each gun equipped with a silencer) and duels where one combatant is supplied with both noisy and silent bullets that are fired in a prescribed order. An interesting variant is the machine-gun duel, a game that introduces the notion of firing rate. Two players are planted a fixed distance apart and supplied with machine guns; the weapon assigned to **A** (**B**) possesses a single-shot hit-probability P_A (P_B) and a firing rate R_A (R_B) shots per unit time. Such a duel has been analyzed by A.D. Groves (Ref.).

Yet other variants of dueling games might include assigning a probability that a hit is not lethal but merely incapacitates the opponent so that his firing power is reduced; mixing live ammunition with blanks according to some probability distribution; multi-person duels, wherein m combatants on one side face n on the other; and handicapped duels—where one player, for example, is deaf (and thus compelled to play a silent duel), while his opponent plays a noisy duel. Further, we propose a "Fire and Flee" duel. In this version, each player is given n bullets and, after observing that some fraction of them have been discharged without felling his opponent, may exercise the option to flee the scene, thereby altering his opponent's monotonic accuracy function and combining the duel with elements of a pursuit game.

Truels

Three-person games, or "truels," are inherently more complex than duels and, sensibly, more consonant with real-world situations. The strongest of three players may not find himself in the most advantageous position—indeed, "survival of the weakest" is often a feature of truels, depending on the relative strength of the adversaries.

The format for truels posits three players, **A, B**, and **C**, with respective firing accuracies $a \geq b \geq c$, deployed on the vertices of an equilateral triangle, each firing at a player on another vertex or at no one. The firing order—a basic constituent of the game—may be sequential in fixed succession, sequential in random succession, simultaneous, or a function of the players' firing accuracies (e.g., the least accurate player shoots first). Further, ammunition, and thus the number of rounds, may be limited or unlimited as well as noisy or silent. Pacts may be allowed and may be abrogated.

Consider the truel wherein each player simultaneously selects (and announces) his own accuracy function. A revolving order of firing is determined by an equitable random device. Each player, in designated turn, then fires a single bullet at either or neither of his two opponents (if two are yet alive). The game proceeds until either a single truelist remains or three successive shots are fired into the air. We assign a game value of $+1$, which is awarded to the survivor (if all three truelists survive, each receives 1/3).

Assuming that no player selects an accuracy less than 0.5, it is apparent that **C** (the least skilled player) never fires at either opponent if both remain (it is to **C**'s advantage to fire into the air until one adversary eliminates the other, thereby affording **C** first shot in the resulting duel). Thus, of the six possible firing echelons, only two are distinct: that in which **A** precedes **B** and that in which **B** precedes **A**. In either case, **A** fires at **B** (no one fires at **C** first, since he poses the lesser threat) if his mathematical expectation thereby produces a value of 1/3 or greater; otherwise, he fires into the air. Player **B** must inquire as to **A**'s expectation from firing (when **A** precedes **B**), and if it exceeds 1/3, **B** fires at **A** regardless of his own expectation. The winner of the **A** versus **B** duel, if it occurs, then engages **C** to resolve the truel.

The expectation of **A**, E($\boldsymbol{A}$), equals the probability $P(\boldsymbol{A})$ of that player surviving, which equals the probability $P(\boldsymbol{A}\rightarrow\boldsymbol{B})$ that **A** kills **B** times the probability $P(\boldsymbol{A}\rightarrow\boldsymbol{C}|\boldsymbol{A}\rightarrow\boldsymbol{B})$ that **A** kills **C**, given that **A** has first dispatched **B**. We can readily formulate these values. If **A** precedes **B** in the order of firing,

$$\begin{aligned}
\mathrm{E}(\boldsymbol{A}) &= P(\boldsymbol{A}) = P(\boldsymbol{A}\rightarrow\boldsymbol{B})P(\boldsymbol{A}\rightarrow\boldsymbol{C}|\boldsymbol{A}\rightarrow\boldsymbol{B}) \\
&= a\sum_{i=0}^{\infty}[(1-a)(1-c)]^i a(1-c)\sum_{i=0}^{\infty}[(1-a)(1-c)]^i \\
&= \frac{a}{a+b-ab}\cdot\frac{a(1-c)}{a+c-ac} = \frac{a^2(1-c)}{(a+b-ab)(a+c-ac)}
\end{aligned} \tag{9-13}$$

Similarly,

$$\begin{aligned}
\mathrm{E}(\boldsymbol{B}) &= P(\boldsymbol{B}) = P(\boldsymbol{B}\rightarrow\boldsymbol{A})P(\boldsymbol{B}\rightarrow\boldsymbol{C}|\boldsymbol{B}\rightarrow\boldsymbol{A}) \\
&= \frac{b^2(1-a)(1-c)}{(a+b-ab)(b+c-bc)}
\end{aligned} \tag{9-14}$$

$$\begin{aligned}
\mathrm{E}(\boldsymbol{C}) &= P(\boldsymbol{C}) = P(\boldsymbol{C}\rightarrow\boldsymbol{A}|\boldsymbol{A}\rightarrow\boldsymbol{B}) + P(\boldsymbol{C}\rightarrow\boldsymbol{B}|\boldsymbol{B}\rightarrow\boldsymbol{A}) \\
&= \frac{ac}{(a+c-ac)(a+b-ab)} + \frac{bc(1-a)}{(a+b-ab)(b+c-bc)}
\end{aligned} \tag{9-15}$$

If **B** precedes **A** in the order of firing,

$$\mathrm{E}(\boldsymbol{A}) = P(\boldsymbol{A}) = \frac{a^2(1-b)(1-c)}{(a+b-ab)(a+c-ac)} \tag{9-16}$$

$$\mathrm{E}(\boldsymbol{B}) = P(\boldsymbol{B}) = \frac{b^2(1-c)}{(a+b-ab)(b+c-bc)} \tag{9-17}$$

$$\mathrm{E}(\boldsymbol{C}) = P(\boldsymbol{C}) = \frac{bc}{(a+b-ab)(b+c-bc)} + \frac{ac(1-b)}{(a+b-ab)(a+c-ac)} \tag{9-18}$$

Initially, **A**'s game expectation is given by the average of Eqs. 9-13 and 9-16, **B**'s expectation by the average of Eqs. 9-14 and 9-17, and **C**'s expectation by the average of Eqs. 9-15 and 9-18.

It is apparent that **B** finds himself in the most unenviable position. Each player strives to choose an accuracy function that is (1) the highest possible lowest value of the three accuracies, or (2) the highest accuracy—that is, 1.00. To be caught in the middle is anathema.

As an illustration, let the three players select accuracies $a = 1.00$, $b = 0.75$, and $c = 0.50$, respectively. If **A** wins a shooting position prior to **B**'s, he eliminates **B** with unity probability and then is shot at by **C**; if **C** misses (probability 1/2), **A** dispatches **C** on the next round. Hence $P(\boldsymbol{A}) = 0.50$, $P(\boldsymbol{B}) = 0$, and $P(\boldsymbol{C}) = 0.50$. If **B** is selected to shoot first, he must fire at **A** although his expectation remains less than 1/3. If successful, he competes with **C** until either survives, with **C** shooting first; if he misses (probability 1/4), he is eliminated by **A**, who then engages **C** to resolve the truel. Player **C**, if awarded first shot, sends it skyward, thereby guaranteeing himself the first opportunity to eliminate the survivor of the **A** versus **B** contest. Equations 9-16, 9-17 and 9-18 yield $P(\boldsymbol{A}) = 1/8$, $P(\boldsymbol{B}) = 9/28$, and $P(\boldsymbol{C}) = 31/56$. Thus, the *a priori* expectations of the three players with these accuracy functions are

$$\mathrm{E}(\boldsymbol{A}) = \frac{1/2 + 1/8}{2} = \frac{5}{16}$$

$$\mathrm{E}(\boldsymbol{B}) = \frac{0 + 9/28}{2} = \frac{9}{56}$$

$$\mathrm{E}(\boldsymbol{C}) = \frac{1/2 + 31/56}{2} = \frac{59}{112}$$

Note that the least effective marksman (**C**) is most likely to survive, and **B** the least likely.

Even for this elementary truel, computation of the optimal mixed strategies involves overwhelming arithmetical labor. Further, the definition of rational behavior is not immediately apparent in this type of three-sided competition; thus "optimal" strategy is not clearly defined. As with most multiperson, multimove games, this truel may not enjoy a unique, noncooperative equilibrium point (see Chapter 2).

We therefore introduce two simplifications that permit a complete description of the ensuing game: (1) each player is supplied with but a single bullet that he may fire at another player or into the air; and (2) the order of firing is announced *before* the accuracy functions are chosen. It is now more convenient to designate the notation **A, B**, and **C** as defining the players in their order of firing (rather than by accuracy functions). Evidently, if **A** fires at either adversary he selects as his target **B** or **C**, as $b > c$ or $c > b$. Player **B** always shoots at **C** if the latter still survives or at **A** otherwise. Player **C**, if not eliminated before his turn, fires at **A** or **B** equiprobably if both remain (independent of their accuracy functions) or at **A** if **B** has been killed.

It is apparent that **A** always adopts the role of a perfect marksman ($a = 1.00$). By killing **B** or **C**, his expectation $E_s(A)$ becomes

$$E_s(\boldsymbol{A}) = \frac{1}{2}\max(1 - b, 1 - c)$$

since his game value is 1/2 if the remaining player misses and 0 otherwise [probability $\min(b, c)$]. By shooting into the air, **A** receives value 1/2 with probability $b + (c/2)(1 - b)$, value 0 with probability $(c/2)(1 - b)$, and value 1/3 with probability $(1 - b)(1 - c)$. His expectation $E_n(A)$ in this case is therefore given by

$$E_n(\boldsymbol{A}) = \frac{1}{12}(2b - c + bc + 4)$$

(We assume that **C** holds no grudges in deciding at whom he shoots.) Thus, **A** eliminates that opponent owning the higher accuracy function whenever $E_s(\boldsymbol{A}) > E_n(\boldsymbol{A})$. The values of b and c whereby this condition holds are indicated in Figure 9-3.

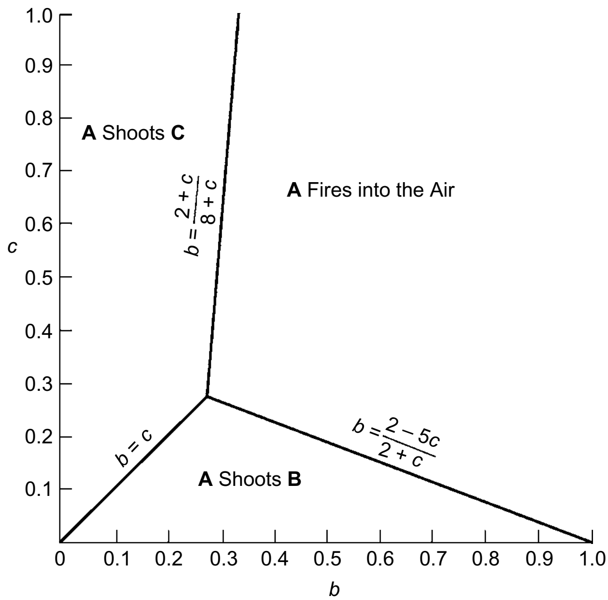

FIGURE 9-3 A truel firing diagram.

The overriding consideration for both **B** and **C** is that each wishes to select an accuracy function smaller than the other, thus diverting the fire of **A** if (b, c) lies within **A**'s firing region. From **C**'s viewpoint, lesser accuracies are even more attractive, since he must still worry over **B**'s shot if **A** fires into the air. Payoffs and conditional expectations for each action are shown in Table 9-3—the notation **A**↑

Table 9-3 Truel Payoffs and Conditional Expectations

Actions	Payoffs			Conditional (to **A**'s Action) Probability of Occurrence	Conditional Expectations		
	A	**B**	**C**		E(**A**)	E(**B**)	E(**C**)
$\mathbf{A}\uparrow \quad \mathbf{B} \rightarrow \mathbf{C}$	1/2	1/2	0	6	$b/2$	$b/2$	0
$\mathbf{A}\uparrow \quad \mathbf{B} \times \mathbf{C} \quad \mathbf{C} \rightarrow \mathbf{A}$	0	1/2	1/2	$(c/2)(1-b)$	0	$(c/4)(1-b)$	$(c/4)(1-b)$
$\mathbf{A}\uparrow \quad \mathbf{B} \times \mathbf{C} \quad \mathbf{C} \times \mathbf{A}$ or $\mathbf{C} \times \mathbf{B}$	1/2	1/2	1/2	$(1/2)(1-b)(1-c)$	$(1/6)(1-b)(1-c)$	$(1/6)(1-b)(1-c)$	$(1/6)(1-b)(1-c)$
$\mathbf{A}\uparrow \quad \mathbf{B} \times \mathbf{C} \quad \mathbf{C} \rightarrow \mathbf{B}$	1/2	0	1/2	$(c/2)(1-b)$	$(c/4)(1-b)$	0	$(c/4)(1-b)$
$\mathbf{A} \rightarrow \mathbf{B} \quad \mathbf{C} \rightarrow \mathbf{A}$	0	0	1	c	0	0	c
$\mathbf{A} \rightarrow \mathbf{B} \quad \mathbf{C} \times \mathbf{A}$	1/2	0	1/2	$1-c$	$(1/2)(1-c)$	0	$(1/2)(1-c)$
$\mathbf{A} \rightarrow \mathbf{C} \quad \mathbf{B} \rightarrow \mathbf{A}$	0	1	0	b	0	b	0
$\mathbf{A} \rightarrow \mathbf{C} \quad \mathbf{B} \times \mathbf{A}$	1/2	1/2	0	$1-b$	$(1/2)(1-b)$	$(1/2)(1-b)$	0

signifies that **A** fires into the air, and **B** × **C** denotes that **B** shoots at but misses **C**. From this table we can write for E($\boldsymbol{B}|\boldsymbol{A}\uparrow$), **B**'s expectation, given that **A** fires into the air,

$$\mathrm{E}(\boldsymbol{B}|\boldsymbol{A}\uparrow) = \frac{1}{12}(4b + c - bc + 2) \tag{9-19}$$

Also

$$\mathrm{E}(\boldsymbol{C}|\boldsymbol{A}\uparrow) = \frac{1}{6}(2c - b - 2bc + 1) \tag{9-20}$$

If collusion were permitted or were mutually advantageous between **B** and **C**, Eqs. 9-19 and 9-20 indicate that $b = 1/3$ and $c = 1.00$ result in an equitable game [E($\boldsymbol{A}$) = E($\boldsymbol{B}$) = E($\boldsymbol{C}$)]. However, the point $(b, c) = (1/3, 1)$ does not represent an equilibrium point, since if either **B** or **C** deviates from this point by lowering his accuracy, the concomitant expectation increases (as **A** then fires at the other player). Consequently, the only "stable" point for **B** and **C** arises from the selection $b = c = 0$. Thus, **B** and **C** declare themselves totally impotent, and the value of the truel is 1/2 to **A** and 1/4 to **B** and **C**. This solution does not preclude the existence of other equilibrium points (e.g., those involving "bluffing" or "deterring"). Since it is possible here for two players to gain even by nonexplicit cooperation and lose by individual action, this truel constitutes a nonzero-sum game. The theory of games has yet to develop a completely satisfactory methodology for the solution of cooperative games.

Pacts (Ref. Kilgour and Brams)

Consider the truel defined by $a = 0.8$, $b = 0.7$, and $c = 0.55$. The respective survival probabilities for the players are $P_A = 0.22$, $P_B = 0.16$, and $P_C = 0.62$. Now suppose that **A** and **B**, dismayed by the supremacy of their weakest competitor, negotiate a pact obliging each of them to fire at **C** until **C** is eliminated (whence they will engage in their own duel). Under that circumstance,

$$P_A = 0.384;\ P_B = 0.565;\ P_C = 0.052$$

averaged over all six permutations of the firing order. **C**'s expectation has now declined precipitously, while **B**'s has risen from lowest to highest (since **C**, in accordance with the SASO [Shoot at the Stronger Opponent] strategy, will fire at **A** as long as both survive), and **A** has benefited considerably—confirming the wisdom of the pact.

Yet in the Hobbesian universe inhabited by game theorists, **B** calculates that he can improve his survival probability further (albeit marginally) by reneging on his agreement, assuming that **A** will initially adhere to the pact. Under this premise, **B** will fire at **A** at his first opportunity, and **A**, in all subsequent turns, will return fire at **B** (assuming that **C** has not dispatched **A** in the interim). Their respective survival probabilities now take on the values

$$P_A = 0.118;\ P_B = 0.568;\ P_C = 0.314$$

However, if **A**, anticipating that **B** will not honor the pact, reneges first (while **B** continues to fire initially at **C**), their respective survival probabilities are

$$P_A = 0.407;\ P_B = 0.262;\ P_C = 0.331$$

Only **C**, with no alternative, will steadfastly adhere to the SASO strategy. **C**'s survival probability will decrease drastically in the event of an honored pact between **A** and **B**, albeit to a lesser degree if either **A** or **B** violates the pact.

This particular example illustrates the general instability that can arise in multiperson games (as opposed to duels), an instability often reflected in the maneuvering of international relationships where a weak nation may gain the advantage over two more powerful competitors, each more fearful of the other.

Truels constitute a unique set of games in that all participants in the truel may maximize their expected payoffs by not firing. This situation does not pertain in duels—where one person's gain is another's loss—or in polyuels with four or more persons. In an N-uel game, it becomes yet more improbable for the player with the highest marksmanship to win out (since $N - 1$ guns will be trained on him). Usually, in an N-uel, combatants can be eliminated with advantage until the game is reduced to a truel.

TOURNAMENTS

Elimination

Of the various procedures for selecting a winner from among many contestants, perhaps the most common is the single elimination, or knockout, tournament. Players are matched pairwise for a single game, and the winners of each round advance to the succeeding round (such games cannot end in a tie). A particular configuration for matching the contestants is known as a draw. With 2^n players in the tournament, there are $\binom{2^n}{2^{n-1}}/2$ ways of setting up the first round, and the total number of draws is, therefore,

$$\prod_{i=2}^{n}\left[\binom{2^i}{2^{i-1}}\Big/2\right]^{n-i+1} = \frac{2^n!}{2^{2n-1}}$$

The tournament comprises $2^n - 1$ games with 2^{2n-1} possible outcomes for each draw (the total number of outcomes for all draws is, of course, $2^n!$).

The probability that a given player wins the tournament following a random draw is simply his probability of winning n matches in succession averaged over all possible draws. Let p_{ij} be the pairwise win probability that the ith player defeats the jth player in a single trial, and as an example consider a tournament of eight players ($n = 3$) possessing the set of pairwise win probabilities shown in Table 9-4. Averaging over the 315 draws, we compute the probabilities $P(i)$ that the ith player wins the tournament:

$$P_i = \frac{1}{315}\sum_{\substack{i \\ i\neq j\neq k\neq m}}\sum_{k<m}\sum\left[p_{ij}(p_{ik}p_{km}+p_{im}p_{mk})\sum_{n}\sum_{\substack{r \\ n\neq r\neq s\neq t \\ \neq i\neq j\neq k\neq m}}\sum_{s<t}\sum p_{in}p_{nr}(p_{ns}p_{st}+p_{nt}p_{ts})\right] \tag{9-21}$$

Table 9-4 Illustrative Set of Pairwise Win Probabilities

		*j*th Player							
		1	2	3	4	5	6	7	8
	1		0.55	0.60	0.65	0.70	0.75	0.80	0.85
	2	0.45		0.55	0.60	0.65	0.70	0.75	0.80
	3	0.40	0.45		0.55	0.60	0.65	0.70	0.75
*i*th Player	4	0.35	0.40	0.45		0.55	0.60	0.65	0.70
	5	0.30	0.35	0.40	0.45		0.55	0.60	0.65
	6	0.25	0.30	0.35	0.40	0.45		0.55	0.60
	7	0.20	0.25	0.30	0.35	0.40	0.45		0.55
	8	0.15	0.20	0.25	0.30	0.35	0.40	0.45	

Evaluating Eq. 9-21 for each i, using the sample pairwise win probabilities, the results are listed in the first column of Table 9-5.

Table 9-5 Probability of a Tournament Win

	Single Elimination Random Draw	Single Elimination Seeded Draw	Single Elimination with Replication Random Draw
P(1)	0.307	0.351	0.388
P(2)	0.226	0.240	0.255
P(3)	0.164	0.160	0.161
P(4)	0.116	0.103	0.097
P(5)	0.081	0.066	0.055
P(6)	0.054	0.042	0.027
P(7)	0.034	0.025	0.013
P(8)	0.020	0.013	0.005

More often, the draw for a tournament is not selected at random but is designed to favor the better players. Such an arrangement, known as a *seeded* draw, is sketched in Figure 9-4. With the same set of pairwise win probabilities, the probabilities $P(i)$ that the ith player emerges victorious become those shown in the second column of Table 9-5. As expected, the better players gain increased chances of winning, and the poorer players decreased chances.

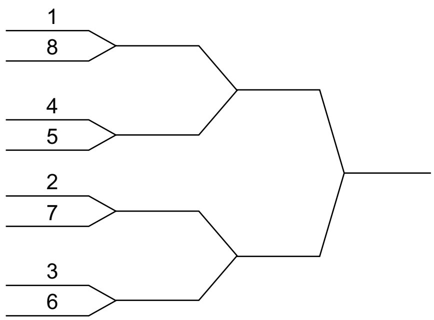

FIGURE 9-4 The seeded draw.

We can also compute the probability of the two best players entering the finals as 0.128 for a random draw and 0.258 for the seeded draw. This difference accounts for the attractiveness of the seeded draw in many professional sports (such as tennis).

If the number of players in a tournament is not exactly 2^n, then one or more pseudo-players are concocted and awarded win-probabilities of zero. A real player thus wins with probability one against a pseudo-player and is said to have a "bye." When i and j are both pseudo-players, $p_{ij} = p_{ji} = 0$ (hence all byes are confined to the first round).

One method of improving the single elimination tournament (insofar as favoring the better players) consists of adding replication to each match. With this procedure each pair of players competes until one has won m games out of a possible $2m - 1$. If p_{ij} is the probability that i defeats j in a single contest, the probability q_{ij} that i wins the best m out of a possible $2m - 1$ is

$$q_{ij} = p_{ij}^m \sum_{k=0}^{m-1} \binom{n+k-1}{k} p_{ji}^k$$

This type of tournament affords yet further advantage to the better players since we can show that, for $p_{ij} \geq 1/2$, $q_{ij} \geq p_{ij}$. Letting each competition be resolved by the best two out of three matches, and with our illustrative set of (single-trial) pairwise win probabilities (see Table 9-4), the distribution of final win probabilities for each player is that of the third column of Table 9-5. The extent to which superior players are favored by replication (or seeded draws) is, of course, strongly dependent upon the particular pairwise win-probabilities of the competing players.

Double elimination, or double knockout, tournaments also constitute a common format for handling a large number of competitors. Here, after the first round, the winning players are grouped in a W bracket, losing players in an L bracket. The W bracket is then conducted as a single-elimination tournament except that the losers of each round drop down to the L bracket. A loser in an L-bracket match is eliminated. Thus one-fourth of the contestants are eliminated at each round following the first. Ultimately, the W-bracket winner contends against the L-bracket winner, needing a single victory to win the tournament, while the L-bracket winner requires two wins to emerge victorious.

Double-elimination tournaments offer the advantage that the third and fourth places can be determined without recourse to a "consolation" match. Further, the higher-rated players (teams) are more likely to advance.

Another format is that of serial elimination—characterized as a single elimination tournament with 2^n players followed by a second single elimination playoff among the winner and those n players beaten by the winner. If the winner does not also win the playoff, an additional game between the original winner and the playoff winner determines the ultimate victor. Serial elimination tournaments with replication can, of course, also be conducted. These tournaments have been investigated by D.T. Searls (Ref.) and by W.A. Glenn (Ref.).

Games Played

Since, in an N-player single-elimination tournament, there is a one-to-one correspondence between the number of games played and the number of losers, there are $N - 1$ games played to resolve a winner. Similarly, the number of games played in an N-player double-elimination tournament will be either $2(N - 1)$ or $2(N - 1) + 1$, depending on whether the winner has no losses or one loss, respectively.

In general, for an N-person, M-tuple-loss tournament, (Ref. Bernander and Bernander) the number of games played is determined from the fact that all players except the winner will lose m games—for a total of $M(N - 1)$ games. The winner will have experienced either 0, 1, …, or $m - 1$ losses. The number of games played, therefore, is

$$M(N-1) + i, \quad \text{where } i = 0 \;\text{ or }\; 1 \;\text{ or }\; 2 \;\text{ or }\; m-1 \qquad \text{(9-22)}$$

The World Series of baseball constitutes a two-team, four-loss elimination tournament. Thus the total number of games played, as per Eq. 9-22, is 4, 5, 6, or 7 (see Chapter 5).

Round-Robin

Contests wherein each player vies against every other player a fixed number of times are not generally as efficient as elimination tournaments. However, the round-robin[8] does exhibit some capability in this regard.

[8] From 17th-century French, *ruban rond* (round ribbon). Petitioners against authorities would append their names in a nonhierarchical circle—thus preventing identification of anyone as the ringleader.

Mathematically, a round-robin tournament is a directed graph (digraph) obtained by choosing a direction for each edge in an undirected complete graph. The score of each node is its "outdegree"—i.e., the number of outward directed graph edges from a given vertex in the digraph. We define a score sequence, $S_1, S_2, \ldots, S_N$, satisfying the following three conditions:

1. $0 \leq S_1 \leq S_2 \leq \cdots \leq S_N$
2. $S_1 + S_2 + \cdots + S_1 \geq S_i\binom{i}{2}$ for $i = 1, 2, \ldots, N - 1$
3. $S_1 + S_2 + \cdots + S_N = \binom{N}{2}$

The number of different score sequences possible in an N-person round-robin tournament are, for $N = 1, 2, \ldots,$

$$1, 1, 2, 4, 9, 22, 59, 167, 890, 1486, \ldots$$

It should be noted that a score sequence does not uniquely resolve a tournament (Ref. Harary and Moser).

As one example, consider a round-robin tournament with $N = 4$ contestants. There are four possible outcomes (with six matches played):

		Games won by player			
		A	**B**	**C**	**D**
	1	3	2	1	0
Outcomes	2	3	1	1	1
	3	2	2	2	0
	4	2	2	1	1

The first two outcomes declare **A** the winner; the third and fourth outcomes eliminate one and two players, respectively, and require further competition.

With five players, a single round-robin procedure may produce any of nine outcomes, five of which uniquely proclaim a winner; with six players, 13 of a possible 22 outcomes designate a winner; a round-robin of seven players admits of 59 possible outcomes, 35 of which specify a winner; and with eight players, there are 100 such outcomes of a possible 167.

Obviously the probability of each outcome is a function of the players' pairwise win probabilities. If $p_{ij} = p_{ji} = 1/2$ for all i and j, the single round-robin tournament with four players has the following distribution of outcomes:

$$P(\text{Outcome } 1) = 3/8 \quad P(\text{Outcome } 3) = 1/8$$
$$P(\text{Outcome } 2) = 1/8 \quad P(\text{Outcome } 4) = 3/8$$

Hence the probability p of determining a winner in this four-player tournament is 0.5; with five players, $p = 0.586$; with six players, $p = 0.627$; a seven-player round-robin is uniquely resolved with probability $p = 0.581$; and with eight evenly matched players, $p = 0.634$. As the number of players increases indefinitely, the probability of obtaining a unique outcome approaches unity.

If one player has a high win probability against each of his adversaries, his probability of winning the tournament is generally greater from a round-robin than from a single elimination procedure.

Pairings

With $2n - 1$ players (or teams) labeled 1, 2, …, $2n - 1$ entered in a round-robin tournament, the general pairings are arranged as in Table 9-6.

Table 9-6 General Round-Robin Pairings

Round	Bye	Pairs				
1	1	2 vs. $(2n - 1)$	3 vs. $(2n - 2)$	4 vs. $(2n - 3)$ …	$(n - 1)$ vs. $(n + 2)$	n vs. $(n + 1)$
2	$n + 1$	$(n + 2)$ vs. 4	$(n + 3)$ vs. 3	$(n + 4)$ vs. 2 …	$(n + 3)$ vs. $(n + 6)$	$(n + 4)$ vs. $(n + 5)$
3	2	1 vs. $(n - 1)$	n vs. 7	$(n + 8)$ vs. 6	$(n + 7)$ vs. $(n + 10)$	$(n + 8)$ vs. $(n + 9)$
4	$2n - 2$					
.						
.						
.						

For $n = 4$, the seven players 1, 2, …, 7 are matched up as shown in Table 9-7.

Table 9-7 Seven-Player Round-Robin Pairings

Round	Bye	Pairs		
1	1	2-7	3-6	4-5
2	5	4-6	7-3	1-2
3	2	1-3	4-7	5-6
4	6	5-7	1-4	2-3
5	3	2-4	5-1	6-7
6	7	6-1	2-5	3-4
7	4	3-5	6-2	7-1

RANDOM TIC-TAC-TOE

An alternative to the pure-strategy forms of Tic-Tac-Toe (Chapter 10) was proposed by F.E. Clark (Ref.). From the set of nine numbers, 1, 2, … 9, **A** selects one at random and places an X in the corresponding square of the 3×3 Tic-Tac-Toe matrix (see Figure 10-1). **B** then randomly selects one of the eight remaining numbers and enters an O in the corresponding square, and so forth. The standard objective pertains: to configure three Xs (Os) in a straight line.

There are $\binom{9}{5} = 126$ possible configurations for **A** if the game is played through all nine moves (though rotational and mirror symmetries leave only 17 unique positions). Of the total number of configurations, only 16 will draw (probability 0.127)—since a draw requires **B** to have at least one O in a corner, and each corner placement leaves four combinations for his other three Os. Also, **A** will lose whenever **B** completes either of the two diagonals, there being six possibilities for each—i.e., 12 losses for **A**.

Further, there are 36 configurations with three Xs and three Os in a straight line. Six columns or rows can be completed by **A**, and two by **B**. Accordingly, of the 36 configurations, **A** will win

$$36\binom{5}{3}^{-1}\left\{\binom{3}{3} + \left[\binom{4}{3} - \binom{3}{3}\right]^2 \binom{4}{3}^{-1}\right\} = 3.6 + 8.1 = 11.7$$

and will lose $36 - 11.7 = 24.3$. **A**'s total losses, then, are $24.3 + 12 = 36.3$, for a loss probability of 0.288; his wins total $126 - 16 - 36.3 = 73.7$, for a win-probability of 0.585.

The game presents a value of 0.390 to **A**. He can offer **B** a 2-to-1 payoff and still retain a positive expectation.

The probability that **A** wins with his third, fourth, or fifth X is 2/21, 37/140, and 71/315, respectively. The probability that **B** wins with his third or fourth O is 37/420 and 1/5, respectively.

INQUIZITION

Solutions to the three following statistical problems are beyond the reach of all but the most agile game theorists. In each case, only partial analysis can be readily achieved.

1. *The Harem-Staffing Problem.* From a supply of n candidates, the Sultan wishes to select m maidens for his harem. The n aspirants are paraded one by one before the Sultan, each being either selected or rejected—and once rejected, cannot be recalled. Since m nymphs *must* be chosen, the final few may be forced upon the Sultan to complete the seraglio complement. A preference ordering of the candidates is assumed although not known *a priori*. What function of the girls' desirability should the Sultan adopt as his objective, and what is his concomitant strategy?

   Note: For $m = 1$, assuming that the Sultan wishes to maximize his probability of engaging the "best" candidate, he assigns a utility of 1 to this girl and 0 to all others. Show that his optimal strategy consists of rejecting the first n/e candidates (for large n) and then selecting the first one who exceeds the best of that sample; if no better one appears, he is stuck with the final girl. Show also that his expected utility for this objective is approximately e^{-1}.

2. *A Truel in the Sun.* Each of three players, **A, B**, and **C** (with respective firing accuracies P_A, P_B, and P_C) fires in cyclical order (or abstains from

firing) only at his successor in the order. After one truelist is eliminated, the remaining two duel until one is eliminated.

Compute the players' winning probabilities when $P_A = P_B = P_C = p = 1/2$. Determine the values of p for which the game degenerates into perpetual abstentions.

3. *Martini Golf.* The object of the game of golf is to impel a small white ball into a slightly larger hole with a minimum number of strokes delivered by means of a set of specialized clubs. We designate by $p_a(n)$ and $p_b(n)$ the probabilities that **A** and **B**, respectively, require n strokes for a given hole. Whenever a player wins a hole (by using fewer strokes than his opponent), he must consume one martini. After j martinis, **A**'s probability distribution $p_a^j(n)$ of holing out in n strokes becomes

$$p_a^j(n) = p_a(n - j)$$

and, similarly,

$$p_b^j(n) = p_b(n - j)$$

Compute the probability that **A** has taken fewer strokes after a round of 18 holes.

If one player is permitted to add strokes deliberately, and the lower score on each hole wins one unit while the lower final score wins m units, what strategy should he follow to maximize his expected gain?

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Chapter | ten

Games of Pure Skill and Competitive Computers

DEFINITIONS

We define games of pure skill to be devoid of probabilistic elements. Such games are not often associated with monetary wagers, although formal admonishments against profiting from a skill are not proclaimed in most cultures. From a rational standpoint, it might be expected that a person should be far more willing to express financial confidence in his skills than on the mindless meanderings of chance. Experience, however, has strongly supported the reverse proposition.

The more challenging games of skill, some of which have resisted solution for centuries, exhibit either high *space complexity* (the number of positions in the search space) or high *decision complexity* (the difficulty encountered in reaching optimal decisions) or both. Generally, such games may be "solved" on three levels:

1. *Ultra-weak.* The game-theoretic value for the initial position has been determined. It might be proved, for example, that the first person to move will win (or lose or draw). Such a proof, *sui generis*, lacks practical value.
2. *Weak.* An algorithm exists that, from the opening position *only*, secures a win for one player or a draw for either against any opposing strategy.
3. *Strong.* An algorithm is known that can produce perfect play from any position against any opposing strategy (assuming not-unreasonable computing resources) and may penalize inferior opposing strategies.

