2

Introduction to Multiobjective Optimization: Interactive Approaches

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Abstract. We give an overview of interactive methods developed for solving nonlinear multiobjective optimization problems. In interactive methods, a decision maker plays an important part and the idea is to support her/him in the search for the most preferred solution. In interactive methods, steps of an iterative solution algorithm are repeated and the decision maker progressively provides preference information so that the most preferred solution can be found. We identify three types of specifying preference information in interactive methods and give some examples of methods representing each type. The types are methods based on trade-off information, reference points and classification of objective functions.

2.1 Introduction

Solving multiobjective optimization problems typically means helping a human decision maker (DM) in finding the most preferred solution as the final one. By the *most preferred solution* we refer to a Pareto optimal solution which the DM is convinced to be her/his best option. Naturally, finding the most preferred solution necessitates the participation of the DM who is supposed to have insight into the problem and be able to specify preference information related to the objectives considered and different solution alternatives, as discussed in Chapter 1. There we presented four classes for multiobjective optimization methods according to the role of the DM in the solution process.

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This chapter is a direct continuation to Chapter 1 and here we concentrate on the fourth class, that is, interactive methods.

As introduced in Chapter 1, in interactive methods, an iterative solution algorithm (which can be called a solution pattern) is formed, its steps are repeated and the DM specifies preference information progressively during the solution process. In other words, the phases of preference elicitation (decision phase) and solution generation (optimization stage) alternate until the DM has found the most preferred solution (or some stopping criterion is satisfied, or there is no satisfactory solution for the current problem setting). After every iteration, some information is given to the DM and (s)he is asked to answer some questions concerning a critical evaluation of the proposed solutions or to provide some other type of information to express her/his preferences. This information is used to construct a more or less explicit model of the DM's local preferences and new solutions (which are supposed to better fit the DM's preferences) are generated based on this model. In this way, the DM directs the solution process and only a part of the Pareto optimal solutions has to be generated and evaluated. Furthermore, the DM can specify and correct her/his preferences and selections during the solution process.

In brief, the main steps of a general interactive method are the following: (1) initialize (e.g., calculate ideal and nadir values and showing them to the DM), (2) generate a Pareto optimal starting point (some neutral compromise solution or solution given by the DM), (3) ask for preference information from the DM (e.g., aspiration levels or number of new solutions to be generated), (4) generate new Pareto optimal solution(s) according to the preferences and show it/them and possibly some other information about the problem to the DM. If several solutions were generated, ask the DM to select the best solution so far, and (6) stop, if the DM wants to. Otherwise, go to step (3).

Because of the structure of an iterative approach, the DM does not need to have any global preference structure and (s)he can learn (see Chapter 15) during the solution process. This is a very important benefit of interactive methods because getting to know the problem, its possibilities and limitations is often very valuable for the DM. To summarize, we can say that interactive methods overcome weaknesses of a priori and a posteriori methods because the DM does not need a global preference structure and only such Pareto optimal solutions are generated that are interesting to the DM. The latter means savings in computational cost and, in addition, avoids the need to compare many Pareto optimal solutions simultaneously.

If the final aim is to choose and implement a solution, then the goal of applying a multiobjective optimization method is to find a single, most preferred, final solution. However, in some occasions it may be preferable that instead of one, we find several solutions. This may be particularly true in case of robustness considerations when some aspects of uncertainty, imprecision or inconsistency in the data or in the model are to be taken into account (but typically, eventually, one of them will still have to be chosen). In what follows, as the goal of using an interactive solution process we consider finding a single most preferred solution.

A large variety of interactive methods has been developed during the years. We can say that none of them is generally superior to all the others and some methods may suit different DMs and problems better than the others. The most important assumption underlying the successful application of interactive methods is that the DM must be available and willing to actively participate in the solution process and direct it according to her/his preferences.

Interactive methods differ from each other by both the style of interaction and technical elements. The former includes the form in which information is given to the DM and the form and type of preference information the DM specifies. On the other hand, the latter includes the type of final solution obtained (i.e., whether it is weakly, properly or Pareto optimal or none of these), the kind of problems handled (i.e., mathematical assumptions set on the problem), the mathematical convergence of the method (if any) and what kind of a scalarizing function is used. It is always important that the DM finds the method worthwhile and acceptable and is able to use it properly, in other words, the DM must find the style of specifying preference information understandable and preferences easy and intuitive to provide in the style selected. We can often identify two phases in the solution process: a learning phase when the DM learns about the problem and feasible solutions in it and a decision phase when the most preferred solution is found in the region identified in the first phase. Naturally, the two phases can also be used iteratively if so desired.

In fact, solving a multiobjective optimization problem interactively is a constructive process where, while learning, the DM is building a conviction of what is possible (i.e., what kind of solutions are available) and confronting this knowledge with her/his preferences that also evolve. In this sense, one should generally speak about a psychological convergence in interactive methods, rather than about a mathematical one.

Here we identify three types of specifying preference information in interactive methods and discuss the main characteristics of each type as well as give some examples of methods. The types are methods based on trade-off information, reference point approaches and classification-based methods. However, it is important to point out that other interactive methods do also exist. For example, it is possible to generate a small sample of Pareto optimal solutions using different weights in the weighted Chebyshev problem (1.8) or (1.10) introduced in Chapter 1 when minimizing the distance to the utopian objective vector. Then we can ask the DM to select the most preferred one of them and the next sample of Pareto optimal solutions is generated so that it concentrates on the neighbourhood of the selected one. See (Steuer, 1986, 1989) for details of this so-called Tchebycheff method.

Different interactive methods are described, for example, in the monographs (Chankong and Haimes, 1983; Hwang and Masud, 1979; Miettinen, 1999; Sawaragi *et al.*, 1985; Steuer, 1986; Vincke, 1992). Furthermore, methods with applications to large-scale systems and industry are presented in (Haimes *et al.*, 1990; Statnikov, 1999; Tabucanon, 1988). Let us also mention examples of reviews of methods including (Buchanan, 1986; Stewart, 1992) and collections of interactive methods like (Korhonen, 2005; Shin and Ravindran, 1991; Vanderpooten and Vincke, 1989).

2.2 Trade-off Based Methods

2.2.1 Different Trade-off Concepts

Several definitions of trade-offs are available in the MCDM literature. Intuitively speaking, a trade-off is an exchange, that is, a loss in one aspect of the problem, in order to gain additional benefit in another aspect. In our multiobjective optimization language, a trade-off represents giving up in one of the objectives, which allows the improvement of another objective. More precisely, how much must we give up of a certain objective in order to improve another one to a certain quantity. Here, one important distinction must be made. A trade-off can measure, attending just to the structure of the problem, the change in one objective in relation to the change in another one, when moving from a feasible solution to another one. This is what we call an *objective trade-off*. On the other hand, a trade-off can also measure how much the DM considers desirable to sacrifice in the value of some objective function in order to improve another objective to a certain quantity. Then we talk about a *subjective trade-off*. As it will be seen, both concepts may be used within an interactive scheme in order to move from a Pareto optimal solution to another. For objective trade-offs, let us define the following concepts:

Definition 1. *Let us consider two feasible solutions* $\mathbf{x}^1$ *and* $\mathbf{x}^2$*, and the corresponding objective vectors* $\mathbf{f}(\mathbf{x}^1)$ *and* $\mathbf{f}(\mathbf{x}^2)$*. Then, the* ratio of change *between* f_i *and* f_j *is denoted by* $T_{ij}(\mathbf{x}^1, \mathbf{x}^2)$*, where*

$$T_{ij}(\mathbf{x}^1, \mathbf{x}^2) = \frac{f_i(\mathbf{x}^1) - f_i(\mathbf{x}^2)}{f_j(\mathbf{x}^1) - f_j(\mathbf{x}^2)}.$$

$T_{ij}(\mathbf{x}^1, \mathbf{x}^2)$ *is said to be a* partial trade-off *involving* f_i *and* f_j *between* $\mathbf{x}^1$ *and* $\mathbf{x}^2$ *if* $f_l(\mathbf{x}^1) = f_l(\mathbf{x}^2)$ *for all* $l = 1, \ldots, k$, $l \neq i, j$. *If, on the other hand, there exists an index* $l \in \{1, \ldots, k\} \setminus \{i, j\}$ *such that* $f_l(\mathbf{x}^1) \neq f_l(\mathbf{x}^2)$*, then* $T_{ij}(\mathbf{x}^1, \mathbf{x}^2)$ *is called the* total trade-off *involving* f_i *and* f_j *between* $\mathbf{x}^1$ *and* $\mathbf{x}^2$*.*

When moving from one Pareto optimal solution to another, there is at least a pair of objective functions such that one of them is improved and the other one gets worse. These trade-off concepts help the DM to study the effect of changing the current solution. For continuously differentiable problems, the finite increments quotient represented by $T_{ij}(\mathbf{x}^1, \mathbf{x}^2)$ can be changed by an infinitesimal change trend when moving from a certain Pareto optimal solution $\mathbf{x}^0$ along a feasible direction $\mathbf{d}$. This yields the following concept.

