Normal Vector Identification and Interactive Tradeoff Analysis Using Minimax Formulation in Multiobjective Optimization

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***Abstract*—In multiobjective optimization, tradeoff analysis plays an important role in determining the best search direction to reach a most preferred solution. This paper presents a new explicit interactive tradeoff analysis method based on the identification of normal vectors on a noninferior frontier. The interactive process is implemented using a weighted minimax formulation by regulating the relative weights of objectives in a systematic manner. It is proved under a mild condition that a normal vector can be identified using the weights and Kuhn–Tucker (K–T) multipliers in the minimax formulation. Utility gradients can be estimated using local preference information such as marginal rates of substitution. The projection of a utility gradient onto a tangent plane of the noninferior frontier provides a descent direction of disutility and thereby a desirable tradeoff direction, along which tradeoff step sizes can be decided by the decision maker using an explicit tradeoff table. Necessary optimality conditions are established in terms of normal vectors and utility gradients, which can be used to guide the elicitation of local preferences and also to terminate an interactive process in a rigorous yet flexible way. This method is applicable to both linear and nonlinear (either convex or nonconvex) multiobjective optimization problems. Numerical examples are provided to illustrate the theoretical results of the paper and the implementation of the proposed interactive decision analysis process.**

***Index Terms*—Multiobjective optimization, normal vector, Pareto-optimality, reliability, tradeoff analysis, water quality.**

I. INTRODUCTION

ONE OF the inherent characteristics associated with complex real-world decision making problems is their inescapably multifarious nature. One of the multifarious features of such problems is their multiple objectives that are usually noncommensurable and are often in conflict. Real-world decision making problems, thus, often lead to a multiobjective optimization problem formulation [3], [5], [7], [18], [28]. The ultimate goal in multiobjective optimization is to seek a most preferred solution from among the set of noninferior solutions.

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In most situations, a decision maker's "global" disutility function is not available. We are thus interested in this paper to develop an efficient interactive solution methodology for generating a most preferred solution. Interactive methods are desirable in certain decision situations where little *a priori* knowledge and experience are known about a decision problem in hand. Recent years have seen an increasing number of papers in literature, reporting applications of interactive methods to optimization problems in engineering design [2], [13], [15], [20], [22], [25]. This growing interest results from the recognition that interactive techniques allow the solution to progress toward a preferred solution through an adaptive process during which the decision maker's preferences are elicited progressively. In this self-learning process, the decision maker is supported to investigate what is achievable and what should be done to arrive at a most preferred solution [1], [14], [17], [19], [23], [24]. This mirrors the common adaptive processes in real-world decision making in engineering and management.

Most interactive methods are based on the elicitation of the decision maker's local preferences. Among the very different forms of local preference information, explicit tradeoffs between different objective functions are widely used in several approaches [4], [6], [8], [16], [21], [26], [29]. Tradeoff rates developed in some typical generating methods are connected with each other [10]. In a tradeoff analysis, the decision maker could provide local preference information such as indifference tradeoffs during the interactive solution process for generating a most preferred solution [4], [23], [30].

A pioneer approach in explicit tradeoff analysis is Geoffrion's method where marginal rates of substitution are used to facilitate interactive tradeoff analysis. In Geoffrion's method, tradeoff direction is determined through the estimation of marginal rates of substitution and tradeoff step size is chosen using a tradeoff table constructed inside the feasible decision space. An empirical criterion is also used to terminate an interactive process. Apart from the applicability of Geoffrion's method limited to convex problems, in its interactive tradeoff process little guidance is given to support the elicitation of the local preferences and no direct relationship between the elicited preferences and the termination criterion is provided. Consequently, the interactive process may be unexpectedly terminated when the decision maker is not consciously aware why this happens and what this means. Similar problems exist in other representative interactive procedures [21].

This paper investigates a new interactive method that also uses local preference information in a form of marginal rates of substitution for estimating tradeoff directions and step sizes. However, this estimation is supported through the identification of normal vectors on a noninferior frontier [9]. In this paper, it is proved under a mild condition that a normal vector can be identified using a weighted minimax formulation. It is shown that a most preferred solution that minimizes the decision maker's disutility is achieved when the normal vector is proportional to the utility gradient at the solution. This optimality condition is examined to establish termination criteria and used to define optimal indifference tradeoffs to support the elicitation of local preferences.

In the proposed method, the optimality condition is checked interactively at each generated noninferior solution. If the optimality condition is not satisfied, a systematic procedure is proposed to guide the decision maker to assess new indifference tradeoffs for estimating marginal rates of substitution. The projection of the estimated marginal rates of substitution onto the tangent plane of the noninferior frontier provides a descent direction of the decision maker's underlying disutility. Tradeoffs among different objectives along this direction provide a most realistic picture about what changes of the objective functions can be expected as a consequence of a designated tradeoff. Having been made aware of the changes, the decision maker may be in a better position to judge what tradeoff steps should be chosen.

The above interactive process is not irreversible, which means that the decision maker can freely search the noninferior frontier while he is also made aware what changes of the objectives are attainable at a generated noninferior solution and when and why this solution could be his most preferred solution, though such irreversibility and freedom may lead to the provision of inconsistent preferences, which should be properly examined and will be dealt with in other papers. Since the interactive process is based on a minimax formulation, both linear and nonlinear (convex or nonconvex) problems can be handled using the new method. A normal vector can be generated at no extra cost if a dual method, such as the primal and dual method and the simplex method, is used to solve the minimax problem. Otherwise, a set of linear equations needs to be solved for generating a normal vector.

The organization of this paper is as follows. Section II is devoted to investigating the identification of normal vector, the establishment of the optimality conditions, and the design of an iterative optimization algorithm for problems with known disutility functions. In Section III, an explicit interactive tradeoff analysis process is proposed, including the estimation of both tradeoff directions and step sizes. Four numerical examples are provided in Section IV to illustrate the main features of the theoretical results of this paper and to demonstrate the implementation of the proposed interactive tradeoff analysis method. Finally, the paper concludes in Section V.

II. Minimax Formulation and Normal Vector Identification

A. Kuhn–Tucker (K–T) Conditions for a Minimax Formulation

A multiobjective optimization problem can, in general, be represented as follows:

$$\min \quad (J_1(x),\, J_2(x),\, \ldots,\, J_k(x)) \tag{1a}$$
$$\text{s.t.} \quad g_j(x) \le 0 \qquad j = 1, 2, \ldots, m \tag{1b}$$

where $x \in R^n$, $J_i(x)$ $(i = 1, 2, \ldots, k)$ are multiple objectives to be minimized and $g_j(x)$ $(j = 1, 2, \ldots, m)$ are constraints. All $J_i(x)$ $(i = 1, 2, \ldots, k)$ and $g_j(x)$ $(j = 1, 2, \ldots, m)$ are assumed to be second-order differentiable. Denote by X the feasible region of (1)

$$X = \{x | g_j(x) \le 0,\ j = 1, 2, \ldots, m\}. \tag{1c}$$

If an implicit disutility function is assumed, then (1) can be interpreted as finding a most preferred solution that minimizes the disutility function, that is

$$\min \quad J = \phi(J_1(x),\, J_2(x),\, \ldots,\, J_k(x)) \tag{2a}$$
$$\text{s.t.} \quad x \in X \tag{2b}$$

where J represents the decision maker's disutility function, which increases with $J_i(x)$ $(i = 1, 2, \ldots, k)$ at any $x \in X$, or

$$\left.\frac{\partial J}{\partial J_i}\right|_x > 0 \qquad \forall\, x \in X. \tag{3}$$

The implication of (3) is that an improvement in any individual objective will lead to a favorable change in the value of the disutility function while keeping other objectives unchanged. Without loss of generality and to simplify the description of the theoretical results, the minimum value of each $J_i(x)$ over X is assumed to be strictly positive in this section. This assumption will be dropped in the next section. Furthermore, it is assumed that at the point where J is optimized the corresponding value of each $J_i(x)$ $(i = 1, 2, \ldots, k)$ is finite.

A solution x^* is said to be noninferior in (1) if there exists no other feasible solution x such that $J_i(x) \le J_i(x^*)$ for all $i = 1, 2, \ldots, k$ with strict inequality for at least one i. Denote by X^* the set of noninferior solutions in the decision space, i.e.

$$X^* = \{x | x \text{ is noninferior solution of (1)}\}. \tag{4}$$

Denote by $\Omega(X^*)$ the set of noninferior solutions in the objective space, which is the projection of X^* onto the objective space, i.e.

$$\Omega(X^*) = \{(J_1(x),\, J_2(x),\, \ldots,\, J_k(x)) | x \in X^*\}. \tag{5}$$

In nondegenerate situations, X^* is of $(k-1)$ dimension where there are k objective functions, as is the dimension of $\Omega(X^*)$. The corresponding geometric representation is that in the $\{J_1, J_2, \ldots, J_k\}$ space a noninferior frontier is always on the boundary of the feasible region where the inner normal vector belongs to R_+^k. The noninferior frontier is a curve when two objectives are present, a surface when three objectives are present, and a $(k-1)$ hypersurface when k objectives are present.

One prominent generating method in obtaining noninferior solutions is the weighted minimax approach [11]. For any noninferior solution of (1), there exists a weighting vector $w = [w_1\ w_2\ \cdots\ w_k]$ satisfying $w_1 = 1$ and $w_i \ge 0\ (i = 2, 3, \ldots k)$ such that the noninferior solution can be generated by the following weighted minimax formulation:

$$\min_x \max_{1 \le i \le k} \{w_i J_i(x)\} \tag{6a}$$
$$\text{s.t.} \quad g_j(x) \le 0 \qquad j = 1, 2, \ldots, m. \tag{6b}$$

If J_i^* is the minimum feasible value of the objective J_i and is not positive, then (6a) can be replaced by (6c) as follows:

$$\min_x \max_{1\le i\le k} \{w_i(J_i(x) - J_i^*)\}. \tag{6c}$$

Note that (6) is capable of generating all noninferior solutions of (1) by regulating the weights without requiring assumptions such as convexity or differentiability of the noninferior frontier.