Many of the games that have succumbed, at one of these levels, to the prowess of computer programs still retain an attraction, at least to the layman—particularly with strong solutions that involve complex strategies beyond the players' memories or with weak solutions on large fields of play. The (weak) solution of Checkers, for example, has not intruded on the average person's practice or understanding of the game.

TIC-TAC-TOE

Possibly the oldest and simplest of the games of pure skill is Tic-Tac-Toe (the name itself apparently derives from an old English nursery rhyme). With variations in spelling and pronunciation, its recorded history extends back to about 3500 B.C.—according to tomb paintings from Egyptian pyramids of that era.

Two players alternately place a personal mark (traditionally Xs and Os for the first and second players, respectively) in an empty cell of a matrix. Winner of the game is that player who configures his marks along an unbroken, complete line horizontally, vertically, or diagonally. In the event that neither player achieves such a line, the game is drawn (known colloquially as a "cat's game").

The generalized form of Tic-Tac-Toe entails a $k \times \ell$ matrix with each player striving for an n-in-a-row formation. Further generalizations expand the game to three or more dimensions and/or non-planar playing surfaces such as a cylinder, a torus, or a Möbius strip.

A 3×3 matrix with $n = 3$ constitutes the common form of Tic-Tac-Toe (Figure 10-1), with **A** designated as the first player, **B** the second. There are 138

1	2	3
4	5	6
7	8	9

FIGURE 10-1 The Tic-Tac-Toe matrix.

unique games with this format, allowing for symmetries and winning positions that end the game before all nine cells are marked; the matrix itself has 8-fold symmetry. Of these, 91 are wins for X, 44 for O, while 3 are drawn. (Without symmetries, there are 255,168 possible games, 131,184 wins for X, 77,904 for O, 46,080 draws.) With rational play on the part of both players, no conclusion other than a draw will be reached.[1] The opening move must be one of three: center (cell 5), corner (cells 1, 3, 7 or 9), or side (cells 2, 4, 6, or 8). Correct response to a center opening is placement of a mark in a corner cell (responding in a side cell leads to a loss). Correct response to a corner opening is occupying the center. To a side opening, the correct response is the center, adjoining corner, or far side. Additional counter-responses to a total of nine moves are readily obtained from only casual study.

A simple but fundamental proof, known as the *strategy-stealing argument*, demonstrates that traditional Tic-Tac-Toe constitutes a draw or win for **A**.

[1]The 3×3 game is sufficiently simple that bright chimpanzees have learned to play with creditable performance. Indeed, some casinos have offered the opportunity to compete against trained chickens.

Postulate a winning strategy for **B**. **A** can then select any first move at random and thereafter follow **B**'s presumed winning strategy. Since the cell marked cannot be a detriment, we have a logical contradiction; ergo a winning strategy for the second player does not exist.

The basic Tic-Tac-Toe game can be extended to matrices larger than 3×3 (Ref. Ma). For $n = 3$, **A** wins on all matrices $k \geq 3$, $l > 3$. For $n = 4$, the game is a draw on a 5×5 matrix but a win for **A** on all matrices $k \geq 5$, $l > 5$ (including the infinite board). The $n = 5$ game, when played on matrices 15×15 or greater, offers a forced win for **A** (Ref. Allis, van der Meulen, and van den Herik) and is known as Go-Moku (Japanese: *gomoku narabe*, line up five) when adapted to the intersections on a (19×19) Go board (Ref. Allis, van den Herik, and Huntjens). With $n = 6$, the game is a draw on a 6×6 matrix and unsolved on all larger matrices. Games with $n > 6$ on larger (and infinite) matrices are drawn under rational decision-making.

An efficient procedure for determining whether a particular Tic-Tac-Toe format leads to a draw is that of the Hales-Jewett pairing strategy (Ref. Hales and Jewett). Figure 10-2 illustrates this strategy for $n = 5$ on a 5×5 matrix.

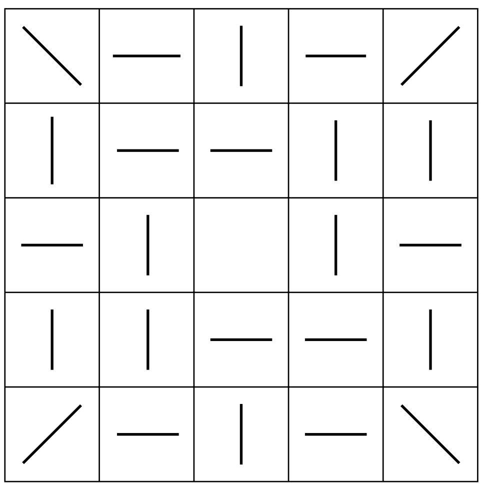

FIGURE 10-2 A Hales-Jewett pairing.

Each square is assigned a direction: north-south, east-west, northeast-southwest, or northwest-southeast. Then, for every square occupied by **A**, **B** occupies the similarly marked square in the direction indicated by the slash. Accordingly, **B** will have occupied at least one square in every possible line of five squares—and will thereby have forced a draw. (Further, **B** can concede to **A** the center square *and* the first move and still draw the game.)

The Hales-Jewett pairing strategy is especially useful for higher values of n on large matrices.

Variants of Tic-Tac-Toe extend back more than two millennia. Following are some of the more noteworthy.

Misère Tic-Tac-Toe

Whoever first places his marks collinearly *loses*. For $n = 3$ on a 3×3 matrix, the "X" player can ensure a draw by occupying the center square and thereafter countering the "O" moves by reflecting them through the center. With an even value for k or l, **B**, using this strategy, has at least a draw for any n.

Wild Tic-Tac-Toe

Players may use *either* an X or O for each move. This variant offers limited appeal since **A** has a forced win by placing a mark in the center cell.

Wild Misère Tic-Tac-Toe

The first player to complete three-in-a-row (with either mark) *loses*. The second player can force a draw by marking a cell directly opposite his opponent's moves, choosing X against O and O against X.[2]

Kit-Kat-Oat

Each participant plays his opponent's mark. The disadvantage lies with **A**; he can achieve a draw, however, by placing his opponent's mark in the center cell as his first move and thereafter playing symmetrically about the center to his opponent's previous move.

Kriegspiel Tic-Tac-Toe

Each player is accorded a personal 3×3 matrix, which he marks privately (unseen by his opponent). A referee observes the two matrices and ensures that corresponding cells are marked on only one matrix by rejecting any moves to the contrary. Since **A**, as first player, possesses a pronounced advantage, it is reasonable to require that he be handicapped by losing a move upon receiving a rejection, while **B** is guaranteed a legal move at each turn. Imposition of this handicap reduces, but does not eliminate **A**'s advantage. Played on 4×4 matrices, the game still offers a slight edge to **A**. A remaining unsolved problem is to ascertain the size of the matrix whereby, under the handicap rule, Kriegspiel Tic-Tac-Toe becomes a fair game.

Tandem Tic-Tac-Toe

Each player possesses his personal 3×3 cellular matrix, which he labels with the nine integers 1, 2, …, 9, each cell being marked with a distinct integer. The marking process occurs one cell at a time—first as a concealed move, which is then exposed simultaneously with his opponent's move. With the information obtained, a second cell of each array is marked secretly, then revealed, and so forth, until all nine cells of both matrices are numbered. For the second phase of the game, **A** places an X in a cell in his matrix *and* in the same-numbered cell in his opponent's matrix. **B** then selects two similarly matching cells for

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[2]Analyzed independently by Solomon Golomb and Robert Abbott.

his two Os, and so forth. If a win (three collinear personal symbols) is achieved in one matrix, the game continues in the other matrix. Clearly, both players strive for two wins or a single win and a tie. **B**'s evident strategy is to avoid matching of the center and corner cells between the two matrices.

Rook-Rak-Row

Each player, in turn, may place one, two, or three marks in a given row or column of the 3×3 matrix (if two cells are marked, they need not be adjacent). Whichever player marks the last cell wins the game. **B** can always force a win. If **A** places a single X, **B** counters with two Os that form a connected right angle. If **A** marks two or three cells, **B** marks three or two cells, respectively, to form a "T," an "L," or a cross composed of five cells.

Cylindrical Tic-Tac-Toe

The top and bottom of a $k \times l$ matrix are connected. For $n = 3$ on a 3×3 matrix, two additional collinear configurations are possible: 4-8-3 and 6-8-1.

Toroidal Tic-Tac-Toe

Both top and bottom *and* left and right sides of the playing matrix are connected (thus forming a torus). All opening moves are equivalent. With matrices of 3×3 or $2 \times l$, $l \geq 3$, **A** can force a win for the $n = 3$ game. For $n = 4$, **A** wins on a $2 \times l$ matrix for $l \geq 7$, while the $n = 5$ game is a draw on any $2 \times l$ matrix.

Möbius Strip Tic-Tac-Toe

Here, cells 1, 2, 3 are connected to cells 9, 8, 7.

Algebraic Tic-Tac-Toe

A first selects a value for any one of the nine coefficients in the system of simultaneous equations

$$\begin{aligned} a_1 x + a_2 y + a_3 z &= 0 \\ b_1 x + b_2 y + b_3 z &= 0 \end{aligned} \tag{10-1}$$

$$c_1 x + c_2 y + c_3 z = 0 \tag{10-2}$$

Then **B** assigns a value to one of the remaining eight coefficients, and so forth. **A** wins the game if the completed system has a non-zero solution.

Winning strategy for **A**: Select *any* value for coefficient b_2 and thereafter play symmetrically to **B**—so that after all nine coefficients have been assigned values we have

$$a_1 = c_1; \quad a_2 = c_2; \quad a_3 = c_3; \quad b_1 = b_3$$

Thence Eqs. 10-1 and 10-2 are equivalent. Since the system of simultaneous equations is homogeneous, it is consistent and admits of infinitely many solutions—including a non-(0, 0, 0) solution. Thus the game is a win for **A**.

Numerical Tic-Tac-Toe

A and **B** alternately select an integer from the set 1, 2, …, 9, placing it in one cell of the 3×3 matrix (Ref. Markowsky). Objective: to achieve a line of three numbers that sums to 15. This game is isomorphic to standard Tic-Tac-Toe—as per the correspondence shown in the magic square of Figure 10-3. **A** is afforded an easy win—on his second move if he begins with the center cell 5, on his third move if he begins with a corner or side cell.

4	3	8
9	5	1
2	7	6

FIGURE 10-3 Tic-Tac-Toe magic square.

Considerably greater complexity ensues if the set of integers is divided between **A** (permitted to place only the odd integers) and **B** (permitted to place only the even integers).

Three-Dimensional Tic-Tac-Toe

Surprisingly, the $3 \times 3 \times 3$ game with $n = 3$ proves uninteresting since **A** has a simple forced win by marking the center cell; failure to follow this strategy allows **B** to do so and thereby usurp the win. The game cannot be drawn since it is impossible to select 14 of the 27 cells without some three being collinear, either orthogonally or diagonally.

Draws *are* possible with $n = 4$ played on a $4 \times 4 \times 4$ lattice.[3] Here the two players alternately place their counters in one of the 64 cubicles. Objective: to achieve a line of (four) connected personal counters either in a row, a column, a pillar, a face diagonal, or a space diagonal.

This game was weakly solved by Oren Patashnik (in 1980) and subsequently (in 1994) strongly solved by Victor Allis using proof-number search, confirming that the first player can effect a win.

Three-Dimensional Misère Tic-Tac-Toe

The $n = 3$ game offers a win for **A**—by applying the same strategy as in planar Misère Tic-Tac-Toe: first marking the center cell and thereafter playing symmetrically opposite his opponent. (Since drawn games are impossible, this strategy forces **B** ultimately to mark a collinear pattern.)

The $n = 4$ misère game, similar to standard three-dimensional $4 \times 4 \times 4$ Tic-Tac-Toe, likewise admits of drawn positions.

Higher-Dimensional Tic-Tac-Toe

Since two-dimensional Tic-Tac-Toe exists logically for two players, we might infer that the three- and four-dimensional versions can be constructed for three and four players, respectively. In d dimensions, a player can achieve a

[3] Marketed under the name Qubic.

configuration with d potential wins so that the $d-1$ succeeding players cannot block all the means to a win.

Two-person, d-dimensional Tic-Tac-Toe offers some mathematical facets. Played in an $n \times n \times \cdots \times n$ (d times) hypercube, the first player to complete any line-up of n marks wins the game—and there are $[(n+2)^d - n^d]/2$ such lines (which for $n = 3$ and $d = 2$—the standard planar game—is equal to 8; for the $n = 4$, $d = 3$ game, there are 76 winning lines). **B** can force a draw (with the appropriate pairing strategy) if $n \geq 3^d - 1$ (n odd) or $n \geq 2^{d+1} - 2$ (n even). However, for each n there exists a d_n such that for $d \geq d_n$, **A** can win (Ref. Ma).

Construction of games in $d \geq 4$ dimensions presents difficulties in our three-dimensional world. It has been proved that a draw exists for $n \geq cd \log d$, where c is a constant (Ref. Moser).

Misère Higher-Dimensional Tic-Tac-Toe

S.W. Golomb has shown that **A** can achieve at least a draw in a d-dimensional hypercube of side n whenever $n \geq 3$ is odd (Ref. Golomb and Hales). **A** marks the center cell initially and thereafter plays diametrically opposite each opponent move. (Thus **A** will never complete a continuous line-up through the center cell, and any other possible line can only be a mirror image of such a line already completed by **B**.)

Random-Turn Tic-Tac-Toe

A coin toss determines which player is to move. The probability that **A** or **B** wins when both play optimally is precisely the probability that **A** or **B** wins when both play randomly (Chapter 9, Random Tic-Tac-Toe).

Moving-Counter Tic-Tac-Toe

This form, popular in ancient China, Greece, and Rome, comprises two players, each with three personal counters. **A** and **B** alternately place a counter in a vacant cell of the 3×3 matrix until all six counters have been played. If neither has succeeded in achieving a collinear arrangement of his three counters, the game proceeds with each player moving a counter at his turn to any empty cell orthogonally adjacent. **A** has a forced win by occupying the center cell as his opening move.

Other versions, which result in a draw from rational play, permit diagonal moves or moves to any vacant cell.

Tic-Tac-Toe with moving counters has also been adapted to 4×4 and 5×5 matrices. In the latter example, invented by John Scarne (Ref.) and called Teeko (Tic-tac-toe chEss chEcKers bingO), two players alternately place four personal counters on the board followed by alternating one-unit moves in any direction. Objective: to maneuver the four personal counters into a configuration either collinear or in a square formation on four adjacent cells (the number of winning positions is 44). Teeko was completely solved by Guy Steele (Ref.) in 1998 and proved to be a draw.

Multidimensional moving-counter games—“hyper-Teeko”—remain largely unexplored.

Finally, quantum Tic-Tac-Toe (Ref. Goff) allows players to place a quantum superposition of numbers ("spooky" marks) in different matrix cells. When a measurement occurs, one spooky mark becomes real, while the other disappears.

Computers programmed to play Tic-Tac-Toe have been commonplace for close to seven decades. A straightforward procedure consists of classifying the nine cells in order of decreasing strategic desirability and then investigating those cells successively until it encounters an empty one, which it then occupies. This process can be implemented by relays alone.

A simple device that converges to an optimal Tic-Tac-Toe strategy through a learning process is MENACE (Matchbox Educable Noughts And Crosses Engine), a "computer" constructed by Donald Michie (Ref.) from about 300 matchboxes and a number of variously colored small glass beads (beads are added to or deleted from different matchboxes as the machine wins or loses against different lines of play, thus weighting the preferred lines). Following an "experience" session of 150 plays, MENACE is capable of coping (i.e., drawing) with an adversary's optimal play.

Another simple mechanism is the Tinkertoy computer, constructed by MIT students in the 1980s, using, as its name implies, Tinker Toys.

More sophisticated is the Maya-II, developed at Columbia University and the University of New Mexico, which adapts a molecular array of YES and AND logic gates to calculate its moves. DNA strands substitute for silicon-based circuitry. Maya-II brazenly always plays first, usurps the center cell, and demands up to 30 minutes for each move.

NIM AND ITS VARIATIONS

A paradigmatic two-person game of pure skill, Nim is dubiously reputed to be of Chinese origin—possibly because it reflects the simplicity in structure combined with subtle strategic moves (at least to the mathematically unanointed) ascribed to Oriental games or possibly because it was known as Fan-Tan (although unrelated either to the mod 4 Chinese counter game or the elimination card game) among American college students toward the end of the 19th century.

The word *Nim* (presumably from *niman*, an archaic Anglo-Saxon verb meaning to take away or steal) was appended by Charles Leonard Bouton (Ref.), a Harvard mathematics professor, who published the first analysis of the game in 1901. In the simplest and most conventional form of Nim, an arbitrary number of chips is apportioned into an arbitrary number of piles, with any admissible distribution. Each player, in his turn, removes any number of chips one or greater from any pile, but only from one pile. That player who removes the final chip wins the game.

In the terminology of Professor Bouton, each configuration of the piles is designated as either "safe" or "unsafe." Let the number of chips in each pile be represented in binary notation; then a particular configuration is uniquely safe if the mod 2 addition of these binary representations (each column being added independently) is zero; this form of addition, known as the *digital sum*, is designated by the symbol $\dot{+}$. For example, three piles consisting of 8, 7, and 15 chips are represented as follows:

$$\begin{aligned} 8 &= 1000 \\ 7 &= 111 \\ 15 &= \underline{1111} \\ \text{Digital sum} &= 0000 \end{aligned}$$

and constitute a safe combination. Similarly, the digital sum of 4 and 7 and 2 is $100 \dot{+} 111 \dot{+} 10 = 1$; three piles of four, seven, and two chips thus define an unsafe configuration.

It is easily demonstrated that if the first player removes a number of chips from one pile so that a safe combination remains, then the second player cannot do likewise. He can change only one pile and he must change one. Since, when the numbers in all but one of the piles are given, the final pile is uniquely determined (for the digital sum of the numbers to be zero), and since the first player governs the number in that pile (i.e., the pile from which the second player draws), the second player cannot leave that number. It also follows, with the same reasoning, that if the first player leaves a safe combination, and the second player diminishes one of the piles, the first player can always diminish some pile and thereby regain a safe combination. Clearly, if the initial configuration is safe the second player wins the game by returning the pattern to a safe one at each move. Otherwise, the first player wins.

Illustratively, beginning with three piles of four, five, and six chips, the first player performs a parity check on the columns:

$$\begin{aligned} 4 &= 100 \\ 5 &= 101 \\ 6 &= \underline{110} \\ \text{Digital sum} &= 111 \end{aligned}$$

Bouton notes that removing one chip from the pile of four, or three chips from the pile of five, or five chips from the pile of six produces even parity (a zero digital sum, a safe position) and leads to a winning conclusion. Observe that from an unsafe position, the move to a safe one is not necessarily unique.

A modified version of Nim is conducted under the rule that the player removing the last chip *loses*. The basic strategy remains unaltered, except that the configuration 1, 1, 0, 0, …, 0 is unsafe and 1, 0, 0, …, 0 is safe.

If the number of counters initially in each pile is selected at random, allowing a maximum of 2^{m-1} chips per pile (m being any integer), the possible number of different configurations N with k piles is described by

$$N = \binom{2^m + k - 2}{k}$$

Frequently Nim is played with three piles, whence

$$N = \frac{(2^m + 1)(2^m)(2^m - 1)}{6} = \frac{2^{m-1}(2^{2m} - 1)}{3}$$

The number of safe combinations N_s in this case is

$$N_s = \frac{(2^{m-1} - 1)(2^m - 1)}{3}$$

Thus, the probability P_s of creating a safe combination initially is given by

$$P_s = \frac{N_s}{N} = \frac{2^{m-1} - 1}{2^{m-1}(2^m + 1)}$$

which is the probability that the second player wins the game, assuming that he conforms to the correct strategy.

Moore's Nim

A generalization of Nim proposed by E.H. Moore (Ref.) allows the players to remove any number of chips (one or greater) from any number of piles (one or greater) not exceeding k. Evidently, if the first player leaves fewer than $k + 1$ piles following his move, the second player wins by removing all the remaining chips. Such a configuration is an unsafe combination in generalized Nim. Since the basic theorems regarding safe and unsafe patterns are still valid, a player can continue with the normal strategy, mod $(k + 1)$, until the number of piles is diminished to less than $k + 1$, whence he wins the game.

To derive the general formula for safe combinations, consider n piles containing, respectively, $c_1, c_2, \ldots, c_n$ chips. In the binary scale of notation,[4] each number c_i is represented as $c_i = c_{i0} + c_{i1}\,2^1 + c_{i2}\,2^2 + \cdots + c_{ij}2^j$. The integers c_{ij} are either 0 or 1 and are uniquely determinable. The combination is safe if and only if

$$\sum_{i=1}^{n} c_{ij} = 0, \quad \operatorname{mod}(k+1), \quad j = 0, 1, 2, \ldots$$

that is, if and only if for every place j, the sum of the n digits c_{ij} is exactly divisible by $k + 1$.

Moore's generalization is referred to as Nim_k. Bouton's game thus becomes Nim_1 and provides a specific example wherein the column addition of the chips in each pile is performed mod 2.

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[4] It is also possible to develop the theory with a quaternary representation of the numbers. Such a procedure, although less familiar than binary representation, involves fewer terms.

Matrix Nim

Another generalization, termed Matrix Nim and designated by Nim^k (nomenclature applied by John C. Holladay (Ref.) to distinguish the game from Moore's version), consists of arranging the k piles of chips into an $m \times n$ rectangular array. For a move in this game, the player removes any number of chips (one or greater) from any nonempty set of piles, providing they are in the same row or in the same column, with the proviso that at least one column remains untouched. The game concludes when all the piles are reduced to zero, the player taking the final chip being the winner. Nim^k with $k = 1$ becomes, as with Moore's generalization, the ordinary form of Nim.

A safe position in Nim^k occurs if the sum of the chips in any column is equal to the sum of the chips in any other column and if the set of piles defined by selecting the smallest pile from each row constitutes a safe position in conventional Nim.

Shannon's Nim

In a Nim variation proposed by Claude Shannon (Ref. 1955), only a prime number of chips may be removed from the pile. Since this game differs from Nim_1 only for those instances of four or more chips in a pile, a safe combination is defined when the last two digits of the binary representations of the number of chips in each pile sum to 0 mod 2. For example, a three-pile game of Nim_1 with 32, 19, and 15 chips provides a safe configuration in Shannon's version, although not in the conventional game. This situation is illustrated as follows:

$$
\begin{array}{rr}
15 = & 1111 \\
19 = & 10011 \\
32 = & \underline{100000} \\
\text{Digital sum} = & 111100
\end{array}
$$

It should be noted that the basic theorems of safe and unsafe combinations do not hold for prime-number Nim_1.

Wythoff's Nim—Tsyan/Shi/Dzi

Obviously, we can impose arbitrary restrictions to create other variants of Nim. For example, a limit can be imposed on the number of chips removed or that number can be limited to multiples (or some other relationship) of another number. A cleverer version, credited to W.A. Wythoff (Ref.) restricts the game to two piles, each with an arbitrary number of chips. A player may remove chips from either or both piles, but if from both, the same number of chips must be taken from each. The safe combinations are the Fibonacci pairs: (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 20), … . The rth safe pair is $([r\tau], [r\tau^2])$, where $\tau = (1 + \sqrt{5})/2$, and $[x]$ is defined as the greatest integer not exceeding x. Note that every integer in the number scale appears once and only once. Although evidently unknown to Wythoff, the Chinese had long been playing the identical game under the name of Tsyan/shi/dzi (picking stones).

Unbalanced Wythoff

One extension of Wythoff's game (Ref. Fraenkel, 1998) defines each player's moves as either (1) taking any (positive) number of counters from either pile, or (2) removing r counters from one pile and $r \le s$ from the other. This choice is constrained by the condition $0 \le$ s—r $<$ r + 2, $r > 0$. That player who reduces both piles to zero wins.

The recursion expressions for this game (for all values of $n \ge 0$) are

$$A_n = \text{mex}\{A_i, B_i\colon i < n\} \quad \text{mex designates the minimum excluded value in the set } \{\bullet\}$$
$$B_n = 2(A_n + n)$$

which generate the safe positions shown in Table 10-1. The player who leaves any of these pairs for his opponent ultimately wins the game.

Table 10-1 Safe Positions for Unbalanced Wythoff

n	A_n	B_n	n	A_n	B_n	n	A_n	B_n
0	0	0	8	10	36	16	20	72
1	1	4	9	11	40	17	21	76
2	2	8	10	13	46	18	23	82
3	3	12	11	14	50	19	24	86
4	5	18	12	15	54	20	25	90
5	6	22	13	16	58	21	27	96
6	7	26	14	17	62	22	28	100
7	9	32	15	19	68	23	29	104

Four-Pile Wythoff

A further extension of Wythoff's game—devised by Fraenkel and Zusman (Ref.)—is played with $k \ge 3$ piles of counters. Each player at his turn removes (1) any number of counters (possibly all) from up to $k - 1$ piles, or (2) the same number of counters from each of the k piles. That player who removes the last counter wins.

For $k = 4$, the first few safe positions are listed in Table 10-2. The player who reduces the four piles to any of these configurations ultimately wins the game.

Table 10-2 Safe Positions for Four-Pile Wythoff

(0, 0, 0, 0)				
(1, 1, 1, 2)				
(3, 3, 3, 5),	(3, 3, 4, 4)			
(6, 6, 6, 9),	(6, 6, 7, 8),	(6, 7, 7, 7)		
(10, 10, 10, 14),	(10, 10, 11, 13),	(10, 10, 12, 12),	(10, 11, 11, 12)	
(15, 15, 15, 20),	(15, 15, 16, 19),	(15, 15, 17, 18),	(15, 16, 16, 18),	(15, 16, 17, 17)
(21, 21, 21, 27),	(21, 21, 22, 26),	(21, 21, 23, 25),	(21, 21, 24, 24),	(21, 22, 22, 25),
			(21, 22, 23, 24),	(21, 23, 23, 23),
. . .				

The Raleigh Game (After Sir Walter Raleigh for no discernible reason)

In this yet further extension of Wythoff's game proposed by Aviezri Fraenkel (Ref. 2006), the format encompasses three piles of chips, denoted by (n_1, n_2, n_3), with the following take-away rules:

1. Any number of chips may be taken from either one or two piles.
2. From a position where two piles have an equal number of chips, a player can move to (0, 0, 0).
3. For the case where $0 < n_1 < n_2 < n_3$, a player can remove the same number of chips m from piles n_2 and n_3 and an arbitrary number from pile n_1—with the exception that, if $n_2 - m$ is the smallest component in the new position, then $m \neq 3$.

The safe positions for Raleigh, (A_n, B_n, C_n), are generated from the recursive expressions

$$A_n = \text{mex}\{A_i, B_i, C_i : 0 \leq 1 \leq n\}$$

$$B_n = A_n + 1$$

$$C_n = \begin{cases} C_{n-1} + 3, & \text{if } A_n - A_{n-1} = 2 \\ C_{n-1} + 5, & \text{if } A_n - A_{n-1} = 3 \ (n \geq 2) \end{cases}$$

and are tabulated in Table 10-3. The player who configures the three piles into one of these positions can force a win.

Table 10-3 Safe Positions for the Raleigh Game

n	A_n	B_n	C_n	n	A_n	B_n	C_n
0	0	0	0	11	27	28	45
1	1	2	3	12	30	31	50
2	4	5	8	13	33	34	55
3	6	7	11	14	35	36	58
4	9	10	16	15	38	39	63
5	12	13	21	16	40	41	66
6	14	15	24	17	43	44	71
7	17	18	29	18	46	47	76
8	19	20	32	19	48	49	79
9	22	23	37	20	51	52	84
10	25	26	42				

Nimesis (Ref. Cesare and Ibstedt)

A, accorded the first move, divides a pile of N counters into two or more equal-sized piles and then removes a single pile to his personal crib. **B**, for his move, reassembles the remaining counters into a single pile and then performs the same operation. **A** and **B** continue to alternate turns until a single counter remains. That player whose crib holds the greater number of counters wins the game.

To compensate for **A**'s moving first, **B** selects the value of N. With N restricted to numbers under 100, **B**'s optimal strategy is to select that N for which both N and

$$\frac{N-(2^{i+1}-1)}{2^{i+1}}$$

are prime for all values of i from 0 to $[\log(N+1)/\log 2]-1$, where $[x]$ denotes the largest integer within x. Only $N = 47$ satisfies these conditions.

For the first step of the game, **A** must, perforce, divide the 47 counters into 47 piles of 1 counter each. **B** will then divide the remaining 46 counters into two piles of 23 counters. Over the five steps defining the complete game, **A** will accrue 5 counters, and **B** will garner $23 + 11 + 5 + 2 + 1 = 42$ counters. Thus **B** has a winning fraction of $42/47 = 0.894$.

With N allowed a value up to 100,000, **B** selects $N = 2879$. Thence **A**'s counters total $1 + 1 + 1 + 1 + 1 + 1 + 22 + 1 + 1 + 1 = 31$, while **B**'s counters add up to $1439 + 719 + 359 + 179 + 89 + 44 + 11 + 5 + 2 + 1 = 2848$. Accordingly, **B**'s winning fraction is $2848/2879 = 0.989$. As the allowable upper limit is raised, the winning fraction for **B**—already high—approaches yet closer to 1.

Tac Tix

Another ingenious Nim variation, one that opens up an entire new class, was proposed by Piet Hein[5] under the name Tac Tix (known as Bulo in Denmark). The game, which follows logically from Matrix Nim, deploys its chips in an $n \times m$ array. Two players alternately remove any number of chips either from any row or from any column, with the sole proviso that all chips taken on a given move be adjoining.

Tac Tix must be played under the rule that the losing player is the one removing the final chip. Otherwise, there exists a simple strategy that renders the game trivial: the second player wins by playing symmetrically to the first player unless the matrix has an odd number of chips in each row and column, in which case the first player wins by removing the center chip on his first move and thereafter playing symmetrically to his opponent. No strategy is known for the general form of Tac Tix. In the 3×3 game the first player can always win with proper play (his first move should take the center or corner chip or the entire central row or column). As a practical contest between sophisticated game theorists, the 6×6 board is recommended.

[5] A Copenhagen resident of Dutch ancestry, Herre Hein is also the creator of Hex.

Chomp

A Nim-type game invented by David Gale (Ref. 1974), Chomp (the name appended by Martin Gardner) postulates an M by N chocolate bar whose upper-left corner square (1, 1) is poisonous. Two players alternately select a square, which is eaten along with all other squares below and to its right. The initial position consists of the whole bar (i, j), where $1 \le i \le M$, and $1 \le j \le N$. Thus the player who selects the square (i_o, j_o) must perforce eat all the remaining squares where both $i \ge i_o$ and $j \ge j_o$. He who eats the poisonous square loses.

The game is celebrated for Gales's Chomp existence theorem—akin to the strategy-stealing argument of Tic-Tac-Toe. Consider that the first player eats square (M, N), the lower-right corner—leaving $MN-1$ squares. This act constitutes either a winning move or a losing move; if the latter, then the second player has a winning countermove. However, any position reachable from the new array with $MN-1$ squares is also reachable from the initial array with MN squares. Hence the first player could have moved there directly. There exists, *ipso facto*, a winning move for the first player.

Nim-Like Games

Numerous games have been proposed that, on cursory inspection, appear difficult to analyze but succumb quickly when their structures are recognized as Nim-like in form. Such games are referred to as NimIn (for Nim Incognito).

Perhaps the simplest type of NimIn involves "coins on a strip." Consider the format depicted in Figure 10-4. **A** and **B** alternate in moving any coin any number of squares leftward. Only a single coin may occupy any one square, and no coin may jump over another. The game ends when all the coins are lined up at the left end of the strip; the player who achieves this configuration wins.

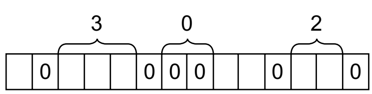

FIGURE 10-4 A simple NimIn game.

The key in this instance is the recognition that the lengths of the gaps between coins—beginning with the rightmost—are simply the sizes of the equivalent Nim piles. Then the initial nim-sums in Figure 10-4 are

$$\begin{array}{r@{\;=\;}r} 2 & 10 \\ 0 & 0 \\ 3 & \underline{11} \\ \text{Digital sum} & 01 \end{array}$$

which constitute an unsafe position. A winning move reduces the gap of three squares to two squares, leaving

$$\begin{aligned} 2 &= 10 \\ 0 &= 0 \\ 2 &= 10 \\ \hline \text{Digital sum} &= 00 \end{aligned}$$

a safe position. (Note that an increase in a gap length is possible by moving the left coin of a pair. However, the winning strategy remains unaffected.)

A somewhat more interesting variant is Stepine's game: played with n coins and a gold piece arrayed at intervals along a line whose left-hand side terminates in a moneybag—illustrated in Figure 10-5 for six coins and the gold

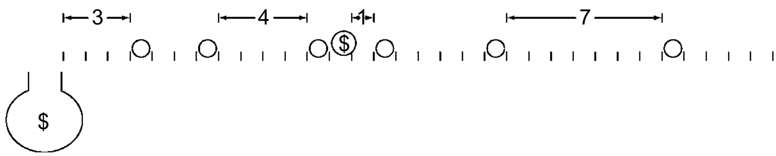

FIGURE 10-5 Stepine's game.

piece. A move (by each of two players in successive turns) consists of shifting any coin or the gold piece any number of empty spaces leftward. When a coin is moved to the leftmost space, it falls into the moneybag. That player who deposits the gold piece in the moneybag takes it home.

Again, the resolution of Stepine's game lies in recognizing its Nim-like structure. Accordingly, the coins (and gold piece) should be considered pairwise beginning at the right. The number of pairs corresponds to the number of piles, and the number of spaces within each pair to the number of chips in that pile in the Nim_1 equivalency. If a coin is left unpaired, the winning strategy includes the money-bag space if the gold piece forms the left side of a pair and excludes the money-bag space if it forms the right side of a pair (if the gold piece is leftmost, it can be moved directly into the money bag). In the example of Figure 10-5, we compute the digital sum of 3, 4, 1, and 7—which is nonzero. A winning move consists of reducing the 3, 1, or 7 interval by one.