Definition 2. *Given a feasible solution* $\mathbf{x}^0$ *and a feasible direction* $\mathbf{d}$ *emanating from* $\mathbf{x}^0$ *(i.e., there exists* $\alpha_0 > 0$ *so that* $\mathbf{x}^0 + \alpha\mathbf{d} \in S$ *for* $0 \le \alpha \le \alpha_0$*), we define the* total trade-off rate *at* $\mathbf{x}^0$*, involving* f_i *and* f_j*, along the direction* $\mathbf{d}$ *as*

$$t_{ij}(\mathbf{x}^0, \mathbf{d}) = \lim_{\alpha \to 0} T_{ij}(\mathbf{x}^0 + \alpha\mathbf{d}, \mathbf{x}^0).$$

If $\mathbf{d}$ *is a feasible direction with the property that there exists* $\bar{\alpha} > 0$ *such that* $f_l(\mathbf{x}^0 + \alpha\mathbf{d}) = f_l(\mathbf{x}^0)$ *for all* $l \notin \{i, j\}$ *and for all* $0 \le \alpha \le \bar{\alpha}$*, then we shall call the corresponding* $t_{ij}(\mathbf{x}^0, \mathbf{d})$ *the* partial trade-off rate*.*

The following result is straightforward:

Proposition 1. *Let us assume that all the objective functions* f_i *are continuously differentiable. Then,*

$$t_{ij}(\mathbf{x}^0, \mathbf{d}) = \frac{\nabla f_i(\mathbf{x}^0)^T \mathbf{d}}{\nabla f_j(\mathbf{x}^0)^T \mathbf{d}}.$$

It must be pointed out that the expression given in Definition 2 for the trade-off rate makes it necessary for direction $\mathbf{d}$ to be feasible. Nevertheless, the characterization given in Proposition 1 for the continuously differentiable case can be extended to (non-feasible) tangent directions.

Now, let us proceed to the definition of subjective trade-off concepts. The term subjective means that the DM's preferences are somehow taken into account. That is, subjective trade-offs are desirable trade-offs for the DM. This idea often implies the existence of an underlying (implicit) value function $v(z_1, \ldots, z_k)$ which defines the DM's subjective preferences among the feasible solutions of the problem. If the objective functions are to be minimized, then v is assumed to be strictly decreasing with respect to each one of its variables. Very frequently, the concavity of v is also assumed. If two alternatives are equally desired for the DM, this means that they lie on the same indifference curve (i.e., an isoquant of the value function), $v(z_1, \ldots, z_k) = v^0$.

This yields the following definition:

Definition 3. *Given two solutions* $\mathbf{x}^1$ *and* $\mathbf{x}^2$*, if* $\mathbf{f}(\mathbf{x}^1)$ *and* $\mathbf{f}(\mathbf{x}^2)$ *lie on the same indifference curve, the corresponding trade-off* $T_{ij}(\mathbf{x}^1, \mathbf{x}^2)$ *whether total or partial, is usually known as the* indifference trade-off *involving* f_i *and* f_j *between* $\mathbf{x}^1$ *and* $\mathbf{x}^2$*.*

Let us assume that all functions f_i are continuously differentiable, and suppose we are studying the indifference trade-offs between f_i and f_j at a fixed point $\mathbf{x}^0$, with objective vector $\mathbf{z}^0 = \mathbf{f}(\mathbf{x}^0)$, which lies on the indifference curve $v(z_1, \ldots, z_k) = v^0$. If $\partial v(\mathbf{z}^0)/\partial z_i \neq 0$, we can express, z_i as an implicit function of the remaining objectives (including z_j):

$$z_i = z_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k). \tag{2.1}$$

This expression allows us to obtain the trade-off rate between two functions when moving along an indifference curve:

Definition 4. *Given a solution* $\mathbf{x}^0$ *and the corresponding* $\mathbf{z}^0$*, the* indifference trade-off rate *or* marginal rate of substitution *(MRS) between* f_i *and* f_j *at* $\mathbf{x}^0$ *is defined as follows:*

$$m_{ij}(\mathbf{x}^0) = \frac{\partial v(\mathbf{z}^0)}{\partial z_j} \bigg/ \frac{\partial v(\mathbf{z}^0)}{\partial z_i}\ .$$

Let us observe that, if the chain rule is applied to expression (2.1), then

$$m_{ij}(\mathbf{x}^0) = - \left.\frac{\partial z_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k)}{\partial z_j}\right|_{\mathbf{z}=\mathbf{z}^0}. \qquad (2.2)$$

Therefore, the marginal rate of substitution between f_i and f_j at $\mathbf{x}^0$ represents the amount of decrement in the value of the objective function f_i that compensates an infinitesimal increment in the value of the objective f_j, while the values of all the other objectives remain unaltered. Alternatively, it can be viewed as the (absolute value of the) slope of the indifference curve v^0 at $\mathbf{f}(\mathbf{x}^0)$, if z_i and z_j are represented on the axes. This, given that v is strictly decreasing, implies that

$$N_v(\mathbf{x}^0) = (-m_{i1}(\mathbf{x}^0), \ldots, -m_{ii-1}(\mathbf{x}^0), -1, -m_{ii+1}(\mathbf{x}^0), \ldots, -m_{ik}(\mathbf{x}^0)) \qquad (2.3)$$

is a normal vector to the indifference curve at $\mathbf{z}^0$ (see Figure 2.1).

2.2.2 Obtaining Objective Trade-offs over the Pareto Optimal Set

When solving a multiobjective optimization problem using an interactive method, it can be important and useful to know the objective trade-offs when moving from a Pareto optimal solution to another one. This knowledge can allow the DM to decide whether to search for more preferred Pareto optimal solutions in certain directions. A key issue for many trade-off based interactive methods is to obtain partial trade-off rates for a Pareto optimal solution. The ε-constraint problem plays a key role in this task.

Given the multiobjective problem, a vector $\varepsilon \in R^{k-1}$, and an objective function f_i to be optimized, let us consider problem (1.3) defined in Chapter 1. We can denote this problem by $P_i(\varepsilon)$. If the feasible set of $P_i(\varepsilon)$ is nonempty and $\mathbf{x}^0$ is an optimal solution, then let us denote by λ^*_{ij} the optimal Karush-Kuhn-Tucker (KKT) multipliers associated with the f_j constraints. Chankong and Haimes (1983) prove (under certain regularity and second order conditions) that if all the optimal KKT multipliers are strictly positive, then $-\lambda^*_{ij}$ is a partial trade-off rate between objectives f_i and f_j, along a direction $\mathbf{d}_j$:

$$-\lambda^*_{ij} = \frac{\partial f_i(\mathbf{x}^0)}{\partial z_j} = t_{ij}(\mathbf{x}^0, \mathbf{d}_j), \qquad j \neq i, \qquad (2.4)$$

where

$$\mathbf{d}_j = \frac{\partial \mathbf{x}(\varepsilon^0)}{\partial \varepsilon_j} \tag{2.5}$$

is an efficient direction, that is, a direction that is tangent to the Pareto optimal set (i.e., frontier) at $\mathbf{x}^0$. Therefore, graphically, $-\lambda^*_{ij}$ can be viewed as the slope of the Pareto optimal frontier at $\mathbf{z}^0$, if functions f_i and f_j are represented on the axes. For objective functions to be minimized, a vector

$$N^*(\mathbf{x}^0) = (-\lambda^*_{i1}, \ldots, -\lambda^*_{ii-1}, -1, -\lambda^*_{ii+1}, \ldots, -\lambda^*_{ik}) \tag{2.6}$$

can be interpreted as a normal vector to the Pareto optimal frontier at $\mathbf{z}^0$ (see Figure 2.1). In fact, this expression matches the traditional definition of a normal vector when the efficient frontier is differentiable.