The weighted minimax formulation given in (6) can be rewritten as the following equivalent form by introducing an auxiliary variable y:

$$\min_{x,\,y} \quad \varphi(y) \tag{7a}$$
$$\text{s.t.} \quad w_i J_i(x) \le y \qquad i = 1, 2, \ldots, k \tag{7b}$$
$$g_j(x) \le 0 \qquad j = 1, 2, \ldots, m \tag{7c}$$

where $\varphi(y)$can be any second-order differentiable and strictly increasing function of y. If J_i^* is not necessarily positive, then (7b) should be replaced by (7d) as follows:

$$w_i(J_i(x) - J_i^*) \le y \qquad i = 1, 2, \ldots, k. \tag{7d}$$

Assuming a regularity of the solution of (7), the corresponding Lagrangian of (7) can be written as follows:

$$L(w) = \varphi(y) + \sum_{i=1}^{k} \lambda_i [w_i J_i(x) - y] + \sum_{j=1}^{m} \mu_j g_j(x) \tag{8}$$

where λ_i $(i = 1, 2, \ldots, k)$ and μ_j $(j = 1, 2, \ldots, m)$ are nonnegative Kuhn–Tucker (K–T) multipliers associated with (7b) and (7c), respectively. The argument w in L indicates the dependency of the Lagrangian function on the weighting vector. The set of the first-order necessary optimum conditions for (8) is

$$\frac{\partial L}{\partial y} = \frac{d\varphi(y)}{dy} - \sum_{i=1}^{k} \lambda_i = 0 \tag{9a}$$
$$\frac{\partial L}{\partial x} = \sum_{i=1}^{k} \lambda_i w_i \frac{\partial J_i(x)}{\partial x} + \sum_{j=1}^{m} \mu_j \frac{\partial g_j(x)}{\partial x} = 0 \tag{9b}$$
$$\lambda_i [w_i J_i(x) - y] = 0 \qquad i = 1, 2, \ldots, k \tag{9c}$$
$$w_i J_i(x) - y \le 0 \qquad i = 1, 2, \ldots, k \tag{9d}$$
$$\mu_j g_j(x) = 0 \qquad j = 1, 2, \ldots, m \tag{9e}$$
$$g_j(x) \le 0 \qquad j = 1, 2, \ldots, m. \tag{9f}$$

When (9) satisfies a mild condition as specified in the implicit function theorem [12] for a given w, i.e., an associated Jacobian matrix of (9) with respect to x, y, λ, and μ is nonsingular, the noninferior solutions in the decision space, the corresponding optimal auxiliary variable, and the optimal K–T multipliers can be expressed locally as functions of w

$$x = x^*(w) \qquad \forall\, x \in X^* \tag{10a}$$
$$y = y(w) \tag{10b}$$
$$\lambda = \lambda(w) \tag{10c}$$
$$\mu = \mu(w). \tag{10d}$$

Similarly, the noninferior frontier in the objective space can be locally characterized by a parametric form

$$J_1^*(w) = J_1[x^*(w)]$$
$$J_2^*(w) = J_2[x^*(w)]$$
$$\cdots$$
$$J_k^*(w) = J_k[x^*(w)].$$

It is assumed in this paper that the noninferior frontier of (1) is simply connected and of $(k-1)$ dimension. In degenerate cases where the noninferior frontier is of a dimension less than $(k-1)$, two or more J_is do not conflict with each other. Thus, a degenerate case can be always converted to a nondegenerate case by reducing the number of J_is to be placed in (6).

B. Normal Vector Identification and Utility Gradient Projection

It is clear that the set X^* contains all most preferred solutions of (1). We can therefore confine the search of a most preferred solution of (1) in X^*. It needs to be emphasized here that the devised algorithm does not require the functional form of $x^*(w)$. A specific noninferior solution that attains a most preferred solution of (1) is sought iteratively or interactively.

A most preferred solution x^* of (1) can always be generated by the weighted minimax formulation in (6) or (7) with the weighting vector $w^* = [1,\, J_1(x^*)/J_2(x^*),\, J_1(x^*)/J_3(x^*),\, \ldots,\, J_1(x^*)/J_k(x^*)]$. Note that each component of w^* is strictly positive due to the assumptions made in (3). Thus, the search for the optimum weighting vector can be confined in $W_+ = \{[1,\, w_2,\, w_3,\, \ldots,\, w_k] | w_i > 0, i = 2, 3, \ldots, k\}$.

Theorem 1: Suppose $x^*(w)$ solves (7) and $[J_1^*(w),\, J_2^*(w),\, \ldots,\, J_k^*(w)]$ is the corresponding noninferior solution in the objective space. Then, the following is satisfied:

$$\sum_{i=1}^{k} \lambda_i w_i \frac{\partial J_i^*(w)}{\partial w_j} = 0 \qquad j = 2, 3, \ldots, k \tag{11}$$

if (9) locally satisfies the condition in the implicit function theorem.

Proof: When (9) locally satisfies the condition in the implicit function theorem, $x^*(w)$, $y(w)$, $\lambda(w)$, and $\mu(w)$ are locally differentiable with respect to w from the assumptions on φ, J_i $(i = 1, 2, \ldots, k)$, and g_j $(j = 1, 2, \ldots, m)$. Substituting parametric forms of x, y, λ, and μ into (9) yields

$$\left.\frac{d\varphi(y)}{dy}\right|_{y(w)} - \sum_{i=1}^{k} \lambda_i(w) = 0 \tag{12a}$$
$$\sum_{i=1}^{k} \lambda_i(w) w_i \left.\frac{\partial J_i(x)}{\partial x}\right|_{x(w)} + \sum_{j=1}^{m} \mu_j(w) \left.\frac{\partial g_j(x)}{\partial x}\right|_{x(w)} = 0 \tag{12b}$$
$$\lambda_i(w)[w_i J_i^*(w) - y(w)] = 0 \qquad i = 1, 2, \ldots, k \tag{12c}$$
$$w_i J_i^*(w) - y(w) \le 0 \qquad i = 1, 2, \ldots, k \tag{12d}$$
$$\mu_j(w) g_j(x(w)) = 0 \qquad j = 1, 2, \ldots, m \tag{12e}$$
$$g_j(x(w)) \le 0 \qquad j = 1, 2, \ldots, m. \tag{12f}$$

When $\{x(w), y(w), \lambda(w), \mu(w)\}$ solves $L(w)$, we must have

$$\begin{aligned}\varphi(y(w)) &+ \sum_{i=1}^{k} \lambda_i(w)[w_i J_i^*(w) - y(w)] \\ &+ \sum_{j=1}^{m} \mu_j(w) g_j(x(w)) \\ &\le \varphi(y(w+dw)) \\ &+ \sum_{i=1}^{k} \lambda_i(w+dw)[w_i J_i^*(w+dw) - y(w+dw)] \\ &+ \sum_{j=1}^{m} \mu_j(w+dw) g_j(x(w+dw)). \qquad (13)\end{aligned}$$

Multiplying both sides of (12a) by $(dy/dw)^T dw$, we have

$$\left.\frac{d\varphi(y)}{dy}\right|_{y(w)} \left(\frac{dy}{dw}\right)^T dw - \sum_{i=1}^{k} \lambda_i(w) \left(\frac{dy}{dw}\right)^T dw = 0. \quad (14)$$

From (14), we further have

$$\begin{aligned}&[\varphi(y(w+dw)) - \varphi(y(w))] \\ &\qquad - \sum_{i=1}^{k} \lambda_i(w)[y(w+dw) - y(w)] = 0. \quad (15)\end{aligned}$$

Using (12e) and (15), (13) can be simplified to, up to the first-order infinitesimal

$$0 \le \sum_{i=1}^{k} \{\lambda_i(w) w_i \, dJ_i^*(w) + d\lambda_i(w)[w_i J_i^*(w) - y(w)]\}. \quad (16)$$

If inequality $w_i J_i^*(w) - y(w) \le 0$ is binding, $d\lambda_i(w)[w_i J_i^*(w) - y(w)] = 0$. If inequality $w_i J_i^*(w) - y(w) \le 0$ is not binding, $\lambda_i(w) = 0$, and there exists a small dw, such that $\lambda_i(w+dw)$ remains zero. Thus, $d\lambda_i(w) = 0$ and $d\lambda_i(w)[w_i J_i^*(w) - y(w)] = 0$. Equation (16) can be further written as

$$0 \le \sum_{i=1}^{k} \lambda_i(w) w_i \, dJ_i^*(w) \quad (17)$$

where

$$\begin{aligned}dJ_i^*(w) = &\sum_{j=2}^{k} \frac{\partial J_i^*(w)}{\partial w_j} dw_j \\ &+ \frac{1}{2} \sum_{j=2}^{k} \sum_{l=2}^{k} \frac{\partial^2 J_i^*(w)}{\partial w_j \partial w_l} dw_j \, dw_l + o(dw^2). \quad (18)\end{aligned}$$

Since w_j $(j = 2, 3, \ldots, k)$ are strictly positive and dw_j $(j = 2, 3, \ldots, k)$ can take any sign, (11) must be held to guarantee that (17) is satisfied. Q.E.D.

From the assumption that the noninferior frontier of problem is of $(k-1)$ dimension, the $(k-1)$-dimensional tangent hyperplane on the noninferior frontier at $x^*(w)$ is spanned by

$$\begin{bmatrix} \frac{\partial J_1^*(w)}{\partial w_2} \\ \frac{\partial J_2^*(w)}{\partial w_2} \\ \vdots \\ \frac{\partial J_k^*(w)}{\partial w_2} \end{bmatrix}, \begin{bmatrix} \frac{\partial J_1^*(w)}{\partial w_3} \\ \frac{\partial J_2^*(w)}{\partial w_3} \\ \vdots \\ \frac{\partial J_k^*(w)}{\partial w_3} \end{bmatrix}, \ldots, \begin{bmatrix} \frac{\partial J_1^*(w)}{\partial w_k} \\ \frac{\partial J_2^*(w)}{\partial w_k} \\ \vdots \\ \frac{\partial J_k^*(w)}{\partial w_k} \end{bmatrix}$$

when $x^*(w)$ is locally differentiable. Then, from Theorem 1, the inner normal vector N on the noninferior frontier at $[J_1^*(w), J_2^*(w), \ldots, J_k^*(w)]$ in the k-dimensional objective space $\{J_1, J_2, \ldots, J_k\}$ is given by

$$N = [w_1\lambda_1, w_2\lambda_2, \ldots, w_k\lambda_k]^T. \quad (19)$$

Denote by ∇J the gradient of J at $[J_1(x), J_2(x), \ldots, J_k(x)]$

$$\nabla J = \left[\frac{\partial J}{\partial J_1}, \frac{\partial J}{\partial J_2}, \ldots, \frac{\partial J}{\partial J_k}\right]^T. \quad (20)$$

Define ΔJ to be the projection of the negative gradient of J, $-\nabla J$, onto the tangent hyperplane at $[J_1(x), J_2(x), \ldots, J_k(x)]$. Projection ΔJ can be calculated using the following equation:

$$\Delta J = [\Delta J_1, \Delta J_2, \ldots, \Delta J_k]^T = -\nabla J + \frac{(\nabla J)^T N}{N^T N} N. \quad (21)$$

Premultiplying both sides of (21) by $(\nabla J)^T$ yields

$$(\nabla J)^T \Delta J = -(\nabla J)^T \nabla J + \frac{[(\nabla J)^T N]^2}{N^T N} \le 0 \quad (22)$$

by using Cauchy–Schwarz inequality. Vector ΔJ thus always represents a descent direction of J at $\{J_1(x), J_2(x), \ldots, J_k(x)\}$ in the $\{J_1, J_2, \ldots, J_k\}$ space. This projection thus provides a desirable tradeoff direction and will be used to determine tradeoff step sizes.

C. Optimality Conditions

Based on the normal vector and the utility gradient, a necessary optimality condition can be established as follows.