Superficially more complicated is Northcott's Nim. Here, the playing field is an $n \times m$ checkerboard. Each row of the board holds a white piece and a black piece. For his turn, White may move any one of his pieces along its row, in either direction, as far as the empty squares allow. Black then moves any one of his pieces under the same rules. The game proceeds in this manner until one player cannot move—and thereby loses.

Northcott's Nim is equivalent to Nim_1 with a pile of chips for each row. The number of chips in a particular pile is equivalent to the number of squares between the black and white pieces in the corresponding row. (Each player's prerogative to increase the space between black and white pieces does not alter the character of the game or its ultimate convergence.)

Unsolved Nim Variants

1. Nim_k played under the restriction that chips may be removed only from a prime number of piles.
2. Wythoff's Nim played with the option of removing simultaneously from the two piles an arbitrary ratio a:b of chips (rather than an equal number).
3. Wythoff's Nim played with three or more piles, where the player is awarded the option of removing the same number of chips simultaneously from each pile.
4. Tac Tix played with upper and lower limits placed on the number of chips that may be removed at each move.
5. Tac Tix played with the option of removing the same number of chips in both directions from a row-column intersection.

Nim Computers

A Nim_1-playing computer should logically be a relatively simple device because of the binary nature of the strategy. Nimatron, the first such computer, was patented in 1940 by Edward Condon (Ref.), former director of the National Bureau of Standards. Built by the Westinghouse Electric Corp, Nimatron was exhibited at the New York World's Fair, where it played 100,000 games, winning about 90,000 (most of its defeats were administered by attendants demonstrating that possibility). At present, it reposes in the scientific collection of the Buhl planetarium in Pittsburgh. One year later, a vastly improved machine was designed by Raymond M. Redheffer (Ref.). Both the Redheffer and Condon computer programs were devised for play with four piles, each containing up to seven chips.

Subsequently, a Nim_1-playing device named Nimrod was exhibited at the 1951 Festival of Britain and later at the Berlin Trade Fair. A different approach to Nim was expressed through a simple relay computer developed at the Research Institute of Mathematical Machines (Prague) in 1960. This machine, programmed with knowledge of the optimal strategy, is pitted against a mathematical neophyte. The designated objective is to achieve a victory while disclosing as little information as possible regarding the correct game algorithm. Hence the computer applies its knowledge sparingly as a function of its opponent's analytic prowess.

SINGLE-PILE COUNTDOWN GAMES

Although not linked by firm evidence, it is likely that Nim is related to a game described by Bachet de Mésiriac in the early 17th century. From a single pile comprising an arbitrary number of chips, each of two players alternately removes from one to a chips. The winner is that player who reduces the pile to zero. Thus, a safe position in Bachet's game occurs when the number of chips in the pile totals 0, mod $(1 + a)$.

Single-pile games[6] can be formulated by specifying any restricted set of integers $S = \{n_1, n_2, \ldots\}$ that prescribes the number of chips that may be taken from the pile at each move. For each such set of integers, there exist positions (states) of the pile that are winning, losing, or tying for the player confronted with a move. Designating the winning and losing states by $W(S)$ and $L(S)$, respectively, we observe that

$$W(S) \cup L(S) = \text{the set of all integers} \geq 0$$

and

$$W(S) \cap L(S) = \emptyset$$

that is, the union of the winning and losing sets includes all nonnegative integers, while the intersection of these sets is empty, under the assumption that the set of tying positions $T(S) = \emptyset$—an assumption valid if and only if 0 is not a member of the subtractive set S and if 1 is a member of S. For greater harmony, it is advisable to include the number 1 in the subtractive set and to designate the winner as that player reducing the pile to zero.

As in the various Nim games, every member of S added to a member of $L(S)$ results in a member of $W(S)$. Thus, a losing or safe position is always attainable from a winning or unsafe position, but not from another safe position.

Two subtractive sets S_1 and S_2 are characterized as *game isomorphic* if they define games with identical losing states. For example, the set of all positive powers of two, $S_1 = \{1, 2, 4, 8, \ldots\}$, is game isomorphic to $S_2 = \{1, 2\}$ since

$$L(S_1) = L(S_2) = \{0, \text{mod } 3\}$$

(no power of 2 is a multiple of 3), and the set of primes, $S_1 = \{1, 2, 3, 5, 7, \ldots\}$, is game isomorphic to $S_2 = \{1, 2, 3\}$, since

$$L(S_1) = L(S_2) = \{0, \text{mod } 4\}$$

(no prime is a multiple of 4). Thus, a single-pile countdown game where only a prime number of chips may be subtracted can be won by the first player to reduce the pile to a multiple of 4.

It can be demonstrated that game isomorphism is closed under union, but not under intersection. For example, consider the two subtractive sets $S_1 = \{1, 4, 5\}$ and $S_2 = \{1, 3, 4, 7\}$. We compute $L(S_1)$ by the paradigm shown in Table 10-4. The sequence of $L(S_1)$ is determined by entering in the $L(S_1)$ column the lowest integer not previously represented; that integer is then added to 1, 4, and 5, respectively, and the sum entered in the corresponding column, etc. Evidently, $L(S_1) = \{0, 2, \text{mod } 8\}$. By the same technique, we can also calculate $L(S_2) = \{0, 2, \text{mod } 8\}$ and show that $L(S_1 \cup S_2) = \{0, 2, \text{mod } 8\}$, whereas $L(S_1 \cap S_2) = \{0, 2, \text{mod } 5\}$.

[6]The succeeding analysis of single-pile games was first outlined in a USC lecture by Professor Solomon Golomb.

Table 10-4 Computation of Losing States $L(S_1)$

	S_1		
$L(S_1)$	1	4	5
0	1	4	5
2	3	6	7
8	9	12	13
10	11	14	15
16	17	20	21
18	19	22	23
24	25	28	29
26			

To complete our understanding of the sets that exhibit game isomorphism, we introduce the states $\lambda(S)$ as all the nonnegative first differences of $L(S)$. We can prove that $\lambda(S)$ cannot intersect S—that is,

$$\lambda(S) \cap S = \emptyset$$

and the converse statement that any number not in $\lambda(S)$ can be adjoined into S. Illustratively, consider the set $S = \{1, 4\}$, which generates the set of losing positions $L(S) = \{0, 2, \text{mod } 5\}$. In this instance, the numbers 3, mod 5, are obtained by first differences of the other members of $L(S)$, so that $\lambda(S) = \{0, 2, 3, \text{mod } 5\}$. All remaining numbers (1, mod 5 and 4, mod 5) can be adjoined to S:

$$S \cup \{1, \text{mod } 5 \;\text{ and }\; 4, \text{mod } 5\} \equiv S^*$$

Thus, any set S^* is game isomorphic to S. For example, the game played with $S = \{1, 4\}$ is equivalent (in the sense of identical losing positions) to that played with $S^* = \{1, 4, 9, 16\}$.

Given a particular subtractive set, it is not always possible to find another set with the property of game isomorphism. As one instance, consider the game whereby, from an initial pile of m chips, each player alternately subtracts a perfect-square number of chips, $S = \{1, 4, 9, 16, \ldots\}$, with the usual objective of removing the final chip to score a win. It is not apparent that a simple formula exists that determines the safe positions $L(S)$ for this game. However, as demonstrated by Golomb, the sequence of safe positions can be generated by an appropriately constructed shift register.

A binary shift register is a simple multistage device whereby, in a given configuration, each stage exhibits either a 1 or a 0. At prescribed intervals the output of each stage of the register assumes the value represented by the output of the previous stage over the previous interval. The outputs from one or more stages are operated on in some fashion and fed back into the first stage. For the single-pile countdown game with $S = \{1, 4, 9, 16, \ldots\}$, the appropriate shift register is of semi-infinite length, since the perfect squares form an infinite series. The outputs of stages 1, 4, 9, 16, … are delivered to a NOR gate and thence returned to the first stage.

Figure 10-6 indicates the perfect-square shift register. Initially, the first stage is loaded with a 1 and the remaining stages with 0s. If at least one 1 enters the

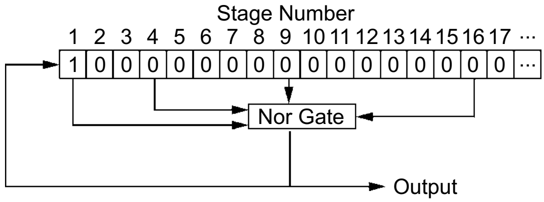

FIGURE 10-6 The perfect-square shift register.

NOR gate, it feeds back a 0 to the first stage as the register shifts; otherwise, a 1 is fed back. The continuing output of the shift register designates the safe and unsafe positions, with 1 representing safe and 0 unsafe. Table 10-5 illustrates the shift-register performance. According to the output sequence, positions 2, 5, and 7 (of the first 8) are safe; that is, if a player reduces the pile of chips to 2, 5, or 7, he possesses a winning game, since his opponent can achieve only an unsafe position.

Table 10-5 The Perfect-Square Shift Register Output

No. of Shifts, *t*	Shift Register Contents after *t* Shifts										Output Symbol after *t*th Shift
1	1	0	0	0	0	0	0	0	0	0 …	0
2	0	1	0	0	0	0	0	0	0	0 …	1
3	1	0	1	0	0	0	0	0	0	0 …	0
4	0	1	0	1	0	0	0	0	0	0 …	0
5	0	0	1	0	1	0	0	0	0	0 …	1
6	1	0	0	1	0	1	0	0	0	0 …	0
7	0	1	0	0	1	0	1	0	0	0 …	1
8	1	0	1	0	0	1	0	1	0	0 …	0
⋮											⋮

Allowing the perfect-square shift register to run for 1000 shifts produces 1s at the positions shown in Table 10-6. Thus, beginning with a pile of 1000 or less, the player who reduces the pile to any of the positions shown in this table can achieve a win.

Table 10-6 The Safe Positions for the Perfect-Square Game

2	5	7	10	12	15	17	20	22	34	39	44	52	57
62	65	67	72	85	95	109	119	124	127	130	132	137	142
147	150	170	177	180	182	187	192	197	204	210	215	238	243
249	255	257	260	262	267	272	275	312	317	322	327	332	335
340	345	350	369	377	390	392	397	425	430	437	442	447	449
464	502	507	510	512	515	517	520	522	554	572	589	595	600
613	619	652	663	665	680	702	704	707	724	770	787	793	827
850	855	860	862	865	867	782	894	918	923	928	935	940	979
990	997												

There are 114 safe positions for a pile of 1000 chips, 578 safe positions for a pile of 10,000 chips, and 910 safe positions for a pile of 20,000 chips. Evidently, as m, the number of initial chips, increases, the percentage of safe positions decreases. The totality of safe positions is infinite, however, as $m \to \infty$ (there exists no largest safe position).

A substantial number of engaging single-pile countdown games can be invented. As one example, we can regulate each player to subtract a perfect-cube number of chips. For this game, the semi-infinite shift register of Figure 10-6 is rearranged with the outputs of stages 1, 8, 27, 64, … fed to the NOR gate. The output of such a shift register indicates safe positions at 2, 4, 6, 9, 11, 13, 15, 18, 20, 22, 24, 34, 37, 39, 41, 43, 46, 48, 50, 52, 55, 57, 59, 62, 69, 71, 74, 76, 78, 80, 83, 85, 87, 90, 92, 94, 97, 99, … .

The shift register method can also be enlisted to solve more elementary countdown games. Consider Bachet's game with $a = 4$—that is, $S = \{1, 2, 3, 4\}$. In this instance we construct a finite four-stage shift register (the number of stages required is specified by the largest member of S) with the outputs of all four stages connected to the NOR gate, as shown in Figure 10-7. Inserting a 1 into the first stage and 0s into the remaining stages and permitting the shift register to run results in the output sequence 00001000010000100…; the safe positions occur at 0, mod 5, as we would expect.

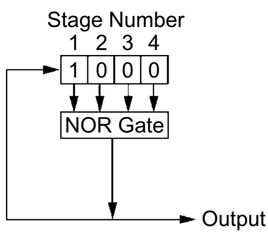

FIGURE 10-7 Shift register for Bachet's game.

Several extensions of Bachet's game are worthy of note. For one example, we might impose the additional constraint that each player must remove at least b chips (but no more than $a > b$) at each move. A win is awarded to that player reducing the pile to $b - 1$ or fewer chips. The safe positions here are 0, 1, 2, …, $b - 1$, mod $(b + a)$. If different limits are assigned to each player, the possessor of the larger limit can secure the winning strategy.

Another offspring of Bachet's game arises from the restriction that the same number of chips (of the set $S = \{1, 2, \ldots, a\}$) cannot be subtracted twice in succession. If a is even, no change is effected by this rule: $L(S) = \{0, \text{mod } (a + 1)\}$. If a is odd, correct strategy is altered in certain critical positions; for example, with $a = 5$ and six chips remaining, the winning move consists of subtracting 3 chips, since the opponent cannot repeat the subtraction of 3. Specifically, for $a = 5$, $L(S) = \{0, 7, \text{mod } 13\}$. For any odd value of a except 1 and 3, $L(S) = \{0, a + 2, \text{mod } (2a + 3)\}$; for $a = 3$, $L(S) = \{0, \text{mod } 4\}$.

Slim Nim (Sulucrus)

A and **B** alternately remove counters from a single pile of N counters ($N > 22$). **A** has the option at each turn of removing 1, 3, or 6 counters; **B** at his turn may remove 2, 4, or 5. Objective: to remove the last counter(s). The game is a win for **B** regardless of which player moves first (Ref. Silverman).

If N is of the form 2, mod 5 or 4, mod 5, **B**, at his first turn, subtracts 2 or 4 chips, respectively, and thereafter counters a move by **A** (1, 3, or 6) with a move complementary to 0, mod 5—that is, 4, 2, or 4.

If N is of the form 3, mod 5, **B** subtracts 5 at his first turn and thereafter again moves to reduce the pile to 0, mod 5—removing 2, 5, 2 against 1, 3, 6, respectively.

Finally, if N is of the form 1, mod 5, **B** removes 5 chips and thereafter subtracts 5 against a move of 1 or 6. Against a reduction of 3 chips by **A, B** is left with 3, mod 5 chips and proceeds as indicated previously.

Single-Factor Nim

Beginning with a pile of N counters, **A** and **B** alternately subtract from the number of counters remaining any single factor of that number except for the number itself. Whoever performs the last subtraction, thereby leaving his opponent with a single counter, wins the game.

For example, with $N = 64$, **A** may subtract 1, 2, 4, 8, 16, or 32 (but not 64) counters. If **A** subtracts 2, then **B** may subtract 1, 2, or 31, and so forth.

Since all factors of odd numbers are odd, subtracting one counter leaves an even number. Therefore, the safe positions are the odd numbers; the opponent will perforce then leave an even number of counters. The winning move for **A** from a pile of 64 counters is to subtract one. **B** will ultimately leave two counters, whence **A** will execute the final coup by leaving a single counter.

If the rules are changed to preclude removing a single counter, the safe positions are now the odd numbers plus the odd powers of 2.

Illustratively, from $N = 64$ (2^6) counters, **A**'s winning move consists of subtracting 32 (2^5) counters. If **B** leaves a number other than a power of 2, **A** then subtracts the largest odd factor, leaving an odd number. Thence **B** is forced to leave an even number and will eventually be faced with a prime number, which admits of no legal move. Alternatively, if **B** subtracts a power of 2, **A** responds by subtracting the next power of 2, ultimately leaving **B** with the losing position of 2^1 counters.

Fibonacci (Doubling) Nim

From a pile of N counters, **A** and **B** alternately may remove up to twice as many counters as was taken on the previous move (**A**, as his first move, is prohibited from removing all the counters). Winner is that player taking the last counter (Ref. uncc.edu).

The key to this game is the representation of N as a sum of distinct Fibonacci numbers (similar to the procedure wherein binary notation expresses a positive integer as the sum of distinct powers of 2). For $N = 20$ counters, for example, we can write the corresponding Fibonacci numbers—using Table 5–1—as

Fibonacci series:	13	8	5	3	2	1
Representation for 20:	1	0	1	0	1	0

since $20 = 13 + 5 + 2$ constitutes a unique sum. The winning move for **A** consists of removing that number of counters indicated by the rightmost 1 in the Fibonacci representation (the smallest summand)—in this instance, two counters. The remaining number, 18, defines a safe position. **B** must then remove up to four counters, leaving 14 to 17, all unsafe positions.

Illustratively, let **B** remove four counters. Then **A**, at his next turn, may take up to eight from the 14 remaining. To determine his next move, **A** expresses 14 as a Fibonacci number:

Fibonacci series:	13	8	5	3	2	1
Representation for 14:	1	0	0	0	0	1

and follows the indicated strategy of removing one counter (the rightmost 1 in the Fibonacci representation). Ultimately, **A** will take the last counter. Note that **B** has a winning strategy iff N is a Fibonacci number.

Nebonacci (Tripling) Nim

Here, **A** and **B**, from the pile of N counters, alternately remove up to *three* times as many counters as was taken on the previous move—with the winner taking the last counter. The key to this game is the Nebonacci sequence (Table 5-1). With $N = 20$ counters, the corresponding Nebonacci numbers are

Nebonacci series:	13	7	4	2	1
Representation for 20:	1	0	1	1	1

since $20 = 13 + 4 + 2 + 1$. **A**'s indicated move consists of removing one counter (the smallest summand), thereby leaving a safe position (19). **B**, as before, is compelled to leave an unsafe position, and **A** will ultimately take the last counter. Again, **B** has a winning strategy iff N is a Nebonacci number.

Multibonacci Nim

For the general game, **A** and **B** alternatively remove from a single pile of N counters m times as many counters as was taken on the previous move. Optimal strategy is specified by the appropriate multibonacci numbers (Table 5-1).

Further generalizations can be implemented by imposing an upper limit F(n) on the number of counters taken by a player when the previous move has taken

N counters. F(N) need not be limited to linear functions as in the multibonacci examples.

Single-Pile Variants

The following variations on the basic theme can claim interesting ramifications.

1. A single-pile countdown game played under a rule constraining one player to remove a number of chips defined by one set of integers while the second player may remove a number of chips described by a different set of integers. Note that safe and unsafe positions are not mutually exclusive in this game.
2. A single-pile game played under the rule that each member of the subtractive set $S = \{n_1, n_2, \ldots\}$ can be used only once. Clearly, the initial number of chips m must be chosen less than or equal to the sum of the set of numbers $n_1 + n_2 + \cdots$.
3. A single-pile game wherein each player may either subtract or add the largest perfect-square (or cube or other function) number of chips contained in the current pile. The number of chips in the pile can obviously be increased indefinitely under this rule. It is also apparent that certain positions lead to a draw—for example, with a pile of two chips, a player will add one, whereas with a pile of three chips, he will subtract one; thus if the pile ever contains two or three chips, the game is drawn. Every drawn state less than 20,000 is reducible to the 2-3 loop except for the loop 37, 73, 137, 258, 514, 998, 37, …, wherein only one player has the option of transferring to the 2-3 oscillation. Some of the winning, losing, and tying states are listed in Table 10-7.

Table 10-7 The Game of Add or Subtract a Square: Winning, Losing, and Tying Positions

L(S)	*W(S)*	*T(S)*
5, 20, 29, 45, 80, 101, 116, 135, 145, 165, 173, 236	1, 4, 9 11, 14, 16, 21, 25, 30, 36, 41, 44, 49, 52, 54, 64, 69, 81, 86, 92, 100, 105, 120, 121, 126, 144	2, 3, 7, 8, 12, 26, 27, 37, 51, 73, 137, 258

THE GRUNDY FUNCTION; KAYLES

Single-pile and other countdown games, such as Nim and Tac-Tix, are susceptible to analysis through certain powerful techniques developed by the theory of graphs. According to that theory, we have a *graph* whenever there exists (1) a set X, and (2) a function Γ mapping X into X. Each element of X is called a *vertex* and can be equated to what we have termed a position, or state, of a game. For a finite graph (X, Γ), we can define a function g that associates an integer $g(x) \geq 0$ with every vertex x. Specifically, $g(x)$ denotes a *Grundy function* on the graph if, for every vertex x, $g(x)$ is the smallest nonnegative integer (not necessarily unique) *not* in the set

$$g(\Gamma x) = \{g(y)|y \in \Gamma x\}$$

It follows that $g(x) = 0$ if $\Gamma x = \emptyset$.

Since, in a graph representation of a countdown game, each vertex represents a state of the game, and since we conventionally define the winner as that player who leaves the zero state for his opponent, the zero Grundy function is associated with a winning vertex. From all other vertices, there always exists a path to a vertex with a zero Grundy function, and from a zero Grundy function vertex there are connections only to vertices with nonzero Grundy functions (this statement is equivalent to the theorem of safe and unsafe positions at Nim). Letting the initial state in a countdown game be represented by x_0, the first player moves by selecting a vertex x_1 from the set Γx_0; then his opponent selects a vertex x_2 from the set Γx_1; the first player moves again by selecting a vertex x_3 from the set Γx_2, etc. That player who selects a vertex x_k such that $\Gamma x_k = \emptyset$ is the winner.

Analogous to the countdown games discussed previously, there is a collection of winning positions (vertices) that lead to a winning position irrespective of the opponent's responses. Specifically, the safe positions $L(S)$ with which a player wishes to confront his adversary are those whereby the digital sum of the individual Grundy functions is zero.

As an example, consider a simplified form of Tac Tix, embodying n distinct rows of chips, with no more than m chips in any row. A legal move consists of removing any integer number of adjoining chips from 1 to j, where $1 \leq j \leq m$. If chips are removed from other than a row end, the consequence is the creation of an additional row (since the chips removed must be adjoining). Two players alternate moves, and the player removing the final chip is declared the winner. For $j = 2$, the game is known as Kayles.

To compute the Grundy functions for Kayles, we begin with $g(0) = 0$; thence $g(1) = 1$, $g(2) = 2$, and $g(3) = 3$, since the two previous Grundy functions cannot be repeated. For $g(4)$, we observe that a row of four chips can be reduced to a row of three, a row of two, a row of two and a row of one, or two rows of one; the respective Grundy functions are 3, 2, the digital sum of 2 and 1 (i.e., 3), and the digital sum of 1 and 1 (i.e., 0). Hence, the vertex associated with a row of four counters is connected to other vertices with Grundy functions of 3, 2, 3, and 0. The smallest integer not represented is 1, and therefore $g(4) = 1$. Table 10-8 presents a tabulation of the Grundy functions for Kayles.

It is apparent that the Grundys here are almost periodic for smaller values of x and become perfectly periodic with period 12 for $x \geq 71$. We are consequently led to inquire as to the type of games associated with periodic Grundy functions. R.K. Guy and C.A.B. Smith (Ref.) have delineated a classification system that can distinguish those games whose Grundy functions are

Table 10-8 Grundy Functions for Kayles

x	$g(x)$	x	$g(x)$	x	$g(x)$	x	$g(x)$	x	$g(x)$	x	$g(x)$	x	$g(x)$
0	0	12	4	24	4	36	4	48	4	60	4	72	4
1	1	13	1	25	1	37	1	49	1	61	1	73	1
2	2	14	2	26	2	38	2	50	2	62	2	74	2
3	3	15	7	27	8	39	3	51	8	63	8	75	8
4	1	16	1	28	5	40	1	52	1	64	1	76	1
5	4	17	4	29	4	41	4	53	4	65	4	77	4
6	3	18	3	30	7	42	7	54	7	66	7	78	7
7	2	19	2	31	2	43	2	55	2	67	2	79	2
8	1	20	1	32	1	44	1	56	1	68	1	80	1
9	4	21	4	33	8	45	8	57	4	69	8	81	8
10	2	22	6	34	6	46	2	58	2	70	6	82	2
11	6	23	7	35	7	47	7	59	7	71	7	83	7

ultimately periodic. They define a sequence of numerals $\alpha_i\alpha_2\alpha_3\ldots$, $0 \leq \alpha_j \leq 7$ for all values of j, such that the jth numeral α_j symbolizes the conditions under which a block of j consecutive chips can be removed from one of the rows of a configuration of chips. These conditions are listed in Table 10-9. A particular sequence of α_js defines the rules of a particular game.

Table 10-9 Classification System for Periodic Grundy Functions

α_j	Conditions for Removal of a Block of *j* Chips
0	Not permitted
1	If the block constitutes the complete row
2	If the block lies at either end of the row, but does not constitute the complete row
3	Either 1 or 2
4	If the block lies strictly within the row
5	Either 1 or 4 (but not 2)
6	Either 2 or 4 (but not 1)
7	Always permitted (either 1 or 2 or 4)

Thus, Kayles is represented by 77, and Nim by 333. The Guy-Smith rule states that if a game is defined by a finite number n of α_js, and if positive integers y and p exist (i.e., can be found empirically) such that

$$g(x+p) = g(x)$$

holds true for all values of x in the range $y \leq x < 2y + p + n$, then it holds true for all $x \geq y$, so that the Grundy function has ultimate period p.

To illustrate this rule, consider the game of Kayles played with an initial configuration of three rows of 8, 9, and 10 chips. Referring to Table 10-8, the binary representations of the appropriate Grundy functions are displayed in the form

$$\begin{array}{r r} g(8) = & 1 \\ g(9) = & 100 \\ g(10) = & 10 \\ \hline \text{Digital sum} = & 111 \end{array}$$

Thus, the vertex $x_0 = (8, 9, 10)$ is not a member of $L(S)$ and hence constitutes a winning position. One winning move consists of removing a single chip from the row of 8 in a manner that leaves a row of 2 and a row of 5. The opponent is then faced with $x_1 = (2, 5, 9, 10)$ and a 0 Grundy function: $g(2) \dotplus g(5) \dotplus g(9) \dotplus g(10) = 10 \dotplus 100 \dotplus 100 \dotplus 10 = 0$. He cannot, of course, find a move that maintains the even parity for the digital sum of the resulting Grundy functions.

Simplified forms of Tac Tix (where the mapping function Γ is restricted to rows only) can be played with values of $j > 2$. Table 10-10 tabulates the Grundy functions up to $g(10)$ for $3 \le j \le 7$. The game defined by $j = 4$ is known as Double Kayles (7777 in the Guy-Smith classification system); its Grundy functions exhibit an ultimate period of 24. In general, for $j = 2^i$, the resulting Kayles-like games have Grundys that ultimately repeat with period $6j$.

Table 10-10 Grundy Functions for Simplified Tac Tix

Grundys	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
$g(0)$	0	0	0	0	0
$g(1)$	1	1	1	1	1
$g(2)$	2	2	2	2	2
$g(3)$	3	3	3	3	3
$g(4)$	4	4	4	4	4
$g(5)$	1	5	5	5	5
$g(6)$	6	6	6	6	6
$g(7)$	3	7	7	7	7
$g(8)$	2	3	8	8	8
$g(9)$	1	2	3	9	9
$g(10)$	6	8	2	3	10

Many other games with ultimately periodic Grundys are suggested by the Guy-Smith classification system. For example, game 31 (where $\alpha_1 = 3$, $\alpha_2 = 1$) specifies the rule that one chip may be removed if it constitutes a complete row or if it lies at either end of a row without being a complete row, while a block of two chips may be removed only if it constitutes a complete row. Some of these games are listed in Table 10-11. The overlined numbers refer to the periodic component of the Grundy functions.

Table 10-11 Some Games with Ultimately Periodic Grundy Functions

Game (Guy-Smith Classification System)	Grundy Functions $g(0), g(1)\ldots$	Period
03	$\overline{0011}$	4
12	$0\overline{1001}$	4
13	$\overline{0110}$	4
15	$0\overline{1101122122}$	10
303030… Nim with $S = \{2i + 1, i = 0, 1, \ldots\}$	$\overline{01}$	2
31	$012\overline{01}$	2
32	$0\overline{102}$	3
33030003… Nim with $S = \{2^i, i = 1, 2, \ldots\}$	$\overline{012}$	3
34	$0101201\overline{03121203}$	8
35	$0\overline{120102}$	6
52	$010\overline{2103}$	4
53	$0112210224\overline{0122112241}$	9
54	$010\overline{1222411}$	7
57	$0\overline{1122}$	4
71	$012\overline{10}$	2
72	$0\overline{1023}$	4

Nim and its variations, as described in previous sections, can also be analyzed with the theory of graphs. If the initial configuration in Nim$_1$, say, consists of n piles of chips, the corresponding graph requires an n-dimensional representation such that the vertex $(x_1, x_2, \ldots, x_n)$ defines the number of chips x_1 in the first pile, x_2 in the second pile, and so on. Allowable moves permit a vertex to be altered by any amount one unit or more in a direction orthogonal to an axis. The Grundy function of the number of chips in each pile equals that number—that is, $g(x) = x$; thus, the Grundy of each vertex is simply $g(x_1) \dotplus g(x_2) \dotplus \cdots \dotplus g(x_n)$. The members of $L(S)$ are those vertices labeled with a 0 Grundy function; the game winner is that player who reaches the vertex $(0, 0, \ldots, 0)$.

It is simpler to demonstrate this intelligence by considering a two-pile game such as Tsyan/shi/dzi (Wythoff's Nim). In this instance, the rules (Γ) permit each player to move one or more units along a line orthogonally toward either axis and also one or more units inward along the diagonal (corresponding to the removal of an equal number of chips from both piles). Grundy functions for the vertices of a Tsyan/shi/dzi game are readily calculated. The vertex (0, 0) is labeled with a 0, since it terminates the game; Grundy functions along the two axes increase by 1 with each outgoing vertex, since connecting paths are allowed to vertices of all lower values. The remaining assignments of Grundys follow the definition that a vertex is labeled with the smallest integer not represented by those vertices it is connected to by the mapping function Γ. Values of the graph to (12, 12) are shown in Figure 10-8. From the vertex (9,7), as illustrated, the mapping function permits moves to any of the positions along the three lines indicated. Those vertices with 0 Grundys are, of course,

12	13	14	15	11	9	16	17	18	19	7	8	10
11	9	10	7	12	14	2	13	17	6	19	15	8
10	11	9	8	13	12	0	15	16	17	18	19	7
9	10	11	12	8	7	13	14	15	16	17	6	19
8	6	7	10	1	2	5	3	4	15	16	17	18
7	8	6	9	0	1	4	5	4	14	15	13	17
6	7	8	1	9	10	3	4	5	13	0	2	16
5	3	4	0	6	8	10	1	2	7	12	14	9
4	5	3	2	7	6	9	0	1	8	13	12	11
3	4	5	6	2	0	1	9	10	12	8	7	15
2	0	1	5	3	4	8	6	7	11	9	10	14
1	2	0	4	5	3	7	8	6	10	11	9	13
0	1	2	3	4	5	6	7	8	9	10	11	12

FIGURE 10-8 Grundy functions for Tsyan/shi/dzi.

the members of $L(S)$ and constitute the safe positions: the Fibonacci pairs $([r\tau], [r\tau^2])$, $r = 1, 2, 3, \ldots$, where $\tau = (1 + \sqrt{5})/2$, and the brackets define the greatest integer not exceeding the enclosed quantity.

Distich, Even Wins

Countdown games can be distinguished by the position that designates the winner. In the game of Distich two players alternately divide a pile of chips, selected from a group of n piles, into two unequal piles.[7] The last player who can perform this division is declared the winner. Strategy for Distich evidently follows the rule of determining those vertices with a 0 Grundy, thus specifying the safe positions $L(S)$. The Grundy for a configuration of n piles is simply the digital sum of the Grundys of each pile. Since a pile of one or two chips cannot be divided into two unequal parts, we have $g(1) = g(2) = 0$. For a pile of three chips, $g(3) = 1$, as 3 can be split only into 2 and 1; the digital sum of the Grundys of 2 and 1 is 0, and 1 is thus the smallest integer not connected to the vertex (3). Table 10-12 tabulates the Grundy functions for Distich up to $g(100)$.

We should note that for Distich, as well as for Tsyan/shi/dzi and Nim, the Grundy functions are unbounded, although high values of $g(x)$ occur only for extremely high values of x. The safe positions for Distich are $L(S) = \{1, 2, 4, 7, 10, 20, 23, 26, 50, 53, 270, 273, 276, 282, 285, 288, 316, 334, 337, 340, 346, 359, 362, 365, 386, 389, 392, 566, \ldots\}$.

As a numerical example, we initiate a Distich game with three piles of 10, 15, and 20 chips. The corresponding Grundy functions are 0, 1, and 0, respectively, and their digital sum is 1; thus, for the vertex (10, 15, 20), the Grundy function is 1. A winning move consists of splitting the pile of 10 into two piles

[7] Initially proposed by P.M. Grundy.

Table 10-12 Grundy Functions for Distich

x	*g*(*x*)	*x*	*g*(*x*)	*x*	*g*(*x*)	*x*	*g*(*x*)	*x*	*g*(*x*)
1	0	21	4	41	5	61	1	81	2
2	0	22	3	42	4	62	3	82	4
3	1	23	0	43	1	63	2	83	5
4	0	24	4	44	5	64	4	84	2
5	2	25	3	45	4	65	3	85	4
6	1	26	0	46	1	66	2	86	3
7	0	27	4	47	5	67	4	87	7
8	2	28	1	48	4	68	3	88	4
9	1	29	2	49	1	69	2	89	3
10	0	30	3	50	0	70	4	90	7
11	2	31	1	51	2	71	3	91	4
12	1	32	2	52	1	72	2	92	3
13	3	33	4	53	0	73	4	93	7
14	2	34	1	54	2	74	3	94	4
15	1	35	2	55	1	75	2	95	3
16	3	36	4	56	5	76	4	96	5
17	2	37	1	57	2	77	3	97	2
18	4	38	2	58	1	78	2	98	3
19	3	39	4	59	3	79	4	99	5
20	0	40	1	60	2	80	5	100	2

of 3 and 7; the digital sum of the four Grundys associated with 3, 7, 15, and 20 is zero. The first player should be the winner of this particular game.

