The Fourier Series

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1 Introduction

1.1 The Importance and History of Fourier Series

The Fourier series was originally developed to solve partial differential equations arising from heat flow and vibration. Before the time of Fourier, people like D'Alembert tried and were able to explain these interesting physical phenomena, specifically vibrations, without using the rigorous ideas of real analysis. When Fourier was posed with the challenge of explaining how to predict the temperature of every point on a disk given a function of the temperature along the disk's boundary, he came up with what we call Fourier series today.

Although Fourier's work was monumental and one of the most influential pieces of physics and mathematics, Fourier was criticized by Laplace and Lagrange for his lack of rigor. A lot of Fourier's claims came with little proof and the definitions of various ideas like "functions" and "graphs" were still undeveloped at the time.

Many mathematicians from Dirichlet to Riemann to Cantor to Fejer understood the importance of Fourier's work and spent the next few decades making it rigorous. Arguably, Fourier analysis was the seed that led to the development of most of the tools we see in real analysis from Cantor's theory to Riemann integrable to the notion of a function. All these ideas had the end goal of helping understand Fourier series and in so doing, formed a major part of modern analysis. Given this historical context, it is quite important to understand the mechanics of Fourier series and the rigorous proof that took many years to develop.

In this paper, we will go over the proof of the piecewise convergence of the Fourier series. We will show how functions that are piecewise continuous and piecewise differentiable can be approximated by a Fourier series. We begin by defining what are called the "Fourier coefficients" and explaining the intuition behind them. Then, we define the Dirichlet kernel and some of its properties. Although the Dirichlet kernel may seem unrelated at first, we show how it can be used to simplify the partial sum of the Fourier series. Afterwards, we analyze convergence in two cases: convergence on points that are and are not differentiable. Finally, we use the convergence on points that are not differentiable to segue into an exploration of an interesting phenomenon related to Fourier series, the Gibbs Phenomenon. The Gibbs phenomenon, counter-intuitively, shows that the Fourier series will always overshoot a jump discontinuity, no matter the resolution.

2 Definitions

2.1 The Fourier Series

Given a 2π-periodic function $f(x)$ that is piecewise continuous and piecewise differentiable on an interval I , its Fourier series on I is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \tag{1}$$

The coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx, \tag{2}$$

for $m \geq 1$—provided these integrals are defined.

As with most series, we usually like to to talk about the partial sums of the the Fourier series. That is,

$$S_N(f,x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx). \tag{3}$$

When we say that the Fourier series can be evaluated at x, it is equivalent to saying that the Fourier Series (1) converges at x.

2.2 The Fourier Coefficients

Before we dive in, let us discuss some intuition for these coefficients and explain why they are correct. We start off with the idea of a Taylor series. The Taylor series for an arbitrary function f(x) is:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... \tag{4}$$

To find a_0 all we need to do is plug in $f(0)$ since all the other terms go to 0 since they have a factor of x. We may do the same for a_1 but this time with $f'(0)$ and so on. We would expect to be able to do the same thing for the coefficients of a Fourier series, after all we are talking about series that can represent a function. It turns it out that it is not at easy as just plugging in some value to $f(x)$, yet we can still rely on f(x).

In order to make this discussion more efficient let us call

$$g(n,x) = a_n cos(nx) + b_n sin(nx). \tag{5}$$

That is, $g(n,x)$ is the terms inside the summation of the Fourier series. We can quickly observe that if we were to take the integral on $[\pi, -\pi]$ of $g(n,x)$ it will evaluate to 0 $n > 1$. In other words,

$$\int_{-\pi}^{\pi} g(n,x)dx = a_n \int_{-\pi}^{\pi} cos(nx)dx + b_n \int_{-\pi}^{\pi} sin(nx)dx = 0 \tag{6}$$

since both sin(nx) and cos(nx) are 2π periodic.