Theorem 2: A necessary condition for an optimal solution of the weighted minimax formulation in (6) or (7) to reach a most preferred solution of (1) is

$$[w_1\lambda_1, w_2\lambda_2, \ldots, w_k\lambda_k] \propto \left[\frac{\partial J}{\partial J_1}, \frac{\partial J}{\partial J_2}, \ldots, \frac{\partial J}{\partial J_k}\right] \quad (23)$$

if (9) satisfies the condition in the implicit function theorem in a neighborhood of the optimal solution, where λ_i $(i = 1, 2, \ldots, k)$ is the corresponding optimal K–T multiplier associated with the ith objective constraint in (7b).

Proof: In abstract, an optimal solution of (2) can be sought using the following nonlinear programming model:

$$\min \quad J = \phi[J_1(x^*(w)), J_2(x^*(w)), \ldots, J_k(x^*(w))] \quad (24a)$$

$$\text{s.t.} \quad w_i \ge 0 \qquad i = 2, 3, \ldots, k \quad (24b)$$

where $x^*(w)$ is the noninferior solution generated by solving (6) or (7).

If an optimal solution of (2) can be reached by a solution of the weighted minimax formulation with all weighting coefficients strictly positive, then the K–T condition of (24) leads to the following:

$$\sum_{i=1}^{k} \frac{\partial J}{\partial J_i} \frac{\partial J_i^*}{\partial w_j} = 0 \qquad j = 2, 3, \ldots, k. \tag{25}$$

From Theorem 1 and (25), both $[\partial J/\partial J_1, \partial J/\partial J_2, \ldots, \partial J/\partial J_k]^T$ and $[w_1\lambda_1, w_2\lambda_2, \ldots, w_k\lambda_k]^T$ are orthogonal to a $(k-1)$-dimensional tangent hyperplane S spanned by

$$\begin{bmatrix} \dfrac{\partial J_1^*(w)}{\partial w_2} \\ \dfrac{\partial J_2^*(w)}{\partial w_2} \\ \cdots \\ \dfrac{\partial J_k^*(w)}{\partial w_2} \end{bmatrix}, \begin{bmatrix} \dfrac{\partial J_1^*(w)}{\partial w_3} \\ \dfrac{\partial J_2^*(w)}{\partial w_3} \\ \cdots \\ \dfrac{\partial J_k^*(w)}{\partial w_3} \end{bmatrix}, \ldots, \begin{bmatrix} \dfrac{\partial J_1^*(w)}{\partial w_k} \\ \dfrac{\partial J_2^*(w)}{\partial w_k} \\ \cdots \\ \dfrac{\partial J_k^*(w)}{\partial w_k} \end{bmatrix}$$

and they thus belong to a one-dimensional (1-D) space and must be proportional, or

$$[w_1\lambda_1, w_2\lambda_2, \ldots, w_k\lambda_k] \propto \left[\frac{\partial J}{\partial J_1}, \frac{\partial J}{\partial J_2}, \ldots, \frac{\partial J}{\partial J_k}\right].$$

Q.E.D.

The geometric interpretation of Theorem 2 is that, at a most preferred solution of (1), the normal vector on the noninferior frontier is proportional to the utility gradient of J with respect to J_i $(i = 1, 2, \ldots, k)$ in the $\{J_1, J_2, \ldots, J_k\}$ space. As (23) given in Theorem 2 is a first-order necessary condition for the optimality, it will be satisfied by all local minimum and maximum points of J on the noninferior frontier.

Let x^t be a solution of (7), ∇J^T the utility gradient at x^t, and N^T the normal vector at x^t. If x^t attains a most preferred solution of (1), then ∇J^T is equal to cN^T at x^t in accordance with Theorem 2 where c is a constant. Substituting ∇J^T by cN^T in (21), ΔJ^t becomes a zero vector at a most preferred solution of (1). Thus, another form of the necessary condition is given by

$$\Delta J^t = 0. \tag{26}$$

Equation (23) or $\Delta J^t = 0$ can be used to test the optimality of the solution generated by the weighted minimax formulation at each iteration. If vector ΔJ^t is not a zero vector, ΔJ^t provides a descent direction on the tangent hyperplane at $\{J_1(x^t), J_2(x^t), \ldots, J_k(x^t)\}$ in the $\{J_1, J_2, \ldots, J_k\}$ space.

Since $\nabla J^T = cN^T$ at a most preferred solution of (1), the third form of the necessary condition is given by

$$\frac{\partial J/\partial J_1}{w_1\lambda_1} = \frac{\partial J/\partial J_2}{w_2\lambda_2} = \cdots = \frac{\partial J/\partial J_k}{w_k\lambda_k}. \tag{27}$$

This form will be used to define so-called optimal indifference tradeoffs to facilitate the elicitation of local preference information.

D. Iterative Optimization With Known Utility Function

For a multiobjective optimization problem with a known disutility function J, an updating scheme needs to be devised for finding an updated weighting vector with which a new noninferior solution can be identified to realize a decrement in J.

One important observation from the necessary optimality condition in (23) is that if the K–T multipliers λ_i $(i = 1, 2, \ldots, k)$ are all strictly positive, the optimization process of the weighted minimax formulation acts as an equalizer to make all w_iJ_i $(i = 2, 3, \ldots, k)$ equal to y and thus equal to J_1. The corresponding geometric interpretation is that the weighted minimax formulation searches an optimal feasible solution along the ray that starts from the origin and is specified by

$$J_1 = w_2J_2 = \cdots = w_kJ_k \tag{28a}$$

if all objective constraints in (7b) are binding. When some objective functions of $J_1, J_2, \ldots, J_k$ do not satisfy the assumption to be strictly positive in their domain, (28a) can be generalized by

$$J_1 - J_1^* = w_2(J_2 - J_2^*) = \cdots = w_k(J_k - J_k^*) \tag{28b}$$

where J_i^* is the minimum value of J_i.

On the basis of the above recognition, the weighting vector can be updated at iteration t using the following formula:

$$w_i^{t+1} = \frac{J_1(x^t) + \alpha\Delta J_1^t}{J_i(x^t) + \alpha\Delta J_i^t} \qquad i = 2, 3, \ldots, k \tag{29a}$$

or more generally

$$w_i^{t+1} = \frac{J_1(x^t) + \alpha\Delta J_1^t - J_1^*}{J_i(x^t) + \alpha\Delta J_i^t - J_i^*} \qquad i = 2, 3, \ldots, k \tag{29b}$$

where α is a step size parameter which can be adjusted on-line to guarantee a decrement of the disutility function J or can be determined by the following 1-D search:

$$\begin{aligned} \min \phi(\alpha) = \phi(J_1(x^t) + \alpha\Delta J_1^t,\, J_2(x^t) + \alpha\Delta J_2^t, \ldots, \\ J_k(x^t) + \alpha\Delta J_k^t). \end{aligned} \tag{30}$$

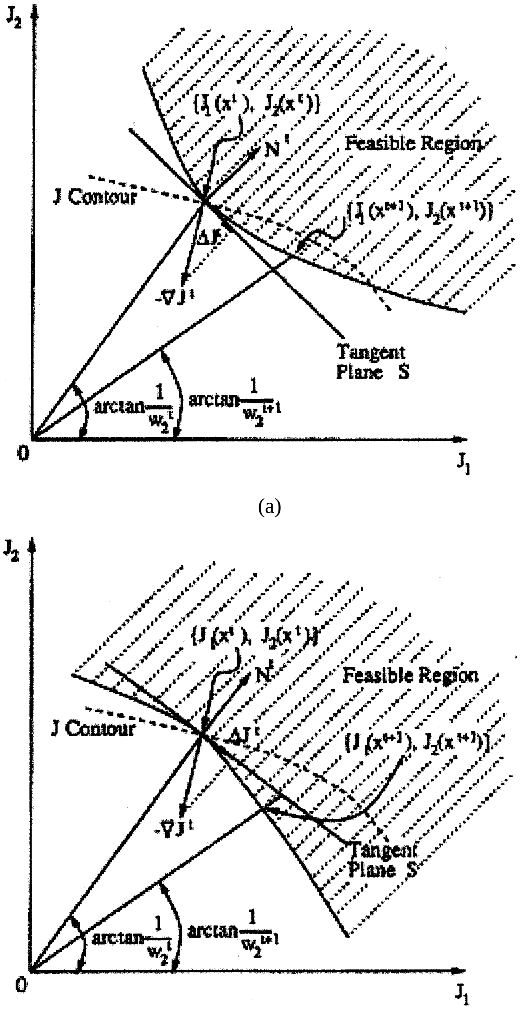
Note here that both solution $\{J_1(x^{t+1}), J_2(x^{t+1}), \ldots, J_k(x^{t+1})\}$ of the following weighted minimax formulation:

$$\min_x \max_{1\le i\le k} \{w_i^{t+1}J_i(x)\} \tag{31a}$$

$$\text{s.t.} \quad g_j(x) \le 0 \qquad j = 1, 2, \ldots, m \tag{31b}$$

and the point $\{J_1(x^t) + \alpha\Delta J_1^t, J_2(x^t) + \alpha\Delta J_2^t, \ldots, J_k(x^t) + \alpha\Delta J_k^t\}$ lie on the ray specified by $J_1 = w_2^{t+1}J_2 = \cdots = w_k^{t+1}J_k$ if all constraints in (7b) are binding (see Fig. 1). When the step size parameter α approaches zero, vector $[J_1(x^{t+1}) - J_1(x^t), J_2(x^{t+1}) - J_2(x^t), \ldots, J_k(x^{t+1}) - J_k(x^t)]^T$ coincides with nonzero vector $[\Delta J_1^t, \Delta J_2^t, \ldots, \Delta J_k^t]^T$ both originating from $\{J_1(x^t), J_2(x^t), \ldots, J_k(x^t)\}$. Therefore, for a sufficiently small step size parameter α, the following convergence condition is satisfied:

$$\begin{aligned} \phi(J_1(x^{t+1}), J_2(x^{t+1}), \ldots, J_k(x^{t+1})) \\ < \phi(J_1(x^t), J_2(x^t), \ldots, J_k(x^t)). \end{aligned} \tag{32}$$



(a)

(b)

Fig. 1. (a) Utility gradient and normal vector in a convex case. (b) Utility gradient and normal vector in a nonconvex case.

The original multiobjective optimization problem can be now solved in a two-level solution structure. For a given weighting vector w, the weighted minimax formulation in (6) or (7) is solved at the lower level using appropriate solution schemes depending on the problem structure of the parametric minimax formulation in (6) or (7). At the upper level, the utility gradient is calculated or estimated and the optimal stopping condition in (23) or one of its variants is checked upon receiving the solutions from the lower level. If (23) is not satisfied, (29) is used to update the value of the weighting vector. Problem (6) or (7) at the lower level is then solved again for the updated value of w. The iteration process continues until (23) is satisfied.

The overall algorithm of the iterative parametric minimax algorithm is now summarized as follows.

Step 1: Select an initial weighting vector w^1 (e.g., equal weights for all objectives), choose a small number ε for the error tolerance in the stopping condition, and set iteration number $t = 1$.

Step 2: For the selected weighting vector w^t, solve weighted minimax problem (6) or (7) and obtain solution $\{x^t, y^t, \lambda^t\}$.