A modification of Distich allows a pile to be divided into any number of unequal parts. Here, the Grundy functions $g(1)$, $g(2)$, $g(3)$, … take the values 0, 0, 1, 0, 2, 3, 4, 0, 5, 6, 7, 8, 9, 10, 11, 0, 12, …—that is, the sequence of positive integers spaced by the values $g(2^i) = 0$, $i = 0, 1, 2, \ldots$

A game of Russian origin bearing a kindred structure with other countdown games is Even Wins. In the original version, two players alternately remove from one to four chips from a single pile initially composed of 27 chips. When the final chip has been removed, one player will have taken an even number, and the other an odd number; that player with the even number of chips is declared the winner. Correct strategy prescribes reducing the pile to a number of chips equal to 1, mod 6 if the opponent has taken an even number of chips, and to 0 or 5, mod 6 if the opponent has an odd number of chips. The theorems of Nim with regard to safe and unsafe positions apply directly. All positions of the pile 1, mod 6 are safe. Since 27 is equivalent to 3, mod 6, the first player can secure the win.

In more general form, Even Wins can be initiated with a pile of any odd number of chips from which the players alternately remove from 1 to n chips. Again, that player owning an even number of chips when the pile is depleted wins the game. The winning strategies are as follows: if n even, and the opponent has an even number of chips, $L(S) = \{1, \text{mod } (n+2)\}$; if the opponent has an odd number of chips, $L(S) = \{0, n+1, \text{mod } (n+2)\}$. For odd n, winning strategy is defined by $L(S) = \{1, n + 1, \bmod (2n + 2)\}$ if the opponent has an even number of chips, and $L(S) = \{0, n + 2, \bmod (2n + 2)\}$ if the opponent has an odd number. If a random odd number of chips is selected to comprise the pile initially, the first player can claim the win with probability $n/(n + 2)$, n even, and probability $(n - 1)/(n + 1)$, n odd.

The type of recursive analysis presented in this section is also applicable, in theory, to such "take-away" games as Tic-Tac-Toe, Chess, Hex, Pentominoes, and, in general, to any competitive attrition game. Beyond the field of countdown games, more extensive applications of Grundy functions are implied by the theory of graphs. A potential area of considerable interest encompasses solutions for the dual control of finite state games. A variant of such games is the "rendezvous" problem where the dual control reflects a cooperative nature. Other examples will likely arise in abundance as Grundy functions become more widely appreciated.

SEEMINGLY SIMPLE BOARD GAMES

The Morris Family (aka Mills, aka Merels, aka Morabaraba)

Extending back to the ancient Egyptians, Morris games (a derivative of "Moorish," a mummery dance) were particularly popular in medieval Europe. The most common form is Nine-Men's Morris[8] played on a board of 24 points (sometimes referred to as a "Mills board") that is formed by three concentric squares and four transversals, as illustrated in Figure 10-9.

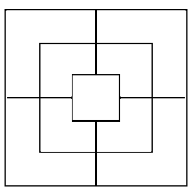

FIGURE 10-9 The Mills board.

Two contending players, each equipped with nine stones of a color, alternately place a stone onto one of the vacant points until all 18 are on board. Then each, in turn, moves a stone to an adjacent point along any line on which it stands. Whenever a player succeeds in placing three stones contiguous on a line, referred to as *closing a mill*, he is entitled to remove from the board any opponent stone that is not part of a mill—or, if all opponent stones are conjoined in mills, he may remove any one mill. When a player is reduced to three stones, he may move a stone from any intersection to any other (empty) intersection. The game's objective: to reduce the number of opponent stones to two or to maneuver to a position wherein the opponent cannot make a legal move.

[8]"A Midsummer Night's Dream": Titania, Queen of the Faeries, laments, "The nine men's morris is fill'd up with mud, …"

An established mill may be “opened” by moving one stone off the common line; it can subsequently be “closed” by returning the stone to its previous position, whence the formation is credited as a new mill. Thus a highly favorable position is the “double mill,” whereby a stone may be shuttled back and forth between two 2-piece groups, forming a mill at each move (and eliminating an opponent stone).

Nine-Men’s Morris was strongly solved (in 1993) by Ralph Gasser of the Institut Für Theoretische Informatik, Switzerland (Ref.). An 18-ply alpha-beta search program with a retrograde analysis algorithm that compiles databases for all 7.7×10^9 legal positions (out of 3^{24} possible states) determined that the game is a draw with correct play. Optimal strategy, however, is deemed to be “beyond human comprehension”—i.e., not reducible to a viable prescription.

Still widely played in Germany and the Scandinavian countries, Nine-Men’s Morris demands a degree of skill well above that of Tic-Tac-Toe without the immense number of strategies associated with classical board games such as Chess or Checkers. It has gained status as one of the games contested in the annual Computer Olympiad at the Ryedale Folk Museum in York, England. Gasser’s AI program, “Bushy,” is considered the world’s strongest Morris-playing computer.

Other formats, such as Three-Men’s-, Six-Men’s-, and Twelve-Men’s-Morris (with corresponding board configurations) are not played extensively—although Morabaraba (with added diagonals and 12 pieces) is popular in southern Africa.

Yashima (Ref. Arisawa)

On the board shown in Figure 10-10 each of two players, Black and White, alternately moves his personal counter to an adjacent unoccupied vertex, *erasing the path traversed to reach that vertex*. That player ultimately unable to move loses the game.

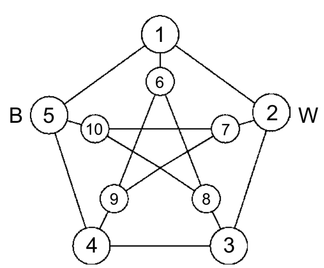

FIGURE 10-10 The Yashima board.

The complete strategy, leading to a win for the first player, is illustrated in Figure 10-11. After White’s first move to vertex (1), the two numbers at each vertex represent Black’s move followed by White’s response. Only White’s winning moves are shown, while Black’s two possible moves are counted. (*) signifies a forced move, and (°) indicates the end of the game (with White’s final move).

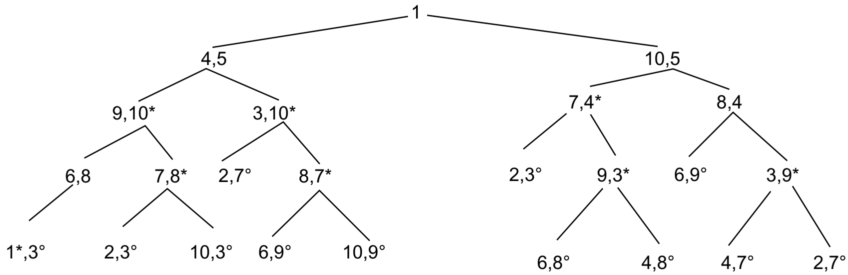

FIGURE 10-11 Complete strategy for Yashima.

The maximal length game consists of six moves by each player. Of the 15 paths, at most 12 can be deleted; at game's end, only four of the 10 vertices can be isolated—six must have at least one remaining path, and these vertices are connected in pairs by three surviving paths. To attain the maximal length game, one player moves along the outer pentagon, the other around the inside star.

Dodgem

An $n \times n$ board is initially set up with $n - 1$ (white) checkers belonging to **A** along the right-bottom edge and $n - 1$ (black) checkers controlled by **B** along the left-upper edge—illustrated in Figure 10-12 for $n = 3$.

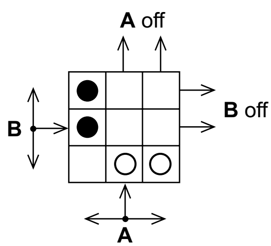

FIGURE 10-12 An $n = 3$ Dodgem board.

A and **B**, in turn, move either of their checkers one square forward (upward for **A**, rightward for **B**) or sideways (left or right for **A**, up or down for **B**) onto any empty square. Checkers depart the board (permanently) only by a forward move. That player left with no legal move—as a result of being blocked in or having both checkers off the board—wins the game. Each player's obvious strategic goal is to block his opponent's progress while pursuing his own.

The deceptively simple 3×3 game, invented by Colin Vout (Ref. Vout and Gray), has been strongly solved by Berlekamp, Conway, and Guy (Ref.), proving to be a win for the first player—not surprising, considering the strategy of starting with the outside checker and always keeping it outside. To offset the advantage of moving first, a scoring system has been suggested that awards the winner points corresponding to the number of moves required by the loser to clear the board of his checkers and then subtracting one point.

Appendix Table L details wins and losses for each of the 1332 legal positions. If **B** plays first, for example, a move from (cf/gh) to any + configuration such as (bf/gh) preserves the winning strategy.

Dodgem obviously offers far greater strategic depth than most other games played on a 3×3 matrix (e.g., Tic-Tac-Toe). The 4×4 and 5×5 games, with optimal play, are never resolved (Ref. desJardins) as both players will repeatedly move their checkers from side to side to block the opponent's winning move. The percentage of draws increases with increasing *n*, strongly suggesting that higher-order Dodgem games are also draws with optimal play. (An additional rule advanced for these games designates as the loser that player who prevents his opponent from moving to a position not previously encountered.)

Hex

The inventor of Tac Tix and Soma Cubes, Piet Hein, is also the creator (in 1942) of Hex, an abstract strategy game that shares a distant kinship with Go. (It was independently invented by John Nash[9] in 1948.) Hex is played on a rhombic-shaped board composed of hexagons. Conventionally, the Hex board has 11 hexagons along each edge, as shown in Figure 10-13 although any reasonable number can be used (because of the resemblance of the Hex board to the tiles found on bathroom floors, the game is sometimes known as "John"). Two opposite sides of the rhombus are labeled Black, while the other two sides are designated as White; hexagons at the corners of the board represent joint property. Black and White alternately place personal counters on any unoccupied hexagon. The objective for each is to complete a continuous path of personal counters between his two assigned sides of the board.

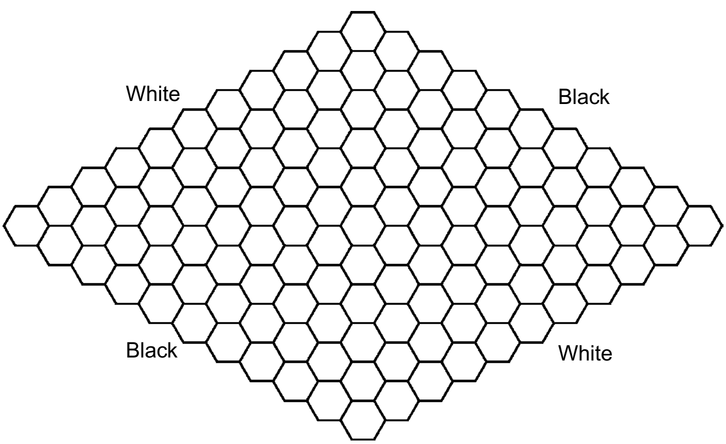

FIGURE 10-13 The 11 × 11 Hex board.

[9]Nobel laureate in Economics, 1994 (shared with game theorists Reinhard Selten and John Harsanyi).

It is self-evident that Hex cannot end in a draw, since a player can block his opponent only by completing his own path across the board. There exists a *reductio ad absurdum* existence proof—similar to that for Tic-Tac-Toe and Gale's Nim game—that the first player always possesses a win, and complete solutions have been computed for all boards up to and including 9×9 Hex (but not for larger boards).

Because the first player has a distinct advantage, the "pie rule" is frequently invoked—that is, the second player is afforded the option of switching positions with the first player after the first counter is emplaced (or after the placement of three counters in some versions). Another method of handicapping allows the second player to connect the shorter width of an $n \times (n - 1)$ parallelogram; here, a pairing strategy results in a win for the second player.

The strategic and tactical concepts underlying this seemingly simple game can be quite profound (Ref. Browne). Its solution is inhibited by a large branching factor, about 100, that precludes exhaustive tree-searching (a printout of the full strategy would be far too unwieldy to be of use).

An interesting analog Hex-playing mechanism (although the game obviously involves digital processes) was designed by Claude Shannon (Ref. 1953) and E.F. Moore. The basic apparatus establishes a two-dimensional potential field corresponding to the Hex board, with White's counters as positive charges and Black's as negative charges. White's sides of the board are charged positively and Black's negatively. The device contained circuitry designed to locate the saddle points, it being theorized that certain saddle points should correspond to advantageous moves. The machine triumphed in about 70% of its games against human opposition when awarded the first move and about 50% of the time as second player. Its positional judgment proved satisfactory although it exhibited weakness in end-game combinatorial play.

Extending Shannon's and Moore's concept, the Hex-playing computer program, Hexy (Ref. Anshelevich), employs a selective α–β search algorithm with evaluation functions based on an electrical-resistor–circuit representation of positions on the Hex board. Every cell is assigned a resistance depending on whether it is empty, occupied by a Black counter, or occupied by a White counter. Electric potentials, applied across each player's boundaries vary with the configuration of Black and White counters. (Hexy runs on a standard PC with Windows.)

Hexy reaped the Hex tournament gold medal at the 5th Computer Olympiad in London (in 2000). At present, no program can surpass the best human players.

Of the several Hex variants, Chameleon[10] offers the greatest interest. Here, Black and White each has the option of placing a counter of *either* color on the board—and each player wins by connecting a line of either color between his two sides of the board. (If a counter creates a connection between both Black's and White's sides (so that all sides are connected), the winner is that player who places the final counter.)

[10] Invented independently by Randy Cox and Bill Taylor.

Misère Hex (aka Reverse Hex aka Rex)

Here the object of the game is reversed. White wins if there is a black chain from left to right. Black wins in the event of a white chain from top to bottom.

It has been proved (Ref. Lagarius and Slator) that on an $n \times n$ board the first player has a winning strategy for n even, and the second player for n odd. A corollary to this proof confirms that the losing player has a strategy that guarantees every hexagon on the board must be filled before the game ends.

Random-Turn Hex

Rather than alternating turns, the players in this version (Ref. Peres et al.) rely on a coin toss to specify who is awarded the next turn. A computer simulation by Jing Yang has shown that the expected duration of this game on an $n \times n$ board is at least $n^{3/2}$. It is interesting to note that as n becomes large, the correlation between the winner of the game and the winner of the majority of turns throughout the game tends to 0.

Triangular Homeohex[11]

The field of play consists of a equilateral triangle of side length n packed with n^2 equivalent triangles of side length 1. Players alternate in placing a personal counter in any empty cell (unit triangle). The first player to connect all three sides of the triangle wins. Corner cells link to both their adjacent sides. The existence proof for a first-player-win at Tic-Tac-Toe and at Chomp applies equally well to Triangular Homeohex.

Bridg-It

Superficially similar to Hex is the game of Bridg-It,[12] created by David Gale (Ref.). The Bridg-It board, shown in Figure 10-14, comprises an $n \times (n + 1)$ rectangular array of dots embedded in a similar $n \times (n + 1)$ rectangular array

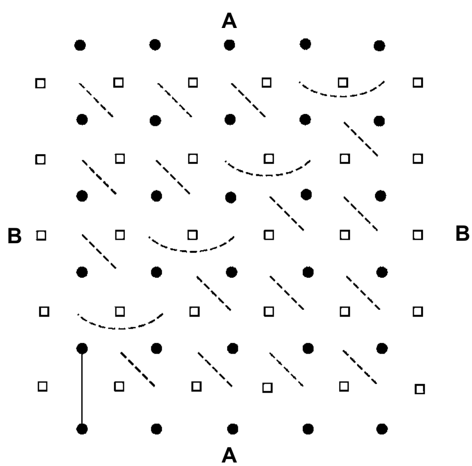

FIGURE 10-14 The Bridg-It board and Oliver Gross's solution.

[11] Suggested by Claude Shannon.
[12] A trade name marketed by Hasenfield Bros., Rhode Island.

of small blocks ($n = 5$ in the game originally proposed). In turn, **A** connects two adjacent dots with an *a*-colored bridge, and **B** connects two adjacent blocks with a *b*-colored bridge. The winner is the player who completes an unbroken line between his two sides of the board.

As with Hex, the existence of the first player's winning strategy is readily proved for Bridg-It. Oliver Gross (Ref. Gardner, 1991) devised the general winning strategy illustrated by the dotted lines in the figure. **A** begins by playing the bridge indicated in the lower-left corner; thereafter, whenever a bridge by **B** crosses the end of a dotted line, **A** plays by crossing the other end of the same line. This pairing strategy—which can be extended to Bridg-It boards of any size—guarantees a win for **A**.

Double-Move Bridg-It

After **A**'s first move, all subsequent moves for **A** and **B** consist of building two bridges.[13]

Whimsical Bridg-It

At each turn, **A** or **B** may build *both* an *a*-colored and a *b*-colored bridge or may choose (at whim) which bridge color to play; the latter option is allowed only once in the game at the cost of a turn. Subsequently, each player builds one bridge of his color per move.[14]

London Bridg-Its

Each player is restricted to a total of m bridges of his personal color. If neither player has won after the $2m$ bridges have been placed, the game proceeds by shifting a bridge to a new position at each move.[15]

Dots and Boxes

This well-known children's game employs an $n \times n$ rectangular grid of dots. Players **A** and **B** alternately draw a line connecting any two adjacent dots (horizontally or vertically). Whenever a line completes the fourth side of a square—forming a unit box—the player drawing that line initials that box and draws another line, continuing his turn with each completed box. That player who ultimately initials the greater number of boxes wins the game.

A complete solution for Dots and Boxes has not been determined. Strategic principles, developed by Elwyn Berlekamp (Ref., 2000) recognize the essential parity of the game and underscore its surprising subtleties and mathematical complexities.

Generally, after half (or slightly more) of the $2n(n + 1)$ possible lines for the n^2-box game have been drawn, a state of "gridlock" is reached—that is, the

[13] Invented independently by Randy Cox and Bill Taylor.
[14] Suggested by Mark Thompson.
[15] Suggested by the author.

next line *must* form the third side of at least one box, leaving the next player with one or more boxes to complete before relinquishing his turn. At this stage the grid will have been divided into "chains" (wherein the completion of any one box in the chain leads to the completion of all boxes in the chain).

An inferior but common strategy has the first player after gridlock draw a line conceding the shortest chain. His opponent then concedes the next shortest chain, and so on. If the number of chains at gridlock is even, the player who surrenders the first chain will win (since he will receive an equal or larger chain for his next move); if that number is odd, he will lose (since his opponent claims the last move).

A superior strategy consists of "double dealing": foregoing the completion of a full chain, accepting instead some fraction of the chain and leaving the remainder for his opponent—thereby gaining control of the game's progression.

An illustration of this principle, credited to Ian Stewart (Ref., 2001) is shown in the 5×5 game of Figure 10-15—where it is **B**'s turn to move.

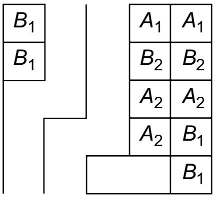

FIGURE 10-15 The 5 × 5 Dots-and-Boxes grid.

Obviously, **B** will claim the two 2×1 rectangles (B_1) in the lower-right and upper-left corners (failure to do so merely surrenders the four boxes to **A** with no recompense). Now, rather than opening one of the four chains, **A** executes a double-dealing move by accepting only two of the boxes in the four-box chain (A_1), leaving a 2×1 rectangle for **B** to fill in (B_2), after which **B** is forced to open the five-box chain. A second double-dealing move by **A** (A_2) accepts three of these five boxes, leaves another 2×1 rectangle for **B**, and forces **B** to open one of the six-box chains. Thus **A** has gained control by declining the last two boxes of every chain (except for the last chain).

At gridlock, let ϕ define the number of chains of three-or-more boxes plus the number of dots in the grid. To gain control of the game in the fashion described, **A** attempts to shape the configuration of lines so that ϕ is even. **B**, of course, strives to ensure that ϕ is odd.

MORE COMPLEX GAMES

Polyominoes

A fascinating class of games can be derived by manipulation of polyominoes on a chessboard (the term *polyomino* was adopted in 1953 by Solomon W. Golomb, then a graduate student at Harvard University). By definition, an *n*-omino covers a rookwise-connected set of *n* squares of the chessboard. Several examples of polyominoes from $n = 1$ to $n = 4$ are pictured in Figure 10-16. We can observe that the monominoe and domino have a unique configuration the tromino can assume one of two forms, and the tetromino any of five. Asymmetrical pieces turned over are not regarded as constituting distinct forms.

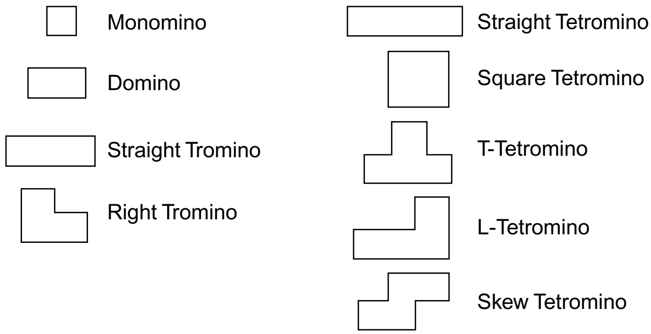

FIGURE 10-16 Monomino, domino, trominoes, and tetrominoes.

The general properties of polyominoes have been extensively investigated by Golomb. He has proved a number of theorems, including:

I. The chessboard can be completely covered by 21 straight trominoes and a single monomino if and only if the monomino is placed on one of the four intersections of the 3rd and 6th ranks with the 3rd and 6th files.
II. Regardless of the placement of the monomino, the remaining 63 squares of the chessboard can be covered with 21 right trominoes.
III. The chessboard can be completely covered by 16 straight, square, T-, or L-tetrominoes, but not by 16 skew tetrominoes (in fact, not even a single edge can be completely covered).
IV. It is impossible to cover the chessboard with 15 T-tetrominoes and one square tetromino, nor can coverage be achieved with 15 L-tetrominoes and one square tetromino, nor with 15 straight or skew tetrominoes (in any combination) and one square tetromino.
V. The five distinct tetrominoes cannot be grouped together so as to form an unbroken rectangle.

There exist 12 distinct (“free”) pentominoes—illustrated in Figure 10-17—and named by the letter each most resembles. (Additionally, there are 18 one-sided and 63 fixed pentominoes.) From a recreational standpoint, the properties of pentominoes are more compelling than those of the other polyominoes.

In Golomb’s Pentomino Game, two players alternately fit onto the chessboard one of the distinct pentominoes until either no pieces remain or none of those remaining will fit on the board. That player unable to place a piece is declared the loser.

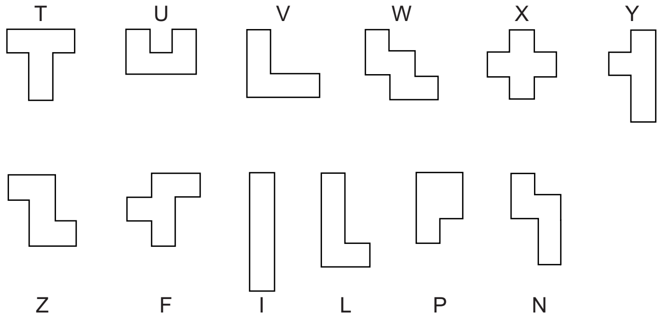

FIGURE 10-17 The 12 pentominoes.

Maximum duration of the Pentomino Game is clearly 12 moves—when each of the 12 pieces has been played (a highly improbable situation). Minimum duration is five moves, an example of which is shown in Figure 10-18 none of the remaining seven pentominoes can be placed on the board.

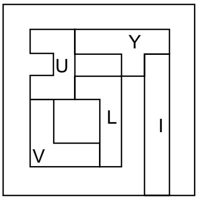

FIGURE 10-18 A minimal pentomino game.

Golomb propounds two basic strategic principles:

1. Try to move in such a way that there will be room for an even number of pieces.
2. If you cannot analyze the situation, try to complicate the position so that your opponent will have even greater difficulty in analyzing it.

The Pentomino Game was computer-solved by Hilarie K. Orman (Ref.) using a backtracking search procedure and proved to be a win for the first player. There are 296 possible opening moves (2308 without discounting symmetries).

Several variations of the basic game have also been suggested by Golomb. In Choose-Up Pentominoes, the two players alternately select a piece from the set of 12 pentominoes until each has six; the game then proceeds in the normal manner, each player allowed to place only a member of his own set on the board. The player who chooses first plays second to compensate for the advantage of first choice. Strategy for Choose-Up Pentominoes differs from that of the standard game in that instead of attempting to leave an even number of moves following his own move, the player strives to leave as many moves for his own pieces and as few as possible for his opponent's. An approximate preference ordering of the values of the 12 pentominoes is *PNUYLVITZFWX*.

In other variations of the basic game, the pentominoes can be distributed at random between the two players, more than one set of the 12 distinct pieces can be used, more than two players can participate, and boards other than the 8×8 chessboard can be introduced.

Of the higher-order n-ominoes, the number and complexity increases exponentially with n, rendering them generally unsuitable for game applications. There are, for example, 35 hexominoes and 108 heptominoes, one of the latter being non-simply connected (i.e., its configuration contains a hole). Further, there are 363-octominoes plus six with holes; 1248 enneominoes[15] plus 37 with holes; and 4460 decominoes plus 195 with holes. A total of 17,073 hendecominoes[16] (Ref. Stein and Ulam) and 63,000 dodecominoes exist, including those multiply connected.

The question of determining the number of distinct n-ominoes as a function of n is identical to a classical unsolved cell growth problem. We consider a square-shaped one-celled animal that can grow in the plane by adding a cell to any of its four sides; we then inquire as to how many distinct connected n-celled animals are possible under isomorphism. Stein, Walden, and Williamson of the Los Alamos Scientific Laboratory have programmed a computer to generate the isomorphism classes of such animals.

Instead of constructing rookwise-connected sets of squares, we can effect the edgewise union of sets of equilateral triangles or of hexagons (only such sets of these three regular polygons can fill a plane). The sets of equilateral triangles, known as polyiamonds, were first explored by T.H. O'Beirne (Ref.). For a given number of cells, fewer distinct shapes are possible than with polyominoes: moniamonds, diamonds, and triamonds can assume only one shape; there are three different-shaped tetriamonds, four pentiamonds, 12 hexiamonds, and 24 heptiamonds.

Solid polyominoes have also been investigated by Golomb and R.K. Guy. Known as "Soma Cubes," they were invented by the prolific Piet Hein, who conceived the first pertinent theorem: The sole irregular solid tromino and the six irregular solid tetrominoes (irregular in that the figure somewhere contains a corner) can be joined together to fashion a $3 \times 3 \times 3$ cube. These seven solid polyominoes comprise a "Soma set," which can be used to devise a multitude of entertaining constructions.

Polyominoids

A related set of structures, "polyominoids"—two-dimensional, edge-connected squares in three-space—was originally proposed by the author (Ref. Epstein). An n-ominoid contains n coplanar or orthogonal squares, thus resembling the floors, walls, and ceilings of a pathological office building. There exist two distinct dominoids (one coplanar domino plus one in three-space) and 11 distinct

[16] "Polyominoes" constitutes a false etymology from "dominoes." Purists would advocate applying strictly Greek-root prefixes rather than the usual deplorable practice of successively mixing Greek and Latin roots.

trominoids (two trominoes plus nine in three-space). The number of n-ominoids rises precipitously with increasing n (Table 10-13).

Table 10-13 Number of n-Ominoids

n	1	2	3	4	5	6	7	8	9	10
Number of n-ominoids	1	2	11	80	780	8,781	104,828	1,298,506	16,462,696	212,457,221

One of the 11 trominoids and four of the 780 pentominoids are illustrated in Figure 10-19 (Ref. *www.geocities.com*).

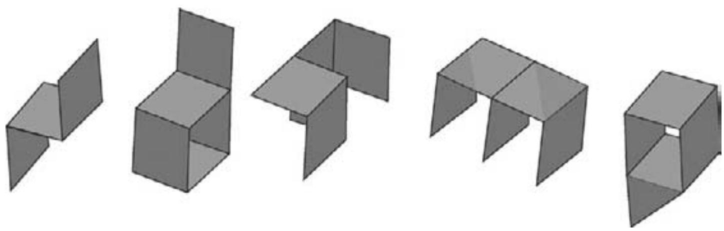

FIGURE 10-19 Polyominoids.

Quadraphage

Initially suggested by the author (Ref. Epstein; Gardner, 1973), the Quadraphage, as its name implies, is a "square-eater"—i.e., a super–chess-like piece that "eats away" whatever square it is placed on. For its move, a q-Quadraphage may consume any q previously uneaten squares on the board.

The Quadraphage may be pitted, in alternating moves, against a chess King—wherein the Quadraphage attempts to entrap the King (which may not traverse an eaten square), preventing it from escaping to the edge of the board.

With $q = 1$, the King, starting from a position on one of the four central squares of the standard 8×8 board, can readily reach an outside square (Ref. Berlekamp, Conway, and Guy, Chapter 19). On boards up to size 33×33, the King, with optimal play, cannot be entrapped by the Quadraphage (Ref. Silverman, 1971). On yet larger boards, it is doomed.

As a King versus Quadraphage game, the Quadraphage moves first—with the objective of forcing the King to an edge of the board. The King's objective is to maximize the number of its moves before reaching an edge. The payoff to the King is one unit for each square obliterated by the Quadraphage, but zero if it becomes entrapped. The game offers yet greater interest when played on the 18×18 cells of a Go board.

Postulating a super-King with $k = 2$ King-moves per turn, it can be proved that such a piece can forever elude the Quadraphage (on an infinite board).

With $q = 2$ (a second-order Quadraphage), a conventional King cannot escape on boards 8×8 and greater. With $q = 3$, the King can be captured on boards 6×6 and greater. And with $q = 4$ (the Quadraphage obliterates four squares per turn), the King can be trapped on all boards 5×5 and greater.

Replacing the King by a Bishop suggests Quadraphage games on an infinite chessboard, but with limitations on the length of the Bishop moves. With $q = 3$, the Bishop can be captured. Similarly, $q = 3$ entraps the Rook on an infinite board.

Checkers (Draughts)

Alquerque, the venerable ancestor of Checkers, dates back to 600 B.C. or earlier. Played on a 5×5 board, each side controls 12 counters (the center square is initially vacant). About 1100 A.D. in southern France, the game was transferred to a chessboard and titled Dames. Some four centuries later, the French introduced the compulsory-taking rule, referring to the game as Jeu Force. With the 1756 publication of "Introduction to the Game of Draughts" by William Payne, the game rapidly acquired a large following in England and Scotland and soon spread throughout the English-speaking world. It arrived in America, mysteriously renamed Checkers.

In standard 8×8 Checkers, seven possible alternatives exist for Black's opening move, with seven alternative ways of responding. Of the 49 two-move openings, 45 lead to draws and four to losses (with optimal play). There are 144 distinct three-move openings that lead to draws. In major tournaments, players are not allowed to choose opening moves but are assigned an opening selected at random (after eliminating those leading to a loss; each starting position is played twice, with the players exchanging colors for the second game). The preponderant majority of games in expert play are drawn.

With 5×10^{20} positions, Checkers poses a daunting obstacle to any computer attack. The first substantial program was designed in the late 1950s and early 1960s by Arthur L. Samuel of IBM (Ref.). Its principal feature was the introduction of a learning technique enabling the machine to improve its skill with playing experience.

Samuel's program and its several successors were rendered hors de combat by Chinook, a project that began in 1989 (at the University of Alberta by a team directed by Jonathan Schaeffer) and, with dozens of computers running almost continuously for 18 years, culminated with the complete (weak) solution of Checkers (Ref. Shaeffer et al., 2007). As anticipated, the computational proof determined that the game is a draw.

Chinook's procedure encompasses three components: (1) an endgame database (backward search) of 3.9×10^{13} positions; (2) a proof-tree manager (forward search) that generates positions whose assessment advances the proof's progress; and (3) a proof solver (forward search) that evaluates each position designated by the manager.

In solving the game of Checkers, this search-intensive ("brute-force") approach, representing a coupling of AI research and parallel computing, has completed a task six orders of magnitude more complex than achieving, for example, a solution of Connect Four.

It seems likely that the techniques refined by Chinook will be extended to other fields—such as biological processes—where real-time access of immense data sets are of value in accommodating parallel searching.

In its playing prowess, Chinook showed steady improvement from its inception, gaining the title of Checkers World Champion by defeating Dr. Marion Tinsley,[17] and then defending its title in subsequent years. It can now claim invincibility, subject only to being tied—well beyond the reach of man or machine.

Checkers Variants

Some of the few Checkers variants that have acquired a following:

Brazilian Checkers. Played under the rules of International Draughts, but on an 8×8 board with 12 pieces on each side.

Canadian Checkers. International Draughts on a 12×12 board with 30 pieces on a side.

Chubby Checkers (suggested by the author). Played on the black squares of a 10×10 checkerboard, the 12 pieces of each side are confined to the concentric 8×8 playing field. Only Kings are accorded access to the full 10×10 board.

International Draughts. Prevalent in Russia and Eastern Europe, often referred to as Polish Draughts (except in Poland, where it is called French Draughts), the game is played on a 10×10 board with 20 men per side. It offers commensurately greater complexity (checkers are allowed to capture diagonally backwards, and Kings, known as "Flying Kings," may travel any number of squares, akin to a chess Bishop that jumps).

Lasca (invented by Emanuel Lasker, former World Chess Champion). Played on a 7×7 checkered board with 11 men on each side, 25 diagonally-connected same-colored squares define the playing field. Jumped pieces are placed under the jumper to form a "tower"; only the top piece of a tower may be captured by a jumper.

Suicide Checkers (aka Anti-Checkers, aka Losing Checkers, aka Poddavki). The objective is the opposite of standard Checkers, the winner being that player who loses all his men.

Turkish Checkers (aka Dama in the Middle East). Each player begins with 16 pieces (on the second and third ranks) that move one space straightforward or sideways and capture by jumping over an adjacent opponent piece. At the eighth rank, a piece promotes to a Flying King (Dama) that moves like a Chess Rook and may jump over a piece, landing on any empty square beyond.

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[17] Unchallenged as the greatest Checker player ever, Dr. Tinsley held the World Championship for over 40 years, losing only five games in that period. Then along came a computer that was even astuter…

Chess

Chess (Chaturanga) first appeared in late 6th-century India according to generally accepted but unscholarly sources, confined to a narrow circle of intellectuals. It traveled to Persia about a century later (there known as Shatranj), crossed into Arab culture after a further century, and thence into the western world.