Based on this, when we take the integral of [1] to get the average value. We expect that

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} g(n,x)dx \\
&= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} g(n,x)dx \\
&= \int_{-\pi}^{\pi} a_0 dx \\
&= 2\pi a_0
\end{aligned} \tag{7}$$

by applying what we found in [6]. The above equation implies our original definition:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \tag{8}$$

(Note that there is a leap of faith in our argument as we move the integral into an infinite series. This leap usually does not hold true, but in this case we assume the trigonometric series behaves "nice" enough for us to make this assumption). Sadly, we cannot proceed with this nice relationship to explain the other terms. We must invoke the idea of orthogonality. In other words,

$$\int_{-\pi}^{\pi} \sin kx \cos nx dx = 0 \text{ this always holds} \tag{9}$$

$$\int_{-\pi}^{\pi} \sin kx \sin nx dx = \begin{cases} 0 & \text{if } k \neq n \\ \pi & \text{if } k = n \neq 0 \\ 0 & \text{if } k = n = 0 \end{cases} \tag{10}$$

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \begin{cases} 0 & \text{if } k \neq n \\ \pi & \text{if } k = n \neq 0 \\ 2\pi & \text{if } k = n = 0 \end{cases} \tag{11}$$

If we look back at our equations for the coefficients and the original definition of the Fourier series, we may begin to understand why the Fourier coefficients are defined the way they are. If we want to find the nth coefficient for the cos term all we need to do is multiply through by $\cos(nx)$ where $n > 0$. When we do so, we get:

$$f(x)\cos(nx) = a_0 \cos(mx) + \sum_{m=1}^{\infty} a_m \cos(mx)\cos(nx) + b_n \sin(mx)\cos(nx). \tag{12}$$

Now, in order to use the orthogonality relationship, we can integrate both sides from $-\pi$ to π to get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos(nx) = &\int_{-\pi}^{\pi} a_0 \cos(mx)dx \\ &+ \sum_{m=1}^{\infty} [a_m \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx + b_n \int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx]. \end{aligned} \tag{13}$$

We can immediately apply our orthogonality conditions now to get:

$$\int_{-\pi}^{\pi} f(x)\cos(nx) = \pi a_n \implies a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{14}$$

We can similarly do the same b_n to get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{15}$$

Another interesting way to think about this is based on inner products. We can think of sin and cos as "orthogonal" functions (hence the name) and think of the integral as the inner product of the two functions.

3 Convergence of Fourier Series

In this section, we set out the prove the following theorem:

Theorem 3.1 *Given a function f(x) that is piecewise continuous and differentiable, the partial sum of the Fourier series, S_N, converges piecewise to f(x) at points of continuity and converges to $\frac{f(x^+)+f(x^-)}{2}$ at points of discontinuity.*

3.1 The Dirichlet Kernel

When we substitute the values of the coefficients from equation [2] to the summation of the partial sum in equation 3, we get the following:

$$\begin{aligned} S_N(f,x) = &\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy \\ &+ \sum_{n=1}^{N} [a_n \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy\cos(nx) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy\sin(nx)]. \end{aligned} \tag{16}$$

This can be simplified to

$$S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1+2\sum_{n=1}^{N}\cos(ny)\cos(nx)+\sin(ny)\sin(nx)]f(y)dy. \tag{17}$$

Our goal is to make the long equation for the partial sums ,S_N, look more compact and easier to deal with. We recall the old fact that

$$cos(A-B) = cos(A)cos(B) + sin(A)sin(B) \tag{18}$$

We can see right away that the element being summed can be written in a simplified form using equation [11] that is,

$$S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1+2\sum_{n=1}^{N}\cos(n(x-y))]f(y)dy. \tag{19}$$

We call this new form of the integrand the Dirichlet kernel. More formally,

$$D_N(x) = \frac{1}{2\pi}(1+2\sum_{n=1}^{N}\cos(nx)). \tag{20}$$

Thus, plugging this in, we get our compact form that we wanted:

$$S_N(f,x) = \int_{-\pi}^{\pi} D_N(x-y)f(y)dy. \tag{21}$$

It is important to notice here that this definition of D_N implies something important about its integral. Since we know that for $n \in \mathbb{N}$ for $n > 0$

$$\int_{-\pi}^{\pi}\cos(nx)dx = 0. \tag{22}$$

It is easy to see that an integral of $D_N(x)$ will yield 1 through the following set of steps:

$$\begin{aligned}
\int_{-\pi}^{\pi} D_N(x) &= \int_{-\pi}^{\pi}\frac{1}{2\pi}(1+2\sum_{n=1}^{N}\cos(nx))dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}1dx + \int_{-\pi}^{\pi}2\sum_{n=1}^{N}\cos(nx)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}1dx + 2\sum_{n=1}^{N}\int_{-\pi}^{\pi}\cos(nx)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}1dx + 2\sum_{n=1}^{N}0 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}1dx = 1.
\end{aligned} \tag{23}$$