Remark 1: If the ray specified by $J_1 = w_2^{t+1} J_2 = \cdots = w_k^{t+1} J_k$ directs to a relative interior point on the noninferior frontier, that relative interior noninferior point is a unique solution generated at Step 2) and all inequalities in (7b) are binding. Due to the assumption that noninferior frontier is simply connected, some inequalities in (7b) can become not binding at solution $\{x^t, y^t\}$ only when the ray specified by $\{J_1(x^{t-1}) + \alpha\Delta J_1^{t-1}, J_2(x^{t-1}) + \alpha\Delta J_2^{t-1}, \ldots, J_k(x^{t+1}) + \alpha\Delta J_k^{t-1}\}$ directs outside of the noninferior frontier. The value of w^t in this case needs to be reassessed to make all constraints in (7b) binding in order to test for the optimum and to perform the next iteration. A simple correction procedure is to reduce the value of the step size parameter α to make the ray specified by $J_1 = w_2^{t+1} J_2 = \cdots = w_k^{t+1} J_k$ direct to a relative interior point on the noninferior frontier.

Remark 2: If the solution scheme at the lower level does not furnish the corresponding K–T multipliers, the K–T multipliers can be found by solving the set of first-order K–T conditions of (7) along with the available knowledge of the identified optimal solution of $\{x^t, y^t\}$. Notice that the set of the first-order K–T conditions of (7) is linear in K–T multipliers when solution $\{x^t, y^t\}$ is known. If $\{x^t, y^t\}$ is a regular point, for given w, the solution for K–T multipliers is unique.

Step 3: Check if the following optimal condition is satisfied:

$$\max_{1\le i\le k}\left\{\frac{\partial J/\partial J_i}{w_i\lambda_i}\right\} - \min_{1\le i\le k}\left\{\frac{\partial J/\partial J_i}{w_i\lambda_i}\right\} \le \varepsilon \tag{33}$$

or

$$\max_{1\le i\le k}\{\Delta J_i\} \le \varepsilon. \tag{34}$$

If yes, the search terminates. Otherwise go to Step 4.

Step 4: Update w using (29). Set $t = t + 1$ and go back to Step 2.

III. Interactive Decision Analysis Process

In most multiobjective optimization problems, the explicit form of a disutility function is unavailable and tradeoffs among objectives can only be based on the decision maker's local preferences. Indifference tradeoffs or marginal rates of substitution are one type of information that could be provided by the decision maker. In this section, we investigate how to conduct explicit tradeoff analysis using the normal vector identified at a generated noninferior solution and the necessary optimality condition established, as developed in the previous section. Both a tradeoff direction and a step size can be determined using the decision maker's local preferences.

A. Optimal Indifference Tradeoffs and Tradeoff Direction

An indifference hypersurface of an underlying disutility function can be expressed as follows:

$$J = \phi(J_1, J_2, \ldots, J_k) = c \tag{35}$$

where c is a constant. If both $[J_1(\hat{x}), J_2(\hat{x}), \ldots, J_k(\hat{x})]$ and $[J_1(\tilde{x}), J_2(\tilde{x}), \ldots, J_k(\tilde{x})]$ lie on the same indifference hypersurface, the corresponding tradeoff between these two points is known as the marginal rate of substitution.

The relationship between the marginal rate of substitution and the underlying disutility function is well known [3]. In abstract, solving (35) for J_i yields

$$J_i = J_i(J_1,\, J_2,\, \ldots,\, J_{i-1},\, J_{i+1},\, \ldots,\, J_{k},\, c). \tag{36}$$

The marginal rate of substitution between J_i and J_j, M_{ij}, can be expressed as

$$M_{ij} = -\frac{\partial J_i(J_1,\, J_2,\, \ldots,\, J_{i-1},\, J_{i+1},\, \ldots,\, J_k,\, c)}{\partial J_j}. \tag{37}$$

Alternatively, applying the chain rule to (35) generates the following:

$$M_{ij} = \frac{\partial J}{\partial J_j} \Big/ \frac{\partial J}{\partial J_i}. \tag{38}$$

Choose the first objective as the reference objective. Denote the vector of the marginal rate of substitution at x by M

$$M = [1,\, M_{12},\, M_{13},\, \ldots,\, M_{1k}]^T \tag{39}$$

and the gradient of J at $\{J_1(x),\, J_2(x),\, \ldots,\, J_k(x)\}$ by ∇J

$$\nabla J = \left[\frac{\partial J}{\partial J_1},\, \frac{\partial J}{\partial J_2},\, \ldots,\, \frac{\partial J}{\partial J_k}\right]. \tag{40}$$

Then, vectors ∇J and M have the following relationship:

$$M = \frac{\nabla J}{\partial J / \partial J_1}. \tag{41}$$

This means that ∇J and M are proportional.

In practice, the explicit form of an disutility function is unknown and the marginal rate of substitution M_{ij} may be approximated by the indifference tradeoff that equals the amount of increment (decrement) of J_i with which the decision maker considers to exactly compensate one unit of decrement (increment) of J_j. Denote by dJ_i a small change in the ith objective. If the first objective is chosen as the reference objective and the following is true

$$\phi(J_1 + dJ_1,\, J_2,\, \ldots,\, J_{i-1},\, J_i - dJ_i,\, J_{i+1},\, \ldots,\, J_k) = c \tag{42}$$

then M_{1j} can be approximated by

$$M_{1i} = -\frac{dJ_1}{dJ_i} \qquad i = 2,\, 3,\, \ldots,\, k. \tag{43}$$

If dJ_1 is fixed at one unit, or $dJ_1 = 1$, then (43) reduces to

$$M_{1i} = -\frac{1}{dJ_i} \qquad i = 2,\, 3,\, \ldots,\, k. \tag{44}$$

The decision maker is requested to provide dJ_i $(i = 2,\, 3,\, \ldots,\, k)$ for a given dJ_1. This may not be an easy task for the decision maker if he does not know the consequences of such a tradeoff.

Fortunately, the necessary condition (23) can be used to guide the elicitation of indifference tradeoffs. From (23) and (41), we have at a most preferred solution

$$\frac{M_{1i}}{w_i \lambda_i} = \frac{M_{11}}{w_1 \lambda_1} \qquad i = 2,\, 3,\, \ldots,\, k. \tag{45}$$

From (43), we have at any most preferred solution

$$dJ_i = -dJ_1 \frac{w_1 \lambda_1}{w_i \lambda_i} \qquad i = 2,\, 3,\, \ldots,\, k. \tag{46}$$

Equation (46) represents an optimal indifference tradeoff between objective 1 and objective i. At a generated noninferior solution x^t, $w_i \lambda_i$ is given as $w_i^t \lambda_i^t$ and therefore dJ_i^t can be calculated using (46) once dJ_1^t is given (e.g., fixed at one unit). The generated dJ_i^t may then be shown to the decision maker. If he agrees that dJ_i^t exactly compensates for dJ_1^t for all $i = 2,\, 3,\, \ldots,\, k$, then the current noninferior solution is his most preferred solution that minimizes his underlying disutility function.

Otherwise, the project of M^t onto the tangent plane of the noninferior frontier at the solution is not zero and provides the descent direction of his underlying disutility function. Similar to (21), the projection is calculated by

$$\overline{\Delta J}^{\,t} = \left[\overline{\Delta J}_1^{\,t},\, \overline{\Delta J}_2^{\,t},\, \ldots,\, \overline{\Delta J}_k^{\,t}\right]^T = -M^t + \frac{(M^t)^T N^t}{(N^t)^T N^t} N^t \tag{47}$$

where $\overline{\Delta J}^{\,t}$ is the new tradeoff direction. If $\overline{\Delta J}_i^{\,t}$ is less than (more than or equal to) zero, it means that objective i should be decreased (increased or kept unchanged) in order to improve the underlying disutility function. Similar to (29), the weighting vector in (7) can be updated at solution x^t as follows:

$$w_i^{t+1} = \frac{J_1(x^t) + \overline{\alpha} \overline{\Delta J}_1^{\,t}}{J_i(x^t) + \overline{\alpha} \overline{\Delta J}_i^{\,t}} \qquad i = 2,\, 3,\, \ldots,\, k \tag{48a}$$

or more generally

$$w_i^{t+1} = \frac{J_1(x^t) + \overline{\alpha} \overline{\Delta J}_1^{\,t} - J_1^*}{J_i(x^t) + \overline{\alpha} \overline{\Delta J}_i^{\,t} - J_i^*} \qquad i = 2,\, 3,\, \ldots,\, k. \tag{48b}$$

B. Determine Tradeoff Step Size Using an Explicit Tradeoff Table

In (48), $\overline{\alpha}$ is the tradeoff step size. $\overline{\alpha}$ could be fixed at a small real number or be regulated in an elaborate way. Since the underlying disutility function is unknown explicitly, $\overline{\alpha}$ cannot be regulated using the 1-D search. A tradeoff table may be used to facilitate the interactive regulation of $\overline{\alpha}$.

Suppose J_i^* and J_i^- are, respectively, the best and worst feasible values of objective i, obtained from the payoff table. $\overline{\alpha}$ is decomposed into two parts: $\overline{\alpha} = \alpha_{\max} \alpha_2$ where $\alpha_{\max}$ denotes the largest permissible step size and α_2 a regulating factor with $\alpha_2 \in [0\ 1]$. Suppose I_s is the index set of objectives that need to be sacrificed at solution x^t, or

$$I_s = \left\{ i \,\middle|\, \overline{\Delta J}_i^{\,t} \geq 0,\ i \in \{1,\, \ldots,\, k\} \right\}. \tag{49}$$

Then, $\alpha_{\max}$ may be determined as follows:

$$\alpha_{\max} = \min_{i \in I_s} \left\{ \frac{|J_i(x^t) - J_i^-|}{\left|\overline{\Delta J}_i^{\,t}\right|} \right\}. \tag{50}$$

α_2 may then be determined using a tradeoff table, as shown in Table I, which shows the changes of objective functions along the direction $\overline{\Delta J}^{\,t}$, where $J_i(\overline{\alpha}) = J_i(x^t) + \overline{\alpha} \overline{\Delta J}_i^{\,t}$.

TABLE I
DETERMINATION OF TRADEOFF STEP SIZE

α_2	$J_1(\overline{\alpha})$	$J_2(\overline{\alpha})$	$\cdots$	$J_k(\overline{\alpha})$
0.1	$J_1(x')+0.1\alpha_{\max}\overline{\Delta J}_1'$	$J_2(x')+0.1\alpha_{\max}\overline{\Delta J}_2'$	$\cdots$	$J_k(x')+0.1\alpha_{\max}\overline{\Delta J}_k'$
0.2	$J_1(x')+0.2\alpha_{\max}\overline{\Delta J}_1'$	$J_2(x')+0.2\alpha_{\max}\overline{\Delta J}_2'$	$\cdots$	$J_k(x')+0.2\alpha_{\max}\overline{\Delta J}_k'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.0	$J_1(x')+1.0\alpha_{\max}\overline{\Delta J}_1'$	$J_2(x')+1.0\alpha_{\max}\overline{\Delta J}_2'$	$\cdots$	$J_k(x')+1.0\alpha_{\max}\overline{\Delta J}_k'$

In Table I, some typical values of $\alpha_2(0.1, 0.2, \ldots, 1.0)$ are used. Note in Table I that $J_i(\overline{\alpha})$ is monotonically either increasing or decreasing. In general, the decision maker may choose a value of α_2 by checking the corresponding values of $J_i(\overline{\alpha})$ $(i = 1, 2, \ldots, k)$, as illustrated in the numerical study in Section IV-C. Heuristics could be used to determine α_2.