Its Japanese form, Shogi ("General's Game"), employs a 9×9 uncheckered board with 11 pieces and 9 Pawns controlled by each player. The Chinese version, Xiangqi ("Elephant Game"), deploys each side's 11 pieces and 5 Pawns on the intersections of 10 horizontal and 9 vertical lines. From its inception, Chess was often regarded with nearly Olympian awe. It still claims supremacy over all other table games, embued with a cryptic mystique—a mystique that, during the middle ages, reached into theological spheres.

Chess rules evolved over nine centuries to 1492, when the Spaniard Ramirez de Lucena introduced castling as a single move, completing what became the modern format. By then the Queen (originally Counselor or Vizier) had augmented its mobility from a King-like one-diagonal-square-at-a-time to become the most powerful figure on the board, and the lowly Pawn was afforded the option of initially advancing two squares (while subject to capture by adjacent Pawns *en passant*).

Chess is a mathematically limited game (comprising a finite number of positions); thus a "perfect" strategy must exist. However, determining this strategy is computationally beyond the pale and will likely remain so failing some revolutionary breakthrough in technology. Such a situation invites computer analysis, and computers now virtually monopolize advances in Chess knowledge. The "silicon beast" flaunts ever more powerful playing programs, databases of hundreds of millions of games, and ever deeper analysis of typical positions.

Chess programs model the game as a tree search wherein each board position corresponds to a node on the game-tree. From the current position each branch represents one of the possible legal moves leading to a new node on the tree, and from each new node further branches lead to other new nodes, *et seriatim*. The tree is thus made up of alternating levels of moves—known as plys—for either player. (With about 35 moves available from the average position [in mid-game] and with the average game lasting 39 moves, a complete game encompasses more than 10^{120} potential positions [close to a billion trillion googol], although *legal* positions number fewer than 10^{50}. Ten thousand high-speed computers, each searching a billion positions a second would wind up woefully short of practicality.)

The groundbreaking game-tree work was formulated in 1950 by Claude Shannon (Ref.), who proposed a playing strategy governed by an evaluation function designed to predict the likely outcome of the game from the current position. The evaluation function need be concerned primarily with strategic knowledge since tactical information is contained essentially in the variations themselves. Shannon's evaluation function, E, encompassed three factors (material, Pawn structure, and mobility):

$$\mathrm{E} = 9(\Delta \mathrm{Q}) + 5(\Delta \mathrm{R}) + 3(\Delta \mathrm{B} + \Delta \mathrm{N}) + \Delta \mathrm{P} - 0.5(\Delta \mathrm{D} + \Delta \mathrm{S} + \Delta \mathrm{I}) + 0.1(\Delta \mathrm{M})$$

where ΔQ, ΔR, ΔB, ΔN, and ΔP are the differences in the number of Queens, Rooks, Bishops, Knights, and Pawns possessed by White over those remaining to Black; ΔD, ΔS, and ΔI represent the differences in doubled Pawns, backward Pawns, and isolated Pawns, respectively; ΔM is the difference in the number of legal moves available.

This function is applied at the terminal nodes of the game tree—that is, at nodes on the level where the search is stopped. A positive value of E defines a position advantageous to White. With White seeking positions that maximize E, and Black countering with positions that minimize E, a minimax solution is theoretically accessible, though Shannon cautioned that such a function—as well as the similar and far more intricate functions that followed in its wake—could claim only "statistical validity."

Two principal search strategies were (correctly) predicted: Type-A programs that apply "brute force" inspection of every possible position over a fixed number of plys; and Type-B programs that prune potential moves according to some selection function and then examine the significant sets over as many plys as practical and only at those positions reflecting a degree of stability. The apparent weakness of a Type-B strategy is its reliance on the ability to determine which moves in a given situation are worthy of detailed consideration—a problem that proved more intractable than increasing the speed of Type-A searches.

(Shannon also referred to a Type-C program that plays in a manner similar to that of the Chess masters—by evaluation based on learning.)

A third strategy, a *quiescence search*, was subsequently designed to evaluate tactical concerns such as checking sequences, Pawn promotions, and sequences of capturing moves.

The first Chess-playing program translated into practice was designed by the eminent British theoretician A.M. Turing in 1951 for the MADAM computer. Pursuing an essentially Type-B strategy, the machine performed at a low level of competence, usually resigning after a series of tactical blunders.

A major breakthrough came in 1956 with the introduction, by Prof. John McCarthy at Stanford University, of the alpha-beta algorithm (Ref. Slagle and Dixon), a protocol that eliminates the examination of "any branch that cannot affect the minimax value of the current node," thereby reducing the branching factor at each node to approximately the square root of the previous value (and doubling the depth of search feasible for a fixed time).

Another technique, known as *iterative deepening*, also proved of value. Here, a computer performs a series of increasingly deeper searches until its allotted time has expired (as opposed to an exhaustive search of a particular ply). The best move is thereby produced for a given time constraint.

Type-B programs held sway into the 1970s, when improved computational speeds led most investigators to reconsider Type-A programs. By the 1990s, processing power had advanced by several orders of magnitude. With current technology, programs exhaustively search through 10 to 20 ply in the middle game—hundreds of millions of positions per second (early computers enabled a search of about 500 positions per second). Yet, around 1998, Type-B programs staged a comeback, defeating several world-class players over the next few years.

In addition to containing a book of openings with more than a million entries, current Chess programs are linked to a dictionary of endgames, thereby overcoming what had long been a principal weakness owing to the extreme depth of search needed to reach the endgame phase. All six-piece endings have now been analyzed (Ref. Nalimov)—requiring a storage capacity of approximately 1.2 terabytes. (All 5-piece endings require slightly more than seven gigabytes, potentially available to a desktop PC.) A computer reaching one of the positions in an endgame database will be able to continue with perfect play, knowing immediately if its position leads to a win, loss, or draw. Moreover, such knowledge could be useful in either avoiding or steering toward such positions.

Far more sophisticated evaluation functions (Ref. Waters) have been developed since Shannon's innovation. "Crafty,"[18] for example (designed around bitboards, 64-bit data structures), embraces appraisals of the following elements, *inter alia*:

> material; King safety; development of pieces; center control; trapped Rooks; Rooks on open files; doubled Rooks; Rooks behind passed Pawns; trapped Bishops; and isolated, backward, and passed Pawns

Material balance, MB, by far the most significant factor in any evaluation function, is based on relative values compiled in the Chess literature: Queen = 900 points, Rook = 500, Bishop = 325, Knight = 300, Pawn = 100; the King has infinite value. Each side's MB is then simply the sum of N_p, the number of pieces of a certain type still in play, and V_p, the value of that piece: $\text{MB} = \Sigma\, N_p\, V_p$.

Assigning relative weights to the various elements comprising the evaluation function is the critical step in determining optimal strategies (Ref. Laramée). A practical limitation on the complexity of the evaluation function lies in the fact that the more time it requires the less time remains for calculating the candidate moves—with a concomitant loss of tactical ability. (At present, it is difficult to enhance an evaluation function to gain as much as can be achieved from an extra ply of search.) This limitation should be alleviated with the advent of greater computer power, an improvement that could also enable nonlinear combinations of the pertinent elements.

The inaugural World Computer-Chess Championship (Stockholm, 1974) was won by the Russian program, KAISSA, a multidomain search process wherein each successive domain examines a subset of moves that were admissible in the preceding domain. Thereafter, the tourney was staged every three years (except for a one-year postponement in 1998) until 2002, when an annual schedule was adopted. The 14th competition (2006) was won by the Israeli program JUNIOR (programmed by Amir Bin and Shy Bushinsky), which analyzes only about three million positions per second but applies highly effective pruning strategies (and can run on a desktop PC). JUNIOR also stands as the current (19th) non-human champion of the world.

May 1997 brought the landmark event in human versus machine Chess, when DEEP BLUE, a complete (Type A) program developed at the IBM Watson

[18] Written by Bob Hyatt, Crafty is a descendant of Cray Blitz, winner of the 1983 and 1986 World Computer-Chess Championships.

Research Center, notched the first match victory over a reigning world champion (Garry Kasparov). Since then, computers such as DEEP JUNIOR and the supercomputer HYDRA (potentially the strongest Chess engine yet conceived, estimated to warrant an ELO rating[19] in excess of 3000), win ever more convincingly over human Chess masters. With a total of 64 gigabytes of RAM, HYDRA can evaluate to a depth of 18 ply. Each of its numerous individual computers incorporates its own field-programmable gate array (FPGA) acting as a Chess co-processor.

[Inevitably, a chess-computer program was caught with its circuits crossed. RYBKA (Czech for "little fish"), WC-CC winner from 2007 through 2010, was found to have plagiarized two other programs, CRAFTY and FRUIT, and, along with its designer Vasik Rajlich, was banned for life from official tournament play.]

With computers thus looming over the chessboard like all-powerful Cyclopses, the demise of classical Chess has become a perennial and oft-lamented prediction among professional Chess masters—although, to the preponderant majority of Chess aficionados, the game still retains (for the foreseeable future) its interest and capacity for creativity.

For world-championship play between humans, a long-standing controversy concerns the manner of scoring draw games. The match comprises $2n$ games (by custom $n = 12$) with 1 point awarded for a win, 1/2 point for a draw, and 0 for a loss. The champion requires a score of n points to retain his title; the challenger needs $n + \frac{1}{2}$ to supersede him.

If champion and challenger are of equal skill, the probability of either winning a single game is $\frac{1}{2}(1 - r)$, where r is the probability of a draw. Let R_n denote the probability that both score n points. Then the champion will retain his title with probability $P_n(r)$ given by

$$P_n(r) = R_n + \tfrac{1}{2}(1 - R_n) = \tfrac{1}{2}(1 + R_n) \tag{10-3}$$

The probability $Q_n(r)$ that the match will result in k wins by the champion, k wins by the challenger, and $2n - 2k$ draws is the multinomial distribution (Eq. 2-12),

$$Q_n(r) = \frac{(2n)!}{(k!)^2(2n - 2k)!}\left(\frac{1-r}{2}\right)^{2k} r^{2n-2k}$$

and

$$R_n(r) = \sum_{k=0}^{n} Q_n(r)$$

For $n = 1$, Eq. 10-3 yields

$$P_1(r) = \tfrac{1}{4}(3r^2 - 2r + 3)$$

[19]The standard scale to assess relative strength of tournament Chess players (after its creator, Árpad Élö). Only 39 humans have ever achieved a rating over 2700; only four have surpassed 2800.

which is minimized for $r = 1/3$, whence $P_1(1/3) = 2/3$. For $n = 2$,

$$P_2(r) = \frac{35}{16}r^4 - \frac{15}{4}r^3 + \frac{21}{8}r^2 - \frac{3}{4}r + \frac{11}{16}$$

which is minimized for $r = 0.253$, thus yielding $P_2(0.253) = 0.614$. For $n = 12$, as in championship matches, P_{12} is minimized with $r = 0.087$, which yields $P_{12}(0.087) = 0.543$. The minimizing value of r approaches 0 as n increases.

For moderate values of n, the champion is accorded a substantial advantage by the system that registers draw games as value 1/2; the figures support those who contend that draws should not play a strategic role in championship play. Equation 10-3 advises the champion to play conservatively—that is, to adopt a strategy leading to large values of r—and urges the challenger to seek wins even at considerable risk. In grandmaster play, $r \sim 2/3$—and the probability that the champion retains his title (against an equally skilled adversary) is $P_{12}(2/3) = 0.571$. Other values of r produce, as examples, $P_{12}(0.9) = 0.635$, $P_{12}(1/2) = 0.557$, and $P_{12}(0) = 0.581$.

The preceding analysis errs slightly in that champion and challenger can be considered of equal ability only in paired games with each playing White and Black in turn.

Compilations of recent international tournaments indicate that White's win probability is 0.31, while Black's is 0.22, leaving 0.47 as the probability of a draw. These data suggest an equitable scoring system of 1 and 0 points to either player for a win and loss, respectively, while White receives 0 and Black is awarded 1/4 point for the draw. Thus, over a series of games, the assignment of the White and Black pieces becomes theoretically irrelevant. For world championship play, drawn games should either be discounted or scored as 1/8 for Black.

Chess Variants

While over 2000 Chess variants have been devised and promoted, only a paltry few have sustained even modest interest from the Chess-playing public. Those that are worth noting include the following.

Cheskers. Invented by Solomon Golomb; each player begins with two Kings, a Bishop (equivalent to a Chess Bishop), and a Camel (that lurches one diagonal and two straight squares) on the first rank, and eight Pawns on the second and third ranks. A player must capture all opponent Kings to win.

CHESS960. To address concerns that more and more computer analysis leaves less and less room for originality (virtually all openings have been optimally determined for 20-some moves), Chess960 (aka Fischerandom Chess[20]) pseudorandomly arranges the starting position of pieces along the first rank. Specifically, the two Bishops are located on opposite colors and can each occupy one of four positions ($4 \times 4 = 16$ permutations); the Queen is placed

[20] Invented in 1996 by ex-World Champion, turncoat, and notorious anti-Semite, Bobby Fischer.

on one of the remaining six squares (6 choices); the two Knights are placed at random on the remaining five squares $\left[\binom{5}{2} = 10 \text{ possible arrangements})\right]$; on the remaining three empty squares, the King is situated between the two flanking Rooks. This process generates $16 \times 6 \times 10 \times 1 = 960$ equiprobable configurations for the starting position. (The order of placement is significant.) Prior to the beginning of each game, one of the 960 configurations is chosen at random, and the contest proceeds with standard rules of play (except for rather complicated castling regulations).

Computers promptly entered the Chess960 lists, oblivious to the irony of engaging in a game designed with "anti-computer" intent. Spike,[21] a Chess engine[22] (not a complete program) developed in Germany, was crowned World Champion at the first Chess960 tournament in 2005 (held in Mainz). It was succeeded the following year by Shredder, another Chess program from Germany, one that had previously garnered ten titles as World Computer-Chess Champion.

Crazyhorse. Captured pieces change color and may re-enter the board on any unoccupied square.

Hostage Chess. Captured pieces are held hostage as exchange against the opponent's captured pieces and then returned to the board.

Kriegspiel. Each player has his own board with the correct positions of his own pieces. The positions of opposing pieces are unknown but deduced. A referee (obviously!) is essential.

Maharajah and the Sepoys. Black has his normal complement of pieces that move conventionally (Pawn promotion and castling are prohibited). White has a single piece, the Maharajah (positioned initially on the King's square), whose moves combine those of a Queen and a Knight. With perfect play, Black wins.

Suicide (Losing) Chess. Capturing, when possible, is mandatory. The player who loses all his pieces wins.

In general, games with off-beat pieces (unconventional moves) and/or non-standard playing fields remain mired in obscurity.[23] The more a proposed variant deviates from classical Chess, the less likely it is to gain popularity.

Go

In contrast to the "artificial" rules that govern Chess and Checkers, the game of Go exhibits a "natural" legalistic structure. It is often described as the most complex

[21] After a character in the "Buffy the Vampire Slayer" TV series.

[22] Computer Chess programs are mostly divided into an *engine* (that computes the best move given a specific position) and, separately, a *user interface*. The two parts communicate via a communication protocol such as Xboard/Winboard or UCI (Universal Chess Interface).

[23] More than 1200 different pieces with non-orthodox moves, aka "fairy pieces," have been invented for the countless concoctions of "fairy Chess."

game ever devised. As such, it has survived without significant change for over four millennia, its invention traditionally credited to the Chinese emperor Yao.[24]

With two players alternately setting black and white stones on the intersections of a 19×19 grid,[25] the deceptively simple objective of Go is (for Black and for White) to form chains of same-colored stones that surround as great an area of the playing field as possible. A vacant point orthogonally adjoining a chain is a *degree of freedom*, referred to as a *liberty*; chains or single stones left with no liberties are captured (removed from the board) and added to the opponent's score at the conclusion of the contest. (Diagonally adjacent stones may cooperate in surrounding territories.) One restriction: the "Ko rule" that prohibits capturing and recapturing moves that lead to immediate repetitive positions. The game is concluded when neither player can increase the number of stones captured or the extent of territories surrounded. Final scoring, then, is the sum of these quantities—albeit to offset Black's advantage of playing first, an additional $6\frac{1}{2}$ points (*komi*) is often credited to White's account.

Go rankings for amateurs delineate a pecking order from 30 *kyu* (novice) up to 1 *kyu* and then from 1 *dan* up to 7 *dan*. A separate scale applies to professionals, ranging from 1 *dan* to 9 *dan* (held by fewer than 200 players worldwide). In a match between two players of unequal rankings, the weaker player is allowed a number of handicap stones equal to the difference between their rankings (these stones are placed on the board at the start of the game).

Six basic concepts shape the general Go strategies:

1. Connectiveness: placing stones in a compact pattern
2. Cutting (isolating) opposing stones
3. Life: creating a pattern of stones that can permanently avoid capture
4. Death: forcing positions wherein opponent stones cannot escape capture
5. Invasion: striving to create a living group within an area controlled by the opponent
6. Reduction: encroaching on opponent's territory to reduce its size while still maintaining connectiveness

These strategic concepts, in practice, can be highly complex and abstract. Moreover, as the game progresses, it becomes ever more complex (at least for the first 100 plys) as additional stones enter the board—in contrast to most capture games (Chess, Checkers) where positions simplify with time.

Because of its unrivaled strategic depth, its larger board, and an elusive and complex architecture, Go is inherently unamenable both to tree-search and to

[24] Known as Wei-ch'i ("Envelopment Game") in Chinese, Go was introduced into Japan around mid-8th century when the Japanese ambassador to China, Kibidaijin, returned home to spread the word.

[25] Oddly, the Go board is not square. Its grid, for reasons known only to the inscrutable Japanese, has a length-to-width ratio of 15 to 14 (1.5 *shaku* long by 1.4 *shaku* wide; one *shaku* is about 30.3 cm.); margins are added to accommodate stones along the edges. Further, black stones are slightly larger than white stones.

accurate and practical evaluation functions. Possible positions number roughly $3^{361} \times 0.12 = 2.1 \times 10^{170}$, most of which result from about $(120!)^2 = 4.5 \times 10^{397}$ different games for a total of 9.3×10^{170} games. As a comparison, there are approximately 10^{80} atoms in the universe. From a typical midgame position, there are about 200 candidate moves; thus a 4-ply search would necessitate examination of 200^4 (1.6×10^9) positions. The comparable situation in Chess would entail examination of 35^4 (1.6×10^6) positions. Further, the average game of Go lasts for some 250 moves, compared to about 39 moves for the average Chess game. A Go computer as powerful as Deep Blue (assessing 500 million positions per second) could not evaluate a single move in a year's time. The game would seem to defy solution within the foreseeable future.

Initial computer forays focused on reduced board sizes. In 1962, H. Remus developed a Go program for the IBM 704 on an 11×11 board, applying the principles of machine learning—weighing the possible number of degrees of freedom of chains, number of stones captured, distance from the last stone set by the opponent, and number and color of stones in certain areas.

An alternative approach combines positional evaluation and tree searching as practiced with Chess. Thorp and Walden (Ref.) have produced such programs for Go on $M \times N$ boards with small values of M and N.

Van der Werf et al. (Ref.), applying an iterative deepening alpha-beta search, have produced solutions for all boards up to size 5×5.

Other early programs incorporated an "influence function" introduced by Albert Zobrist (Ref.), Black stones are assigned a value of $+50$, White stones a value of -50, and empty points a zero value. The neighbors of each positive-valued point gain $+1$ in their influence values and, similarly, the neighbors of negative-valued points tally -1 to *their* influence values. This process is then repeated another three times, thus "radiating" Black and White influences numerically. The point with the highest value indicates the program's next move.

Subsequent generations of programs conceptualized abstract representations of the Go board to analyze groupings of stones. Bruce Wilcox (Ref.) developed the theory of sector lines, partitioning the board into zones for individual analysis. Another advance involved pattern recognition to identify typical positions and identify moves—as exemplified by the Goliath program, the 1991 World Computer-Go Champion.

Current Go programs incorporate many of these techniques plus searches on local groups and overall positions, using both patterns and abstract data structures in addition to applying principles of combinatorial game theory, Monte Carlo techniques, and cognitive modeling (cognitive science and Go might be said to share a genetic inheritance). Some programs further emphasize neural networks for candidate-move generation. Databases for full-board openings (*fuseki*) and corner openings (*joseki*) are also common features. Yet, with all these elements, even the most proficient Go computers—e.g., Go4++, Many Faces of Go (written by David Fotland), Go Intellect (1994 Computer-Go Champion), Fungo, Gnugo, and Handtalk—rank about 6 or 7 *kyu* and pose no threat to accomplished Go players. Handtalk III, developed by Prof. Zhixing Chen at the University of Guangzhou, reigned as World Computer-Go Champion 1995–1997. The current (2006) champion, KCC Igo, along with Handtalk III, were awarded 4-*kyu* diplomas, still well below the level of professional players.

Programming a computer to play professional-level Go remains a staggeringly complex task. Game theorists predict that, even with Moore's Law ("Computer power doubles every 18 months") and a corresponding improvement in search algorithms, such a Go program (*shodan*) will not be feasible until the early 22nd century. Chaucer's caveat seems apt: "I warne you well, it is no childes pley."

Go Variants

Possibly owing to the game's deep historical roots, Go practitioners tend to view variants prejudicially. The following few can claim some adherents:

DoNotPass Go (suggested by the author). The board is divided into four quadrants (the two mid-line axes are divided between Black and White). Black plays black stones in two diagonally-connected quadrants and white stones in the other two, scoring as Black and White, respectively. White plays the complementary rule.

GoLightly (suggested by the author). Each player may enter either a black *or* a white stone on the board. Captured pieces of either color are credited to the captor's account. The player making the last legal move of the game chooses which territory (black- or white-surrounded) to score. GoLightly is usually played on boards smaller than 19×19.

Kakomú Go. The edges of the board are lined with Black stones (so White cannot form eyes along an edge). White wins with a group that cannot be captured and loses if all his stones are dead.

No-Go. Each player has the option of rejecting his opponent's move. The opponent then makes a second move that cannot be rejected.

Phantom Go. Equivalent to Kriegspiel. Each player's board reflects only his stones and his deductions for the locations of his opponent's stones. A referee prevents illegal moves. Phantom Go is played more frequently on smaller boards.

Twin Move Go. After Black's first move, each player is accorded two (independent) consecutive moves.

GAMES COMPUTERS PLAY

Reversi or Othello (After Shakespeare's Drama)

Played on an uncheckered 8×8 board, this game features 64 checkers that are *dark* on one side and *light* on the other. Correspondingly, the two players are designated DARK and LIGHT.

Initially, four counters are placed on the four center squares of the board, two showing light sides and connected diagonally, the other two showing dark sides and also connected diagonally. DARK moves first, placing a counter dark side up in a position that connects directly—either orthogonally or diagonally—to another dark counter and with one or more contiguous light counters between itself and a second dark counter (Figure 10-20). These light counters are then reversed (turned over) to show their dark side. DARK can then use them in subsequent moves unless LIGHT has meanwhile reversed them back.

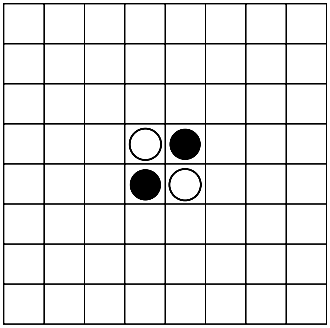

FIGURE 10-20 Othello starting position.

World championship tournaments for Othello have been held annually since 1977, with Japanese players winning 21 of the 30 contests. Although a strong solution for Othello remains just out of reach (it entails up to 10^{28} legal positions and a game-tree complexity of about 10^{58}), computer programs such as *The Moor* and *Logistello* handily defeat the topmost human players. (*Logistello's* evaluation function is based on disc patterns and features over a million numerical parameters that are tuned using linear regression.)

Experts lean to the view that perfectly-played Othello results in a draw (Ref. Allis, 1984).

Connect-Four

A Tic-Tac-Toe–like game played on a *vertical* board seven columns across and six rows high, Connect-Four pairs two players who begin with 21 counters each of a personal color, alternately placing one on the board. A counter may be entered into a given column *only* in the lowest unoccupied row of that column. The objective is to achieve a line of four personal counters either orthogonally or diagonally. If all 42 counters have been entered (filling the board) without achieving a line of four, the game is drawn.

Connect-Four has been solved by Victor Allis (Ref., 1988) with a Shannon Type-C program, proving to be a win for the first player.

THE SHADOW OF THINGS TO COME

With the dawn of the 21st century, human versus computer matches have come to be considered as unsporting as human versus automobile races. Computer versus computer will increasingly take over center stage—likely with clusters of computers linked through the Internet, each assigned to search a particular section of the game tree. In 2004, Project ChessBrain set a world record for the largest number of computers (2070) simultaneously playing a game of Chess. (Yet, massively parallel architectures affect only linear increases in computer power—valuable for many games but inadequate for those, like Go, with extremely high complexity.)

Faster hardware and additional processors will steadily improve game-playing program abilities. Computer-memory capacity is increasing exponentially, with a new generation born every three years. Humans, at least for the time being, are constrained by the snail's pace of biological evolution.

The logical structure of future computers is difficult to forecast. To date, the techniques of game-playing programs have contributed little or nothing to the field of artificial intelligence. Perhaps as a consequence, the present trend is toward an imitation of the human brain structure. Yet a human brain contains approximately 10 billion neurons, logic and memory units, and possesses the capabilities of instinct, intuition, and imagination, which permit intelligent, adaptive, and creative behavior. A contemporary computer, by comparison, encompasses the equivalent of about 100 million neurons; it cannot act intuitively by the accepted meaning of intuition. Human Chess players employ a qualitative and functional analysis; computers might well be advised to seek a different *modus operandi*.

There is no profound reason why the ultimate format of game-playing computers should resemble a human counterpart any more than interplanetary rockets should adhere to aviary principles. Whatever the course of development, the limits of computers with respect to human functions have yet to be imagined.

BOARD BAFFLERS

Graham's version. Player **A** (Odd) owns the set of integers, 1, 3, 5, 7, 9; **B** (Even) owns 2, 4, 6, 8. Beginning with **A**, the players alternately place one of their numbers on an empty cell of the standard 3×3 matrix. The winner is that player who completes a line—orthogonally or diagonally—whose numbers total 15.

Nygaard's version. A variation of Graham's game that additionally awards a win to the player who places any three of his numbers (odd for **A**, even for **B**) in a line.

Prove that, with optimal play, Graham's game is a win for **A**, while Nygaard's game is a draw.

2. *Plus-or-minus Tic-Tac-Toe.* **A** and **B** alternately place a number—without replacement—from the set 1, 2, …, 15 on any empty cell of the 3×3 matrix

and assign a plus or minus to that number. The winner is the player who completes a line, orthogonally or diagonally, whose numbers sum to zero. Determine the optimal strategies.

3. *Knights of the Square Table*. How multipronged must a Quadraphage be to entrap two Knights on an $n \times n$ board? How much more difficult is it to trap two Knights rather than one?

4. *Quadrastructor*. To abet his escape from a q-powered Quadraphage, the King engages the services of a Quadrastructor, another super–chess-like piece that can restore an obliterated square or, alternatively, construct a square beyond the original $n \times n$ board. (For obvious reasons, the King cannot move onto a new or reconstructed square until its next turn—after the Quadrastructor has leapt to another space.)

   Conjecture: To cope with the Quadrastructor (and achieve its stated objective), a q-powered Quadraphage must morph itself into a $(q + 1)$-powered Quadraphage.

5. Show that the total number of squares on an $n \times n$ checkerboard is expressed by

$$n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2 = (1/6)\, n(n+1)(2n+1)$$

   which, for $n = 8$, equals 204.

6. *Versa-Tile Hex*. Black's and White's counters, if turned over, exhibit the opposite color. For each move, Black/White places a personal counter on any unoccupied hexagon. OR turns over one of his opponent's counters. A reversed counter cannot be re-reversed. Show that this game is also a win for the first player.

   Allowing a reversed counter to be re-reversed after one or more intervening moves (Vice-versa-Tile Hex) does not alter the basic strategy or outcome of the game.

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Chapter | eleven

Fallacies and Sophistries

PSYCHOLOGY AND PHILOSOPHY

To Alice's insistence that "one *can't* believe impossible things," the White Queen demurred: "I daresay you haven't had much practice. When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

The White Queen's sophistry is particularly applicable to gambling—a phenomenon that continues to be associated with superstition and metaphysics. Anthropologists have often commented on the striking resemblance between the uneducated gambler and the primitive. Before declaring war, the primitive consults his crystal ball, tea leaves, buzzard entrails, or other divination media, and if the gods signify success, war follows simply as an administrative action. If, subsequently, the war becomes a losing cause and he flees the battlefield, it is not from cowardice but rather because the "signs" have been erroneously read or interpreted and the gods really favored the opposition. Thus his losses are due to "fate"—a predestined decision from which there is no appeal. Similarly, a gambler assigns his successes to a combination of "luck" and prowess and his failures to the disfavor of the gods of chance. Such a belief is an apparently ineradicable irrationality of the unenlightened gambler.

The ubiquitous belief in luck and numerology is not confined to the unsophisticated gambler. All too many eminent geniuses have, in senility or in moments of sad aberration, succumbed to the irrationalities that beset frailer men.[1] Huygen's faith in "six" as the perfect number led him to the conviction that the planet Saturn could possess but a single moon. Hobbes's geometry "plussed or minused you to heaven or hell." Leibnitz applied geometrical principles to ethics and politics, and geometry to the selection of the King of Poland

[1]Appropriate is the comment of Frederick the Great, writing of Prince Eugen and Marlborough: "What a humbling reflection for our vanity (when) the greatest geniuses end up as imbeciles…"

as well as to the deployment of the French army (for Louis XIV). John Craig expounded his "Mathematical Principle of Christian Theology." Sir James Jeans advocated his conviction that the cosmos was the creation of a divine mathematician. Some of these creeds arise directly from supernaturalism and others from the more sophisticated appeal of the Cartesian image of an admirable science of numbers that would solve all human problems, a doctrine that yet finds ardent devotees in contemporary psychology.

There has been no dearth of attempts to establish axiomatic principles for intuitive mathematics. Perhaps the first such effort emerged from the concept of "moral" expectation as opposed to mathematical expectation. Buffon held that the moral value of any sum varies inversely with the total wealth of the individual who acquires it. Cramer considered it to vary as the square root of total wealth. D. Bernoulli maintained that the moral value of a fortune varies as the logarithm of its magnitude: that is, the personal value of any small monetary increment dx is inversely proportional to the total sum x in the possession of the individual (this idea is familiar to psychophysicists in the form of Fechner's law).

We have emphasized the advantages of the objective utility function and objective probability for gambling phenomena. The introduction of intuition and morality into the workings of gambling theory is but another trap for the unwary set by the unenlightened.

FALLACIES

Without mathematical context, we list here some of the fallacies prevalent in gambling as a result of psychological influences. The uneducated gambler exemplifies, par excellence, the processes of intuitive logic. He is often charged with emotion and directed by passionate, primitive beliefs; thus his acts may often contrast with the dictates of his objective knowledge. He injects a personal element into an impersonal situation.

Culled from many observations and laboratory experiments,[2] the following 13 fallacies appear to be most prevalent and most injurious (in terms of profit and loss) to the susceptible gambler:

1. A tendency to overvalue wagers involving a low probability of a high gain and undervalue wagers involving a relatively high probability of low gain. This tendency accounts for some of the "long-shot" betting at race tracks.
2. A tendency to interpret the probability of successive independent events as additive rather than multiplicative. Thus the chance of throwing a given number on a die is considered twice as large with two throws of the die as it is with a single throw.
3. After a run of successes a failure is inevitable, and vice versa (the Monte Carlo fallacy).

[2] A large percentage of these experiments were conducted, curiously, with children. Is it the wage scale or the naïveté of juveniles that accounts for this practice?

4. The psychological probability of the occurrence of an event exceeds its mathematical probability if the event is favorable, and conversely. For example, the probability of success of drawing the winning ticket in a lottery and the probability of being killed during the next year in an automobile accident may both be one chance in 10,000; yet the former is considered much more probable from a personal viewpoint.
5. The prediction of an event cannot be detached from the outcomes of similar events in the past, despite mathematical independence.
6. When a choice is offered between a single large chance and several small chances whose sum is equal to the single chance, the single large chance is preferred when the multiple chances consist of repeated attempts to obtain the winning selection from the same source (with replacement); however, when there is a different source for each of the multiple chances, they are preferred.
7. The value of the probability of a multiple additive choice tends to be underestimated, and the value of a multiplicative probability tends to be overestimated.
8. When a person observes a series of randomly generated events of different kinds with an interest in the frequency with which each kind of event occurs, he tends to overestimate the frequency of occurrence of infrequent events and to underestimate that of comparatively frequent ones. Thus one remembers the "streaks" in a long series of wins and losses and tends to minimize the number of short-term runs.
9. A tendency to overestimate the degree of skill involved in a gambling situation involving both skill and chance.
10. A strong tendency to overvalue the significance of a limited sample selected from a relatively large population.
11. Short-term outcomes will exhibit the same frequencies as long-term outcomes.
12. The concept of "luck" is conceived as a quantity stored in a warehouse, to be conserved or depleted. A law of conservation of luck is implied, and often "systems" are devised to distribute the available luck in a fortuitous manner. Objectively, luck is merely an illusion of the mind.
13. The sample space of "unusual" events is confused with that of low-probability events. For one example, the remarkable feature of a Bridge hand of 13 Spades is its apparent regularity, not its rarity (all hands are equally probable). For another, if one holds a number close to the winning number in a lottery, one tends to feel that a terribly bad stroke of misfortune has caused one *just* to miss the prize. Bertrand Russell's remark that we encounter a miracle every time we read the license number of a passing automobile is encompassed by this fallacy. The probability of an unusual occurrence should be equated to the ratio of the number of unusual (by virtue of symmetry or other aesthetic criteria) events to the total number of events.