While we are here, it would be nice to show another way to define $D_N(x)$ that will be helpful later on. Using another trigonometric identity,

$$2sin(A)cos(B) = sin(A+B) + sin(A-B), \tag{24}$$

we can see that if we were to compute the product of $\sin(\frac{x}{2})$ and $D_N(x)$ we will get:

$$\sin\left(\frac{x}{2}\right)D_N(x) = \frac{1}{2\pi}[\sin\left(\frac{x}{2}\right)) + \sum_{n=1}^{N}2\cos(nx)\sin\left(\frac{x}{2}\right)]. \tag{25}$$

Using equation [15] this can be simplified to

$$\sin\left(\frac{x}{2}\right) D_N(x) = \frac{1}{2\pi}[\sin\left(\frac{x}{2}\right) + \sum_{n=1}^{N} \sin((n+\frac{1}{2})x) - \sin\left((n+\frac{1}{2})x\right)]. \tag{26}$$

We quickly recognize that the equation on the right is a telescoping since we are dealing with integer multiples of n. Our sum ends up turning into

$$\sin\left(\frac{x}{2}\right) D_N(x) = \frac{1}{2\pi} \sin\left((N+\frac{1}{2})x\right). \tag{27}$$

Thus, we now have a new way of writing $D_N(x)$ as

$$D_N(x) = \frac{1}{2\pi} \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}. \tag{28}$$

3.2 Showing Convergence

One of the main goals of proving the Fourier series is to show that it actually converges to the function of interest. If the Fourier series did not converge to the function of interest there would be little or no use for Fourier series. Although it would be nice to see that the Fourier transform converges to all functions we set out to approximate, this is not the case and it is not binary. We need to define the specific type of convergence and impose the conditions on f to ensure the specific convergence. The fact that we brought up these two ideas shows how broad and complex the convergence of Fourier series is. Definitely, not something we can cover completely in this paper. Instead, we will focus on putting conditions on f and dealing with point-wise convergence (that is the partial sums converge to each point they are attempting to approximate).

We start by restricting f(x) to be 2π period and piecewise continuously differentiable on I. Before we proceed let us recall a fact and make it more general. In equation 14, we showed that

$$\int_{-\pi}^{\pi} D_N(x) = 1.$$

Recall that D_N is 2π-periodic function. This implies more generally that

$$\int_{-\pi}^{\pi} D_N(x-y) = 1. \tag{29}$$

Now, we immediately use this fact to make an interesting, yet subtle, remark about the convergence of the Fourier series to f. To quantify this convergence, we will find the difference between the partial sums and the function to attempt to find some relationship that can help us determine whether $S_N(x)$ is getting "closer" to f(x). If we compute the difference, we can make some interesting simplifications,

$$\begin{aligned} S_N(f,x) - f(x) &= \int_{-\pi}^{\pi} D_N(x-y)f(y)dy - f(x) \\ &= \int_{-\pi}^{\pi} D_N(x-y)f(y)dy - (1)f(x) \\ &= \int_{-\pi}^{\pi} D_N(x-y)f(y)dy - \int_{-\pi}^{\pi} D_N(x-y)dyf(x) \\ &= \int_{-\pi}^{\pi} D_N(x-y)(f(y)-f(x))dy. \end{aligned} \tag{30}$$

If we didn't prove equation 14 and the generalization for the integral of the Dirichlet kernel, we would not have been able to bring f(x) into the integral and parentheses as we did.

Since f is differentiable at some points, let us choose one of these points, x, and define a function $g(y)$ such that

$$g(y) = \frac{f(y)-f(x)}{x-y}. \tag{31}$$

We know $g(y)$ is piecewise continuous on the interval $[-\pi, \pi]$ since f(x)(a constant) and f(y) is piecewise continuous, their sum is also piecewise continuous. In addition, the only candidate for discontinuity is when y approaches x; however, since we chose x to be differentiable, we know this limit exists, since, in the limit, it is equivalent to the derivative of f(x).