Suppose attainment levels are given for objectives. If any objective earmarked for sacrifice becomes just above its attainment level at a step size $\tilde{\alpha}_2$, then a value smaller than $\tilde{\alpha}_2$ should be selected for α_2. Another heuristic results from the observation that the value of J_i shown in Table I is generally different from the actual objective value at an expected noninferior solution due to the nonlinearity of the noninferior frontier. The larger the value of α_2, the bigger the difference may be. Consequently, along the direction $\overline{\Delta J}^t$ an objective J_i earmarked for improvement may decrease (improve) faster than along the actual noninferior frontier. In general, when $J_i(\overline{\alpha})$ becomes just better than J_i^* at $\tilde{\alpha}_2$, a value smaller than $\tilde{\alpha}_2$ should be selected for α_2. This is demonstrated in Section IV-C.

C. Interactive Tradeoff Analysis Process

The algorithm for interactive tradeoff analysis includes four steps similar to those given in Section II-D. Since the disutility function is unknown, however, the details of these steps are different.

Step 1: The decision maker provides an initial weight or expected value for each objective (e.g., equal weights). In the latter case, a canonical weight for the objective can be generated [11]. Choose a small number ε for the error tolerance in the stopping condition, and set interaction number $t = 1$.

Step 2: For the obtained weighting vector w^t, solve weighted minimax problem (6) or (7) and obtain solution $\{x^t, y^t, \lambda^t\}$. Calculate the normal vector at the solution using (19).

Step 3: Estimate the marginal rate of substitution by acquiring indifference tradeoffs from the decision maker. Use the optimal indifference tradeoffs to assist the preference elicitation. If the decision maker is satisfied with the optimal indifference tradeoffs, the most preferred solution is already achieved and the interactive process terminates. Otherwise, new indifference tradeoffs need to be provided by the decision maker, the marginal rates of substitution are approximated using (43), and (47) can be used to calculate the projection $\overline{\Delta J}^t$ of the obtained marginal rates of substitution, which will provide a new tradeoff direction.

Step 4: Use (50) to calculate the maximum step size $\alpha_{\max}$. A tradeoff table (see Table I) can be constructed once $\alpha_{\max}$ and $\overline{\Delta J}^t$ are generated. The decision maker may use the table to select the regulating factor α_2. Once $\overline{\alpha} = \alpha_{\max}\alpha_2$ is determined, calculate a new weighting vector using (48), set $t = t + 1$ and then go back to Step 2.

IV. NUMERICAL EXAMPLES

In this section, four examples will be examined to show both the prominent features of the theoretical results and the interactive tradeoff analysis process developed in the previous sections. Different ways of calculating normal vectors will be discussed. The first simple linear example is used to show the iterative solution algorithm both analytically and graphically. The second example is used to illustrate the step-by-step procedure of the proposed iterative solution algorithm with the primal and dual method used to solve a minimax problem. The third example is selected to show that the proposed approach is capable of dealing with nonconvex problems. The final example is a little more complicated and has a nonsmooth noninferior frontier. It is fully examined to demonstrate both the convergence of the iterative procedure for a given nonseparable utility function and the explicit interactive tradeoff analysis process with unknown disutility function.

A. An Illustrative Example

Example 1: Consider the simple linear optimization problem, shown in the equation at the bottom of the page, having two variables, two objectives, and four constraints only.

To demonstrate the basic principles of the new iterative algorithm graphically, an underlining utility function is assumed to be a quadratic function defined as follows:

$$J(J_1, J_2) = 1800 - (30 - f_1(x))^2 - (15 - f_2(x))^2.$$

$$\begin{aligned}
\max \quad & J(J_1, J_2) = \{J_1(x) = 5x_1 - 2x_2,\ J_2(x) = -x_1 + 4x_2\} \\
\text{s.t.} \quad & x \in X, \qquad x = [x_1\ x_2]^T \\
& X = \left\{ x \,\middle|\, \begin{array}{ll} g_1(x) = -x_1 + x_2 - 3 \le 0, & g_2(x) = x_1 + x_2 - 8 \le 0 \\ g_3(x) = x_1 - 6 \le 0, & g_4(x) = x_2 - 40,\ x_1, x_2 \ge 0 \end{array} \right\}
\end{aligned}$$

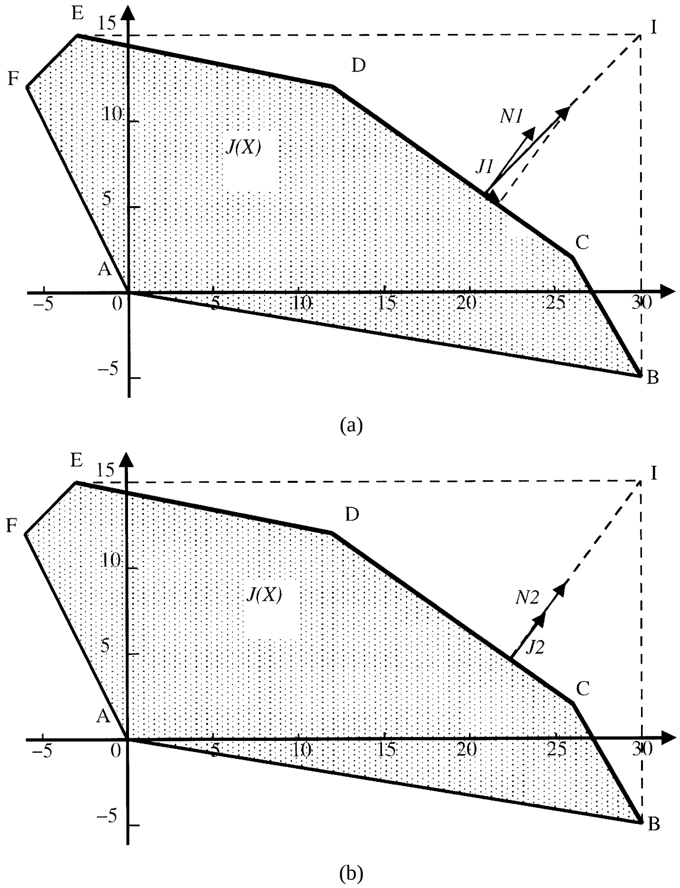


Fig. 2. (a) Feasible objective space and the initial solution. (b) Feasible objective space and the best solution.

The gradient of the utility function is given by

$$\nabla J = \left[\frac{\partial J}{\partial J_1}, \frac{\partial J}{\partial J_2}\right]^T = [2(30 - f_1),\ 2(15 - f_2)]^T.$$

In the objective space, the contour of the above utility function for a given utility value is a circle with point [30 15] being its center. The reason for assuming this simple utility function is that all its gradients point to the center, thus facilitating the graphical illustration of the algorithm.

The feasible objective space $J(X)$ of the problem is shown in Fig. 2(a) as enclosed by the lines $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$. The single objective optimal solutions, denoted by $\hat{x}^1$ and $\hat{x}^2$ and obtained by maximizing objectives $J_1(x)$ and $J_2(x)$ in X, respectively, are given as follows:

$$\hat{x}^1 = [\hat{x}_1^1\ \hat{x}_2^1]^T = [6\ 0]^T, \qquad \hat{J}^1 = \left[\hat{J}_1^1\ \hat{J}_2^1\right]^T = [30\ {-6}]^T$$

$$\hat{x}^2 = [\hat{x}_1^2\ \hat{x}_2^2]^T = [1\ 4]^T, \qquad \hat{J}^2 = \left[\hat{J}_1^2\ \hat{J}_2^2\right]^T = [-3\ 15]^T.$$

At points $\hat{x}^1$ and $\hat{x}^2$, we have $J(\hat{J}^1) = 1359$ and $J(\hat{J}^2) = 711$. It is clear from Fig. 2(a) that the efficient frontier of the problem, denoted by $\Omega(X^*)$, is composed of the line segments $\overline{ED}$, $\overline{DC}$, and $\overline{CB}$, or the equation shown at the bottom of the page.

The iterative procedure for solving the above problem using the new algorithm given in Section II-C is illustrated both analytically and graphically as follows. Suppose the two objectives are initially given equal weights, or $W^1 = [w_1^1\ w_2^1] = [1\ 1]$. Note that the best values for $J_1(x)$ and $J_2(x)$ in X are given as $J_1^* = \hat{J}_1^1 = 30$ and $J_2^* = \hat{J}_2^2 = 15$, respectively. It should also be noted in the following analysis that maximizing $J_1(x)$ [or $J_2(x)$] is equivalent to minimizing $(-J_1(x))$ [or $(-J_2(x))$]. Then, the following minimax problem can be formulated following (7) by assuming $\varphi(y) = y$:

$$\begin{aligned} &\max\ y \\ &\quad \text{s.t.}\quad 30 - (5x_1 - 2x_2) \le y \\ &\qquad\quad 15 - (-x_1 + 4x_2) \le y \\ &\qquad\quad x \in X, \qquad x = [x_1\ x_2]^T. \end{aligned}$$

Suppose the dual variables (simplex multipliers) of the above two objective constraints for $J_1(x)$ and $J_2(x)$ are defined by λ_1 and λ_2, respectively. Then, the optimal solution of the above problem is generated using the simplex method by

$$\begin{aligned} [x_1^1\ x_2^1\ y^1]^T &= [5.25\ 2.75\ 9.25]^T \\ J^1 &= [J_1^1\ J_2^1]^T = [20.75\ 5.75]^T \\ [\lambda_1^1\ \lambda_2^1]^T &= [0.4167\ 0.5835]^T \\ \nabla J^1 &= 18.5 \times [1\ 1]^T, \qquad J(J^1) = 1628.875. \end{aligned}$$

Note that $\lambda_1^1 = 0.4167$ and $\lambda_2^1 = 0.5835$ are the dual prices (values of dual variables) of the two objective constraints at the optimal solution, which can be generated using most simplex-based software packages. Using (19) and (21), we can generate the normal vector N^1 and the projection ΔJ^1 of ∇J^1 onto the tangent line at J^1 as follows:

$$\begin{aligned} N^1 &= [w_1^1\lambda_1^1\ w_2^1\lambda_2^1]^T = [0.4167\ 0.5835]^T \\ &= 0.4167 \times [1\ 1.4]^T \\ \Delta J^1 &= [\Delta J_1^1\ \Delta J_2^1]^T = \nabla J^1 - \frac{(\nabla J^1)^T N^1}{(N^1)^T N^1} N^1 \\ &= [3.5\ {-2.5}]^T. \end{aligned}$$

As illustrated in Fig. 2(a), N^1 is orthogonal to $\overline{DC}$ and ΔJ^1 is on $\overline{DC}$. To find the step size, we construct the following 1-D search problem:

$$\max_{\alpha} J(\alpha) = 1800 - (30 - (20.75 + 3.5\alpha))^2 - (15 - (5.75 - 2.5\alpha))^2.$$