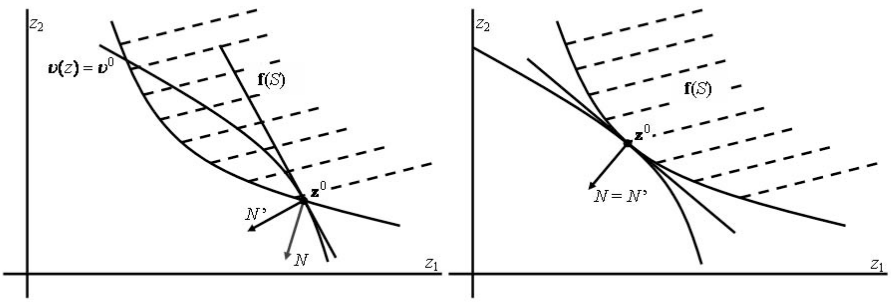

Fig. 2.1. On the left, N, given by the optimal KKT multipliers, is the normal vector to the Pareto optimal frontier at $\mathbf{z}^0$, and N', given by the MRS, is the normal vector to the indifference curve at $\mathbf{z}^0$. On the right, the convergence condition given in (2.7) holds.

If the strict positivity conditions on the multipliers is removed, a more general result is also proved in (Chankong and Haimes, 1983). Namely, if all the optimal multipliers are strictly positive, then $-\lambda^*_{kj}$ is a partial trade-off rate between objectives f_k and f_j, along direction $\mathbf{d}_j$, as defined in (2.5). If, on the other hand, there are optimal multipliers equal to zero, then $-\lambda^*_{ij}$ is a total trade-off rate between f_i and f_j, along direction $\mathbf{d}_j$, as defined in (2.5).

Let us consider a Pareto optimal solution $\mathbf{x}^0$ and the objective vector $\mathbf{z}^0 = \mathbf{f}(\mathbf{x}^0)$. If the Pareto optimal set is connected as well as of full dimensionality $(k-1)$ and continuously differentiable, there is an alternative way of obtaining a normal vector to the Pareto optimal frontier at $\mathbf{z}^0$. It does not require the second order sufficiency conditions to be satisfied and works by means of solving problem (1.9) formulated in Chapter 1 with

$$w_i = \frac{1}{z_i^0 - z_i^{\star\star}}, \quad i = 1, \ldots, k,$$

where $\mathbf{z}^{\star\star}$ is the utopian objective vector. If λ^*_{ij} are the optimal KKT multipliers of this problem associated with the function constraints, then the vector

$$N = (-w_1\lambda^*_{i1}, \ldots, -w_k\lambda^*_{ik})$$

is a normal vector to the Pareto optimal frontier at $\mathbf{z}^0$ (Yang and Li, 2002).

2.2.3 The Use of Trade-offs within Interactive Methods

Among the different trade-off based interactive methods reported in the literature there are two most commonly used schemes:

- to determine at each iteration objective trade-offs, which are shown to the DM, who must give some kind of answer about the desirability of such trade-offs, or
- to ask the DM to provide subjective trade-offs, which are used to find a Pareto optimal solution with a better value of the DM's (implicit) value function.

The Zionts-Wallenius (Z-W) method (Zionts and Wallenius, 1976) belongs to the first group. In this method, the DM is shown several objective trade-offs at each iteration, and (s)he is expected to say whether (s)he likes, dislikes or is indifferent with respect to each trade-off. More elaborated information is required from the DM in the ISWT method (Chankong and Haimes, 1983), where several objective trade-offs are shown at each iteration to the DM who must rank each one of them in a scale from −10 to 10, depending on its desirability (or from −2 to 2, as suggested by Tarvainen (1984)).

In the second group, there are three important methods. The Geoffrion-Dyer-Feinberg (GDF) method (Geoffrion *et al.*, 1972) uses a Frank–Wolfe algorithm in order to perform a line search using the subjective trade-off information given by the DM to determine the search direction. In the SPOT method (Sakawa, 1982), the subjective trade-offs given by the DM are also used to determine a search direction, but a proxy function is used to calculate the optimal step length. Finally, the GRIST method(Yang, 1999) uses the normal vector in order to project the direction given by the subjective trade-offs onto the tangent plane to the Pareto optimal frontier.

All these five methods will be briefly described in the following section. In most of them, the relation between the objective and the subjective trade-offs is very important in order to determine the final solution. Namely, at a final solution, that is, a Pareto optimal solution that maximizes the DM's value function, the indifference curve of the value function must be tangent to the Pareto optimal frontier. This implies that if the indifference trade-off rate is defined like in (2.2) and the objective partial trade-off rate is defined as in (2.4), then in a final solution the relations

$$m_{ij}(\mathbf{x}^0) = \lambda^*_{ij} \quad j = 1, \ldots, k \quad j \neq i \tag{2.7}$$

must hold, which in turn implies that the normal vectors (2.3) and (2.6) must coincide (see Figure 2.1). Note that, in this case, the i-th component of both the vectors is equal to 1, and that is why they must exactly coincide. If the normal vector to the Pareto optimal frontier is defined as in (2.6), then the equality given in expression (2.7) or the equality between the normal vectors should be replaced by a proportionality condition.

2.2.4 Some Trade-off Based Methods

In this section, we will briefly give the basic ideas of the previously mentioned methods. For further details, the reader may follow the references given. See also (Miettinen, 1999) for other trade-off based interactive techniques not mentioned here.

The Z-W method was originally proposed in (Zionts and Wallenius, 1976), and it is based on piecewise linearizations of the problem and the use of the properties of Pareto optimal solutions of linear problems. The assumptions of this method are the following:

- An implicit value function v exists, and it is assumed to be concave.
- The objective functions and the feasible region are convex.

Although not strictly necessary from the theoretical point of view, the authors mention that the additive separability of the objective functions is convenient for practical reasons. Another version of the algorithm exists for a class of pseudoconcave value functions (Zionts and Wallenius, 1983). As the method is based on piecewise linear approximations of the functions, we will briefly describe it for MOLP problems representing one of these piecewise approximations. The idea of the method is the following: A Pareto optimal solution is found using the weighting method (see Section 1.3.1 in Chapter 1). Then adjacent Pareto optimal vertices to the current solution are identified and the corresponding trade-offs are shown to the DM, who is asked to say whether (s)he prefers each of them to the current solution or not. Making use of this information, the weights are actualized, and a new solution is found.

The interactive surrogate worth trade-off method (ISWT) is an interactive version of the original surrogate worth trade-off method (Haimes and Hall, 1974). The basic idea lies on the concept of surrogate worth, which is a valuation by the DM of the desirability of the trade-offs obtained at a Pareto optimal solution. The interactive method was first reported in (Chankong and Haimes, 1978), and both versions are also described in (Chankong and Haimes, 1983). The basic assumptions of this method are the following:

- An implicit value function v exists, and it is assumed to be continuously differentiable and monotonically decreasing.
- All the functions are twice continuously differentiable.
- The feasible region S is compact.
- Optimal KKT multipliers provide partial trade-offs.

This method proceeds as follows. A Pareto optimal solution is determined using the ε-constraint problem (see Section 1.3.2 in Chapter 1). The objective trade-offs at the current solution are obtained and shown to the DM, who is asked to assess their desirability using a scale from -10 to 10. This information is used to actualize the vector of upper bounds, and to produce a new solution.

The basic idea of the interactive Geoffrion, Dyer and Feinberg (GDF) algorithm (Geoffrion *et al.*, 1972) is the following. The existence of an implicit value function v is assumed, which the DM wishes to maximize over the feasible region. The Franke-Wolfe algorithm is applied to solve the intermediate problems formed. The assumptions of the GDF method are:

- The feasible region S is compact and convex.
- Objective functions are continuously differentiable and convex.
- An implicit value function v exists, and is assumed to be continuously differentiable, monotonically decreasing and concave.

In the GDF method, given the current solution, the DM is asked to provide marginal rates of substitution, which are used to determine an ascent direction for the value function. Then, the optimal step-length is approximated using an evaluation scheme, and the next iteration is generated.

The sequential proxy optimization technique (SPOT) is an interactive algorithm developed by Sakawa (1982). The basic idea of this method is to assume the existence of an implicit value function v of the DM, that has to be maximized over the feasible region. This maximization is done using a feasible direction scheme. In order to determine the optimal step-length, a proxy function is used to simulate locally the behavior of the (unknown) value function (the author proposes several options for this function). The ε-constraint problem is used to determine Pareto optimal solutions and to obtain trade-off information at the current solution, and the DM is asked to provide marginal rates of substitution. The assumptions of this method are the following:

- The implicit value function v exists, and is continuously differentiable, strictly decreasing and concave.
- All objective functions f_i are convex and twice continuously differentiable.
- The feasible region S is compact and convex.
- Optimal KKT multipliers provide partial trade-offs.