Now we can see similarly that

$$p(y) = \frac{f(y) - f(x)}{2\pi \sin((x-y)/2)} \tag{32}$$

is also piecewise differentiable, following from the fact that g(y) is piecewise differentiable. Now, we can rewrite our equation for $S_N(f,x) - f(x)$ as

$$\begin{aligned} S_N(f,x) - f(x) &= \int_{-\pi}^{\pi} D_N(x-y)(f(y) - f(x))dy \\ &= \int_{-\pi}^{\pi} \sin((n+1/2)(x-y))(p(x)))dy. \end{aligned} \tag{33}$$

This comes straight from substituting our alternative definition of D_N from the previous section. Now we have an interesting function that will help us determine the convergence of S_N to f(x).

3.3 The Riemann-Lebesgue Lemma and Its Implications

Looking at the equation we came up with above, right away we can make some intuitive remarks before we dive into a formal proof. As N gets larger in this equation, the frequency of the sin term increases, this leads to more "wiggles" that have an average of 0. As these wiggles increase and get thinner (i.e. more compact) any variation from wiggle to wiggle becomes negligible. Effectively, due to this thinning and similarity the positive and negative parts of the wiggle begin to "cancel out" and end up leading the sum to evaluate to 0.

Now comes the time for the formal proof, which as the title of this section suggests will invoke, the Riemann-Lebesgue Lemma to make our musings in the previous paragraph precise.

We begin by recalling that $p(x)$ defined in the previous section in equation [25] is continuous and 2π periodic.

Now, we introduce the Riemann-Lebesgue Lemma.

Theorem 3.2 *Riemann-Lebesgue Lemma*
Let f be Riemann integrable on [a,a+2π]

$$\lim_{n\to\infty} \int_{a}^{a+2\pi} f(t)\sin(nt)dt = 0$$

The Riemann-Lebesgue lemma basically formalizes the intuition that we discussed above. It says that as we increase the frequency of the sin(nx) term by taking the limit of n, the entire integral vanishes on the interval of $[a, a+2\pi]$.

Thus, now using the Riemann-Lebesgue lemma, we can make the following observation that

$$\begin{aligned} \lim_{N\to\infty} S_N(f,x) - f(x) &= \lim_{N\to\infty} \int_{-\pi}^{\pi} \sin((n+1/2)(x-y))(p(x)))dy \\ &= 0 \end{aligned} \tag{34}$$

where $a = -\pi$ and n is our limiting variable.

Now, we have proved that the partial sums of the Fourier series converges point-wise to points where f is differentiable. In the case that f is discontinuous at the point x, we take a slightly different approach. We recall that

$$\int_{-\pi}^{\pi} D_N(x)dx = 1;$$

thus, since $D_N(x)$ is even, we may split the integral around the origin to get

$$\int_0^\pi D_N(x)dx = \int_{-\pi}^0 D_N(x)dx = \frac{1}{2}. \tag{35}$$

Before, to get our desired result, we simply just subtracted $S_N(x)$ by $f(x)$, but now since we are dealing with an x that has a jump, we consider the average of the two "sides" of the jump. That is

$$S_N(x) - \frac{f(x^+) + f(x^-)}{2}.$$

Using the 2π periodicity of D_N we can bring $\frac{f(x^+)+f(x^-)}{2}$ into the integral by splitting the integral and using the equality in [35], to write

$$\begin{aligned} S_N(x) - \frac{f(x^+) + f(x^-)}{2} &= \frac{1}{2}\int_{x-\pi}^{x} D_N(x-y)[f(y) - f(x^-)]dy \\ &+ \frac{1}{2}\int_{x}^{x+\pi} D_N(x-y)[f(y) - f(x^+)]dy. \end{aligned} \tag{36}$$

Now that we have these integrals, our argument converges to our last argument since we are dealing with the respective values of f(x) on each side of the jump. This way, we can recreate our function $p(x)$ for each of the integrals above. Then, we may reapply the Riemann Lebesgue Lemma to both integrals to get that $S_N(x)$ converges to the average of the jump.