The optimal solution of the above problem is $\alpha^1 = 0.5$.

$$\Omega(X^*) = \left\{ (J_1\ J_2) \,\middle|\, \begin{array}{ll} (\overline{ED})\colon J_1 + 5J_2 = 72; & (\overline{DC})\colon J_1 + 1.4J_2 = 28.8 \\ (\overline{CB})\colon J_1 + 0.5J_2 = 27, & (J_1\ J_2) \in J(X) \end{array} \right\}$$

Using (29b), the weight for $J_2(x)$ can be updated as follows:

$$\begin{aligned} w_2^2 &= \frac{J_1^* - (J_1^1 + \alpha^1 \Delta J_1^1)}{J_2^* - (J_2^1 + \alpha^1 \Delta J_2^1)} \\ &= \frac{30 - (20.75 + 0.5 \times 3.5)}{15 - (5.75 + 0.5 \times (-2.5))} = 0.7143. \end{aligned}$$

We can then formulate a new minimax problem as follows:

$$\begin{aligned} &\max\ y \\ &\ \text{s.t. } 30 - (5x_1 - 2x_2) \le y \\ &\quad 0.7143(15 - (-x_1 + 4x_2)) \le y \\ &\quad x \in X, \qquad x = [x_1\ x_2]^T. \end{aligned}$$

The optimal solution of the above problem is given by

$$\begin{aligned} [x_1^2\ x_2^2\ y^2]^T &= [5.5\ 2.5\ 7.5]^T \\ J^2 &= [J_1^2\ J_2^2]^T = [22.5\ 4.5]^T \\ [\lambda_1^2\ \lambda_2^2]^T &= [0.3378\ 0.6622]^T \\ \nabla J^2 &= 15 \times [1\ 1.4]^T, \qquad J(J^2) = 1633.5. \end{aligned}$$

Using (19) and (21), we get the normal vector N^2 and the projection ΔJ^2 of ∇J^2 onto the tangent line at J^2 as follows:

$$\begin{aligned} N^2 &= [w_1^2 \lambda_1^2\ w_2^2 \lambda_2^2]^T = [0.3378\ 0.4730]^T \\ &= [1\ 1.4]^T \times 0.3378 \\ \Delta J^2 &= [\Delta J_1^2\ \Delta J_2^2]^T = \nabla J^2 - \frac{(\nabla J^2)^T N^2}{(N^2)^T N^2} N^2 = [0\ 0]^T. \end{aligned}$$

Since $\Delta J^2 = [0\ \ 0]^T$, we can confirm that J^2 is the best solution. The above results are illustrated in Fig. 2(b), where N^2 is proportional to ∇J^2. Obviously, J^2 is the optimal point in the objective space as it is the point in $J(X)$ which is the closest to the point I.

B. Iterative Optimization for Nonseparable Problems

Example 2: Consider the following two-objective nonlinear optimization problem with a known disutility function:

$$\min \left\{ J_1 = 8 + x_1 + x_2 + x_3,\ J_2 = \sum_{i=1}^{3} (x_i + c_i)^2 \right\} \tag{51a}$$

$$\text{s.t. } \sum_{i=1}^{3} [e^{a_i x_i} + b_i x_i^2] \le 10 \tag{51b}$$

$$x_1, x_2, x_3 \le 0. \tag{51c}$$

The disutility function is given by

$$J = 150 e^{J_1 - 8} + J_2. \tag{52}$$

In (51a) and (51b), the parameters are given by

$$\begin{aligned} &a_1 = 2, \quad a_2 = 1, \quad a_3 = 3 \\ &b_1 = 1, \quad b_2 = 3, \quad b_3 = 2 \\ &c_1 = 1, \quad c_2 = 2, \quad \text{and} \quad c_3 = 3. \end{aligned} \tag{53}$$

The above problem is highly nonlinear and nonseparable. It can be verified that both J_1 and J_2 are strictly positive under constraints (51b) and (51c) and both $\partial J / \partial J_1 (=150 e^{J_1 - 8})$ and $\partial J / \partial J_2 (=1)$ are strictly positive. From (6) a weighted minimax problem is formulated by

$$\min \max \left\{ 8 + x_1 + x_2 + x_3,\ w_2 \sum_{i=1}^{3} (x_i + c_i)^2 \right\} \tag{54a}$$

$$\text{s.t. (51b) and (51c).} \tag{54b}$$

Choosing $\varphi(y)$ to be y^2, the following equivalent problem of (54) can be formed:

$$\min\ y^2 \tag{55a}$$

$$\text{s.t. } 8 + x_1 + x_2 + x_3 \le y \tag{55b}$$

$$w_2 \sum_{i=1}^{3} (x_i + c_i)^2 \le y$$

$$\text{(51b) and (51c).} \tag{55c}$$

The auxiliary weighted minimax formulation in (55) is convex and separable with respect to y, x_1, x_2, and x_3. It can be solved using the primal–dual method. The dual function of (55) is

$$\begin{aligned} H(\lambda_1, \lambda_2, \mu) = \min \Bigg\{ & y^2 + \lambda_1 [8 + x_1 + x_2 + x_3 - y] \\ & + \lambda_2 \left[w_2 \sum_{i=1}^{3} (x_i + c_i)^2 - y \right] \\ & + \mu \left[\sum_{i=1}^{3} [e^{a_i x_i} + b_i x_i^2] - 10 \right] \Bigg\}. \end{aligned} \tag{56}$$

For given values of λ_1, λ_2, and μ, the above dual function can be solved through decomposition at the lower level.

Subproblem 0: Solving

$$\min\ y^2 - \lambda_1 y - \lambda_2 y \tag{57a}$$

$$\text{s.t. } y \ge 0 \tag{57b}$$

yields the following optimal solution

$$y^* = (\lambda_1 + \lambda_2)/2. \tag{58}$$

Subproblem i ($i = 1, 2, 3$): Solving

$$\min\ \lambda_1 x_i + \lambda_2 w_2 (x_i + c_i)^2 + \mu [e^{a_i x_i} + b_i x_i^2] \tag{59a}$$

$$\text{s.t. } x_i \le 0 \tag{59b}$$

yields optimal solution x_i^* that satisfies

$$\mu a_i e^{a_i x_i^*} = -(\lambda_1 + 2\lambda_2 w_2 c_i) - 2(\lambda_2 w_2 + \mu b_i) x_i^*. \tag{60}$$

Denote the iteration number in the primal–dual method by s. The values of λ_1, λ_2, and μ are adjusted at the second level by maximizing the dual function

$$\lambda_1^{s+1} = \max\{0,\ \lambda_1^s + \alpha_1 [8 + x_1 + x_2 + x_3 - y]\} \tag{61}$$

$$\lambda_2^{s+1} = \max \left\{ 0,\ \lambda_2^s + \alpha_1 \left[w_2 \sum_{i=1}^{3} (x_i + c_i)^2 - y \right] \right\} \tag{62}$$

$$\mu^{s+1} = \max \left\{ 0,\ \mu^s + \alpha_1 \left[\sum_{i=1}^{3} (e^{a_i x_i} + b_i x_i^2) - 10 \right] \right\} \tag{63}$$

where α_1 is a step size parameter. The primal–dual solution process in solving (55) continues until the optimal conditions $\lambda_1(\partial H/\partial)\lambda_1 = 0$, $\lambda_2(\partial H/\partial\lambda_2) = 0$ and $\mu(\partial H/\partial\mu) = 0$ are met.

Each time after the solution of (55) is obtained for a given weighting coefficient w_2, a new value of w_2 is calculated at the third level using (29) as follows:

$$w_2^{t+1} = \frac{J_1(x^t) + \alpha\Delta J_1^t}{J_2(x^t) + \alpha\Delta J_2^t} \tag{64}$$

where α is a step size parameter and ΔJ_1^t and ΔJ_2^t are calculated using (21)

$$[\Delta J_1^t,\, \Delta J_1^t] = \left\{-\left[\frac{\partial J}{\partial J_1},\, \frac{\partial J}{\partial J_2}\right]^T + \frac{\lambda_1^t \frac{\partial J}{\partial J_1} + w_2^t\lambda_2^t\frac{\partial J}{\partial J_2}}{(\lambda_1^t)^2 + (w_2^t\lambda_2^t)^2}[\lambda_1^t,\, w_2^t\lambda_2^t]^T\right\}\Bigg|_{x^t}. \tag{65}$$

The initial value of w_2 is set to 1 and the step size parameters α_1 and α are selected to be equal to 0.1 and 1.0, respectively. The stopping criterion is that the absolute value of the difference between $(\partial J/\partial J_1)/\lambda_1$ and $(\partial J/\partial J_2)/w_2\lambda_2$ is less than 0.0005. The iteration process converges very fast and ends at the tenth iteration with optimal solution $w_2 = 1.278\,322$, $x_1 = -1.340\,131$, $x_2 = -0.967\,588\,9$, $x_3 = -1.571\,213$, and $J = 6.323\,923$.

The above example illustrates the implementation procedure of the iteration algorithm with a known disutility function. The noninferior frontier of the above example is smooth. Since the primal–dual method can be used, the dual variables λ_1 and λ_2 are obtained in the method. The projection is directly calculated using (65). In the following examples, the noninferior frontier is either nonconvex or nonsmooth and a nondual method is used to solve a minimax problem. The set of linear equations for the K–T conditions needs to be constructed to generate the optimal values of the dual variables λ_1 and λ_2 for calculating a normal vector.

C. Iterative Optimization for a Nonconvex Problem

Example 3: Consider the following engineering design problem. Both the unreliability $J_1(x)$ and the production cost $J_2(x)$ of a two-component series system are to be minimized

$$J_1(x) = x_1 + x_2 - x_1x_2 \tag{66a}$$
$$J_2(x) = 1.5 - 0.5x_1 - 0.45x_2 \tag{66b}$$

where x_1 and x_2 represent the unreliabilities of components 1 and 2, respectively, and they satisfy the following constraint:

$$0 \le x_1, \quad x_2 \le 1. \tag{67}$$

A cost-unreliability minimization problem is formulated as follows:

$$\begin{aligned} &\min\ \phi[J_1(x),\, J_2(x)] \\ &\text{s.t.}\ (67). \end{aligned} \tag{68}$$

The following underlying disutility function is assumed in this example to act as a pseudodecision-maker

$$\phi[J_1(x),\, J_2(x)] = e^{2J_1} + 2(J_2)^2 \tag{69}$$

to provide the marginal rates of substitutions.

Choosing $\varphi(y)$ to be y^2, the following weighted minimax problem can be formed for (68)

$$\min\ y^2 \tag{70a}$$
$$\text{s.t.}\ x_1 + x_2 - x_1x_2 \le y \tag{70b}$$
$$w_2(1.5 - 0.5x_1 - 0.45x_2) \le y \tag{70c}$$
$$0 \le x_1, \quad x_2 \le 1. \tag{70d}$$

Note that the noninferior frontier in this example is *nonconvex*. The initial value of w_2 is set to 1. For a given weighting coefficient w_2, optimal x_1, and x_2 are found for (70). At the same time, the K–T multipliers λ_1 and λ_2 associated with (70b) and (70c) are also obtained by solving the linear equations of the K–T conditions. The normal vector N^t at $\{J_1(x^t),\, J_2(x^t)\}$ is then identified using (19).

