Note on the Pressure in the Euler and Navier-Stokes Equations

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1 Overview

In this brief note, we'll discuss how we can solve for the pressure scalar field p in terms of the velocity vector field u in the Euler and Navier-Stokes equations. In an effort to keep this note (very) brief and not obscure what's important, we'll assume familiarity with the methods and definitions of analysis necessary here, and provide some excellent resources for readers to fill in the gaps.

2 Introduction

We'll consider here the *incompressible Navier-Stokes equations*, which are a system of $d+1$ differential equations describing the time evolution of a vector field $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ which gives the velocity of a continuum Newtonian fluid at point (x, t) in space time. If $p : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a scalar field giving the pressure at (x, t), then we have the following.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u \\ \nabla \cdot u = 0 \end{cases} \tag{1}$$

[1].The constant $\nu > 0$ is the *kinematic viscosity*, and "$\nabla \cdot u$" is the incompressibility constraint. See [2] for the derivation of these equations from the laws of Newtonian mechanics, as well as further discussion of the equations. We'll note that, when $\nu = 0$, the above become the *Euler equations* for ideal fluids. There are notable distinctions between the dynamics of these different systems of equations, but we'll note that the result we're to discuss also holds identically for the Euler equations.

One interesting thing about this system is that it's overdetermined. We have $d+1$ equations with d unknown scalar functions on $\mathbb{R}^d$ (those being, for

[1] A few notes on terminology and notation. As far as terminology goes, roughly speaking, a fluid is *incompressible* if it's 'volume preserving', and *Newtonian* if its viscosity is unaffected by shear forces. For notation, we have, as usual, that operators "$\nabla \cdot$" and "Δ" are the divergence and Laplace operators respectively, and the nonlinear operator "$(u \cdot \nabla)$" defined by $(u \cdot \nabla)w := \sum_{i=1}^{d} u_i \partial_{x_i} w$.

$u = (u_1, ..., u_d)$, our components u_i of vector field u and the scalar field p), so we may try to solve for p in terms of u, which is what we will do in this note.

3 Solving for the Pressure

3.1 Preliminary Considerations and Definitions

Here we'll briefly outline some of what needs to be established for what follows, that being definitions of the Fourier transform and Fourier multipliers, Riesz transforms, and the inverse Laplace operator. Readers familiar with these may skip this subsection.

First, we'll note which convention for the Fourier transform we're using here

Definition. *For Schwartz functions $f : \mathcal{S}(\mathbb{R}^d)$, the Fourier transform $\widehat{f}$ is defined by the integral*

$$\widehat{f}(x) = \frac{1}{(\sqrt{2\pi})^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx \tag{2}$$

For our discussion, putting the normalization constant in front of the integral will remove the need to keep track of unnecessary constants later.

Now, we say that an operator T on $\mathbb{R}^d$ functions is a *Fourier multiplier operator* with *symbol* $m : \mathbb{R}^d \to \mathbb{R}$ if $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$. From this definition is follows that, using the definition of Fourier transform above, that the following holds.

Proposition. *The partial derivative operator ∂_k on $\mathbb{R}^d$ functions ($1 \leq k \leq d$) is a Fourier multiplier operator with symbol $i\xi_k$.*

Now, we'll define the two operators we'll need for this discussion, those being the *Riesz transforms* and the *inverse Laplace operator*.

Definition. *For $f : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, define the Riesz transform operator R_j as the Fourier multiplier with symbol $i\frac{\xi_j}{|\xi|}$, which may be given by the principle value integral*

$$R_j f(x) = \frac{1}{\pi \omega_{d-1}} \int_{\mathbb{R}^d} f(x-y) \frac{y_j}{|y|} dy \tag{3}$$

Where $\omega_d = \frac{\pi^{d/2}}{\Gamma((d/2)+1)}$ is the volume of the d dimensional unit ball.

From the above definition, it follows that $R_i R_j$ is a Fourier multiplier operator with symbol $\frac{-\xi_i \xi_j}{|\xi|^2}$ which is given by the integral operator

$$R_i R_j f(x) = \frac{\delta_{ij}}{d} f(x) + \frac{1}{\omega_d} p.v \int_{\mathbb{R}^d} f(x-y) \frac{y_i y_j - \frac{\delta_{ij}|y|^2}{d}}{|y|^{d+2}} dy \tag{4}$$

where δ_{ij} is the Kronecker delta symbol, equal to 1 if $i = j$ and 0 otherwise.

Lastly, we'll define the inverse Laplace operator Δ^{-1} by the following convolution with the Newtonian potential.

Definition. *For $f : \mathbb{R}^d \to \mathbb{R}$, define $\Delta^{-1} f$ as the unique solution to the Poisson equation $-\Delta u = f$, which is given by the convolution with Newtonian potential. That is, $(\Delta^{-1} f)(x) = (\mathcal{N} * f)(x)$, where*

$$\mathcal{N}(x) = \begin{cases} \frac{-1}{2\pi} \log |x|, d = 2 \\ \frac{-1}{d(2-d)\omega_d} |x|^{2-x}, d \geq 3 \end{cases} \tag{5}$$

is the Newtonian potential

.That a unique solution does exist for the Poisson equation given sufficient assumptions on f (so that the above definition makes sense) is a classic result in differential equations. See, for example, [4]. It follows from this that Δ^{-1} is a Fourier multiplier operator with symbol $\frac{-1}{|\xi|^2}$, a fact that we'll use in what follows.