Thus, we have now shown that if f is periodic and piecewise differentiable, then at each point x where f(x) is continuous, the partial sums converge to f(x) and each point where x is not differentiable, the partial sums converge to $\frac{f(x^+)+f(x^-)}{2}$. This concludes the proof and illustrates one of the many ways a function can have a convergent Fourier series.

4 Exploration - The Gibbs Phenomenon

At the end of the last section, we dealt with points where f is discontinuous. It turns out that these points hold some very interesting properties. Earlier, we proved that the partial sums converge to $\frac{f(x^+)+f(x^-)}{2}$. Thus, the series does not necessarily converge to the function at those points unless f is defined to be $\frac{f(x^+)+f(x^-)}{2}$ at the discontinuity.

When we examine a neighborhood around the discontinuity and begin using a higher and higher n to approximate the function. We find that the Fourier partial sums consistently overshoot the point. This phenomenon is called the Gibbs phenomenon. In this exploration, we will closely examine the error as n increases. One would think that the error would go to 0 as n increases but it turns out that the error only decreases in the interval where f(x) is continuous but not on the entire interval. We will examine this by taking possibly the simplest discontinuous function.

Let us try to approximate the step function, often denoted as $h(x - x_0)$ where x_0 is where the step occurs, with Fourier series. Let us deal with $h(x)$, that is when we have a step at 0 (that goes from 0 to 1). However, one may notice a problem right away. In the proof in the previous section, we assumed our function was 2π periodic, however the the step function is clearly not 2π periodic. To deal with this, let us cut up the the step function so that we only have the step function, h(x), on the interval $[-\pi, \pi]$. Once we have this section, we paste it back to back so as to create a 2π periodic step function.

Now that we have our conditions satisfied, we can dive in. Using our definition of $S_N(x)$, we recall

$$S_N(f,x) = \int_{-\pi}^{\pi} D_N(x-y)f(y)dy.$$

In this case, since we are dealing with the step function, we can write it in a more specific form

$$S_N(f,x) = \int_{-\pi}^{0} D_N(x-y)(0)dy + \int_0^\pi D_N(x-y)(1)dy = \int_0^\pi D_N(x-y)dy. \tag{37}$$

We may change the variables in our equation by adding a new variable $v = x - y$ to get

$$S_N(f,x) = \int_{x-\pi}^{x} D_N(v)dv.$$

This looks very similar to step we made when analyzing the convergence of the Fourier series of a jump discontinuity. In fact, if we return to equation [35], plug it in and play with the bounds of integration, we can get an equality for this expression. That is,

$$S_N(f,x) = \frac{1}{2} + \int_{0}^{x} D_N(v)dv - \int_{-\pi}^{-\pi+x} D_N(v)dv. \tag{38}$$

Now we wish to choose some sequence of points that limit to 0 so that we can analyze the behavior of the Fourier transform as the bounds are tightened, that is when N is increased.

When we choose this sequence, it may seem arbitrary, but as we work out the proof, you will see why such a definition of the sequence was helpful. If we were to randomly choose a sequence that converges to 0, we would probably go with $x_N = \frac{1}{N}$, but we will go with a slightly modified version of $x_N = \frac{k}{N+\frac{1}{2}}$ where k is some constant that is greater than 0.

Before we proceed with using this sequence, we make the observation that $D_N(-\pi) = \frac{(-1)^N}{(2\pi)}$. One can see this easily by plugging in $-\pi$ into our second definition of D_N. Now, this finding implies that $D_N(y) < \frac{1}{2}$ when $-\pi \leq y \leq -\pi + x_N$ for an n sufficiently large. We have used our sequence x_N here to constrain the bounds of y. We know the inequality is true since $D_N(y)$ is continuous around $-\pi$ and its value at $-\pi$ is less than $\frac{1}{2}$.

By the inequality, it follows that

$$\left| \int_{-\pi}^{-\pi+x_N} D_N(y)dy \right| \leq \frac{x_N}{2} \leq \frac{k}{2n}. \tag{39}$$

One may think of this geometrically. We know the height is at most $\frac{1}{2}$ and the width is x_N, so we expect the integral to be less than their product. The second part of the inequality is achieved by simply substituting in the value of x_N.

Now, we define a new kernel, we call it E that is defined as follows

$$E_N(x) = \frac{sin((n+\frac{1}{2})x)}{\pi x}. \tag{40}$$

We will see shortly why this kernel is useful. It will allow us to do some interesting algebraic footwork and lead us to remarkable insight about the problem.