In addition to those enumerated above, there exist other fallacies more directly associated with superstition than with intuitive logic. For example, there is the belief that the gambler's "attitude" affects the results of a chance event—a pessimistic frame of mind (I'm going to lose $X) biases the outcome against the gambler. Another example is attributing animistic behavior to inanimate objects: "The dice are hot." The concept of luck permeates and pervades the experience of the superstitious gambler. He has lucky days,[3] lucky numbers, lucky clothes (worn during previous winning sessions), and lucky associates (he firmly believes that certain individuals are inherently more lucky than others and are more likely to achieve gambling success or, perhaps, to contaminate others with their power). He believes in a "sixth sense" that anticipates or compels the direction to success. He may carry amulets, talismans, fascina, charms, or mascots, consult *sortes* or oracles, and avoid black cats, ladders, or turning his back to the moon while gambling.

It is peculiar that our modern society, having consigned its seers, witches, water diviners, graphologists, geomancers, anthroposophists, astrologers, spiritualists, phrenologists, and pendulum swingers to the dark intellectual limbo, yet preserves a medieval approach to the laws of chance. With the gambler resides the last vestige of codified superstition.

Gobbledygook

"Vibration Science"

A relatively recent concoction, vibration science postulates that "numbers are self-predicting via the parenting relationship of the creative force vibrations and their unions automatically spawn." Comprehension of this incomprehensible statement, so it is claimed, enables the "prediction of winning numbers."

"Ostension"

Certain phenomena (or noumena)—ghosts, angels, UFOs, alien visitations, Loch Ness monsters—are recognized by true believers as real though unprovable. Ergo, it is acceptable–even imperative—to concoct "evidence" that corroborates such phenomena. This "justifiable" action is referred to as "ostension."

PARANORMAL PHENOMENA

Throughout our cultural development no social group of substance has failed to evolve a body of beliefs and actions that transcends ordinary experience. The type of noumenon currently fashionable is that encompassed by the field of parapsychology. To its adherents, parapsychology offers the fervor and personal involvement of mysticism and fundamentalist religion; to its detractors it is a beacon for those who prefer certainty to truth, for those prone to consult supernatural powers for the resolution of uncertainties, and for those ignorant of cause-and-effect relationships. Satisfying the Spinozistic yearning for an

[3] Indeed, our modern calendar was preceded (before 2000 B.C.) by an astrological list of lucky and unlucky days to mark the occurrence of noteworthy events.

extra-human quality, it clearly fulfills a need. As Voltaire might have said, "If parapsychology did not exist, it would have been necessary to invent it."

Various forms of ostensibly paranormal activity have been touted over many centuries. Fire walking, water divining, lycanthropy, shamanism, spiritualism, poltergeists, levitation, alchemy, materialization, faith healing, and oriental thaumaturgy have each experienced periods of popular appeal. In the 17th century, animism was a prevalent doctrine; doctors proclaimed a belief in "animal spirits" as the motive force of blood circulating in the human body. In the Victorian era, psychical "research" emphasized communication with the dead. Nonphysical communication between living persons was a logical extension, and in 1882 the term *telepathy* was coined by F.W.H. Myers (about seven years earlier Mark Twain had proposed the phrase *mental telegraphy*). The seeds of parapsychology theory were planted.

Gamblers were now offered the dream of "beating the game" through the exercise of personal, psychic powers. Cards could be read from the back, Roulette numbers predicted, and dice levitated to produce the desired outcome. Fortunes were available to those who possessed the "gift"; the "mind" could now conquer the universe.

The Statistical Approach

Perhaps the most significant turn in the investigation of parapsychology (a term coined in 1889 by psychologist Max Dessoir) was the introduction of the statistical approach. While no reason exists to assume that paranormal phenomena admit of statistical measures, this approach has its ingenious qualities. The obliging nature of the random number tables is well proven. And the field of statistics bows to no master for the ability to furnish subterfuge, confusion, and obscuration.

The first modern statistical experiments to test the existence of telepathic communication were performed in 1912 by J.E. Coover (Ref.) at Stanford University. About 100 students were induced to endure 14,000 trials of divining cards drawn from conventional 52-card decks. The results of the poorly controlled experiments were slightly above chance expectations, although Dr. Coover denied any particular significance.

A similar series of card-guessing trials, sponsored by the Society for Psychical Research, was undertaken by Ina Jephson in London in 1929. Sir Ronald Fisher was consulted to establish the scale of probability for partial successes (e.g., predicting the Ace of Spades instead of the Ace of Clubs). Each day, for a total of five days, 240 people guessed a card at random from the 52-card deck and then, following a shuffling operation, guessed again, the operation being performed five times. Of the 6000 guesses, 245 were correct (chance expectation = 115.4) in addition to many partial successes. Later repeated under more rigidly controlled conditions, the experiments exhibited no obvious paranormality.

In the United States, the first university experiments attempting telepathic communication with a "vocabulary" of playing cards were initiated in 1926 by Dr. G.H. Estabrooks of Harvard. A total of 1660 trials was performed wherein card color was conjectured; 938 successes resulted (expectation = 830).

Dr. and Mrs. J.B. Rhine (Ref.) commenced their experiments in 1927 at the Psychology Laboratory of Duke University. The Rhines replaced the 52-card deck with the 25 Zener cards: five kinds of cards with five of each kind as shown in Figure 11-1. Using these cards, Dr. Rhine conducted or supervised a truly

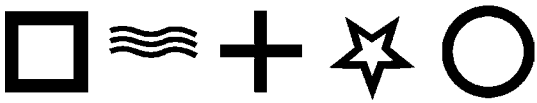

FIGURE 11-1 The Zener cards.

formidable number of experiments (over 10,000 cases) testing for telepathy, clairvoyance, telegnosis (knowledge of distant happenings), precognition (divination of the future), and psychokinesis or PK (mental control over physical objects).

Rhine was responsible for the term “extrasensory perception” (ESP), which subsumes telepathy, clairvoyance, and precognition. Long-time head of the Duke Parapsychology Laboratory, he established himself as the leading American panjandrum for the existence of paranormal phenomena.

His experiments begin with an “agent” who notes each of the Zener cards sequentially and transmits telepathically to a “percipient” who records his “received signal.” Subsequently, the recorded version of the percipient is compared with the actual order of the cards. Sometimes agent and percipient are in the same room and can communicate verbally; at other times they are separated and communicate by lights or buzzers. The guesses have been attempted with and without the presence of observers, at distances from a few feet to across continents, and with delays ranging up to several years. The same set of cards has been utilized for the investigation of precognition and postcognition by examining the guesses for a possible forward or backward displacement (i.e., the percipient may be guessing one card ahead or one behind the actual card in the sequence being transmitted).

In England, the leading investigator of ESP phenomena for many years was British mathematician S.G. Soal who, despite his academic background, stood accused of altering and faking data. It is noteworthy that the first of Soal’s experimentation (1935–1939), conducted with 160 subjects recording 128,000 guesses, produced results well within predictions from chance alone. Subsequently, following the suggestion of another ESP advocate, Whately Carington, Dr. Soal turned to data mining for evidence of displacement and claimed to unearth significant proof of precognition and postcognition.

ESP experimenters range from those whose working environments are suitable for parlor games to those equipped with highly complex (and expensive) laboratory equipment. Dr. Andrija Puharich (a medical parapsychological researcher) operated a laboratory in Maine where he attempted to analyze the nature of telepathic communication by measuring interference patterns between electric or magnetic fields and ESP waves (the agent or percipient is seated in a Faraday cage!). The Communications Science Laboratory of AFCRL (Air Force Cambridge Research Laboratories) at Hanscom Field, Mass., at one time conducted a government-sponsored program for investigating paranormal phenomena. At SRI International, “remote-viewing” experiments were pursued in the 1970s—subjects attempted to "see" a remote place through the eyes of another person; the CIA initially expressed interest, then backpedaled.

In the Soviet Union, the subject was labeled "biological radio communication"; laboratories devoted to its existence and exploitation were set up in Moscow, Leningrad, and Omsk (with results classified Secret).

The world's first chair of parapsychology was established at the University of Utrecht after the Netherlands Ministry of Education had officially recognized the subject as a branch of psychology. The Princeton Engineering Anomalies Research (PEAR) laboratory gave up the ghost in early 2007 after nearly three decades of PK and ESP experiments that mostly embarrassed the university. And there still exist numerous small concentrations of parapsychology experimenters, each guessing the Zener cards, each rolling dice while concentrating, muscles taut, on directing the outcome, and each publishing their results, however questionable, in receptive journals.

Attitudes to ESP

What reasonable posture can we assume in the face of this overwhelming mass of "evidence"? To skewer the subject with systematic rejection would be hubristic, although we cannot accept it comfortably. As Aristotle declared, in speaking of oracular dreams, "It is neither easy to despise such things, nor yet to believe them." Yet, were there any credible evidence for the existence of extrasensory perception, scores of reputable scientists would then leap into such a fascinating field. None such has been forthcoming.

The very existence of ESP would revolutionize our understanding of the physical universe and would demand a drastic overhaul of modern philosophy. But scientists and philosophers unquestionably retain a negative view of ESP, many with such intensity that they find it difficult to repress a Nietzschean outburst against determined believers. Likely, their reasoning is similar to Hume's in his argument regarding miracles:

> A miracle is a violation of the laws of nature; and as a firm and unalterable experience has established these laws, the proof against a miracle, from the very nature of the fact, is as entire as any argument from experience can possibly be imagined . . . [N]o testimony is sufficient to establish a miracle unless the testimony be of such a kind that its falsehood would be more miraculous than the fact which it endeavors to establish.[4]

Are the miracles of Rhine, Soal, et al., less miraculous than the possibility of fraud, self-deception, or error?

Mathematical Critiques of ESP Experiments

From its inception, the field of ESP has been a magnet for criticism directed at its methods of statistical analysis, its standards of measurement, and/or the integrity and sophistication of is investigators. In the 1930s and 1940s, the statistical techniques employed by Rhine and other ESP researchers were widely

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[4] From *An Enquiry Concerning Human Understanding.*

but often erroneously disparaged. Indisputably, statistical evaluations of ESP experiments have been of a quality far beyond the standards of the experiments themselves. As a consequence, the mathematical soundness of the analysis of the ESP and PK experiments has been cited to prove the soundness of the experimental procedures (*ignoratio elenchi*). In addition to the fact that one rigorous element of a structure does not automatically imply a totally sound structure, parapsychologists have shown themselves all too tolerant of worms in their statistical apples.

The first problem in examining the existence of ESP by telepathic transmission of card symbols is to distinguish between prediction and guesswork. A scale must be established that measures the results in terms of their departure from chance occurrences. Following a technique used by psychometrists, ESP investigators have adopted the term *critical ratio*, which provides a numerical evaluation of the significance of a series of "guesses" in terms of the number of standard deviations above (or below) the expectation resulting from chance. That is,

$$\text{Critical ratio} = \frac{\text{observed deviation from chance expectation}}{\text{one standard deviation}}$$

(Initially, probable error was used instead of the standard deviation; thus the critical ratio would be increased by 0.6745.)

This definition is a convenient and sensible one, particularly if the probability of the various correct guesses follows a binomial distribution, for then the probability that the results are due solely to chance is obtainable directly from tables of the probability integral. Indeed, a binomial distribution is assumed in the evaluation of ESP experiments with card guessing.

Consider a percipient predicting cards drawn (without replacement) from a deck of N cards composed of N/M kinds. Then an implicit assumption in application of the binomial distribution is that each kind may be represented in the deck by any number from zero to N, whence the probability of n correct guesses is expressed by

$$P(n) = \binom{N}{n}\frac{M^n}{N^n}\left(\frac{N-M}{N}\right)^{N-n} = \binom{N}{n}\frac{M^n\,(N-M)^{N-n}}{N^n} \tag{11-1}$$

Given knowledge of the deck's composition, it is highly likely that the percipient's predictions will closely correspond to that composition (i.e., he will tend to guess N/M kinds of cards and M of each kind). The probability distribution of correct guesses for this circumstance can be derived as

$$P(n) = \sum_{m=0}^{N-n} (-1)^m \binom{n+m}{n} S_{n+m} \tag{11-2}$$

where

$$S_n = \sum \left(\frac{M^k}{k_1!(N-1)_{n-1}} \prod_{j=1}^{k-1} (N - jM) \prod_{i=1}^{n-1} \frac{1}{k_{i+1}!} \left\{ \frac{[(M-1)_i]^2}{(i+1)!} \right\}^{k_{i+1}} \right) \tag{11-3}$$

with $k = \sum_{i=1}^{n} k_i$ and $n = \sum_{i=1}^{n} ik_i$. The summation in Eq. 11-3 is over those partitions of n restricted by $k \leq N/M$ and $i \leq M$.

Equations 11-1 and 11-2 are evaluated for the Zener pack ($N = 25$, $M = 5$) with results tabulated in Table 11-1. The difference between the two distributions is not excessive, the exact distribution being skewed slightly toward the extreme values.

Table 11-1 Probabilities for Guessing Zener Cards

Number of Correct Guesses	Probability Due to Chance (Binomial Distribution)	Probability Due to Chance (Exact Distributions)
0	3.778×10^{-3}	4.286×10^{-3}
1	2.361×10^{-2}	2.545×10^{-2}
2	7.084×10^{-2}	7.336×10^{-2}
3	1.358×10^{-1}	1.366×10^{-1}
4	1.867×10^{-1}	1.843×10^{-1}
5	1.960×10^{-1}	1.919×10^{-1}
6	1.633×10^{-1}	1.603×10^{-1}
7	1.108×10^{-1}	1.101×10^{-1}
8	6.235×10^{-2}	6.331×10^{-2}
9	2.944×10^{-2}	3.090×10^{-2}
10	1.178×10^{-2}	1.291×10^{-2}
11	4.015×10^{-3}	4.654×10^{-3}
12	1.171×10^{-3}	1.453×10^{-3}
13	2.928×10^{-4}	3.942×10^{-4}
14	6.273×10^{-5}	9.310×10^{-5}
15	1.150×10^{-5}	1.914×10^{-5}
16	1.797×10^{-6}	3.423×10^{-6}
17	2.378×10^{-7}	5.312×10^{-7}
18	2.643×10^{-8}	7.115×10^{-8}
19	2.434×10^{-9}	8.247×10^{-9}
20	1.826×10^{-10}	7.981×10^{-10}
21	1.087×10^{-11}	7.079×10^{-11}
22	4.939×10^{-13}	4.011×10^{-12}
23	1.611×10^{-14}	4.011×10^{-13}
24	3.355×10^{-16}	0
25	3.555×10^{-18}	1.604×10^{-15}

Assigning a binomial distribution to card-guessing phenomena results in a platykurtic bias of the probabilities. For the Zener deck, the binomial distribution is characterized by a standard deviation of

$$\sigma = \sqrt{Npq} = \sqrt{25(1/5)(4/5)} = 2.000$$

where $p = 1/5$ is the probability of a correct guess, and $q = 1 - p$. For the exact distribution of guesses (knowing the composition of the Zener pack), the standard deviation is

$$\sigma = 2.041$$

Thus, the assumption of a binomial distribution injects an error of 2% in the critical ratio, favoring the existence of ESP.

R.C. Read (Ref.) examined an ESP experiment wherein each card is exposed to view following the attempt by the percipient to predict its symbol. Under this rule, an intelligent strategy is to select for the next card that symbol (not necessarily unique) represented most frequently among the remaining cards. With an $n \times n$ deck (n kinds of cards with n of each kind), the expected number of correct guesses is then

$$\mathrm{E} = \frac{1}{n^2}\sum_{m=1}^{m^2}\frac{A_m}{\binom{n^2-1}{m-1}} \tag{11-4}$$

where A_m is the coefficient of t^m in the expression

$$n(1+t)^{n^2} - \sum_{s=0}^{n-1}\sum_{r=0}^{s}\left[\binom{n}{r}t^r\right]^n$$

For the Zener deck, Eq. 11-4 results in E = 8.647, an expectation considerably greater than that (5) obtained without the sequential exposure of each card.

We could improve the significance of these experiments and their interest—and blunt Read's criticism—if each of the 25 cards was independently selected with probability 1/5, both agent and percipient remaining uninformed regarding the specific composition of the deck. (The Zener deck would occur with probability $25!\,(5!)^{-5}\,5^{-25} = 0.00209$.)

Another source of error arises from incorrect treatment of the forward or backward displacement of the predictions. Rigorous experimental technique dictates that the object of a series of guesses be established *a priori*. If the series of trials is sufficiently long, it becomes more feasible to permit *a posteriori* decisions as to the object of the experiment. However, in a relatively short sequence, the expectation of the number of correct predictions increases with the flexibility of the *a posteriori* decisions. For example, a series of 25 guesses with the Zener cards, restricted to zero displacement, yields an expectation of

$$\mathrm{E} = \sum_{i=0}^{25} ip(i) = 5$$

for the number of matches [with either the binomial or exact distribution used for $p(i)$, the probability of guessing exactly i cards correctly].

Now, if we are allowed to decide *a posteriori* that we will search the recorded data for evidence of a forward displacement of one card (precognition), no displacement (telepathy), or a backward displacement of one card (postcognition), and select that which is maximum, the expected number of correct guesses is expressed by

$$\mathrm{E} = \sum_{i=0}^{25} \sum_{j=0}^{i} \sum_{k=0}^{j} ip(i)p(j)p(k)\lambda_{ijk} = 0.6740$$

where

$$\lambda_{ijk} = \begin{cases} 6 & \text{for } i \neq j \neq k \\ 3 & \text{for } i \neq j = k \quad \text{or } i = j \neq k \\ 1 & \text{for } i = j = k \end{cases}$$

and the exact probability distribution of Table 11-1 is used for $p(i)$, $p(j)$, and $p(k)$. Obviously, if displacements greater than plus or minus one card are allowed, the expected number of correct guesses rises further. On the other hand, as the length of the series of trials increases, the expectation decreases, approaching the value (5) obtained by establishing the data analysis procedure *a priori*.

Inadequate shuffling of the Zener cards provides yet another potential source of higher scores in telepathic experiments. From the analysis presented in Chapter 7, we can note that more shuffles are required to randomize odd numbered deck lengths than a deck of $2n$ cards. Thus, special precautions are indicated when employing a 25-card deck. Although the mental feat of predicting cards by estimating the shifts in position after shuffling is a prodigious one, it is not above the ability of some practiced individuals. Further, only a limited success is required to achieve a significant improvement in the results.

Another error common in ESP experimentation arises when the number of trials of card guessing is not established *a priori*. With the trials concluded at the option of the experimenter (usually when the percipient becomes “tired” and his success rate declines; this circumstance is referred to as the “headache phenomenon”), unusual results can be guaranteed. To illustrate this concept, let us examine the null hypothesis H_0 that the mean number of correct guesses in a series of trials is that of chance expectation, as opposed to the hypothesis H_1 that the mean number of guesses is greater (say) than chance expectation. Then, if it is assumed that the series of guesses is normally distributed, the conventional statistical test based on a sample of fixed size—n trials—dictates that we must reject H_0 in favor of H_1 if and only if the number of correct guesses S_n is such that

$$S_n - np > Cn^{1/2} \tag{11-5}$$

where p is the chance probability of achieving a correct guess, and C is some constant. The probability $P(C)$ of rejecting H_0 despite its truth is

$$P(C) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-x^2/2} dx$$

By selecting a suitably large value of C, $P(C)$ can be rendered as small as desired.

Now it follows from the law of the iterated logarithm that with unity probability the inequality of Eq. 11-5 is valid for infinitely many values of n if the sampling is continued indefinitely, regardless of the extent to which $P(C)$ is decreased. Thus, the experimenter must eventually encounter a value of n whereby the inequality holds; terminating the experiment at that point compels us to reject the null hypothesis H_0 (and thereby disregard truth). It is evident that when optional stopping is permitted, the usual statistical procedures for hypothesis testing cannot be applied. We should insist on establishing beforehand the sample size or, at least, restraining it to be within reasonable limits. With the flexibility inherent in the option to alter at will the number of trials, it is not surprising that ESP is described (by parapsychologists) as a phenomenon visited upon some people sometimes.

We should note that the consequences of optional stopping are not relevant in games of known probability, as discussed in Chapter 3. In those instances, it is not a statistical hypothesis under examination, but the running sum of a binary sequence, which is automatically concluded if at any time it reaches some preestablished value.

Other Criticisms

It is sometimes difficult to understand how the elementary points expressed here can be ignored by ESP investigators. The accusation of fraud has been raised, notably by Dr. George Price (Ref.) in *Science* magazine; and, for certain graduate students participating in the experiments, certain *soi-disant* "psychics," and certain individuals with vested interests in the existence of ESP, the perpetration of outright fraud likely explains many outstanding results. Several examples of collusion have been uncovered by statistical analyses of the series of correct "guesses" that detected significant periodicities, indicating a high probability of prearranged "guesses" at predetermined positions on a printed form. Subconsciously motivated scoring errors in favor of the basic premise have been detected by hidden cameras. However, in many instances, the unquestionable sincerity and stature of the experimenters must render the possibility of duplicity exceedingly small. Without impugning an experimenter's integrity, we would be well advised to recall Dicey's dictum that men's interests give a bias to their judgments far oftener than they corrupt their hearts. This bias, which can descend to the level of blatant axe-grinding, is often accompanied by incredible naïveté concerning the safeguards necessary to ensure a controlled experiment.

Aside from fraud and bias—conscious or unconscious—workers in the paranormal field appear susceptible to such logic viruses as "the sheep-goat syndrome" and "the decline complaint." In the former instance, positive results are obtained from believers, while skeptics engender negative results. In the latter case, testing of "successful" subjects is discontinued when the level of "success" declines—attributing the decline to the boring nature of the process rather than to a natural regression toward the mean. Similar is the justification of loose controls on the subject, with the rationale that "controls interfere with the psychic realm."

Finally, it should be noted that the existence of telepathic ability would enable reliable communication links without transfer of energy (thereby violating the Heisenberg principle). Information theory is unequivocal in demonstrating that any system providing a finite capacity for transmitting information can, with appropriate coding, transmit with any desired degree of accuracy. According to Shannon's mathematical theory of communication, the capacity C, in bits per trial, of a communication system composed of telepathic transmissions of the five Zener symbols is

$$C = \log_2 5 + \frac{n}{25}\log_2\left(\frac{n}{25}\right) + \frac{25-n}{25}\log_2\left(\frac{25-n}{100}\right)$$

where n (≥ 5) is the mean number of correct guesses per 25 trials (the formula applies only for equiprobability in selecting any of the five symbols). With five correct guesses per 25 trials, C, of course, is zero. For $n = 6$, $C \simeq 0.0069$ bits per trial, and for $n = 7$, $C \simeq 0.026$ bits per trial. We could consequently realize a communication system with a data rate of several bits per hour—adequate for communicating with submerged Polaris submarines, for example. If the principles of communication theory are admitted, we could increase the data rate by an array of ESP "transmitters" linked for simultaneous emission.

Although but one of many exercises in futility in the search for certainty, ESP has perhaps developed the most convoluted and elaborate illogic.[5] The common technique of ESP practitioners, contrary to scientific tradition, is to concoct a speculative thesis and then attempt to confirm its truth. For example, one such hypothesis, propounded to mollify those with a rudimentary knowledge of

[5]A whimsical featherweight contender to ESP is the "science" of *pataphysics*. Its adherents, including Eugène Ionesco and Raymond Queneau, founded a society, the College of Pataphysics, in Paris. As defined by its creator Alfred Jarry, "Pataphysics is the science of imaginary solutions."

physical phenomena, is that highly complex organisms (the human brain) do not obey the known laws of physics. The prodigious amount of data accumulated in the pursuit of such speculations serves to demonstrate what sumptuous logical superstructures the human mind can build upon a foundation of shifting sands.

Some Additional ESP Experiments

A limited series of ESP and PK experiments was supervised by the author and a small group of associates.[6] Insofar as possible a dispassionate inquiry into the existence of ESP and PK was launched with *tabula rasa* objectivity; the viewpoint was neither credulous nor incredulous. We were, however, unable to devise a satisfactory methodology for mounting a rational assault on the irrational. Equipment from cooperating engineering laboratories was utilized, as well as a polygraph and an EEG. Since interpretation of results obtained through these two instruments is not rigidly deterministic, their readouts were not accepted without supporting evidence. No religious, psychological, or preternatural overtones were permitted gratuitously in the hypotheses. Those experiments involving statistical elements were checked for possible sources of biased mechanization—in Chapter 6 we have seen the significant deviations from chance occurring from poor-quality dice (cf. Wolf's and Weldon's dice data).

The statistical experiments in telepathy and clairvoyance, conducted with the percipient in complete isolation (a soundproof and lightproof chamber), produced results well within one standard deviation of chance expectation. Telepathic trials with the agent and percipient in the same room communicating orally (to signal each card in sequence) yielded, especially with pairs that had worked together for long periods, increasing success with a decreasing number of symbols. That is, binary choices (guessing the color of cards in a conventional deck) were quite successful, five-level choices (the Zener pack) less so, and 26-level and 52-level choices produced only chance results. Of course, as the number of choices increases, the amount of information required from sensory cues increases correspondingly; and it is likely that the subconscious transmission and reception of sensory cues are more efficient when given an established rapport between two individuals closely associated.

For the PK experiments, a cesium-beam atomic clock was requisitioned. Concentrating on the molecular streams, various individuals attempted to change the resonant frequency by altering the vibrations of only a few molecules. The results were negative.

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[6] Initially constituted as the Yoknapatawpha Martini and Metaphysical Society, the group was disbanded in the face of persistently nonmystical results.

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Epilogue

A well-known short story by Robert Coates describes the incredibly chaotic events transpiring from suspension of the law of averages.[1] We have demonstrated—convincingly, we hope—that the law of averages, aka the law of large numbers, can be relied upon in dealing with the phenomena of gambling. We can, in fact, and with a modicum of smugness do, advise: You can bet on it.

In order to synthesize even an elementary framework of gambling theory, we have resorted to the disciplines of probability theory, statistics, and game theory as well as sampling occasional fruits from other branches of mathematics. With these analytic tools, it is possible to evolve the basic principles of gambling theory and thence to derive optimal game strategies and optimal courses of action for individual participation in gambling situations as functions of initial bankroll, objective, etc. As a caveat, it should be stated that the theorems proved herein refer to numbers—that is, abstract quantities; applications to dice, coins, cards, and other paraphernalia are entirely adventitious.

The games considered in this book range from the simplest strategic contests, such as coin matching, to intricate nonstrategic games of skill, such as Chess, or the extreme complexity of social phenomena, as represented by economic conflict. Elementary coin, wheel, and dice games, *inter alia*, have been analyzed completely and their strategies specified. In general, however, games whose outcomes depend not merely on natural statistical fluctuations, but on the actions of other players and wherein a player must select from among relatively complex strategies, pose mathematical problems yet unsolved. Poker (a strategic game based on imperfect information and inverted signaling) and Bridge (a strategic

[1] Robert M. Coates, "The Law," *New Yorker*, 23, Part 4 (November 29, 1947), pp. 41–43. All car owners in the Greater New York area decide simultaneously on an evening's excursion over the Triborough Bridge; restaurants find only a single item on the menu being ordered; a small notions store is inundated by hundreds of customers, each in single-minded pursuit of a spool of pink thread. Eventually, by legislating human behavior, Congress amends the oversight of never having had incorporated the law of averages into the body of federal jurisprudence.

game involving information exchange) provide two examples where detailed optimal strategies cannot be enumerated because of the astronomical amount of attendant computation.

We have also considered games of pure chance, where the opponent is an impartial, godlike statistician; games of mixed chance and skill, such as Blackjack, where the dependence between plays implies the value of a memory; games of pure skill, such as Nim and Pentominoes, where the opponent is another player, and games involving nonstationary statistics, such as horse racing or stock-market speculation. Gambling on contests of pure skill is comparatively rare at present (although it was common in past eras), possibly because a hierarchy of the participants is soon established in a closed community. It would appear that gambling on outcomes determined solely by chance, or on judgments of which horse or dog is the speediest, offers greater appeal. While these categories are not presumed to exhaust the subject, they do comprise a substantive encapsulation of most gambling phenomena.

The subjective element often inherent in games, particularly in those of an economic nature, is considered by means of the fundamental concept of utility functions. Yet the extent to which utility theory describes actual human behavior is debatable. The assumption that individuals act objectively in accordance with purely mathematical dictates to maximize their gain or utility cannot be sustained by empirical observation. Indeed, people more often appear to be influenced by the skewness of the probability distribution; hence the popularity of lotteries and football pools with highly negative mathematical expectations. Perhaps the nature of human preference precludes a normative theory of utility.

An alternate lens through which we can scrutinize the subject of gambling is that constructed by the parapsychologists. We have reviewed the evidence for extrasensory perception and have classified the phenomenon as a *trompe l'oeil*. Similarly, we have dismissed as fantasy the value of propitiating the gods of fortune through esoteric rituals.

"Gambling," said Dr. Theodore Reik, "is a sort of question addressed to destiny." To answer the question requires a hazardous ascent up the slippery ladder of probability, gambling theory, and statistical logic. What treasures await the successful climber, we will not speculate. Information, as we have shown, can only enhance our expectation.

Rien ne va plus.

Appendices

Table A The Reddere Integral

t	.00	.02	.04	.06	.08
0.00	.399	.389	.379	.370	.360
0.10	.351	.342	.333	.324	.315
0.20	.307	.299	.290	.282	.274
0.30	.267	.259	.252	.245	.237
0.40	.230	.224	.217	.210	.204
0.50	.198	.192	.186	.180	.174
0.60	.169	.163	.158	.153	.148
0.70	.143	.138	.133	.129	.125
0.80	.120	.116	.112	.108	.104
0.90	.100	.0968	.0933	.0899	.0865
1.00	.0833	.0802	.0772	.0742	.0714
1.10	.0686	.0659	.0634	.0609	.0584
1.20	.0561	.0538	.0517	.0495	.0475
1.30	.0455	.0436	.0418	.0400	.0383
1.40	.0367	.0351	.0336	.0321	.0307
1.50	.0293	.0280	.0267	.0255	.0244
1.60	.0232	.0222	.0211	.0201	.0192
1.70	.0183	.0174	.0166	.0158	.0150
1.80	.0143	.0136	.0129	.0123	.0116
1.90	.0111	.0105	.00996	.00945	.00896
2.00	.00849	.00805	.00762	.00722	.00683
2.10	.00647	.00612	.00579	.00547	.00517
2.20	.00489	.00462	.00436	.00411	.00388
2.30	.00366	.00345	.00325	.00307	.00289
2.40	.00272	.00256	.00241	.00227	.00213
2.50	.00200	.00188	.00177	.00166	.00156
2.60	.00146	.00137	.00129	.00121	.00113
2.70	.00106	.000993	.000929	.000870	.000814
2.80	.000761	.000711	.000665	.000621	.000580
2.90	.000542	.000505	.000471	.000440	.000410
3.00	.000382				

Programmed by Norman Wattenberger, Casino Vérité.