The idea of the iterations is the following. The ε-constraint problem is used to determine a Pareto optimal solution. The DM is asked to give the MRSs, which are used to find a search direction. The optimal step-length is approximated using the proxy function, and the vector of bounds is updated so as to find the next iteration.

Finally, we introduce the gradient based interactive step trade-off method (GRIST) proposed by Yang (1999). This interactive technique has been designed to deal with general (non necessarily convex) differentiable problems with a differentiable, connected and full dimensional $(k-1)$ Pareto optimal set. The main idea consists of a projection of the vector determined by the marginal rates of substitution given by the DM onto a tangent plane to the Pareto optimal frontier at the current iteration. This projection is proved to be an increasing direction of the DM's underlying value function. Then a reference point is obtained following this direction, which is in turn projected onto the Pareto optimal frontier to generate the next iteration. The assumptions of this method are the following ones:

- The implicit value function v exists, and it is continuously differentiable and strictly decreasing.
- All objective functions are continuously differentiable.
- The feasible region S is compact.
- The Pareto optimal set is differentiable, connected and of full dimensionality $(k-1)$.
- All the solutions generated are regular.

In GRIST, given the current solution, the DM is asked to give the MRSs. The corresponding vector is projected onto the tangent plane to the Pareto optimal frontier, and a step-length is approximated by an evaluation scheme. The point obtained is projected onto the Pareto optimal set, and this yields a new solution.

2.2.5 Summary

Finally, let us point out some of the most outstanding features of the methods described.

Convergence. The mathematical convergence of all the methods to the optimum of the implicit value function can be proved, given that the value function satisfies for each method the assumptions mentioned in Section 2.2.4. (Moreover, for MOLP problems, the Z-W method can be proved to converge in a finite number of iterations.)

Information. The preference information required from the DM can be regarded as not very hard for the Z-W method (the DM has to say whether the trade-offs proposed are desirable or not), hard for the SPOT, GDF and GRIST methods (the DM has to give MRSs at the current iteration) and very hard for the ISWT method (where a surrogate worth for each trade-off has to be given). Besides, in some methods the step-length has to be estimated evaluating different solutions.

Consistency. In all the methods the consistency of the responses of the DM is vital for the real convergence of the method. In the special case of the SPOT method, there are hard consistency tests for choosing the proxy functions and their parameters. Although all the methods (except Z-W) allow to revisit solutions and to go back in the process, they usually do not perform well with inconsistent answers.

Type of problem. The Z-W method handles convex problems for which a piecewise linearization can be carried out, although it is mainly used in practice for linear problems. The GDF method is designed for convex problems. The SPOT and ISWT methods do not assume convexity, but second order sufficient conditions must be satisfied at the iterations. Finally, the GRIST method assumes that the Pareto optimal set is differentiable, connected and full dimensional. These conditions may be very hard to be assured a priori.

Pareto optimality. The Pareto optimality of the final solution is not guaranteed in the GDF method. All the other methods assure that the final solution is Pareto optimal (although only extreme Pareto optimal solutions are obtained with the Z-W method).

2.3 Reference Point Approaches

2.3.1 Fundamental Assumptions of Reference Point Approaches

Reference point approaches have a long history and multiple practical applications (Wierzbicki, 1977, 1980, 1999; Wierzbicki *et al.*, 2000). However, we shall limit the description here to their fundamental philosophy, a short indication of their basic features and of some contemporary, new developments related to this class of approaches. During over 30 years of development of reference point approaches, including their diverse applications, several methodological postulates describing desirable features of the decision process supported by these approaches have been clarified, most of them expressing lessons learned from the practice of decision making. These postulates are:

1) Separation of preferential and substantive models. This indicates the conviction that in a good decision support system, we should carefully distinguish between the subjective part of knowledge represented in this system, concerning the preferences of the user, thus called a *preferential model* (including, but understood more broadly than a preference model of the DM) of the decision situation, and the objective part, representing in this system some selected knowledge about pertinent aspects of the decision situation – obviously selected never fully objectively, but formulated with objectivity as a goal – called a *substantive model* (sometimes *core model*) of the decision situation. For example, objective trade-offs, defined in Section 2.2 are part of the substantive model, while subjective trade-offs belong to the preferential model. Typically, a substantive model has the following general form:

$$\mathbf{y} = \mathbf{F}(\mathbf{x}, \mathbf{v}, \mathbf{a}); \ \mathbf{x} \in \mathbf{S}, \tag{2.8}$$

where

- $\mathbf{y}$ is a vector of outcomes (outputs) y_j, used for measuring the consequences of implementation of decisions;
- $\mathbf{x}$ is a vector of decisions (controls, inputs to the decision making process), which are controlled by the user;
- $\mathbf{v}$ is a vector of external impacts (external inputs, perturbations), which are not controlled by the user;
- $\mathbf{a}$ is a vector of model parameters;

- $\mathbf{F}$ is a vector of functions (including such that are conventionally called objectives and constraints), describing the relations between decisions $\mathbf{x}$, impacts $\mathbf{v}$, parameters $\mathbf{a}$, and outcomes $\mathbf{y}$;
- S is the set of feasible decisions.

The form of (2.8) is only slightly more complex but essentially equivalent to what in the other parts of this book is written simply as $\mathbf{z} = \mathbf{f}(\mathbf{x})$, where $\mathbf{z}$ denotes objectives selected between outcomes $\mathbf{y}$. This short notation is a great oversimplification of the actual complexity of models involving multiple objectives or criteria. However, even the form of (2.8) is misleading by its compactness, since it hides the actual complexity of the underlying knowledge representation: a large model today may have several millions of variables and constraints, even when the number of decision and outcome variables is usually much smaller (Makowski, 2005).

Additionally, the substantive model includes constraint specification (symbolically denoted by $\mathbf{x} \in \mathbf{S}$) that might have the form of feasible bounds on selected model outcomes (or be just a list of considered decision options in a discrete case). While the reference point approach is typically described for the continuous case (with a nonempty interior of S, thus an infinite number of solutions in this set), it is as well (or even better) applicable to the discrete case, with a finite number of decision options. The reason for this is that the reference point approach is specifically designed to be effective for nonconvex problems (which is typical for the discrete case).

The actual issue of the separation of preferential and substantive models is that the substantive model should not represent the preferences of the DM, except in one aspect: *the number of decision outcomes in this model should be large enough* for using them in a separate representation of a preferential structure $P(\mathbf{x}, \mathbf{y})$ of the user, needed for selecting a manageable subset of solutions (decisions) that correspond best to user's preferences. The separate representation of preferential structure (the structure of preferential information) can have several degrees of specificity, while the reference point approaches assume that this specification should be as general as possible, since a more detailed specification violates the sovereign right of a DM to change her/his mind.

The most general specification contains a selection of outcomes y_j that are chosen by the DM to measure the quality of decisions (or solutions), called objectives (values of objective functions) or criteria (quality measures, quality indicators) and denoted by z_j, $j = 1, \ldots, k$. This specification is accompanied by defining a partial order in the space of objectives – simply asking the DM which objectives should be maximized and which minimized (while another option, stabilizing some objectives around given reference levels, is also possible in reference point approaches (Wierzbicki *et al.*, 2000)). Here we consider the simplest case when all the objectives are to be minimized.

The second level of specificity in reference point approaches is assumed to consist of specification of reference points – generally, desired objective function values. These reference points might be interval-type, double, including aspiration levels, denoted here by z_j^a (objective function values that the DM would like to achieve) and reservation levels z_j^r (objective values that should be achieved according to the DM). Specification of reference levels is treated as an alternative to trade-off or weighting coefficient information that leads usually to linear representation of preferences and unbalanced decisions as discussed below, although some reference point approaches (Nakayama, 1995) combine reference levels with trade-off information.

The detailed specification of preferences might include full or gradual identification of value functions, see Section 2.2 on trade-off methods or (Keeney and Raiffa, 1976; Keeney, 1992). This is avoided in *reference point approaches that stress learning instead of value identification*. According to the reference point philosophy, the DM should learn during the interaction with a decision support system (DSS), hence her/his preferences might change in the decision making process and (s)he has full, sovereign right or even necessity to be inconsistent.