Guided by the underlying disutility function in (69), the pseudodecision-maker provides the marginal rate of substitution M^t. The proportionality of N^t and M^t is checked at each iteration to see if the optimum stopping condition in (23) is satisfied. Specifically, the stopping criterion is that the absolute value of the difference between $1/\lambda_1$ and $M_{12}/w_2\lambda_2$ is less than 0.01. If the stopping criterion is not satisfied, a new value of w_2 is calculated using (29) as follows:

$$w_2^{t+1} = \frac{J_1(x^t) + \alpha\Delta J_1^t}{J_2(x^t) + \alpha\Delta J_2^t} \tag{71}$$

where α is set equal to 1 and ΔJ_1^t and ΔJ_2^t are calculated using (21) as follows:

$$[\Delta J_1^t,\, \Delta J_2^t] = \left\{-[1,\, M_{12}]^T + \frac{\lambda_1^t + w_2^t\lambda_2^t M_{12}}{(\lambda_1^t)^2 + (w_2^t\lambda_2^t)^2}[\lambda_1^t,\, w_2^t\lambda_2^t]^T\right\}\Bigg|_{x^t}. \tag{72}$$

The iteration process converges very fast and ends at the eighth iteration to a most preferred solution with $w_2 = 0.140\,066$, $x_1 = 0.149\,773\,8$, $x_2 = 0.055\,307\,3$, $\lambda_1 = 0.027$, $\lambda_2 = 0.365$. The marginal rate of substitution M^8 at the most preferred solution is equal to $[1,\, 1.889]^T$. The error measure $((1/\lambda_1) - (M_{12}/w_2\lambda_2))$ is 0.0025. At the most preferred solution, the corresponding system unreliability is 0.196 797 5 and the associated cost is 1.400 224 8.

D. Iterative Optimization for Problems With Nonsmooth Noninferior Frontiers

Example 4: A decision problem chosen for this study is a modified Bow River Valley water quality management problem, which is a nonlinear three-objective optimization problem [7], [24]–[26]. The first objective $J_1(x)$ represents DO level at Robin State Park, $J_2(x)$ the percentage return on equity at Pierce-Cannery, and $J_3(x)$ the addition to the tax rate at Bowville. There are three decision variables, representing the treatment levels of waste discharges at the Pierce-Cannery, Bowville, and Plympto, denoted by x_1, x_2, and x_3, respectively. The mathematical formulation of the problem is given as follows:

$$\max J_1(x) = 2.0 + 0.524(x_1 - 0.3) + 2.79(x_2 - 0.3) + 0.882(w_1 - 0.3) + 2.65(w_2 - 0.3)$$

$$\max J_2(x) = 7.5 - 0.012\left(\frac{59}{1.0 - x_1^2} - 59\right)$$

$$\min J_3(x) = 1.8 \times 10^{-3}\left(\frac{532}{1.09 - x_2^2} - 532\right)$$

$$\text{s.t. } x \in \Omega, \qquad x = [x_1, x_2, x_3]^T$$

$$\Omega = \left\{ x \;\middle|\; \begin{array}{l} g_1(x)\text{: } 4.75 + 2.27(x_1 - 0.3) \geq 6.0, \\ g_2(x)\text{: } 5.1 + 0.177(x_1 - 0.3) \\ \quad +0.978(x_2 - 0.3) + 0.216(w_1 - 0.3) \\ \quad +0.768(w_2 - 0.3) \geq 6.0, \\ g_3(x)\text{: } 2.50 \times 10^{-3}\left(\dfrac{450}{1.09 - x_3^2} - 450\right) \leq 1.5, \\ g_4(x)\text{: } 1.0 + 0.0332(x_1 - 0.3) \\ \quad +0.0186(x_2 - 0.3) + 3.34(x_3 - 0.3) \\ \quad +0.0204(w_1 - 0.3) + 0.78(w_2 - 0.3) \\ \quad +2.62(w_3 - 0.3) \geq 3.5, \\ g_5(x)\text{: } x_1 \geq 0.3, \quad g_6(x)\text{: } x_2 \geq 0.3, \\ g_7(x)\text{: } x_3 \geq 0.3, \quad g_8(x)\text{: } x_1 \leq 1.0, \\ g_9(x)\text{: } x_2 \leq 1.0, \quad g_{10}(x)\text{: } x_3 \leq 1.0 \\ w_i = \dfrac{0.39}{1.39 - x_i^2}, \quad i = 1, 2, 3 \end{array} \right\}.$$

By optimizing each of the three objective functions within the decision space Ω, the three single-objective optimal solutions can be generated, denoted by $\hat{x}^1$, $\hat{x}^2$, $\hat{x}^3$ that maximize $J_1(x)$ and $J_2(x)$ and minimize $J_3(x)$, respectively. The payoff table is listed as in Table II. Note that the noninferior frontier of the problem is *nonsmooth* at certain points because of the linear and nonlinear constraints.

The weighted minimax formulation for the above problem can then be given by

$$\min \; y \tag{73a}$$
$$\text{s.t. } w_1(6.79 - J_1(x)) \leq y \tag{73b}$$
$$w_2(6.28 - J_2(x)) \leq y \tag{73c}$$
$$w_3(J_3(x) - 1.04) \leq y \tag{73d}$$
$$x \in \Omega, \qquad x = [x_1, x_2, x_3]^T.$$

In this section, an explicit utility function is constructed by assuming that the decision maker would be satisfied with a noninferior solution at which any two of the three objectives are as close to their targeted levels as possible. Suppose the targeted level is 6.79 for DO level at Robin State Park $J_1(x)$, 6.0 for the percentage return on equity $J_2(x)$, and 1.04 for the addition to the tax rate at Bowville $J_3(x)$. The utility function representing the above preferences may then be defined as follows:

$$J = 100.0 - [(6.79 - J_1(x))^2(6.0 - J_2(x))^2 + (6.79 - J_1(x))^2(J_3(x) - 1.04)^2 + (6.0 - J_2(x))^2(J_3(x) - 1.04)^2]. \tag{74}$$

TABLE II
PAYOFF TABLE FOR THE WATER QUALITY MANAGEMENT PROBLEM

	$J_1(\hat{x}^i)$	$J_2(\hat{x}^i)$	$J_3(\hat{x}^i)$
$\hat{x}^1$	6.79	0.34	9.68
$\hat{x}^2$	6.35	6.28	9.68
$\hat{x}^3$	4.86	0.34	1.04

A large decision support system was used to solve the weighted minimax problem [27], where a nonlinear scalar optimization problem is solved using a modified sequential linear programming algorithm. An initial solution x^0 was obtained by assigning equal weights to the three objectives. From x^0, we search for the most preferred solution that could best satisfy the decision maker's preferences. For identification of a normal vector, a set of K–T conditions [9], [27] is constructed and solved at each noninferior solution generated. The golden section search is used to find $\alpha_{\max}$. Parameter α_2 is initially assigned to one at each iteration and is on-line adjusted whenever necessary to guarantee the improvement of J. The iterative calculation procedure is shown in Table III.

The above procedure quickly converges to the maximum utility as shown by column 4 of Table III, though the regulation of α_2 is needed at iterations 3 and 4. The termination condition was $dJ^t < [0.02, 0.02, 0.02]^T$. Solutions x^1, x^2, x^3, x^4, and x^5 in Table V are all noninferior solutions. The utility at $x^5 = [0.8945, 0.8308, 0.8133]^T$ is approximately the maximum and x^5 can thus be suggested as a most preferred solution.

E. Explicit Interactive Tradeoff Analysis With Unknown Utility Function

In this subsection, we demonstrate how explicit interactive tradeoff analysis can be conducted without assuming a utility function, as given in (74). The optimal indifference tradeoffs will be used to guide the elicitation of local preferences. To start the interactive process, an initial noninferior solution x^0 is generated by assigning equal weights to the three objectives as in the last subsection. At x^0, the values of the three variables, the three objectives, and the normal vector are given by

$$x^0 = [0.9617, 0.9558, 0.7674]^T$$
$$J(x^0) = [6.0253, 3.9215, 4.4687]^T$$
$$N^0 = [0.6235, 0.0447, 0.1217]^T.$$

First Interaction: Suppose J_1 is chosen as the reference objective. At x^0, the optimal indifference tradeoff vector for one unit change of J_1 is given by

$$dJ^0 = [1, 13.98, 5.13]^T \tag{75}$$

which means that if x^0 was the most preferred solution, then one unit change of DO level at Robin State Park (J_1) should be exactly offset solely by 13.98 units of the percentage return on equity at Pierce-Cannery (J_2), and also solely by 5.13 units of

TABLE III
ITERATIVE OPTIMIZATION PROCEDURE

t	x^t	J^t	$J(x^t)$	$\nabla J(x^t)$	N^t	dJ^t	$\alpha_{\max}$	α_2	W^{t+1}
0	$\begin{bmatrix}0.9617\\0.9558\\0.7674\end{bmatrix}$	$\begin{bmatrix}6.0253\\3.9215\\4.4687\end{bmatrix}$	39.81	$\begin{bmatrix}0.6176\\1.2886\\0.8448\end{bmatrix}$	$\begin{bmatrix}0.6235\\0.0446\\0.1216\end{bmatrix}$	$\begin{bmatrix}-0.2208\\1.2286\\0.6814\end{bmatrix}$	1.77	1.0	$\begin{bmatrix}1\\0.9234\\0.9923\end{bmatrix}$
1	$\begin{bmatrix}0.8706\\0.9337\\0.7775\end{bmatrix}$	$\begin{bmatrix}5.5458\\6.0757\\3.4301\end{bmatrix}$	91.12	$\begin{bmatrix}0.1562\\-0.0121\\0.0815\end{bmatrix}$	$\begin{bmatrix}0.5218\\0.0944\\0.1384\end{bmatrix}$	$\begin{bmatrix}-0.0030\\-0.0409\\0.0393\end{bmatrix}$	20.01	1.0	$\begin{bmatrix}1\\1.0435\\1.3010\end{bmatrix}$
2	$\begin{bmatrix}0.9234\\0.9097\\0.7847\end{bmatrix}$	$\begin{bmatrix}5.4461\\5.2252\\2.6921\end{bmatrix}$	92.35	$\begin{bmatrix}0.0969\\0.0761\\0.0861\end{bmatrix}$	$\begin{bmatrix}0.4905\\0.0575\\0.1691\end{bmatrix}$	$\begin{bmatrix}-0.0227\\0.0621\\0.0448\end{bmatrix}$	18.46	1.0	$\begin{bmatrix}1\\0.7890\\1.4285\end{bmatrix}$
3	$\begin{bmatrix}0.8611\\0.8681\\0.7959\end{bmatrix}$	$\begin{bmatrix}4.9737\\6.1769\\1.8889\end{bmatrix}$	97.50	$\begin{bmatrix}0.0280\\-0.0146\\0.0580\end{bmatrix}$	$\begin{bmatrix}0.4436\\0.0854\\0.2179\end{bmatrix}$	$\begin{bmatrix}-0.0140\\-0.0227\\0.0374\end{bmatrix}$	18.44	1.0	$\begin{bmatrix}1\\0.8189\\1.8288\end{bmatrix}$
	$\begin{bmatrix}0.9216\\0.8218\\0.8037\end{bmatrix}$	$\begin{bmatrix}4.8045\\5.2649\\1.3519\end{bmatrix}$	97.43					0.5	$\begin{bmatrix}1\\0.8118\\2.1688\end{bmatrix}$
4	$\begin{bmatrix}0.8806\\0.8414\\0.8009\end{bmatrix}$	$\begin{bmatrix}4.8301\\5.9568\\1.5486\end{bmatrix}$	99.00	$\begin{bmatrix}0.0103\\0.0036\\0.0395\end{bmatrix}$	$\begin{bmatrix}0.4317\\0.0728\\0.2553\end{bmatrix}$	$\begin{bmatrix}-0.0146\\-0.0006\\0.0248\end{bmatrix}$	20.43	1.0	$\begin{bmatrix}1\\0.7617\\2.2030\end{bmatrix}$
	$\begin{bmatrix}0.9947\\0.7853\\0.8133\end{bmatrix}$	$\begin{bmatrix}4.8536\\1.1618\\1.0654\end{bmatrix}$	12.21					0.5	$\begin{bmatrix}1\\0.7850\\2.5966\end{bmatrix}$
5	$\begin{bmatrix}0.8945\\0.8308\\0.8133\end{bmatrix}$	$\begin{bmatrix}4.7930\\5.7662\\1.4381\end{bmatrix}$	99.14	$\begin{bmatrix}0.0086\\0.0196\\0.0333\end{bmatrix}$	$\begin{bmatrix}0.4296\\0.0653\\0.2715\end{bmatrix}$	$\begin{bmatrix}-0.0140\\0.0161\\0.0182\end{bmatrix}$	18.52	1.0	$\begin{bmatrix}1\\0.7476\\2.5540\end{bmatrix}$