Table B Yahtzee Strategies

Still to get		1s 3k	2s 4k	3s FH	4s LS	5s HS	6s YZ	CH			
Sum of 1s–6s		0									
Yahtzee scored		No									
A	B	C	D	A	B	C	D	A	B	C	D
11111	11111	11111	YZ	12222	2222	2222	2s	14445	444	444	4s
11112	1111	1111	1s	12223	222	222	2s	14446	444	444	4s
11113	1111	1111	1s	12224	222	222	2s	14455	55	55	1s
11114	1111	1111	1s	12225	222	222	2s	14456	44	44	CH
11115	1111	1111	1s	12226	222	222	2s	14466	66	66	CH
11116	1111	1111	1s	12233	33	33	2s	14555	555	555	5s
11122	111	11122	FH	12234	1234	1234	LS	14556	55	55	CH
11123	111	111	1s	12235	22	22	2s	14566	66	66	CH
11124	111	111	1s	12236	22	22	2s	14666	666	666	6s
11125	111	111	1s	12244	44	44	2s	15555	5555	5555	5s
11126	111	111	1s	12245	22	22	2s	15556	555	555	5s
11133	111	11133	FH	12246	22	22	2s	15566	66	66	CH
11134	111	111	1s	12255	55	55	2s	15666	66	666	6s
11135	111	111	1s	12256	22	22	2s	16666	6666	6666	6s
11136	111	111	1s	12266	66	66	2s	22222	22222	22222	YZ
11144	111	11144	FH	12333	333	333	3s	22223	2222	2222	2s
11145	111	111	1s	12334	33	1234	LS	22224	2222	2222	2s
11146	111	111	1s	12335	33	33	3s	22225	2222	2222	2s
11155	111	11155	FH	12336	33	33	3s	22226	2222	2222	2s
11156	111	111	1s	12344	44	1234	LS	22233	222	22233	FH
11166	111	11166	FH	12345	12345	12345	HS	22234	222	222	2s
11222	222	11222	FH	12346	1234	1234	LS	22235	222	222	2s
11223	22	22	1s	12355	55	55	1s	22236	222	222	2s
11224	22	22	1s	12356	5	235	1s	22244	222	22244	FH
11225	22	22	1s	12366	66	66	1s	22245	222	222	2s
11226	22	22	1s	12444	444	444	4s	22246	222	222	2s
11233	33	1133	1s	12445	44	44	1s	22255	222	22255	FH
11234	1234	1234	LS	12446	44	44	1s	22256	222	222	2s
11235	5	11	1s	12455	55	55	1s	22266	222	22266	FH
11236	6	11	1s	12456	5	456	1s	22333	333	22333	FH
11244	44	44	1s	12466	66	66	1s	22334	33	234	2s
11245	5	11	1s	12555	555	555	5s	22335	33	33	2s
11246	4	11	1s	12556	55	55	1s	22336	33	33	2s
11255	55	55	1s	12566	66	66	CH	22344	44	44	2s
11256	5	11	1s	12666	666	666	6s	22345	2345	2345	LS
11266	66	66	1s	13333	3333	3333	3s	22346	22	234	2s
11333	333	11333	FH	13334	333	333	3s	22355	55	55	2s
11334	33	1133	1s	13335	333	333	3s	22356	22	22	2s
11335	33	1133	1s	13336	333	333	3s	22366	66	66	2s
11336	33	1133	1s	13344	44	44	3s	22444	444	444	FH
11344	44	44	1s	13345	33	345	3s	22445	44	44	2s
11345	345	44	1s	13346	33	33	3s	22446	44	44	2s
11346	4	34	1s	13355	55	55	3s	22455	55	55	2s
11355	55	55	1s	13356	33	33	3s	22456	22	22	2s
11356	5	11	1s	13366	66	66	3s	22466	66	66	2s
11366	66	66	1s	13444	444	444	4s	22555	555	555	FH
11444	444	444	FH	13445	44	44	1s	22556	55	55	2s
11445	44	44	1s	13446	44	44	1s	22566	66	66	2s
11446	44	44	1s	13455	55	55	1s	26666	666	666	FH
11455	55	55	1s	13456	3456	3456	SS	23333	3333	3333	3s
11456	5	456	1s	13466	66	66	CH	23334	333	333	3s
11466	66	66	Is	13555	555	555	5s	23335	333	333	3s
11555	555	555	FH	13556	55	55	CH	23336	333	333	3s
11556	55	55	1s	13566	66	66	CH	23344	44	44	3s
11566	66	66	1s	13666	666	666	6s	23345	2345	2345	LS
11666	666	666	FH	14444	4444	4444	4s	23346	33	234	3s

(*continued*)

Table B (Continued)

Still to get		1s 3k	2s 4k	3s FH	4s LS	5s HS	6s YZ	CH			
Sum of 1s–6s Yahtzee scored		0 No									
A	B	C	D	A	B	C	D	A	B	C	D
23355	55	55	3s	33333	33333	33333	YZ	34566	66	3456	SS
23356	33	33	3s	33334	3333	3333	3s	34666	666	666	3k
23366	66	66	CH	33335	3333	3333	3s	35555	5555	5555	5s
23444	444	444	4s	33336	3333	3333	3s	35556	555	555	5s
23445	2345	2345	LS	33344	333	33344	FH	35566	66	66	CH
23446	44	44	CH	33345	333	333	3s	35666	666	666	3k
23455	2345	2345	LS	33346	333	333	3s	36666	6666	6666	6s
23456	23456	23456	HS	33355	333	33355	FH	44444	44444	44444	YZ
23466	66	66	CH	33356	333	333	3s	44445	4444	4444	4s
23555	555	555	5s	33366	333	33366	FH	44446	4444	4444	4s
23556	55	55	CH	33444	444	444	FH	44455	444	444	FH
23566	66	66	CH	33445	44	44	3s	44456	444	444	4s
23666	666	666	6s	33446	44	44	CH	44466	444	444	FH
24444	4444	4444	4s	33455	55	55	CH	44555	555	555	FH
24445	444	444	4s	33456	33	3456	SS	44556	55	55	CH
24446	444	444	4s	33466	66	66	CH	44566	66	66	CH
24455	55	55	CH	33555	555	555	FH	44666	666	666	FH
24456	44	44	CH	33556	55	55	CH	45555	5555	5555	5s
24466	66	66	CH	33566	66	66	CH	45556	555	555	3k
24555	555	555	5s	33666	666	666	FH	45566	66	66	CH
24556	55	55	CH	34444	4444	4444	4s	45666	666	666	3k
24566	66	66	CH	34445	444	444	4s	46666	6666	6666	6s
24666	666	666	6s	34446	444	444	4s	55555	55555	55555	YZ
25555	5555	5555	5s	34455	55	55	CH	55556	5555	5555	5s
25556	555	555	5s	34456	44	3456	SS	55566	555	555	FH
25566	66	66	CH	34466	66	66	CH	55666	666	666	3k
25666	666	666	3k	34555	555	555	5s	56666	6666	6666	6s
26666	6666	6666	6s	34556	55	3456	SS	66666	66666	66666	YZ

Notes: Column A lists the 252 possible five-dice configurations. Column B designates dice to be fixed assuming that the configuration shown in Column A represents the first throw of the dice. Column C designates dice to be fixed assuming the configuration of Column A represents the second roll of the dice. Column D indicates the scoring category to be selected assuming the Column A configuration represents the final throw of the dice.

Courtesy of Phil Woodward.

Table C (A) EORs—Dealer's Up-Card 2

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-1.86	-2.38	-2.93	-0.03	-0.72	-0.30	0.21	0.66	2.32	-1.92	-38.28	-38.30	-38.31
16	-1.37	-1.83	-2.56	-4.47	-0.37	0.09	0.61	1.06	2.57	-1.43	-17.75	-17.80	-17.80
15	-0.73	-1.22	-1.96	-3.74	-3.82	0.30	0.79	1.21	2.54	-0.98	-12.29	-12.35	-12.36
14	-0.36	-0.60	-1.31	-2.98	-3.10	-3.17	0.97	1.35	2.50	-0.80	-6.85	-6.91	-6.92
13	-0.26	-0.21	-0.65	-2.20	-2.37	-2.48	-2.54	1.50	2.46	-0.64	-1.41	-1.47	-1.48
12	-0.12	-0.07	-0.22	-1.43	-1.63	-1.80	-1.90	-2.05	2.42	-0.47	4.21	4.03	4.01
							DOUBLE						
11	1.00	0.98	1.05	1.86	1.63	1.09	0.65	0.18	-2.53	1.67	23.47	23.31	23.29
10	0.93	0.81	0.91	1.73	1.76	1.52	1.26	0.71	-2.22	-0.73	17.76	17.68	17.67
9	1.46	0.46	0.53	1.49	1.53	1.77	1.69	0.80	-2.29	-0.58	-1.37	-1.34	-1.34
8	1.00	1.15	0.30	1.35	1.56	1.70	1.54	0.85	-2.30	-0.23	-18.42	-18.32	-18.31
							SOFT DOUBLE						
A–9	1.93	1.97	2.16	3.35	3.49	2.94	1.21	-0.53	-3.67	-1.84	-27.72	-27.98	-28.01
A–8	-2.82	1.75	1.89	3.18	3.26	2.69	1.59	0.46	-2.71	-1.19	-14.22	-14.37	-14.39
A–7	-2.13	-3.18	1.61	2.93	2.43	1.95	2.09	1.47	-1.68	-0.47	-0.17	-0.19	-0.19
A–6	-0.76	-1.25	-1.61	1.62	1.78	1.73	1.70	1.47	-1.29	0.49	-0.78	-0.70	-0.69
A–5	-0.23	-0.72	-1.04	-0.59	1.70	1.60	1.54	1.28	-1.21	1.30	-5.26	-5.12	-5.11
A–4	0.55	-0.19	-0.50	0.05	-0.60	1.57	1.46	1.20	-1.33	1.78	-7.33	-7.21	-7.19
A–3	1.02	0.55	0.07	0.68	0.03	-0.76	1.39	1.11	-1.47	1.81	-9.56	-9.45	-0.44
A–2	1.01	1.04	0.83	1.30	0.64	-0.16	-0.98	1.02	-1.62	1.79	-11.92	-11.85	-11.84
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9–9	-1.45	-1.94	2.43	4.03	2.81	2.19	2.03	-0.10	-3.20	-1.06	7.98	7.58	7.54
8–8	-1.31	-1.68	1.87	2.02	2.82	3.31	1.37	1.74	-2.27	-1.05	37.43	36.92	36.85
7–7	-0.08	-1.26	-1.77	2.59	3.34	2.21	2.75	1.75	-2.31	-0.44	16.54	16.30	16.26
6–6	0.35	-0.46	-1.51	1.57	4.84	5.43	5.11	4.23	-5.03	0.55	4.19	4.18	4.15
3–3	1.25	1.30	1.75	2.77	0.99	-0.64	-1.09	1.54	-2.05	0.33	0.16	0.29	0.29
2–2	0.75	1.08	1.56	2.41	1.99	1.00	-1.93	-2.99	-1.00	0.13	4.17	3.45	3.35
A–A	2.02	2.06	2.23	3.22	2.96	2.28	1.70	1.25	-5.07	2.56	39.30	30.02	38.98

Table C (B) EORs—Dealer's Up-Card 3

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-1.86	-2.37	-3.64	-1.07	-0.67	-0.23	0.31	2.05	2.35	-1.91	-41.45	-41.45	-41.45
16	-1.30	-2.07	-3.99	-4.46	-0.30	0.23	0.68	2.27	2.60	-1.45	-21.24	-21.18	-21.17
15	-0.70	-1.46	-3.25	-3.72	-3.73	0.46	0.88	2.27	2.57	-1.01	-15.47	-15.42	-15.42
14	-0.34	-0.82	-2.50	-2.95	-3.01	-3.01	1.07	2.27	2.54	-0.86	-9.69	-9.66	-9.66
13	-0.20	-0.41	-1.75	-2.19	-2.30	-2.34	-2.45	2.29	2.50	-0.68	-3.72	-3.84	-3.85
12	-0.06	-0.25	-1.25	-1.41	-1.56	-1.65	-1.81	-1.41	2.46	-0.47	2.21	1.97	1.94
							DOUBLE						
11	0.94	0.95	1.49	2.01	1.76	1.32	0.85	-1.62	-2.30	1.50	26.09	25.86	25.83
10	0.79	0.78	1.32	1.89	1.93	1.96	1.43	-1.29	-2.01	-0.76	20.53	20.39	20.38
9	1.37	0.36	1.13	1.71	1.93	2.14	1.55	-1.30	-2.07	-0.63	2.04	1.98	1.98
8	0.91	1.12	1.05	1.81	1.92	1.75	1.31	-1.27	-2.08	-0.29	-14.41	-14.42	-14.42
7	0.52	0.86	2.28	1.93	1.65	1.28	0.86	-1.46	-2.01	0.14	-28.31	-28.32	-28.32
							SOFT DOUBLE						
A-9	1.80	1.98	3.19	3.87	3.90	2.77	1.00	-2.24	-3.62	-1.80	-23.56	-23.92	-23.96
A-8	-2.79	1.82	2.99	3.62	3.06	2.54	1.94	-1.31	-2.68	-1.16	-10.48	-10.73	-10.76
A-7	-2.13	-2.90	2.70	2.75	2.27	2.39	2.39	-0.32	-1.68	-0.44	3.11	2.99	2.98
A-6	-0.77	-1.21	-0.90	1.97	1.94	1.92	1.67	-0.57	-1.12	0.43	2.52	2.58	2.59
A-5	-0.29	-0.66	-0.19	-0.26	1.82	1.76	1.50	-0.76	-1.03	1.20	-1.79	-1.68	-1.67
A-4	0.45	-0.16	0.42	0.35	-0.39	1.68	1.41	-0.86	-1.14	1.66	-3.79	-3.69	-3.67
A-3	0.88	0.58	1.01	0.94	0.19	-0.58	1.32	-0.97	-1.25	1.64	-5.90	-5.83	-5.83
A-2	0.91	1.06	1.78	1.50	0.74	-0.04	-0.99	-1.06	-1.38	1.62	-7.98	-8.08	-8.10
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9-9	-1.66	2.16	3.67	3.35	2.81	3.28	2.35	-2.44	-3.13	-1.02	11.62	11.22	11.17
8-8	-1.35	-2.09	1.55	3.27	4.04	4.11	1.35	-1.01	-2.22	-1.01	40.61	40.10	40.04
7-7	-0.40	-2.06	-0.99	4.53	4.52	3.14	3.50	-1.40	-2.55	-0.64	21.65	21.12	21.05
6-6	0.28	-0.31	0.73	2.22	5.17	6.10	5.42	0.06	-5.02	0.44	11.92	11.57	11.53
3-3	1.24	1.12	2.82	3.25	1.29	-0.21	-1.02	-1.70	-1.77	0.31	5.68	5.29	5.25
2-2	0.55	0.99	1.97	2.79	2.42	1.37	-1.91	-5.89	-0.59	0.07	8.40	7.26	7.12
A-A	1.93	1.98	2.43	2.98	2.58	2.01	1.54	-0.88	-4.23	2.34	41.79	41.55	41.52

Table C (C) EORs—Dealer's Up-Card 4

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-1.78	-3.09	-4.73	-1.01	-0.61	-0.03	1.71	1.98	2.35	-1.89	-44.63	-44.64	-44.64
16	-1.52	-3.47	-3.98	-4.35	-0.21	0.35	1.93	2.23	2.61	-1.41	-24.49	-24.51	-24.51
15	-0.92	-2.74	-3.24	-3.61	-3.64	0.59	1.98	2.25	2.59	-1.01	-18.38	-18.43	-18.44
14	-0.52	-2.02	-2.50	-2.87	-2.93	-2.89	2.04	2.29	2.56	-0.84	-12.10	-12.29	-12.31
13	-0.36	-1.53	-1.74	-2.10	-2.21	-2.23	-1.60	2.29	2.53	-0.63	-5.85	-6.16	-6.20
12	-0.23	-1.28	-1.21	-1.31	-1.45	-1.53	-1.43	-1.13	2.50	-0.44	0.20	-0.10	-0.14
							DOUBLE						
11	0.89	1.38	1.78	2.23	2.10	1.61	-0.83	-1.51	-2.27	1.41	28.55	28.38	28.36
10	0.74	1.17	1.61	2.16	2.49	2.22	-0.47	-1.18	-1.99	-0.79	23.18	23.09	23.08
9	1.31	0.93	1.49	2.22	2.45	2.14	-0.46	-1.19	-2.05	-0.68	5.29	5.29	5.29
8	0.87	1.87	1.66	2.29	2.14	1.67	-0.68	-1.18	-2.07	-0.35	-10.61	-10.55	-10.54
7	0.56	2.03	2.82	2.10	1.77	1.23	-1.13	-1.37	-2.02	0.06	-24.04	-23.97	-23.96
							SOFT DOUBLE						
A-9	1.78	-3.01	3.73	4.26	3.74	2.47	-0.79	-2.13	-3.56	-1.82	-19.63	-19.88	-19.91
A-8	-2.63	2.92	3.44	3.39	2.94	2.79	0.12	-1.22	-2.65	-1.18	-6.97	-7.13	-7.15
A-7	-1.92	-1.60	2.52	2.55	2.74	2.65	0.52	-0.28	-1.67	-0.49	6.02	6.08	6.09
A-6	-0.78	-0.52	-0.30	2.25	2.27	2.01	-0.23	-0.49	-1.14	0.35	5.65	5.84	5.86
A-5	-0.27	0.16	0.25	0.07	2.14	1.85	-0.40	-0.68	-1.05	1.09	1.44	1.71	1.75
A-4	0.42	0.72	0.83	0.64	-0.01	1.77	-0.49	-0.78	-1.15	1.51	-0.46	-0.21	-0.18
A-3	0.88	1.47	1.37	1.18	0.52	-0.43	-0.57	-0.86	-1.27	1.49	-2.30	-2.21	-2.20
A-2	0.93	1.94	2.10	1.73	1.06	0.09	-2.84	-0.97	-1.40	1.55	-4.20	-4.34	-4.36
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
T-T	2.91	5.19	6.55	7.54	6.53	4.12	-1.94	-4.49	-5.64	-3.87	-37.70	-37.07	-36.99
9-9	-1.48	3.68	3.28	3.59	4.26	3.88	0.03	-2.16	-3.52	-0.99	15.47	15.11	14.88
8-8	-1.72	-2.33	3.08	4.71	5.00	4.48	-1.16	-0.59	-2.62	-0.96	43.65	43.35	43.31
7-7	-0.64	-1.28	0.56	5.47	5.62	3.44	0.21	-0.89	-2.96	-0.65	26.74	26.27	26.21
6-6	0.28	0.57	0.21	1.44	4.16	4.78	-0.01	-1.08	-2.54	-0.19	19.68	10.21	19.15
4-4	1.74	3.69	2.44	2.08	0.63	0.26	-0.73	-1.59	-2.43	1.23	-5.30	-5.73	-5.78
3-3	1.49	2.39	4.01	4.41	2.34	0.35	-3.80	-1.28	-2.63	0.62	10.92	10.49	10.44
2-2	0.61	2.66	3.86	4.67	4.47	2.77	-4.24	-4.83	-2.64	0.57	11.03	10.91	10.89
A-A	1.84	2.22	2.65	3.06	2.83	2.38	0.03	-0.77	-4.12	2.24	44.68	44.19	44.13

Table C (D) EORs—Dealer's Up-Card 5

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-2.53	-4.15	-4.66	-0.92	-0.43	1.39	1.75	2.01	2.35	-1.86	-47.83	-47.81	-47.81
16	-2.99	-3.51	-3.95	-4.29	0.01	1.59	1.96	2.24	2.58	-1.37	-28.38	-28.28	-28.27
15	-2.29	-2.79	-3.22	-3.56	-3.44	1.69	2.03	2.29	2.56	-0.97	-21.74	-21.80	-21.80
14	-1.82	-2.06	-2.47	-2.80	-2.75	-1.91	2.10	2.30	2.54	-0.76	-15.12	-15.32	-15.35
13	-1.57	-1.55	-1.70	-2.01	-2.02	-1.39	-1.59	2.29	2.53	-0.58	-8.70	-8.91	-8.94
12	-1.32	-1.28	-1.17	-1.22	-1.27	-0.88	-1.13	-1.39	2.52	-0.40	-2.30	-2.51	-2.53
							DOUBLE						
11	1.32	1.64	2.03	2.57	2.37	-0.06	-0.86	-1.45	-2.21	1.32	31.08	30.85	30.82
10	1.14	1.43	1.92	2.72	2.72	0.33	-0.52	-1.17	-1.94	-0.79	25.86	25.70	25.68
9	1.81	1.29	2.05	2.75	2.42	0.11	-0.50	-1.19	-2.01	-0.68	8.64	8.55	8.54
8	1.62	2.44	2.19	2.51	2.03	-0.36	-0.74	-1.20	-2.03	-0.37	-6.67	-6.71	-6.72
7	1.77	2.57	3.00	2.21	1.66	-0.76	-1.17	-1.40	-1.97	0.00	-19.47	-19.52	-19.53
							SOFT DOUBLE						
A-9	2.81	3.54	4.15	4.08	3.36	0.62	-0.80	-2.09	-3.46	-1.82	-15.26	-15.61	-15.65
A-8	-1.22	3.32	3.22	3.24	3.12	0.94	0.03	-1.21	-2.56	-1.21	-3.08	-3.26	-3.29
A-7	-0.57	-1.61	2.36	3.00	2.89	0.80	0.39	-0.29	-1.61	-0.55	9.51	9.55	9.55
A-6	-0.07	0.07	0.10	2.56	2.34	0.08	-0.32	-0.54	-1.13	0.31	8.85	9.03	9.05
A-5	0.59	0.60	0.62	0.48	2.19	-0.06	-0.47	-0.71	-1.05	0.97	4.93	5.15	5.17
A-4	1.31	1.11	1.15	1.00	0.11	-0.11	-0.53	-0.78	-1.15	1.35	3.30	3.37	3.38
A-3	1.74	1.82	1.68	1.53	0.63	-2.27	-0.59	-0.87	-1.27	1.40	1.64	1.48	1.46
A-2	1.82	2.28	2.39	2.07	1.14	-1.76	-2.83	-1.02	-1.38	1.43	-0.39	-0.62	-0.66
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
T-T	4.90	6.28	7.42	7.27	5.90	0.83	-1.88	-4.34	-5.65	-3.79	-29.63	-29.20	-29.15
9-9	-0.01	3.03	3.32	4.73	4.38	1.30	0.02	-2.28	-3.36	-1.04	19.80	19.43	19.38
8-8	-2.07	-1.13	4.27	5.35	5.12	1.40	-1.40	-0.56	-2.54	-0.84	46.90	46.61	46.58
7-7	0.01	-0.04	1.20	6.18	5.82	-0.21	0.05	-0.94	-2.82	-0.76	31.89	31.56	31.51
6-6	2.07	1.67	1.31	2.65	4.53	1.04	-0.25	-1.32	-2.82	-0.41	25.76	25.12	25.04
4-4	3.96	4.03	2.51	2.68	0.82	-3.21	-0.77	-1.55	-2.35	0.93	2.21	1.47	1.38
3-3	3.25	3.19	5.15	5.73	2.88	-3.25	-4.06	-1.52	-2.92	0.29	17.09	16.44	16.36
2-2	2.00	3.81	4.98	5.94	5.20	-0.92	-4.50	-5.07	-2.93	0.27	16.74	16.43	16.39
A-A	2.09	2.42	2.75	3.29	3.16	0.87	-0.01	-0.76	-3.98	2.13	46.64	46.09	46.03

Table C (E) EORs—Dealer's Up-Card 6

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
						HIT							
17	-3.56	-4.08	-4.50	-0.75	1.07	1.39	1.61	1.92	2.25	-2.12	-52.07	-52.06	-52.05
16	-2.96	-3.40	-3.77	-4.08	1.35	1.69	1.92	2.24	2.68	-3.72	-27.51	-27.65	-27.67
15	-2.20	-2.73	-3.08	-3.40	-2.24	1.80	2.01	2.27	2.67	-3.12	-20.75	-21.06	-21.10
14	-1.62	-1.92	-2.37	-2.68	-1.74	-1.85	2.04	2.27	2.66	-2.78	-14.22	-14.54	-14.58
13	-1.36	-1.32	-1.54	-1.95	-1.22	-1.40	-1.65	2.30	2.65	-2.44	-7.72	-8.04	-8.08
12	-1.10	-1.04	-0.92	-1.11	-0.71	-0.94	-1.20	-1.40	2.63	-2.10	-1.23	-1.53	-1.57
						DOUBLE							
11	1.59	1.90	2.27	2.88	0.61	-0.21	-0.78	-1.44	-2.12	1.66	33.61	33.45	33.43
10	1.42	1.73	2.36	3.02	0.87	0.23	-0.47	-1.16	-1.87	-0.49	28.85	28.80	28.80
9	2.14	1.77	2.43	2.77	0.47	0.11	-0.37	-1.22	-1.98	-0.21	12.05	12.09	12.09
8	2.15	2.91	2.26	2.45	0.05	-0.32	-0.49	-1.14	-2.04	0.28	-2.95	-2.85	-2.83
7	2.21	2.67	2.95	2.11	-0.37	-0.77	-0.95	-1.25	-2.00	1.40	-16.99	-16.83	-16.81
						SOFT DOUBLE							
A-9	3.28	3.93	3.93	3.85	1.58	0.49	-0.85	-2.10	-3.42	-0.43	-12.61	-12.76	-12.78
A-8	-0.80	3.17	3.06	3.59	1.36	1.01	0.09	-1.32	-2.64	0.38	-1.70	-1.66	-1.65
A-7	-0.79	-1.71	2.78	3.34	1.19	0.84	0.65	-0.30	-1.83	1.31	9.41	9.68	9.71
A-6	0.32	0.29	0.31	2.71	0.47	0.11	-0.09	-0.40	-1.09	0.63	12.33	12.65	12.69
A-5	0.72	0.71	0.72	0.56	0.33	-0.05	-0.26	-0.58	-0.97	1.73	7.70	7.96	8.00
A-4	1.45	1.18	1.20	1.05	-1.80	-0.10	-0.31	-0.66	-1.07	2.27	6.16	6.15	6.15
A-3	2.07	1.90	1.69	1.53	-1.33	-2.29	-0.43	-0.80	-1.18	2.38	4.17	4.10	4.09
A-2	2.16	2.53	2.40	1.99	-0.88	-1.85	-2.68	-0.92	-1.31	2.49	2.03	1.88	1.86
						DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES							
T-T	5.82	7.06	7.05	6.88	2.71	0.62	-1.94	-4.31	-5.68	-1.18	-24.74	-23.96	-23.87
9-9	-0.82	3.05	4.32	5.03	1.89	1.24	0.07	-2.49	-3.54	1.78	18.95	18.84	18.83
8-8	-0.98	0.66	5.58	6.46	2.61	1.71	-1.33	-0.69	-2.86	-2.56	56.61	56.40	56.38
7-7	1.13	0.56	1.70	6.70	2.30	-0.44	0.49	-0.64	-2.66	-1.16	40.43	40.22	40.19
6-6	2.51	2.13	1.69	2.84	0.69	1.05	0.16	-1.16	-2.64	0.66	31.23	30.90	30.86
4-4	4.02	4.03	2.59	2.68	-2.92	-3.50	-0.53	-1.37	-2.19	3.78	4.33	3.87	3.81
3-3	3.96	3.63	5.59	5.79	-1.22	-3.48	-3.89	-1.31	-2.71	1.79	21.97	21.61	21.57
2-2	2.40	4.49	5.52	6.18	1.07	-1.19	-4.27	-5.03	-2.73	1.75	21.80	21.65	21.63
A-A	2.41	2.66	2.99	3.77	1.64	0.68	-0.01	-0.79	-3.91	2.29	48.90	48.39	48.33

Table C (F) EORs—Dealer's Up-Card 7

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-2.84	-3.31	-3.71	0.78	1.17	1.47	1.67	2.01	1.61	-3.65	-37.73	-37.69	-37.68
16	-1.93	-2.44	-2.78	-2.33	1.80	2.10	2.32	2.77	0.59	-1.88	6.06	6.06	6.06
15	-0.81	-1.89	-2.33	-1.96	-1.94	2.12	2.29	2.70	0.70	-0.95	10.13	10.11	10.10
14	0.03	-0.75	-1.77	-1.59	-1.61	-1.68	2.25	2.65	0.81	-0.76	14.19	14.15	14.15
13	0.15	0.12	-0.61	-1.10	-1.29	-1.40	-1.53	2.60	0.91	-0.58	18.23	18.19	18.19
12	0.26	0.25	0.28	-0.04	-0.85	-1.08	-1.25	-1.23	1.01	-0.39	22.23	22.22	22.22
							SOFT HIT						
17	-1.67	-2.16	-2.62	-0.27	0.31	0.85	1.22	1.81	1.13	-1.98	15.86	16.00	16.02
							DOUBLE						
11	2.26	2.62	3.02	2.22	0.80	-0.90	-1.82	-2.89	-2.04	2.87	17.62	17.26	17.21
10	2.25	2.73	3.15	2.30	1.70	0.04	-1.46	-2.65	-1.91	-0.41	13.78	13.63	13.61
9	2.95	2.56	2.96	1.95	1.14	0.50	-0.99	-2.85	-1.94	-0.46	-6.63	-6.72	-6.73
8	2.59	3.23	2.68	1.56	0.64	-0.01	-0.64	-2.52	-1.91	0.10	-27.16	-27.05	-27.03
							SOFT DOUBLE						
A-7	-2.52	-2.81	2.93	2.14	1.82	1.48	1.34	-0.56	-0.70	-1.04	-18.51	-18.14	-18.10
A-6	-0.61	-0.37	-0.14	2.23	1.35	0.52	0.04	-0.87	-0.70	0.66	-6.88	-6.80	-6.79
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9-9	-0.87	1.67	1.89	1.20	1.65	1.59	0.90	-0.83	-0.85	-2.84	-3.75	-3.56	-3.54
8-8	2.19	2.58	6.50	5.39	0.75	0.15	-2.71	-0.90	-3.46	-0.11	73.42	73.53	73.54
7-7	0.61	0.43	1.28	3.55	3.56	1.89	-1.41	-1.43	-2.03	-0.36	27.76	27.33	27.28
6-6	-1.14	-1.85	-3.33	-3.71	2.83	3.24	3.24	3.49	-0.41	-1.13	-3.64	-4.65	-4.78
3-3	2.67	2.93	3.27	0.79	-2.65	-4.61	-4.73	0.31	0.42	0.36	10.24	10.00	9.97
2-2	0.72	2.31	3.54	1.72	0.42	-0.71	-4.33	-4.67	0.28	-0.12	9.57	9.53	9.52
A-A	3.21	3.56	4.10	3.72	2.53	0.23	-0.86	-2.05	-4.57	3.83	29.94	29.81	29.79

Table C (G) EORs—Dealer's Up-Card 8

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-2.34	-2.86	-2.47	1.38	1.66	1.84	2.18	1.78	0.28	-2.29	-12.38	-12.39	-12.40
16	-1.39	-2.44	-2.20	-2.36	1.72	1.93	2.38	0.20	0.80	-1.05	5.22	5.21	5.21
15	-0.31	-1.35	-1.76	-2.05	-2.04	1.91	2.32	0.33	0.90	-0.63	9.00	8.98	8.98
14	0.04	-0.26	-0.74	-1.61	-1.77	-1.90	2.29	0.45	0.99	-0.46	12.77	12.75	12.75
13	0.15	0.12	0.28	-0.61	-1.39	-1.62	-1.53	0.57	1.08	-0.29	16.53	16.51	16.51
12	0.26	0.25	0.56	0.39	-0.39	-1.23	-1.30	-3.09	1.17	-0.13	20.27	20.27	20.37
							SOFT HIT						
18	-1.69	-2.20	0.10	-0.01	0.42	0.65	1.09	2.02	0.46	-2.21	-6.70	-6.65	-6.64
							DOUBLE						
11	2.13	2.53	1.64	1.88	1.07	-0.35	-2.09	-1.32	-2.06	2.73	12.35	12.16	12.14
10	2.24	2.62	1.71	1.96	1.46	0.58	-1.15	-1.06	-1.93	-0.63	8.85	8.86	8.86
9	2.86	2.28	1.38	1.55	0.80	0.46	-0.76	-0.38	-1.96	-0.34	-12.64	-12.53	-12.52
							SOFT DOUBLE						
A-7	-2.80	-3.08	1.73	1.80	1.51	1.35	0.95	2.22	-0.42	-1.98	-13.78	-13.65	-13.64
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9-9	-0.50	2.36	1.46	1.52	1.91	1.91	1.93	0.35	-1.98	-3.01	12.22	12.64	12.69
8-8	1.34	2.18	3.91	4.24	-0.17	-0.45	-2.42	-1.39	-1.75	-0.21	43.78	43.64	43.62
7-7	0.04	-0.65	-1.53	2.35	2.71	2.60	-2.38	-1.91	-0.32	0.04	-1.15	-1.72	-1.80
6-6	-0.44	-1.84	-4.40	-4.43	2.40	3.25	3.00	3.49	-0.16	-0.39	-14.24	-14.98	-15.08
3-3	2.38	3.33	2.50	0.95	-2.91	-4.55	-4.76	0.20	0.63	0.33	-1.31	-1.41	-1.43
2-2	1.31	1.90	1.66	1.93	0.83	-0.64	-4.57	-4.63	0.52	0.13	-1.68	-1.72	-1.72
A-A	3.03	3.51	2.77	3.20	2.74	1.41	-0.96	-0.32	-4.72	3.49	25.43	25.52	25.53
							LATE SURRENDER						
17	-0.31	-0.55	0.29	0.28	0.63	0.83	1.21	1.05	-0.55	-1.24	-11.91	-11.84	-11.83
16	0.95	1.79	2.43	2.62	-1.07	-1.04	-0.95	-0.73	-0.97	-0.12	-4.27	-4.19	-4.18
15	-0.02	0.90	2.17	2.49	2.55	-1.18	-1.09	-0.88	-1.13	-0.41	-8.41	-8.35	-8.34
14	-0.31	-0.08	1.29	2.22	2.43	2.45	-1.27	-1.04	-1.31	-0.43	-12.86	-12.83	-12.82
13	-0.30	-0.42	0.34	1.37	2.16	2.28	2.31	-1.21	-1.51	-0.47	-17.63	-17.64	-17.64
12	-0.32	-0.44	0.05	0.42	1.25	1.96	2.13	2.37	-1.73	-0.51	-22.76	-22.82	-22.82

Table C (H) EORs—Dealer's Up-Card 9

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-1.81	-2.13	-2.51	1.31	1.56	1.79	1.40	-0.10	0.49	-1.47	-13.17	-13.09	-13.08
16	-0.90	-1.25	-2.10	-2.48	1.60	1.96	-0.22	0.39	0.95	-0.79	3.17	3.31	3.33
15	-0.34	-0.29	-1.14	-2.09	-2.22	1.92	-0.07	0.50	1.03	-0.40	6.71	6.84	6.85
14	-0.01	0.21	-0.17	-1.14	-1.88	-1.91	0.07	0.61	1.12	-0.25	10.24	10.36	10.37
13	0.10	0.47	0.30	-0.19	-0.93	-1.57	-3.60	0.72	1.20	-0.10	13.75	13.87	13.88
							SOFT HIT						
18	-1.49	-1.83	0.35	0.25	0.61	0.75	1.78	1.03	-0.07	-1.18	8.41	8.30	8.28
							DOUBLE						
11	2.07	1.13	1.25	1.56	0.84	-0.11	0.17	-1.24	-2.06	2.59	7.25	7.05	7.03
10	2.12	1.10	1.27	1.59	1.11	0.30	0.97	-0.45	-1.90	-0.38	2.75	2.77	2.77
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9-9	-0.29	1.44	1.20	1.16	1.50	1.55	2.12	-0.26	-1.69	-1.64	10.49	10.51	10.51
8-8	0.29	-0.70	2.95	3.19	-1.21	-1.30	-2.18	0.70	-0.11	0.13	13.01	12.48	12.41
7-7	-0.06	-2.43	-2.20	1.36	2.44	3.04	-1.52	0.03	-0.25	0.34	-14.13	-14.49	-14.53
6-6	0.13	-2.52	-4.40	-4.62	1.92	2.32	3.24	3.95	0.04	-0.17	-26.10	-26.74	-26.82
3-3	1.90	2.09	2.06	1.19	-2.63	-4.44	-4.58	0.62	0.81	0.56	-14.21	-14.34	-14.35
2-2	1.75	0.30	0.93	1.24	1.05	-0.28	-4.25	-4.01	0.71	0.42	-14.36	-14.54	-14.56
A-A	3.16	2.40	2.38	2.90	2.10	1.54	1.91	-0.15	-4.88	3.30	22.68	22.74	22.75
							LATE SURRENDER						
17	-0.30	0.54	0.47	0.42	0.75	0.97	0.81	-0.79	-0.45	-1.08	-7.89	-7.75	-7.73
16	0.47	1.69	2.58	2.87	-0.84	-0.81	-0.60	-0.84	-1.06	-0.26	0.87	0.91	0.92
15	-0.02	0.83	1.78	2.68	2.84	-0.94	-0.73	-0.99	-1.23	-0.56	-2.88	-2.86	-2.85
14	-0.31	0.35	0.91	1.90	2.66	2.71	-0.87	-1.15	-1.40	-0.60	-6.93	-6.91	-6.91
13	-0.33	0.08	0.46	1.05	1.84	2.48	2.78	-1.32	-1.60	-0.65	-11.25	-11.27	-11.28
12	-0.35	0.12	0.16	0.57	0.95	1.64	2.55	2.29	-1.81	-0.70	-15.88	-15.96	-15.97