2) Nonlinearity of preferences. According to a general conviction that human preferences have essentially a nonlinear character, including a preference for balanced solutions. Any linear approximation of preferences (e.g., by a weighted sum distorts them, favoring unbalanced solutions. This is in opposition to the methods taught usually as "the basic approaches" to MCDM. These methods consist of determining (by diverse approaches, between which the AHP (Saaty, 1982) is one of the most often used) weighting coefficients and solving the weighted problem (1.2) discussed in Chapter 1. Such a linear aggregation might be sometimes necessary, but it has several limitations as discussed in Chapter 1. The most serious ones are the following:

- The weighted sum tends to promote decisions with unbalanced objectives, as illustrated by the Korhonen paradox mentioned in Chapter 1. In order to accommodate the natural human preference for balanced solutions, a nonlinear aggregation is necessary.
- The weighted sum is based on a tacit (unstated) assumption that a trade-off analysis is applicable to all objective functions: a worsening of the value of one objective function might be compensated by the improvement of the value of another one. While often encountered in economic applications, this compensatory character of objectives is usually not encountered in interdisciplinary applications.

Educated that weighting methods are basic, the legislators in Poland introduced a public tender law. This law requires that any institution preparing a tender using public money should publish beforehand all objectives of ranking the offers and all weighting coefficients used to aggregate the objectives. This legal innovation backfired: while the law was intended to make public tenders more transparent and accountable, the practical outcome was opposite because of effects similar to the Korhonen paradox. Organizers of the tenders soon discovered that they are forced either to select the offer that is the cheapest and worst in quality or the best in quality but most expensive one. In order to counteract, they either limited the solution space drastically by diverse side constraints (which is difficult but consistent with the spirit of the law) or added additional poorly defined objectives such as the degree of satisfaction (which is simple and legal but fully inconsistent with the spirit of the law, since it makes the tender less transparent and opens a hidden door for graft).

To summarize, a linear weighted sum aggregation is simple but too simplistic in representing typical human preferences that are often nonlinear. Using this simplistic approach may result in adverse and unforeseen side-effects.

3) Holistic perception of objectives. The third basic assumption of reference point approaches is that the DM selects her/his decision using a holistic assessment of the decision situation. In order to help her/him in such a holistic evaluation, a DSS should compute and inform the DM about relevant ranges of objective function values. Such ranges can be defined in diverse ways, while the two basic ones are the following:

- Total ranges of objective functions involve the definition of the lower z_j^{lo} and the upper bound z_j^{up}, over all feasible decisions $\mathbf{x} \in \mathbf{S}$ $(j = 1, \ldots, k)$.
- Pareto optimal ranges of objectives are counted only over Pareto optimal solutions. The lower bound is the utopian or ideal objective vector $z_j^\star$ and is typically equal to z_j^{lo}. The upper bound is the nadir objective vector z_j^{nad} (as discussed in Preface and Chapter 1).

Generally, $z_j^{nad} \leq z_j^{up}$ and the nadir objective vector is easy to determine only in the case of biobjective problems (Ehrgott and Tenfelde-Podehl, 2000) (for continuous models; for discrete models the determination of a nadir point is somewhat simpler). No matter which ranges of objectives we use, it is often useful to assume that all objective functions or quality indicators and their values $f_j(\mathbf{x})$ for decision vectors $\mathbf{x} \in S$ are scaled down to a relative scale by the transformation:

$$z_j^{rel} = f_j^{rel}(\mathbf{x}) = (f_j(\mathbf{x}) - z_j^{lo})/(z_j^{up} - z_j^{lo}) \times 100\%.$$

4) Reference points as tools of holistic learning. Another basic assumption of reference point approaches is that reference (aspiration, reservation) levels and points are treated not as a fixed expression of preferences but as a tool of adaptive, holistic learning about the decision situation as described by the substantive model. Thus, even if the convergence of reference point approaches to a solution most preferred by the DM can be proved (Wierzbicki, 1999), this aspect is never stressed. More important aspects relate to other properties of these approaches. Even if the reference points might be determined in some objective fashion, independently of the preferences of the DM, we stress again a diversity of such objective determinations, thus making possible comparisons of resulting optimal solutions.

5) Achievement functions as ad hoc approximations of value. Given the partial information about preferences (the partial order in the objective space) and their assumed nonlinearity, and the information about the positioning of reference points inside known objective function ranges, the simplest ad hoc approximation of a nonlinear value function consistent with this information and promoting balanced solutions can be proposed. Such an ad hoc approximation takes the form of achievement functions discussed later; see (2.9)–(2.10). (A simple example of them was also introduced in Chapter 1 as problem (1.11). Note that that problem was formulated so that it was to be minimized but here a different variant is described where the achievement function is maximized.)

Achievement functions are determined essentially by max-min terms that favour solutions with balanced deviations from reference points and express the Rawlsian principle of justice (concentrating the attention on worst off members of society or on issues worst provided for (Rawls, 1971)). These terms are slightly corrected by regularizing terms, resulting in the Pareto optimality of the solutions that maximize achievement functions. See also (Lewandowski and Wierzbicki, 1989) for diverse applications where the partial order in the objective space (called also the dominance relation) is not assumed a priori but defined interactively with the DM.

6) Sovereignty of the DM. It can be shown (Wierzbicki, 1986) that achievement functions have the property of full controllability. This means that any Pareto optimal solution can be selected by the DM by modifying reference points and maximizing the achievement function. This provides for the full sovereignty of the DM. Thus, a DSS based on a reference point approach behaves analogously to a perfect analytic section staff in a business organization (Wierzbicki, 1983). The CEO (boss) can outline her/his preferences to the staff and specify the reference points. The perfect staff will tell the boss that her/his aspirations or even reservations are not attainable, if this is the case; but the staff computes in this case also the Pareto optimal solution that comes closest to the aspirations or reservations. If, however, the aspirations are attainable and not Pareto optimal (a better decision might be found), the perfect staff will present to the boss the decision that results in the aspirations and also a Pareto optimal solution corresponding to a uniform improvement of all objectives over the aspirations. In a special case when the aspirations or reservations are Pareto optimal, the perfect staff responds with the decision that results precisely in attaining these aspirations (reservations) – and does not argue that another decision is better, even if such a decision might result from a trade-off analysis performed by the staff. (Only a computerized DSS, not a human staff, can behave in such a perfect fashion.)

7) Final aims: intuition support versus rational objectivity. To summarize the fundamental assumptions and philosophy of reference point approaches, the basic aim when supporting an individual, subjective DM, is to enhance her/his power of intuition (Wierzbicki, 1997) by enabling holistic learning about the decision situation as modelled by the substantive model. The same applies, actually, when using reference point approaches for supporting negotiations and group decision making (Makowski, 2005).

2.3.2 Basic Features of Reference Point Approaches

The large disparity between the opposite ends of the spectrum of preference elicitation (full value function identification versus a weighted sum approach) indicates the need of a middle-ground approach, simple enough and easily adaptable but not too simplistic. An interactive decision making process in a DSS using a reference point approach consists typically of the following steps:

- The decision maker (DM) specifies reference points (e.g., aspiration and reservation levels for all objective functions). To help her/him in starting the process, the DSS can compute a *neutral solution,* a response to reference levels situated in the middle of objective function ranges (see problem (1.6) in Chapter 1);
- The DSS responds by maximizing an achievement function, a relatively simple but nonlinear aggregation of objective functions interpreted as an ad hoc and adaptable approximation of the value function of the DM based on the information contained in the estimates of the ranges of objective functions and in the positioning of aspiration and reservation levels inside these ranges;
- The DM is free to modify the reference points as (s)he will. (S)he is supposed to use this freedom to learn about the decision situation and to explore interesting parts of the Pareto optimal set;
- Diverse methods can be used to help the DM in this exploration (we comment on them later), but the essential point is that they should not limit the freedom of the DM.