3.2 The Pressure

Our ultimate goal is to obtain some explicit expression for the pressure scalar field p in terms of u_i, the components of our velocity vector field. To do this, we'll first derive, from our equations (1), a Poisson equation involving the pressure scalar field. From here, we my take convolution with the Newtonian potential directly and compute, or we may take Fourier transforms and get an expression in terms of the iterated Riesz transforms, which may then be written out explicitly as integral operators. We'll take the second approach here for two reasons. Firstly, Fourier transform methods are "nicer" given that our manipulations can be rigorously justified. Secondly, in this case it gives us our relationship in terms of well known and well studied singular integral operators. For details on the first approach, see [1].

We apply the divergence operator to (1) to get the following.

$$-\Delta p = \nabla \cdot [\partial_t u + (u \cdot \nabla)u + \nu \Delta u] = \nabla \cdot \partial_t u + \nabla \cdot (u \cdot \nabla)u + \nabla \cdot \nu \Delta u \tag{6}$$

We can use incompressibility to simplify things. By the fact that partial derivative operators commute combined with incompressibility, the first and last term in the RHS go to zero. From here we can further use incompressibility to simplify as follows.

$$\begin{aligned} -\Delta p = \nabla \cdot ((u \cdot \nabla)u) &= \sum_{i,j} (\partial_{x_j} u_i \partial_{x_i} u_j + u_i \partial_{x_i} \partial_{x_j} u_j) \\ &= \sum_{i,j} \partial_{x_j} u_i \partial_{x_i} u_j = \sum_{i,j} \partial_{x_i} \partial_{x_j} (u_i u_j) \end{aligned} \tag{7}$$

Now, inverting the Laplacian gives us the following expression for the pressure in terms of the inverse Laplace operator.

$$p = (\Delta^{-1}) \sum_{i,j} \partial_{x_i} \partial_{x_j} (u_i u_j) \tag{8}$$

Now, we'll first assume $p, u_i \in \mathcal{S}(\mathbb{R})^d$ so that we may take Fourier transforms. Using what we've established about Fourier multiplier operators before, we get that

$$\frac{-1}{|\xi|^2}\widehat{p}(\xi) = \sum_{i,j} \xi_i \xi_j \widehat{[u_i u_j]}(\xi) \tag{9}$$

. Then multiplying by $\frac{-1}{|\xi|^2}$ gives

$$\widehat{p}(\xi) = \sum_{i,j} \frac{\xi_i \xi_j}{|\xi|^2} \widehat{[u_i u_j]}(\xi) \tag{10}$$

, and taking inverse Fourier transforms gives us that

$$p(x) = \sum_{i,j} R_i R_j[(u_i u_j)](x) \tag{11}$$

. Now we can extend this to general $L^p(\mathbb{R}^d)$ by noting that, since $\mathcal{S}(\mathbb{R})^d$ is dense in $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$, that an operator T defined on $\mathcal{S}(\mathbb{R}^d)$ has a continuous extension to $L^p(\mathbb{R}^d)$ T is bounded on $L^p(\mathbb{R}^d)$. The *Hörmander-Mikhlin multiplier theorem* gives us the L^p boundedness for a broad range of Fourier multiplier operators given their symbols obey certain estimates which are easily verified for the Riesz transforms.

Theorem (Mikhlin Multiplier Theorem). *If T is a Fourier multiplier operator with symbol $m : \mathbb{R}^d \to \mathbb{R}$ obeying the estimates $|\nabla^k m(\xi)| \leq C|\xi|^{-k}$, for some constant $C > 0$ and all $1 \leq k \leq d+2$, then T is bounded in all L^p, for $1 < p < \infty$*

. [2]. This gives us that the RHS of (11) is well defined for all L^p, so that we can say the following.

Theorem. *Given vector field $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$, $u(x) = (u_1(x), ..., u_d(x))$ satisfying (1) for some scalar field $p : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$, with $u_i \in L^p(\mathbb{R})$ for all $1 \leq i \leq d$, then p is uniquely given by*

$$\begin{aligned} p(x) &= \sum_{i,j} R_i R_j[(u_i u_j)](x) \\ &= \sum_{i,j} \left[\frac{\delta_{ij}}{d} u_i u_j(x) + \frac{1}{\omega_d} p.v \int_{\mathbb{R}^d} u_i u_j(x-y) \frac{y_i y_j - \frac{\delta_{ij}|y|^2}{d}}{|y|^{d+2}} dy \right] \end{aligned} \tag{12}$$

4 Final Remarks

Again as noted in the introduction, I've kept these notes terse as to not obscure the main ideas, but at the cost of potentially assuming too much of the

[2] Specifically, the Hörmander-Mikhlin multiplier theorem says that such operators belong to a certain class of *singular integral operators* known as *Calderon-Zygmund operators*, which are known to be bounded in all L^p, and the Riesz transforms are also easily verified directly to belong to this class. For details, see, for example, [3],[7].

reader. For review on general analysis concepts and definitions (Fourier transforms, boundedness of operators, Schwartz class, etc.) see, for example, [5]. We hope these notes will motivate readers to explore the extensive literature on the very interesting topics covered here. For more on the Euler and Navier-Stokes equations, see for example [1],[2], and [6], and for more on singular integral operators, see, for example, [3] and [7].

5 References

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