the addition to the tax rate at Bowville (J_3). The optimal indifference tradeoffs at x^0 can be equivalently written as follows:

$$[6.0253,\ 3.9215,\ 4.4687]^T \Leftrightarrow [6.0253 - 1,\ 3.9215 + 13.98,\ 4.4687]^T \quad (76a)$$

$$[6.0253,\ 3.9215,\ 4.4687]^T \Leftrightarrow [6.0253 - 1,\ 3.9215,\ 4.4687 - 5.13]^T \quad (76b)$$

where "⇔" reads "is indifferent to." Note that J_1 and J_2 are for maximization and J_3 is for minimization.

Suppose at the solution x^0, the decision maker does not agree with the above optimal indifference tradeoffs and instead provides the following ones:

$$[6.0253,\ 3.9215,\ 4.4687]^T \Leftrightarrow [6.0253 - 1,\ 3.9215 + 0.48,\ 4.4687]^T \quad (77a)$$

$$[6.0253,\ 3.9215,\ 4.4687]^T \Leftrightarrow [6.0253 - 1,\ 3.9215,\ 4.4687 - 0.73]^T. \quad (77b)$$

Note that by employing the above indifference tradeoffs we have actually assumed that the decision maker is implicitly referring to the utility function given by (74). This is purely for illustrative purpose. In reality, the decision maker does not necessarily need to rely on a utility function for conducting this interactive analysis, though he must be conscious about the meanings of the tradeoffs provided.

Given the indifference tradeoffs of (77), the marginal rates of substitution are approximated as follows:

$$M^0 = [1,\ 1/0.48,\ 1/0.73]^T = [1,\ 2.08,\ 1.37]^T.$$

The project of M^0 onto the tangent plane at $J(x^0)$ is then calculated using (47)

$$\overline{\Delta J}^0 = [-0.3575,\ 1.9895,\ -1.1033]^T.$$

The maximum step size $\alpha^0_{\max}$ is calculated using (50)

$$\alpha^0_{\max} = \frac{J_1^0 - J_1^-}{\left|\overline{\Delta J}^0_1\right|} = \frac{6.0253 - 4.86}{0.3575} = 3.26.$$

Note that at this interaction only the first objective needs to be sacrificed as shown in $\overline{\Delta J}^0$. The regulating factor α_2 is selected by constructing Table IV.

It is clear in Table IV that objectives 2 and 3 are improved along $\overline{\Delta J}^0$ much faster than along the noninferior frontier. When $\alpha_2 = 0.4$, the value of J_2 is already 6.5156, larger than the maximum feasible value of J_2 (6.28). This means that it is advisable to set α_2 to a value smaller than 0.4, e.g., 0.3.

TABLE IV
DETERMINATION OF TRADEOFF STEP SIZE

α_2	$J_1(\bar{\alpha})$	$J_2(\bar{\alpha})$	$J_3(\bar{\alpha})$
0	6.0253	3.9215	4.4687
0.1	5.9088	4.5700	4.1091
0.2	5.7922	5.2185	3.7494
0.3	5.6757	5.8670	3.3898
0.4	*5.5592*	***6.5156***	*3.0301*
0.5	*5.4426*	*7.1641*	*2.6704*
0.6	*5.3261*	*7.8126*	*2.3108*
0.7	*5.2096*	*8.4611*	*1.9511*
0.8	*5.0931*	*9.1096*	*1.5915*
0.9	*4.9765*	*9.7582*	*1.2318*
1.0	*4.8600*	*10.4067*	*0.8722*

The weighting vector w_i^1 can then be calculated using (48b) as follows:

$$W^1 = [1,\ 2.6984,\ 0.4742].$$

Solving (66) with w_i replaced by w_i^1 yields the following solution x^1:

$$\begin{aligned} x^1 &= [0.8901,\ 0.9362,\ 0.7764]^T \\ J(x^1) &= [5.6124,\ 5.8302,\ 3.5261]^T \\ N^1 &= [0.5297,\ 0.0833,\ 0.1362]^T. \end{aligned}$$

By checking the values of the three objectives, the obtained objective values are slightly different from what were predicted in Table IV due to the nonlinearity of the noninferior frontier.

Second Interaction: At x^1, the optimal indifference tradeoff vector for one unit change of J_1 is given by

$$dJ^1 = [1,\ 6.36,\ 3.89]^T. \tag{78}$$

Suppose the decision maker does not agree with the above optimal indifference tradeoffs and instead provides the following ones:

$$\begin{aligned} &[5.6124,\ 5.8302,\ 3.5261]^T \\ &\quad \Leftrightarrow [5.6124 - 1,\ 5.8302 + 5.69,\ 3.5261]^T \end{aligned} \tag{79a}$$

$$\begin{aligned} &[5.6124,\ 5.8302,\ 3.5261]^T \\ &\quad \Leftrightarrow [5.6124 - 1,\ 5.8302,\ 3.5261 - 2.08]^T. \end{aligned} \tag{79b}$$

Then, the marginal rates of substitution M^1, the projection $\overline{\Delta J}^1$, and maximum step size are obtained as follows:

$$\begin{aligned} M^1 &= [1,\ 1/5.69,\ 1/2.08]^T = [1,\ 0.1757,\ 0.4808]^T \\ \overline{\Delta J}^1 &= [-0.0555,\ 0.01,\ -0.2096]^T \\ \alpha_{\max}^1 &= \frac{J_1^1 - J_1^-}{\left|\overline{\Delta J}_1^1\right|} = \frac{5.6124 - 4.86}{0.0555} = 13.55. \end{aligned}$$

The regulating factor α_2 is selected by constructing Table V.

It can be seen in Table V that objective 3 decreases along $\overline{\Delta J}^1$ slightly faster than along the noninferior frontier. When $\alpha_2 = 0.9$, the value of J_3 is already reduced to 0.9657, smaller than the minimum feasible value (1.04) of J_3. This means that it is advisable to set α_2 to a value smaller than 0.9, e.g., 0.8. The weighting vector W^2 can then be calculated using (48b) as follows:

$$W^2 = [1,\ 5.1617,\ 8.4631].$$

TABLE V
DETERMINATION OF TRADEOFF STEP SIZE

α_2	$J_1(\bar{\alpha})$	$J_2(\bar{\alpha})$	$J_3(\bar{\alpha})$
0	5.6124	5.8302	3.5261
0.1	5.5371	5.8433	3.2416
0.2	5.4619	5.8564	2.9571
0.3	5.3866	5.8696	2.6727
0.4	5.3114	5.8827	2.3882
0.5	5.2362	5.8959	2.1037
0.6	5.1609	5.9090	1.8192
0.7	5.0857	5.9221	1.5347
0.8	5.0105	5.9352	1.2503
0.9	*4.9352*	*5.9484*	***0.9657***
1.0	*4.8600*	*5.9615*	*0.6813*

Solving (66) with w_i replaced by w_i^2 yields the following solution x^2:

$$\begin{aligned} x^2 &= [0.8881,\ 0.8987,\ 0.7882]^T \\ J(x^2) &= [5.266,\ 5.8576,\ 2.4345]^T \\ N^2 &= [0.4706,\ 0.0751,\ 0.18]^T. \end{aligned}$$

Third Interaction: At x^2, the optimal indifference tradeoff vector for one unit change of J_1 is given by

$$dJ^2 = [1,\ 6.26,\ 2.61]^T \tag{80}$$

which states that, at the attained objective levels given by $J(x^2)$, one unit change of DO level at Robin State Park (J_1) should be exactly offset solely by 6.26 units of the percentage return on equity at Pierce-Cannery (J_2) and also solely by 2.61 units of the addition to the tax rate at Bowville (J_3) if the solution x^2 is regarded as a most preferred solution. If the decision maker agrees with the above optimal indifference tradeoffs, then the interaction terminates and x^2 will be confirmed as the most preferred solution. Otherwise, the decision maker should provide new indifference tradeoffs and the interaction continues in a way similar to the previous two interactions.

V. CONCLUDING REMARKS

A new explicit interactive tradeoff analysis method has been developed in this paper, which is based on the identification of normal vectors on a noninferior frontier. In this method, only local preference information such as marginal rates of substitution is required for conducting the proposed interactive analysis. The manner of information exchange between the analyst and the decision maker is straightforward, and the designed solution scheme is rigorous and efficient. This method is applicable to both linear and nonlinear multiobjective optimization problems where the noninferior frontiers could be nonconvex or nonsmooth at finite points and the underlying disutility functions could be nonlinear, nonseparable, or unknown.

Compared with other interactive methods, the identification of normal vectors equips this new method with the following unique features. First, the established optimality conditions can be used not only for checking in a rigorous yet flexible way whether a most preferred solution is achieved, but also for supporting the elicitation of the local preferences through the definition of the optimal indifference tradeoffs. Second, the tradeoff analysis among multiple objectives is designed to take place on the tangent planes of the noninferior frontier. The decision maker is thus provided with a clear picture, closest to the noninferior frontier, about what would be the consequences of a designated tradeoff. As a result, the decision maker would be in a better position to control the decision analysis process (tradeoff step sizes) and may therefore have a better chance to arrive at his most preferred solution. The four numerical examples have demonstrated the main features as well as the potential of the method to solve general multiobjective optimization problems.

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