In order to formulate an achievement function, we first count achievements for each individual objective function by transforming it (piece-wise linearly) (for objective functions to be minimized) as

$$\sigma_j(z_j, z_j^a, z_j^r) = \begin{cases} 1 + \alpha(z_j^a - z_j)/(z_j^a - z_j^{lo}), \; if \;\, z_j^{lo} \leq z_j \leq z_j^a \\ (z_j^r - z_j)/(z_j^r - z_j^a), \quad if \;\, z_j^a < z_j \leq z_j^r, \\ \beta(z_j^r - z_j)/(z_j^{up} - z_j^r), \quad if \;\, z_j^r < z_j \leq z_j^{up} \end{cases} . \tag{2.9}$$

The coefficients α and β are typically selected to assure the concavity of this function (Wierzbicki *et al.*, 2000); but the concavity is needed only for problems with a continuous (nonempty interior) set of solutions, for an easy transformation to a linear programming problem. The value $\sigma_j = \sigma_j(z_j, z_j^a, z_j^r)$ of this achievement function (where $z_j = f_j(\mathbf{x})$ for a given decision vector $\mathbf{x} \in S$) signifies the satisfaction level with the quality indicator or objective j for this decision vector. If we assign the values of satisfaction from -1 to 0 for $z_j^r < z_j \leq z_j^{up}$, values from 0 to 1 for $z_j^a < z_j \leq z_j^r$, values from 1 to 2 for $z_j^{lo} \leq z_j \leq z_j^a$, then we can just set $\alpha = \beta = 1$. After this transformation of all objective function values, we might use then the following form of the overall achievement function to be maximized[1]:

$$\sigma(\mathbf{z}, \mathbf{z^a}, \mathbf{z^r}) = \min_{\mathbf{j=1,\dots,k}} \sigma_\mathbf{j}(\mathbf{z_j}, \mathbf{z_j^a}, \mathbf{z_j^r}) + \rho \sum_{\mathbf{j=1,\dots,k}} \sigma_\mathbf{j}(\mathbf{z_j}, \mathbf{z_j^a}, \mathbf{z_j^r}), \tag{2.10}$$

where $\mathbf{z} = \mathbf{f}(\mathbf{x})$ is the objective vector and $\mathbf{z^a} = (\mathbf{z_1^a}, \dots, \mathbf{z_k^a})$ and $\mathbf{z^r} = (\mathbf{z_1^r}, \dots, \mathbf{z_k^r})$ the vectors of aspiration and reservation levels, respectively. Furthermore, $\rho > 0$ is a small regularizing coefficient (as discussed in Chapter 1.

There are many possible forms of achievement functions besides (2.9)–(2.10), as shown in (Wierzbicki *et al.*, 2000). All of them, however, have an important property of partial order approximation: their level sets approximate closely the positive cone defining the partial order (Wierzbicki, 1986). As indicated above, the achievement function has also a very important theoretical property of controllability, not possessed by value functions nor by weighted sums: for sufficiently small values of ρ, given any point $\mathbf{z}^*$ in the set of (properly) Pareto optimal objective vectors, we can always choose such reference levels that the maximum of the achievement function (2.10) is attained precisely at this point. In fact, it suffices to set aspiration levels equal to the components of $\mathbf{z}^*$. Conversely, if $\rho > 0$, all maxima of the achievement function (2.10) correspond to Pareto optimal solutions (because of the monotonicity of this function with respect to the partial order in the objective space.) Thus, the behaviour of achievement functions corresponds in this respect to value functions and weighted sums. However, let us emphasize that this is not the case in the distance norm used in goal programming (see Section 1.6.3 in Chapter 1), since the norm is not monotone when passing zero. As noted above, precisely the controllability property results in a fully sovereign control of the DSS by the user.

Alternatively, as shown in (Ogryczak, 2006), we can assume $\rho = 0$ and use the nucleolar minimax approach. In this approach, we consider first the minimal, worst individual objective-wise achievement computed as in (2.9)–(2.10) with $\rho = 0$. If two (or more) solutions have the same achievement value, we order them according to the second worst individual objective-wise achievement and so on.

There are many modifications, variants and extensions (Wierzbicki *et al.*, 2000) or approaches related to the basic reference point approach, mostly designed for helping the search phase in the Pareto optimal set. For example,

- the Tchebycheff method (Steuer, 1986) was developed independently but actually is equivalent to using weighting coefficients implied by reference levels;

[1] Even if in this book objectives are supposed to be typically minimized, achievements are here maximized.

- Pareto Race (Korhonen and Laakso, 1986) is a visual method based on reference points distributed along a direction in the objective space, or the REF-LEX method for nonlinear problems (Miettinen and Kirilov, 2005);
- the satisficing trade-off method, or the NIMBUS method, both described in a later section, or the 'light beam search' method (Jaszkiewicz and Słowiński, 1999), or several other approaches were motivated by the reference point approach.

In this section, we have presented some of the basic assumptions and philosophy of reference point approaches, stressing their unique concentration on the sovereignty of the subjective DM. Next we concentrate on classification-based methods.

2.4 Classification-Based Methods

2.4.1 Introduction to Classification of Objective Functions

According to the definition of Pareto optimality, moving from one Pareto optimal solution to another implies trading off. In other words, it is possible to move to another Pareto optimal solution and improve some objective function value(s) only by allowing some other objective function value(s) to get worse. This idea is used as such in classification-based methods. By classification-based methods we mean methods where the DM indicates her/his preferences by classifying objective functions. The idea is to tell which objective functions should improve and which ones could impair from their current values. In other words, the DM is shown the current Pareto optimal solution and asked what kind of changes in the objective function values would lead to a more preferred solution. It has been shown by Larichev (1992) that the classification of objective functions is a cognitively valid way of expressing preference information for a DM.

We can say that classification is a very intuitive way for the DM to direct the solution process in order to find the most preferred solution because no artificial concepts are used. Instead, the DM deals with objective function values that are as such meaningful and understandable for her/him. The DM can express hopes about improved solutions and directly see and compare how well the hopes could be attained when the next solution is generated.

To be more specific, when classifying objective functions (at the current Pareto optimal solution) the DM indicates which function values should improve, which one are acceptable as such and which ones are allowed to impair. In addition, desirable amounts of improvement or allowed amounts of impairments may be asked from the DM. There exist several classification-based interactive multiobjective optimization methods. They differ from each other, for example, in the number of classes available, the preference information asked from the DM and how this information is used to generate new Pareto optimal solutions.

Let us point out that closely related to classification is the idea of expressing preference information as a reference point (Miettinen and Mäkelä, 2002; Miettinen *et al.*, 2006). The difference is that while classification assumes that some objective function must be allowed to get worse, a reference point can be selected more freely. Naturally, it is not possible to improve all objective function values of a Pareto optimal solution, but the DM can express preferences without paying attention to this and then see what kind of solutions are feasible. However, when using classification, the DM can be more in control and select functions to be improved and specify amounts of relaxation for the others.

As far as stopping criteria are concerned, classification-based methods share the philosophy of reference point based methods (discussed in the previous section) so that the DM's satisfaction is the most important stopping criterion. This means that the search process continues as long as the DM wants to and the mathematical convergence is not essential (as in trade-off based methods) but rather the psychological convergence is emphasized (discussed in the introduction). This is justified by the fact that DMs typically want to feel being in control and do not necessarily want the method to tell them when they have found their most preferred solutions. After all, the most important task of interactive methods is to support the DM in decision making.

In what follows, we briefly describe the step method, the satisficing trade-off method and the NIMBUS method. Before that, we introduce some common notation.

Throughout this section, we denote the current Pareto optimal solution by $\mathbf{z}^h = \mathbf{f}(\mathbf{x}^h)$. When the DM classifies the objective functions at the current solution, we can say that (s)he assigns each of them into some class and the number of classes available varies in different methods. In general, we have the following classes for functions f_i $(i = 1, \dots, k)$

- $I^{<}$ whose values should be improved (i.e., decrease) from the current level,
- $I^{\leq}$ whose values should improve till some desired aspiration level $\hat{z}_i < z_i^h$,
- $I^{=}$ whose values are acceptable in the current solution,
- $I^{\geq}$ whose values can be impaired (i.e., increase) till some upper bound $\varepsilon_i > z_i^h$ and,
- $I^{\diamond}$ whose values are temporarily allowed to change freely.

The aspiration levels and the upper bounds corresponding to the classification are elicited from the DM, if they are needed. According to the definition of Pareto optimality, a classification is feasible only if $I^{<} \cup I^{\leq} \neq \emptyset$ and $I^{\geq} \cup I^{\diamond} \neq \emptyset$ and the DM has to classify all the objective functions, that is, $I^{<} \cup I^{\leq} \cup I^{=} \cup I^{\geq} \cup I^{\diamond} = \{1, \dots, k\}$.

2.4.2 Step Method

The step method (STEM) (Benayoun *et al.*, 1971) uses only two classes. STEM is one of the first interactive methods introduced for multiobjective optimization and it was originally developed for MOLP problems. However, here we describe variants for nonlinear problems according to Eschenauer *et al.* (1990); Sawaragi *et al.* (1985); Vanderpooten and Vincke (1989).