Table C (I) EORs—Dealer's Up-Card T

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-0.66	-1.75	-2.52	1.24	1.47	1.04	-0.58	0.06	0.65	-0.88	-16.70	-16.55	-16.53
16	-0.29	-0.80	-1.73	-2.57	1.64	-0.71	-0.06	0.55	1.12	-0.50	-0.19	-0.02	-0.00
15	0.19	-0.32	-0.73	-1.75	-2.23	-0.54	0.09	0.66	1.20	-0.17	3.36	3.52	3.54
14	0.44	0.17	-0.26	-0.77	-1.41	-4.21	0.22	0.77	1.28	-0.08	6.89	7.05	7.07
13	0.46	0.40	0.21	-0.26	-0.43	-3.22	-3.48	0.88	1.36	0.01	10.40	10.58	10.60
12	0.45	0.40	0.47	0.24	0.04	-2.06	-2.52	-2.86	1.44	0.10	13.89	14.10	14.12
							SOFT HIT						
18	-0.94	-1.93	0.56	0.47	0.31	1.45	0.69	-0.34	0.00	-0.28	3.49	3.46	3.46
							DOUBLE						
11	0.76	0.77	0.79	1.11	0.84	1.53	0.74	-0.61	-1.89	1.61	5.86	5.97	5.99
10	0.58	0.58	0.64	0.88	0.88	1.64	0.85	-0.05	-1.02	-1.93	-3.07	-3.29	-3.31
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9-9	-1.64	0.85	0.96	0.92	0.40	1.65	-0.01	-0.67	0.07	-2.73	-13.34	-13.72	-13.77
8-8	-1.27	-1.49	2.16	2.98	-1.42	-2.14	-0.38	1.14	0.37	-1.07	6.45	6.09	6.05
7-7	-1.62	-2.23	-2.19	1.06	1.49	3.72	0.20	0.44	0.01	-1.01	-18.71	-18.98	-19.01
6-6	-1.13	-1.82	-2.80	-2.99	0.81	2.09	3.10	4.07	-0.11	-0.89	-33.22	33.44	-33.47
3-3	0.25	1.51	0.98	0.20	-2.15	-4.23	-4.29	1.32	1.75	-0.58	-21.34	-21.49	-21.51
2-2	-0.61	-0.01	0.63	0.72	0.29	0.07	-3.30	-3.57	1.29	-0.63	-21.71	-21.77	-21.78
A-A	1.87	2.11	2.00	2.47	2.12	2.80	2.69	1.06	-4.92	2.57	24.47	24.81	24.85
							LATE SURRENDER						
17	0.76	0.65	0.64	0.51	0.73	0.56	-1.17	-0.80	-0.46	-0.02	-8.24	-8.10	-8.08
16	0.93	1.38	2.32	3.07	-0.74	-0.51	-0.77	-1.01	-1.24	0.28	4.02	4.00	3.99
15	0.45	0.89	1.32	2.25	3.13	-0.68	-0.91	-1.12	-1.32	-0.05	0.48	0.46	0.46
14	0.18	0.42	0.91	1.41	2.49	3.03	-1.06	-1.29	-1.50	-0.07	-3.31	-3.35	-3.35
13	0.18	0.16	0.45	0.95	1.62	2.37	2.61	-1.47	-1.70	-0.08	-7.38	-7.44	-7.45
12	0.22	0.16	0.15	0.46	1.18	1.47	1.92	2.17	-1.91	-0.09	-11.74	-11.84	-11.85

Table C (J) EORs—Dealer's Up-Card A

	2	3	4	5	6	7	8	9	T	A	m_2	m_6	m_8
							HIT						
17	-1.54	-2.48	-3.09	0.48	0.61	-1.39	-0.37	0.57	1.94	-0.53	-8.40	-8.08	-8.05
16	-0.92	-1.66	-2.53	-3.18	-1.43	-0.41	0.53	1.40	2.06	-0.02	14.39	14.79	14.83
15	-0.15	-0.88	-1.76	-2.61	-4.93	-0.23	0.67	1.48	2.01	0.37	17.02	17.37	17.41
14	0.26	-0.13	-0.99	-1.78	-4.17	-3.79	0.79	1.56	1.96	0.41	19.60	19.94	19.98
13	0.28	0.25	-0.19	-0.95	-3.19	-3.09	-2.83	1.62	1.91	0.45	22.15	22.50	22.54
12	0.30	0.32	0.25	-0.12	-2.22	-2.17	-2.19	-2.05	1.85	0.46	24.66	25.04	25.09
							SOFT HIT						
18	-1.24	-2.19	0.08	-0.20	1.22	-0.03	-0.91	0.19	0.88	-0.42	0.39	0.62	0.64
							DOUBLE						
11	1.79	1.92	2.17	2.65	3.00	1.76	0.45	-0.90	-3.64	1.74	-2.14	-2.98	-3.08
10	1.74	1.85	2.18	2.72	3.31	2.25	0.94	0.18	-3.28	-2.03	-8.16	-9.09	-9.21
							DOUBLE AFTER SPLIT, SPLIT ACES ONCE, SPLIT OTHER PAIRS THREE TIMES						
9-9	-0.71	1.26	1.23	1.19	2.23	0.61	0.61	2.12	-1.51	-2.51	-3.77	-3.61	-3.62
8-8	-0.34	0.25	3.41	4.29	-1.71	0.73	2.29	3.27	-2.49	-2.22	14.60	14.58	14.55
7-7	-1.08	-1.47	-1.21	2.19	2.33	5.73	1.09	1.17	-1.82	-1.47	-21.34	-21.16	-21.16
6-6	-1.01	-2.42	-3.92	-4.24	2.15	3.02	4.34	5.55	-0.62	-1.00	-29.85	-30.11	-30.15
3-3	0.91	2.37	2.26	1.30	-2.74	-3.37	-2.82	2.45	0.08	-0.67	-18.26	-17.87	-17.86
2-2	0.90	0.41	1.21	1.63	0.67	0.47	-2.50	-1.97	0.01	-0.77	-18.69	-18.11	-18.10
A-A	2.88	3.07	3.40	4.01	4.60	3.37	1.84	-0.05	-6.46	2.71	14.99	13.61	13.44
							LATE SURRENDER						
17	0.16	-0.07	-0.22	-0.50	-0.04	-2.16	-1.53	-0.92	1.20	0.39	-2.46	-2.28	-2.26
16	0.90	1.58	2.26	2.68	-0.63	-1.03	-1.36	-1.65	-0.72	0.15	1.92	1.78	1.77
15	0.18	0.91	1.67	2.34	2.96	-1.20	-1.56	-1.86	-0.86	-0.00	-1.81	-1.94	-1.95
14	-0.01	0.21	1.02	1.69	2.58	2.36	-1.75	-2.07	-1.01	0.00	-5.75	-5.92	-5.94
13	0.05	0.02	0.28	1.00	1.93	1.94	1.77	-2.31	-1.17	-0.00	-9.95	-10.19	-10.22
12	0.08	0.02	0.01	0.25	1.25	1.26	1.31	1.17	-1.34	0.03	-14.41	-14.77	-14.81

Table D Player's Exact Expectation When Holding 16 vs. a 10

	Player's Cards	Expectation by Drawing	Expectation by Standing	Gain by Drawing over Standing
2-Card Holding	6, 10	-0.5069	-0.5430	0.0360
	8, 8	-0.5118	-0.5183	0.0065
	7, 9	-0.5120	-0.5180	0.0060
3-Card Holding	6, 9, A	-0.5117	-0.5374	0.0257
	6, 6, 4	-0.5368	-0.5597	0.0229
	2, 6, 8	-0.5215	-0.5425	0.0211
	3, 6, 7	-0.5291	-0.5374	0.0083
	3, 3, 10	-0.5444	-0.5439	-0.0005
	2, 4, 10	-0.5491	-0.5448	-0.0043
	7, 8, A	-0.5172	-0.5116	-0.0057
	2, 7, 7	-0.5265	-0.5159	-0.0106
	5, 10, A	-0.5496	-0.5360	-0.0136
	3, 4, 9	-0.5557	-0.5415	-0.0142
	2, 5, 9	-0.5595	-0.5422	-0.0174
	3, 5, 8	-0.5665	-0.5372	-0.0292
	5, 5, 6	-0.5859	-0.5560	-0.0299
	4, 4, 8	-0.5683	-0.5375	-0.0308
	4, 5, 7	-0.5797	-0.5330	-0.0467
4-Card Holding	4, 10, A, A	-0.5453	-0.5348	-0.0106
	2, 3, 10, A	-0.5423	-0.5422	-0.0001
	2, 2, 2, 10	-0.5438	-0.5513	-0.0075
	5, 9, A, A	-0.5556	-0.5298	-0.0259
	2, 4, 9, A	-0.5547	-0.5395	-0.0153
	3, 3, 9, A	-0.5491	-0.5375	-0.0117
	2, 2, 3, 9	-0.5519	-0.5483	-0.0036
	6, 8, A, A	-0.5171	-0.5314	-0.0144
	2, 5, 8, A	-0.5653	-0.5355	-0.0298
	3, 4, 8, A	-0.5618	-0.5358	-0.0260
	2, 2, 4, 8	-0.5639	-0.5443	-0.0196
	2, 3, 3, 8	-0.5583	-0.5442	-0.0141
	7, 7, A, A	-0.5222	-0.5048	-0.0174
	2, 6, 7, A	-0.5262	-0.5362	0.0101
	3, 5, 7, A	-0.5728	-0.5302	-0.0425
	4, 4, 7, A	-0.5752	-0.5326	-0.0427
	2, 2, 5, 7	-0.5743	-0.5414	-0.0329
	2, 3, 4, 7	-0.5715	-0.5412	-0.0303
	3, 3, 3, 7	-0.5659	-0.5389	-0.0270
	3, 6, 6, A	-0.5289	-0.5583	0.0294
	2, 2, 6, 6	-0.5309	-0.5693	0.0384
	4, 5, 6, A	-0.5813	-0.5552	-0.0261
	2, 3, 5, 6	-0.5780	-0.5659	-0.0121
	2, 4, 4, 6	-0.5802	-0.5658	-0.0144
	3, 3, 4, 6	-0.5745	-0.5662	-0.0083
	5, 5, 5, A	-0.6309	-0.5489	-0.0820
	2, 4, 5, 5	-0.6296	-0.5616	-0.0681
	3, 3, 5, 5	-0.6242	-0.5605	-0.0637
	3, 4, 4, 5	-0.6262	-0.5632	-0.0630
	4, 4, 4, 4	-0.6284	-0.5645	-0.0638

Table E Optimal Strategy for Full Disclosure

	Total	Cards	Maximum Drawing Numbers		Doubling Down		Splits
			t^*	t_s^*	Hard	Soft	
	4	2, 2	13	17	10, 11	17, 18	A, 7, 8
	5	2, 3	12	17	10, 11	13, 14, 15, 16, 17, 18, 19	A, 2, 3, 6, 7, 8, .9
	6	2, 4	11	17	9, 10, 11		A, 2, 3, 6, 7, 8, 9
		3, 3	11	17	8, 9, 10, 11		A, 2, 3, 4, 6, 7, 8, 9
		2, 5	16	17	10, 11		A, 2, 3, 7, 8
	7	3, 4	16	17	9, 10, 11		A, 2, 3, 7, 8
		2, 6	16	17	10, 11		A, 7, 8, 9
	8	3, 5 or 4, 4	16	17	10, 11		A, 2, 3, 7, 8, 9
		2, 7 or 3, 6	16	18	10, 11		A, 8, 9
	9	4, 5	15	18	10, 11		A, 8, 9
		2, 8	15	18	11		A, 8
	10	3, 7 or 5, 5	15	18	11		A
		4, 6	16	18	11		A
Dealer's Hard Hand		2, 9	15	17			A
		3, 8	14	17			A
	11	4, 7	13	17			A
		5, 6	13	18			A
		2, 10					
	12	3, 9 or 4, 8	11	17	8, 9, 10, 11	13, 14, 15, 16, 17, 18, 19	A, 2, 3, 4, 6, 7, 8, 9, 10
		5, 7 or 6, 6	11	17	8, 9, 10, 11	13, 14, 15, 16, 17, 18, 19, 20	A, 2, 3, 4, 6, 7, 8, 9, 10
	13		11	17	7, 8, 9, 10, 11	13, 14, 15, 16, 17, 18, 19, 20	A, 2, 3, 4, 6, 7, 8, 9, 10
	14		11	17	5, 6, 7, 8, 9, 10, 11	13, 14, 15, 16, 17, 18, 19, 20	A, 2, 3, 4, 6, 7, 8, 9, 10
	15		11	17	5, 6, 7, 8, 9, 10, 11	13, 14, 15, 16, 17, 18, 19, 20	A, 2, 3, 4, 6, 7, 8, 9, 10
	16		11	17	5, 6, 7, 8, 9, 10, 11	13, 14, 15, 16, 17, 18, 19, 20	A, 2, 3, 4, 6, 7, 8, 9, 10
	17		16	17			2, 3, 6, 7, 8
	18	8, 10	17	18			2, 3, 7, 8, 9
		9, 9	17	18			3, 6, 7, 8, 9
	19	9, 10	18	18			9
	20	10, 10	19	19			
		A, A	15	17	11		A, 8
		A, 2	14	17	10, 11		A, 7, 8
		A, 3	13	17	10, 11		A, 7, 8
Dealer's Soft Hand		A, 4	12	17	10, 11		A, 2, 6, 7, 8, 9
		A, 5	12	17	9, 10, 11	17, 18	A, 2, 3, 6, 7, 8, 9
		A, 6	16	17			A, 2, 3, 6, 7, 8
		A, 7	17	18			2, 3, 7, 8, 9
		A, 8	18	18			9
		A, 9	19	19			

Table F Mathematical Expectations as a Function of Dealer's Cards

Dealer's Cards ↓ →	A	2	3	4	5	6	7	8	9	10
A	-0.1201									
2	-0.0560	0.0185								
3	-0.0166	0.0776	0.2744							
4	0.0411	0.1455	0.1682	0.0662						
5	0.1064	0.0637	0.0741	-0.0325	-0.1613					
6	0.3068	0.0147	-0.0350	-0.1657	-0.2141	0.3693				
7	0.1020	-0.0299	-0.1660	-0.2156	0.3691	0.4571	0.5376			
8	-0.1344	-0.1153	-0.2159	0.3483	0.4549	0.4817	0.5034	0.5733		
9	-0.4593	-0.2142	0.3451	0.3915	0.4815	0.5574	0.5734	0.3214	0.1155	
10	-0.9633	0.2911	0.3659	0.4685	0.5546	0.5687	0.3309	0.1244	-0.1153	-0.4377

Table G Double-Exposure Optimal Strategy

Player's Hand	Dealer's Total																					
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	A-A	A-2	A-3	A-4	A-5
20	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	H	–	–	–	–	–
19	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	H	H	–	–	–	–	–
18	–	–	–	–	–	–	–	–	–	–	–	–	–	–	H	H	H	–	–	–	–	–
17	–	–	–	–	–	–	–	–	–	–	–	–	–	H	H	H	H	–	–	–	–	–
16	–	–	–	H	–	–	–	–	–	–	–	–	–	H	H	H	H	–	–	–	–	–
15	–	–	–	H	H	H	–	–	–	–	–	–	–	H	H	H	H	–	–	–	–	–
14	–	–	–	H	H	H	H	–	–	–	–	–	–	H	H	H	H	H	–	–	–	–
13	–	–	–	H	H	H	H	H	–	–	–	–	–	H	H	H	H	H	H	–	–	–
12	–	–	–	H	H	H	H	H	–	–	–	–	–	H	H	H	H	H	H	H	H	–
11	db	db	db	db	db	db	H	H	db	db	db	db	db	H	H	H	H	H	db	db	db	db
10	db	db	db	db	db	H	H	H	db	db	db	db	db	H	H	H	H	H	H	db	db	db
9	H	db	db	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H	db	db	db	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H	db	db	db	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H	db	db	db	H	H	H	H	H	H	H	H	H
soft 20	–	–	–	–	–	–	–	–	–	dbs	dbs	dbs	dbs	–	–	–	H	–	–	–	–	–
soft 19	–	–	–	–	–	–	–	–	dbs	dbs	dbs	dbs	dbs	–	–	H	H	–	–	–	–	–
soft 18	dbs	dbs	dbs	–	H	H	H	H	dbs	db	db	db	db	–	H	H	H	H	H	H	H	–
soft 17	H	db	db	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H
soft 16	H	db	db	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H
soft 15	H	H	db	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H
soft 14	H	H	H	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H
soft 13	H	H	H	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	H	H

A-A	S	S	S	S	S	S	S	H	S	S	S	S	S	H	H	H	H	S	S	S	S	S
T-T	–	–	–	–	–	–	–	–	–	S	S	S	S	–	–	–	H	–	–	–	–	–
9-9	S	S	S	–	S	–	–	–	S	S	S	S	S	–	S	H	H	–	–	–	–	S
8-8	S	S	S	S	S	–	–	–	S	S	S	S	S	S	H	H	H	–	–	S	S	S
7-7	S	S	S	H	H	H	H	H	S	S	S	S	S	S	H	H	H	H	–	–	–	–
6-6	S	S	S	H	H	H	H	H	S	S	S	S	S	S	H	H	H	H	H	H	H	–
5-5	db	db	db	db	db	H	H	H	db	db	db	db	S	H	H	H	H	H	H	db	db	db
4-4	H	H	H	H	H	H	H	H	S	S	S	S	S	H	H	H	H	H	H	H	H	H
3-3	S	S	S	H	H	H	H	H	S	S	S	S	S	S	H	H	H	H	H	H	H	H
2-2	S	S	S	H	H	H	H	H	S	S	S	S	S	S	H	H	H	H	H	H	H	H

– = Stand
db = Double down; if doubling not permitted, then hit
dbs = Double down; if doubling not permitted, then stand
H = Hit
S = Split

Courtesy of Stanford Wong.

Table H Double-Exposure Hi-Lo Strategy Numbers (TCs)

Player's Hand	Dealer's Total																					
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	A-A	A-2	A-3	A-4	A-5
20	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	H	–	–	–	–	–
19	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	H	H	–	–	–	–	–
18	–	–	–	–	–	–	–	–	–	–	–	–	–	–	H	H	H	–	–	–	–	–
17	–	–	–	-7	-6	-8	–	–	–	–	–	–	–	H	H	H	H	-5	-6	-8	-10	–
16	–	–	–	0	0	-2	-8	-8	–	–	–	–	–	H	H	H	H	-3	-4	-6	-7	–
15	-10	–	–	5	4	1	-2	-4	–	–	–	–	–	H	H	H	H	-1	-2	-4	-5	-9
14	-9	-10	–	H	H	10	1	-2	–	–	–	–	–	H	H	H	H	0	0	-2	-4	-7
13	-6	-8	-8	H	H	H	H	1	–	–	–	–	–	H	H	H	H	3	1	0	-2	-5
12	-2	-4	-4	H	H	H	H	7	-8	-10	–	–	–	H	H	H	H	6	4	2	0	-1
11	db	db	db	-6	-4	-1	H	H	db	db	db	db	db	10	H	H	H	0	-1	-2	-4	-6
10	-8	-9	db	-4	-1	3	H	H	db	db	db	db	db	8	H	H	H	4	2	0	-1	-3
9	1	0	-3	6	H	H	H	H	-7	-10	db	db	db	H	H	H	H	H	H	8	5	4
8	9	6	4	H	H	H	H	H	0	-3	-7	-9	db	H	H	H	H	H	H	H	H	10
7	H	H	H	H	H	H	H	H	5	0	-2	-6	db	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	7	2	-1	-4	db	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	7	3	0	-4	db	H	H	H	H	H	H	H	H	H
	–	–	–	H	H	H	H	H	8	3	0	-3	db	H	H	H	H	–	–	–	–	–
soft 21	–	–	–	–	–	–	–	–	–	10	6	3	1	–	–	–	–	–	–	–	–	–
soft 20	8	6	6	–	–	–	–	–	2	0	-2	-4	-9	–	–	–	H	–	–	–	10	9
soft 19	6	3	3	–	–	–	–	–	-2	-5	-8	-10	dbs	–	–	H	H	–	–	–	8	7
soft 18	-2	-5	-7	–	H	H	H	H	0	0	0	0	0	–	H	H	H	H	H	9	4	2
soft 17	3	-1	-6	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	9	7
soft 16	5	0	-5	H	H	H	H	H	db	db	db	db	db	H	H	H	H	H	H	H	10	9
soft 15	6	1	-1	H	H	H	H	H	-10	db	db	db	db	H	H	H	H	H	H	H	10	10
soft 14	6	2	0	H	H	H	H	H	-6	db	db	db	db	H	H	H	H	H	H	H	10	9
soft 13	6	3	2	H	H	H	H	H	-4	-8	db	db	db	H	H	H	H	H	H	H	10	9

A-A	S	S	S	-7	-6	-6	-4	H	S	S	S	S	S	4	7	8	7	-3	-4	-5	-6	-8
T-T	8	6	6	–	–	–	–	–	2	0	-2	-4	-9	–	–	–	H	–	–	–	10	9
9-9	-3	-5	-5	7	S	–	–	–	-9	S	S	S	S	–	S	H	H	8	6	3	1	0
8-8	S	S	S	S	S	H/–	H/–	H/–	S	S	S	S	S	S	-3*	H	H	5	2	0	-2	-3
7-7	-2	-4	-8	8	H/–	H/–	H/–	H/–	-9	S	S	S	S	S	H	H	H	H/–	H/–	8	4	2
6-6	-3	-5	-7	H/–	H/–	H/–	H/–	H/–	-10	S	S	S	S	2*	H	H	H	H/–	H/–	8	4	1
5-5	*– Never split—play as total of ten*																					
4-4	5	1	0	H	H	H	H	H	-6	S	S	S	S	-9*	-8*	H	H	H	H	H	9	9
3-3	0	-2	-5	8	H	H	H	H	-9	S	S	S	S	S	-10*	H	H	H	H	8	5	3
2-2	0	-2	-5	5	H	H	H	H	-9	S	S	S	S	S	H	H	H	H	H	8	5	3

– = Stand
db = Double down; if doubling not permitted, then hit
dbs = Double down; if doubling not permitted, then stand
H = Hit
S = Split
H/– = Hit or stand, never split
number = Stand (or double or split) at a count per deck equal to or greater than the number; hit (or do not split) at a count per deck less than the number
*Split if count per deck is less than the number; otherwise hit

Courtesy of Stanford Wong.

Table I Blackarat Strategy (Single Deck), Bonus Pays 5 to 1

Player's Hand	Dealer's Up-Card									
	0	1	2	3	4	5	6	7	8	9
OPTIMAL HIT/STAND STRATEGY										
0	H	H	H	H	H	H	H	H	H	H
1	H	H	H	H	H	H	H	H	H	H
2	H	H	H	H	H	H	H	H	H	H
3	H	H	H	H	H	H	H	H	H	H
4	H				H	H	H	H	H	H
5							H	H	H	
6										
7										
OPTIMAL SPLIT STRATEGY										
0-0	S	S	S	S	S	S	S	S	S	S
1-1	S	S	S	S	S	S	S	S	S	S
2-2	S	S	S	S	S	S	S	S	S	
3-3		S								
4-4										
5-5	S	S	S	S	S	S	S	S	S	S
6-6	S	S	S	S	S	S	S	S	S	S
7-7	S	S	S	S	S	S	S	S	S	S
8-8	S	S	S	S	S	S	S	S	S	
9-9		S	S	S						
PLAYER EXPECTATIONS										
0	-.316	.137	.095	.060	.010	-.085	-.198	-.289	-.378	-.461
1	-.505	-.153	-.225	-.259	-.296	-.334	-.400	-.437	-.474	-.474
2	-.438	-.041	-.092	-.167	-.204	-.245	-.342	-.374	-.386	-.442
3	-.431	-.049	-.077	-.153	-.227	-.265	-.338	-.325	-.376	-.413
4	-.370	.037	.041	.060	-.100	-.182	-.231	-.300	-.334	-.370
5	-.296	.146	.146	.186	.186	-.052	-.227	-.264	-.292	-.304
6	-.129	.323	.321	.337	.354	.333	.069	-.163	-.161	-.146
7	.020	.462	.462	.462	.486	.486	.446	.210	-.028	.012
8	.428	.876	.874	.856	.872	.893	.853	.855	.646	.274
9	.336	.778	.762	.762	.762	.802	.762	.762	.762	.525

Basic Player Expectation = −2.262%
With 5 to 1 bonus = −0.478%

Programmed by Norman Wattenberger, Casino Vérité.

Table J Blackarat Strategy (Six Decks), Bonus Pays 5 to 1

Player's Hand	Dealer's Up-card									
	0	1	2	3	4	5	6	7	8	9
OPTIMAL HIT/STAND STRATEGY										
0	H	H	H	H	H	H	H	H	H	H
1	H	H	H	H	H	H	H	H	H	H
2	H	H	H	H	H	H	H	H	H	H
3	H	H	H	H	H	H	H	H	H	H
4	H	H			H	H	H	H	H	H
5							H	H	H	
6										
7										
OPTIMAL SPLIT STRATEGY										
0-0	S	S	S	S	S	S	S	S	S	S
1-1	S	S	S	S	S	S	S	S	S	
2-2	S	S	S	S	S	S	S	S	S	
3-3		S								
4-4										
5-5	S	S	S	S	S	S	S	S	S	S
6-6	S	S	S	S	S	S	S	S	S	S
7-7	S	S	S	S	S	S	S	S	S	S
8-8	S	S	S	S	S	S	S	S	S	
9-9		S	S	S						
PLAYER EXPECTATIONS										
0	-.352	.119	.072	.026	-.031	-.108	-.197	-.290	-.383	-.471
1	-.503	-.149	-.235	-.270	-.306	-.342	-.382	-.418	-.454	-.483
2	-.435	-.008	-.088	-.170	-.208	-.249	-.307	-.355	-.395	-.441
3	-.432	-.028	-.062	-.149	-.235	-.271	-.313	-.340	-.378	-.414
4	-.367	.062	.048	.045	-.096	-.182	-.226	-.281	-.329	-.370
5	-.306	.153	.153	.159	.159	-.073	-.235	-.271	-.305	-.307
6	-.141	.330	.326	.326	.326	.319	.090	-.134	-.145	-.153
7	.003	.462	.462	.462	.466	.466	.459	.227	-.004	-.002
8	.460	.925	.923	.918	.919	.923	.916	.916	.689	.431
9	.322	.771	.768	.768	.768	.775	.768	.768	.768	.536

Basic Player Expectation = −2.844%
With 5 to 1 bonus = −0.437%.

Programmed by Norman Wattenberger, Casino Vérité.

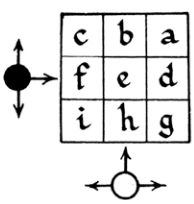
Table K Expectations for Several Blackarat Bonuses

	9-9 bonus pays	House Edge =
Single Deck:		
	5 to 1	.478%
	4 to 1	.930%
	3 to 1	1.383%
	2 to 1	1.835%
Six Decks:		
	5 to 1	.437%
	4 to 1	1.005%
	3 to 1	1.574%
	2 to 1	2.143%

Table L Outcomes of Positions in Dodgem

Left's Position (columns); Right's Position (rows)

	fi	ci	cf	hi	hf	hc	ei	ef	ec	bi	bf	bc	eh	bh	be	gi	gf	gc	di	df	dc	ai	af	ac	i	f	c	gh	ge	gb	dh	de	db	ah	ae	ab	h	e	b	dg	ag	ad	g	d	a
a	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−				−	−	−	−	−	−	−	−	−				−	−	−	−			*	*	
b	−	−	−	−	−	−	−	−	−				−			−	−	−	−	−	−	−	−	−	−	−	−	−	−		−	−		−	−		−	−		−	−	−	*	*	*
c	−			−	−		−	−		−	−		−	−	−	−	−		−	−		−	−		−	−		−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	*	*	*
ab	−	−	−	−	−	−	−	−	−				−			−	−	−	−	−	−				−	−	−	−	−		−	−					*	*		*			+	+	
ac	−			−	−		−	−		−	−		−	−	−	−	−		−	−					−	−		−	−	−	−	−	−				*	*	−	*			+	+	
bc	−			−	−		−	−					−			−	−		−	−		−	−		−	−		−	−		−	−		−	−		*	*		*	*	*	+	+	+
d	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−				−	−	*	−	−	−	−	−	*				*	*	○	*	−	*		+		+		+
e	−	−	−	−	−	−				−	−	−		−		−	−	−	−	−	−	−	−	−	−	−	*	−		*	−		*	−		*	*		+	*	*	*	+	+	+
f		−		−		−	−		−	−		−	−	−	−	−		*	−		*	−		*	−		○	−	−	−	−	−	−	−	−	−	*	*	*	*	*	*	+	+	+
ad	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−							*	−	*	*	−	*							+	*	○				+		
ae	−	−	−	−	−	−				−	−	−		−		−	−	−	−	−	−				*	−	−	*		−	*		−				+		*	+			+	+	
af		−		−		−	−		−	−		−	−	−	−	−		−	−		−				*		−	*	*	−	*	*	−				+	+	*	+			+	+	
bd	−	−	−	−	−	−	−	−	−				−			−	−	−				*	*	−	*	−	−	*	−					+	+		+	*			+		+		+
be	−	−	−	−	−	−										−	−	−	−	−	−	−	−	−	*	−	−	*			*			*			+			+	+	+	+	+	+
bf		−		−		−	−		−				−			−		−	−			−	−	−	*		*	*	*		*	*		*	*		+	+		+	+	+	+	+	+
cd	−			−	−		−	−		−	−		−	*	*	−	−					*	*		*	−		*	−	*				+	+	*	+	*	*		+		+		+
ce	−			−	−					−	−			*		−	−		−	−		−	−		*	−		*		+	*		+	*		+	+		+	+	+	+	+	+	+
cf				−			−			−			−	−	−	−			−			−			*			*	*	*	*	*	*	*	*	*	+	+	+	+	+	+	+	+	+
g	−	−	−	−	−	−	−	*	*	−	−	*	−	*	*				*	+	+	*	*	+	*	−	*				*	+	+	+	+	*	○	*	+			+		+	+
h	−	−	*				*	+	+	−	*	*			*	−	*	*	−	*	*	−	*	*	−	*	+		+	+		+	+		+	+		+	+	+	+	+	+	+	+
i			+		−	*		+	*		+	*	−	−	−		+	+		+	+		+	+		+	*	*	*	*	*	*	*	*	*	*	+	+	+	+	+	+	+	+	+
ag	−	−	−	−	−	−	−	*	−	−	−	−	*	−	*				*	+	*				−	*	−				+	+	+				*	+	*					+	
ah	−	−	−				−	−	−	−	−	−			−	−	*	−	−	*	−				−	*	*		+	*		+	*					+	+	+			+	+	
ai			−		*	−		*	−		−	−	*	−	−		+	*		+	*					+	*	+	+	*	+	+	*				+	+	+	+			+	+	
bg	−	−	−	−	−	−	−	*	−				*						*	+	*	*	+	*	−	*	−				+	+		+	+		*	+				+		+	+
bh	−	−	−				−	−	−							−	*	−	−	*	−	−	*	−	−	*	−		+			+			+			+		+	+	+	+	+	+
bi			−		*	−		*	−				*				+	*		+	*		+	*		+	*	+	+		+	+		+	+		+	+		+	+	+	+	+	+
cg	−			−	−		−	*		*	*		*	+	+				*	+		*	+		−	*					+	+	+	+	+	+	*	+	+			+		+	+
ch	−						−	−		−	*				+	−	*		−	*		−	*		−	*			+	+		+	+		+	+		+	+	+	+	+	+	+	+
ci					*			*			+		*	*	*		+			+			+			+		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
de	−	−	−	−	−	−				*	*	−		*		*	−	−				+	+	*	+	*	*	+		*				+		+	+		+		+		+		+
df		−		−		−	−		−	*		*	−	*	*	*		−				+		+	+		*	+	*	*				+	+	+	+	+	+		+		+		+
ef		−		−		*				*		○		+		*		*	*		*	*		*	+		+	+		+	+		+	+		+	+		+	+	+	+	+	+	+
dg	−	−	−	−	*	−	*	−	*	*	*	*	○	*	+							+	+	+	*	*	*							+	+	+	+	+	+						+
dh	−	−	−				−	−	−	*	*	*			*	−	*	*				+	+	+	−	*	+		+	+					+	+		+	+		+		+		+
di			−		+	*		*	−		*	*	*	*	*		+	*					+	+		+	+	+	+	+				+	+	+	+	+	+		+		+		+
eg	−	*	*	−	−	*				+	+	*		+					+	+	+	+	+	+	*	−	*				+		+	+		+	+		+			+		+	+
eh	−	−	*							*	+	+				*	+	*	*	+	*	*	+	*	−	−	*			+			+			+			+	+	+	+	+	+	+
ei			*		+	*					+	*		+			+	+		+	+		+	+		*	+	+		+	+		+	+		+	+		+	+	+	+	+	+	+
fg		−		−		*	*		*	*		+	*	+	+				+		+	+		+	*		+				+	+	+	+	+	+	+	+	+			+		+	+
fh		−					−		+	−		*			+	*		+	*		+	*		+	+		+		+	+		+	+		+	+		+	+	+	+	+	+	+	+
fi						+			+			+	+	+	+			+			+			+			+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
gh	−	*	*				*	*	*	+	+	+			+				+	+	+	+	+	+	−	*	+					+	+		+	+		+	+			+		+	+
gi			*		+	+		+	*		+	+	+	+	+					+	+		+	+		+	+				+	+	+	+	+	+	+	+	+			+		+	+
hi			+					+	+		+	+			+		+	+		+	+		+	+		+	+		+	+		+	+		+	+		+	+	+	+	+	+	+	+



\+ = Win for black (left)
− = Win for white (right)
* = Win for first player
○ = Win for second player

Courtesy of Elwyn Berlekamp.

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