Since we have a new kernel and previously what we did with the kernel was give a bound for its area, it might seem fruitful to find a bound for the difference of areas of the kernels.

We recall that

$$D_N(x) = \frac{1}{2\pi} \frac{sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}.$$

Now, we take the difference of the two kernels to get

$$\begin{aligned} D_N - E_N &= \frac{1}{2\pi} \frac{sin(\left(n+\frac{1}{2}\right)x)}{\sin\left(\frac{x}{2}\right)} - \frac{sin((n+\frac{1}{2})x)}{\pi x} \\ &= \frac{2}{\pi} sin\left(\left(n+\frac{1}{2}\right)x\right)\left(\frac{1}{\sin\left(\frac{x}{2}\right)} - \frac{2}{x}\right) \\ &\leq \frac{1}{\sin\left(\frac{x}{2}\right)} - \frac{2}{x}. \end{aligned} \tag{41}$$

One can see directly that the limit as x approaches 0, squeezes the difference to 0 (using L'Hospital's rule), that is ,

$$\lim_{x\to\infty}\left(\frac{1}{\sin(\frac{x}{2})}-\frac{2}{x}\right)=0. \tag{42}$$

Now, since we know the difference limits to 0, we can do something similar to what we did with the Drichilet kernel by itself and say that for some $N \in \mathbb{N}$ that $D_N(x_N) < \frac{1}{2}$ and $E_N(x_N) < \frac{1}{2}$ for all $n > N$. This implies

$$\left|\int_0^{x_N} D_N(x)dx - \int_0^{x_N} E_N(x)dx\right| < \frac{x_N}{2} < \frac{k}{2n}. \tag{43}$$

We can also play around with the integral of the new kernel, E_N alone. For example, if we make the substitution $u = ((n + 1/2) * x)$ to the integral of E_N, we get an interesting expression,

$$\int_0^{x_N} E_N(x)dx = \int_0^k \frac{sin(u)}{\pi u}dx = \frac{1}{\pi}\int_0^k \frac{sin(u)}{\pi u}dx. \tag{44}$$

The mechanics of the substitution are $du = (n + 1/2)dx$ and changing the bounds of integration from x_N to $(x_N)(n + 1/2) = k$.

This interesting integral that we have just bumped into is well studied and has a name, the "Integral Sine Function". The integral sine function is often represented as

$$Si(k) = \int_0^k \frac{sin(u)}{\pi u}dx. \tag{45}$$

It is known that Si(k) achieves its max value at $k = \pi$ where $Si(\pi) = 1.85194$.

Now, we put everything that we have done together. If we take the sequence

$$x_N = \frac{2\pi}{2n+1} \tag{46}$$

we had originally and going back to our original expression for the partial sums, we had:

$$S_N(f,x) = \frac{1}{2} + \int_0^x D_N(v)dv - \int_{-\pi}^{-\pi+x} D_N(v)dv$$

If we reorganize this slightly and substitute the following equations:

$$\left|\int_{-\pi}^{-\pi+x_N} D_N(y)dy\right| \leq \frac{x_N}{2} \leq \frac{k}{2n}$$

$$\left|\int_0^{x_N} D_N(x)dx - \int_0^{x_N} E_N(x)dx\right| < \frac{x_N}{2} < \frac{k}{2n}$$

we get

$$S_N(h,x) = \frac{1}{2} + \int_0^{x_N} E_N(x)dx \tag{47}$$

Now, we can use our substitutions to get:

$$\begin{aligned}\lim_{n\to\infty} S_N(h,x_N) &= \frac{1}{2} + \int_0^{x_N} E_N(x)dx \\ &= \frac{1}{2} + \frac{Si(k)}{\pi} = 1.08949\end{aligned} \tag{48}$$

Thus, we see that as the bounds (i.e. x_N) get tighter and tighter, the Fourier series persistently overshoots the jump in h(x). Based on our calculation above, this overshoot is approximately 9% of the jump. It turns out that this happens for all jump discontinuities that you try to approximate with Fourier series. It has a very interesting history as it was not immediately recognized with the development of signal processing. This phenomenon is generally called the Gibbs phenomenon.