In STEM, the DM is assumed to classify the objective functions at the current solution $\mathbf{z}^h$ into those that have acceptable values $I^{\geq}$ and those whose values are too high, that is, functions that have unacceptable values $I^{<}$. Then the DM is supposed to give up a little in the value(s) of some acceptable objective function(s) in order to improve the values of some unacceptable objective functions. In other words, the DM is asked to specify upper bounds $\varepsilon_i^h > z_i^h$ for the functions in $I^{\geq}$. All the objective functions must be classified and, thus, $I^{<} \cup I^{\geq} = \{1, \ldots, k\}$.

It is assumed that the objective functions are bounded over S because distances are measured to the (global) ideal objective vector. STEM uses the weighted Chebyshev problem 1.8 introduced in Chapter 1 to generate new solutions. The weights are used to make the scales of the objective functions similar. The first problem to be solved is

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\ldots,k} \left[\frac{e_i}{\sum_{j=1}^k e_j} (f_i(\mathbf{x}) - z_i^\star) \right] \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{2.11}$$

where $e_i = \frac{1}{z_i^\star} \frac{z_i^{\text{nad}} - z_i^\star}{z_i^{\text{nad}}}$ as suggested by Eschenauer *et al.* (1990). Alternatively, we can set $e_i = \frac{z_i^{\text{nad}} - z_i^\star}{\max \left[|z_i^{\text{nad}}|, |z_i^\star| \right]}$ as suggested by Vanderpooten and Vincke (1989). (Naturally we assume that the denominators are not equal to zero.) It can be proved that the solution of (2.11) is weakly Pareto optimal. The solution obtained is the starting point for the method and the DM is asked to classify the objective functions at this point.

Then the feasible region is restricted according to the information given by the DM. The weights of the relaxed objective functions are set equal to zero, that is $e_i = 0$ for $i \in I^{\geq}$. Then a new distance minimization problem

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\ldots,k} \left[\frac{e_i}{\sum_{j=1}^k e_j} (f_i(\mathbf{x}) - z_i^\star) \right] \\ &\text{subject to} \quad f_i(\mathbf{x}) \leq \varepsilon_i^h \ \text{ for all } \ i \in I^{\geq}, \\ &\qquad\qquad\quad f_i(\mathbf{x}) \leq f_i(\mathbf{x}^h) \ \text{ for all } \ i \in I^{<}, \\ &\qquad\qquad\quad \mathbf{x} \in S \end{aligned}$$

is solved.

The DM can classify the objective functions at the solution obtained and the procedure continues until the DM does not want to change the current solution. In STEM, the idea is to move from one weakly Pareto optimal solution to another. Pareto optimality of the solutions could be guaranteed, for example, by using augmentation terms as discussed in Chapter 1 (see also (Miettinen and Mäkelä, 2006)). The idea of classification is quite simple for the DM. However, it may be difficult to estimate how much the other functions should be relaxed in order to potentiate the desired amounts of improvement in the others. The next method aims at resolving this kind of a difficulty.

2.4.3 Satisficing Trade-off Method

The satisficing trade-off method (STOM) (Nakayama, 1995; Nakayama and Sawaragi, 1984) is based on ideas very similar to those in reference point approaches. As its name suggests, it concentrates on finding a satisficing solution (see Chapter 1).

The DM is asked to classify the objective functions at $\mathbf{z}^h$ into three classes. The classes are the objective functions whose values should be improved $I^{\leq}$, the functions whose values can be relaxed $I^{\geq}$ and the functions whose values are acceptable as such $I^{=}$. The DM is supposed to specify desirable aspiration levels for functions in $I^{\leq}$. Here, $I^{\leq} \cup I^{\geq} \cup I^{=} = \{1, \ldots, k\}$.

Because of so-called *automatic trade-off*, the DM only has to specify desirable levels for functions in $I^{\leq}$ and the upper bounds for functions in $I^{\geq}$ are derived from trade-off rate information. The idea is to decrease the burden set on the DM so that the amount of information to be specified is reduced. Functions are assumed to be twice continuously differentiable. Under some special assumptions, trade-off information can be obtained from the KKT multipliers related to the scalarizing function used (corresponding to Section 2.2).

By putting together information about desirable function values, upper bounds deduced using automatic trade-off and current acceptable function values, we get a reference point $\bar{\mathbf{z}}^h$. Different scalarizing functions can be used in STOM but in general, the idea is to minimize the distance to the utopian objective vector $\mathbf{z}^{\star\star}$. We can, for example, solve the problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{i=1,\ldots,k} \left[\frac{f_i(\mathbf{x}) - z_i^{\star\star}}{\bar{z}_i^h - z_i^{\star\star}} \right] + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x})}{\bar{z}_i^h - z_i^{\star\star}} \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{2.12}$$

where we must have $\bar{z}_i > z_i^{\star\star}$ for all $i = 1, \ldots, k$. It can be proved that the solutions obtained are properly Pareto optimal . Furthermore, it can be proved that the solution $\mathbf{x}^*$ is satisficing (see Chapter 1) if the reference point is feasible (Sawaragi *et al.*, 1985). This means that $f_i(\mathbf{x}^*) \leq \bar{z}_i^h$ for all i. Let us point out that if some objective function f_i is not bounded from below in S, then some small scalar value can be selected as $z_i^{\star\star}$. Other weighting coefficients can also be used instead of $1/(\bar{z}_i^h - z_i^{\star\star})$ (Nakayama, 1995).

The solution process can be started, for example, by asking the DM to specify a reference point and solving problem (2.12). Then, at this point, the

DM is asked to classify the objective functions and specify desirable aspiration levels for the functions to be improved. The solution process continues until the DM does not want to improve or relax any objective function value.

In particular, if the problem has many objective functions, the DM my appreciate the fact that (s)he does not have to specify upper bound values. Naturally, the DM may modify the calculated values if they are not agreeable. It is important to note that STOM can be used even if the assumptions enabling automatic trade-off are not valid. In this case, the DM has to specify both aspiration levels and upper bounds.

2.4.4 NIMBUS Method

The NIMBUS method is described in (Miettinen, 1999; Miettinen and Mäkelä, 1995, 1999, 2000; Miettinen *et al.*, 1998) and it is based on the classification of the objective functions into up to five classes. It has been developed for demanding nonlinear multiobjective optimization. In NIMBUS, the DM can classify objective functions at $\mathbf{z}^h$ into any of the five classes introduced at the beginning of this section, that is, functions to be decreased $I^<$ and to be decreased till an aspiration level $I^\leq$, functions that are satisfactory at the moment $I^=$, functions that can increase till an upper bound $I^\geq$ and functions that can change freely $I^\diamond$ and $I^< \cup I^\leq \cup I^= \cup I^\geq \cup I^\diamond = \{1, \ldots, k\}$. We assume that we have the ideal objective vector and a good approximation of the nadir objective vector available.

The difference between the classes $I^<$ and $I^\leq$ is that the functions in $I^<$ are to be minimized as far as possible but the functions in $I^\leq$ only till the aspiration level (specified by the DM). There are several different variants of NIMBUS but here we concentrate on the so-called synchronous method (Miettinen and Mäkelä, 2006). After the DM has made the classification, we form a scalarizing function and solve the problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\substack{i \in I^< \\ j \in I^\leq}} \left[\frac{f_i(\mathbf{x}) - z_i^\star}{z_i^{\text{nad}} - z_i^{\star\star}}, \frac{f_j(\mathbf{x}) - \hat{z}_j}{z_j^{\text{nad}} - z_j^{\star\star}} \right] + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x})}{z_i^{\text{nad}} - z_i^{\star\star}} \\ \text{subject to} & f_i(\mathbf{x}) \leq f_i(\mathbf{x}^h) \ \text{ for all } \ i \in I^< \cup I^\leq \cup I^=, \\ & f_i(\mathbf{x}) \leq \varepsilon_i \ \text{ for all } \ i \in I^\geq, \\ & \mathbf{x} \in S, \end{array} \tag{2.13}$$

where $\rho > 0$ is a relatively small scalar. The weighting coefficients $1/(z_j^{\text{nad}} - z_j^{\star\star})$ (scaling the objective function values) have proven to facilitate capturing the preferences of the DM well. They also increase computational efficiency (Miettinen *et al.*, 2006). By solving problem (2.13) we get a provably (properly) Pareto optimal solution that satisfies the classification as well as possible.

In the synchronous NIMBUS method, the DM can ask for up to four different Pareto optimal solutions be generated based on the classification once expressed. This means that solutions are produced that take the classification information into account in slightly different ways. In practice, we form a reference point $\bar{\mathbf{z}}$ based on the classification information specified as follows: $\bar{z}_i = z_i^\star$ for $i \in I^<$, $\bar{z}_i = \hat{z}_i$ for $i \in I^\leq$, $\bar{z}_i = z_i^h$ for $i \in I^=$, $\bar{z}_i = \varepsilon_i$ for $i \in I^\geq$ and $\bar{z}_i = z_i^{\mathrm{nad}}$ for $i \in I^\diamond$. (This, once again, demonstrates the close relationship between classification and reference points.) Then we can use reference point based scalarizing functions to generate new solutions. In the synchronous NIMBUS method, the scalarizing functions used are those coming from STOM (problem (2.12) in Section 2.4.3), reference point method (problem (1.11) defined in Chapter 1) and GUESS (Buchanan, 1997). See (Miettinen and Mäkelä, 2002) for details on how they were selected. Let us point out that all the solutions produced are guaranteed to be properly Pareto optimal.

Further details and the synchronous NIMBUS algorithm are described in (Miettinen and Mäkelä, 2006). The main steps are the following: Once the DM has classified the objective functions at the current solution $\mathbf{z}^h$ and specified aspiration levels and upper bounds, if needed, (s)he is asked how many new solutions (s)he wants to see and compare. As many solutions are generated (as described above) and the DM can select any of them as the final solution or as the starting point of a new classification. It is also possible to select any of the solutions generated so far as a starting point for a new classification. The DM can also control the search process by asking for a desired number of intermediate solutions to be generated between any two interesting solutions found so far. In this case, steps of equal length are taken in the decision space and corresponding objective vectors are used as reference points to get Pareto optimal solutions that the DM can compare.

The starting point can be, for example, a neutral compromise solution (see problem (1.6) in Chapter 1) or any point specified by the DM (which has been projected to the Pareto optimal set). In NIMBUS, the DM expresses iteratively her/his desires. Unlike some other methods based on classification, the success of the solution process does not depend entirely on how well the DM manages in specifying the classification and the appropriate parameter values. It is important to note that the classification is not irreversible. Thus, no irrevocable damage is caused in NIMBUS if the solution obtained is not what was expected. The DM is free to go back or explore intermediate points. (S)he can easily get to know the problem and its possibilities by specifying, for example, loose upper bounds and examining intermediate solutions.

The method has been implemented as a WWW-NIMBUS® system operating on the Internet (Miettinen and Mäkelä, 2000, 2006). Via the Internet, the computing can be centralized to one server computer and the WWW is a way of distributing the graphical user interface to the computers of each individual user and the user always has the latest version of the method available. The most important aspect of WWW-NIMBUS® is that it is easily accessible and available to any academic Internet user at http://nimbus.it.jyu.fi/. For a discussion on how to design user interfaces for a software implementing a classification-based interactive method, see (Miettinen and Kaario, 2003). (When the first version of WWW-NIMBUS® was implemented in 1995 it was a pioneering interactive optimization system on the Internet.) Another implementation IND-NIMBUS® for MS-Windows and Linux operating systems also exists (Miettinen, 2006) (see Chapter 12). Many successful applications, for example, in the fields of optimal control, optimal shape design and process design have shown the usefulness of the method (Hakanen *et al.*, 2005, 2007; Hämäläinen *et al.*, 2003; Heikkola *et al.*, 2006; Miettinen *et al.*, 1998).

2.4.5 Other Classification-Based Methods

Let us briefly mention some more classification-based methods. Among them are the interactive reference direction algorithm for convex nonlinear integer problems (Vassilev *et al.*, 2001) which uses three classes $I^{\leq}$, $I^{\geq}$ and $I^{=}$ and the reference direction approach for nonlinear problems (Narula *et al.*, 1994) using the same three classes and generating several solutions in the reference direction (pointing from the current solution towards the reference point). Furthermore, the interactive decision making approach NIDMA (Kaliszewski and Michalowski, 1999) asks for both a classification and maximal acceptable global trade-offs from the DM.

A method where NIMBUS (see Subsection 2.4.4) is hybridized with the feasible goals method (Lotov *et al.*, 2004) is described in (Miettinen *et al.*, 2003). Because the feasible goals method produces visual interactive displays of the variety of feasible objective vectors, the hybrids introduced help the DM in getting understanding of what kinds of solutions are feasible, which helps when specifying classifications for NIMBUS. (There are also classification-based methods developed for MOLP problems which we do not touch here.)

2.5 Discussion

Due to the large variety of interactive methods available in the literature, it is a hard task to choose the most appropriate method for each decision situation. Here, a "decision situation" must be understood in a wide sense: a DM, with a given attitude (due to many possible facts) facing (a part of) a decision problem. This issue will be discussed in detail in Chapter 15. In order to accommodate the variety of methods in a single decision system, some authors have already proposed the creation of open architectures or combined systems (Gardiner and Steuer, 1994; Kaliszewski, 2004; Luque *et al.*, 2007b). Some of such integrated systems have already been implemented. For example, MKO and PROMOIN are described in Chapter 12. It is also worth pointing out that some relations among the different types of information (like weights, trade-offs, reference points etc.) that the interactive methods may require from the DM, are investigated in (Luque *et al.*, 2007a).

One direction for developing new, improved, methods is hybridizing advantages of different methods in order to overcome their weaknesses. For example, hybridizing a posteriori and interactive methods has a lot of potential. The

DM can, for example, first get a rough impression of the possibilities of the problem and then can interactively locate the most preferred solution. One approach in this direction was already mentioned with NIMBUS (Lotov *et al.*, 2004). Another idea is to combine reference points with an algorithm that generates an approximation of the Pareto optimal set (Klamroth and Miettinen, 2008). This means that only those parts of the Pareto optimal set are approximated more accurately that the DM is interested in and (s)he can conveniently control which parts to study by using a reference point. Steuer *et al.* (1993) provide an example of hybridizing ideas of two interactive methods by combining ideas of the Tchebycheff method and reference point approaches.

When dealing with human DMs, behavioural issues cannot be ignored. For example, some points of view in this respect are collected in (Korhonen and Wallenius, 1996). Let us also mention an interesting practical observation mentioned by (Buchanan, 1997). Namely, DMs seem to be easily satisfied if there is a small difference between their hopes and the solution obtained. Somehow they feel a need to be satisfied when they have almost achieved what they wanted for even if they still were in the early steps of the learning process. In this case they may stop iterating 'too early.' Naturally, the DM is allowed to stop the solution process if the solution really is satisfactory but the coincidence of setting the desires near an attainable solution may unnecessarily increase the DM's satisfaction (see also Chapter 15).

2.6 Conclusions

We have characterized some basic properties of interactive methods developed for multiobjective optimization and considered three types of methods based on trade-offs, reference points and classification. An important requirement for using interactive methods is that the DM must have time and interest in taking part in the iterative solution process. On the other hand, the major advantage of these methods is that they give the DM a unique possibility to learn about the problem considered. In this way, the DM is much better able to justify why the final solution is the most preferred one.

As has been stressed, a large variety of methods exists and none of them can be claimed to be superior to the others in every aspect. When selecting a solution method, the opinions of the DM are important because (s)he must feel comfortable with the way (s)he is expected to provide preference information. In addition, the specific features of the problem to be solved must be taken into consideration. One can say that selecting a multiobjective optimization method is a problem with multiple objectives itself.

When dealing with interactive methods, the importance of user-friendliness is emphasized. This is also a topic for future research. Methods must be even better able to correspond to the characteristics of the DM. If the aspirations of the DM change during the solution process, the algorithm must be able to cope with this situation.

DMs want to feel in control of the solution process and, consequently, they must understand what is happening. Thus, the preferences expressed must be reflected in the Pareto optimal solutions generated. But if the DM needs support in identifying the most preferred region of the Pareto optimal set, this should be available, as well. Thus, the aim is to have methods that support learning so that guidance is given whenever necessary. The DM can be supported by using visual illustrations (see, e.g. Chapters 8 and 9) and further development of such tools is essential. In particular, when designing DSSs for DMs, user interfaces play a central role. Special-purpose methods for different areas of application that take into account the characteristics of the problems are also important.

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