Homework 1

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Problem 1

Show that the autocovariance function can be written as

$$\gamma(s,t) \ = \ E[(x_s \ - \ \mu_s)(x_t \ - \ \mu_t)] = E(x_s x_t) - \mu_s \mu_t$$
$$E[(x_s \ - \ \mu_s)(x_t \ - \ \mu_t)] = E[(x_s x_t \ - \ \mu_s x_t \ - \mu_t \ x_s \ - \ \mu_s \mu_t)]$$
$$E(x_s x_t) - \ \mu_s E(x_t) \ - \mu_t E(\ x_s) \ - \ \mu_s \mu_t)$$

Where

$$E(x_t) = \mu_t$$

and

$$E(x_s) = \mu_s$$

Therefore,

$$E(x_s x_t) - \ \mu_s \mu_t \ - \ \mu_s \mu_t - \ \mu_s \mu_t = E(x_s x_t) - \mu_s \mu_t$$

Problem 2

part (a)

Show that the model can be written as

$$x_t \ = \ \delta \ t + \sum_{k \ = \ 1}^{t} \omega_t$$

From

$$x_t \ = \ \delta + x_{t-1} + \omega_t$$

Then,

$$x_1 \ = \ \delta + x_0 + \omega_1 = \ \delta + \omega_1$$
$$x_2 \ = \ \delta + \delta + x_1 + \omega_1 + \omega_2 = 2\delta + \omega_1 + \omega_2$$
$$x_3 \ = \ \delta + \delta + \delta + x_1 + x_2 + \omega_1 + \omega_2 + \omega_3 = 3\delta + \omega_1 + \omega_2 + \omega_3$$

If

$$x_{t-1} = (t-1)\delta + \sum_{k \ = \ 1}^{t-1} \omega_k$$

Therefore,

$$x_t = t\delta + \sum_{k \ = \ 1}^{t} \omega_t$$

As a result, the model can be written as

$$x_t \ = \ \delta \ t + \sum_{k \ = \ 1}^{t} \omega_t$$

part (b)

The mean function

$$x_t \; = \delta \; t + \sum_{k\;=\;1}^{t} \omega_t$$

$$E(x_t) \;\; = \;\; E(\delta \; t + \sum_{k\;=\;1}^{t} \omega_t)$$

$$E(x_t) \; = \delta \; t + \sum_{k\;=\;1}^{t} E(\omega_t)$$

Therefore, the mean function is

$$E(x_t) \; = \delta \; t$$

The autocovariance function:

$$\gamma(s,t) \; = Cov(x_s, x_t) = E[(x_s \; - \; \delta_s)(x_t \; - \; \delta_t)]$$

$$E(\sum_{j\;=\;1}^{s} \omega_j \sum_{k\;=\;1}^{t} \omega_k)$$

$$E[(w_1 + w_2 + ... + w_s)(w_1 + w_2 + ... + w_s + w_{s+1} + ... + w_t)]$$

Therefore, the autocovariance function is

$$min \, \{s,t\} \;\; \sigma_w^2$$

Consider the case

$$s \leq t$$

The autocovariance function is

$$\sum_{j\;=\;1}^{s} w_j^2 = s\sigma_w^2$$

part (c)

The series is not stationary since the mean function is not constant and depends on time, and the autocovariance function does not depend on the lag

$$h = |s - t|$$

part (d)

from (b)

$$\rho_x(t-1,t) = \frac{(t-1)\sigma_w^2}{\sqrt{(t-1)\sigma_w^2}\sqrt{t\sigma_w^2}}$$

Therefore,

$$\sqrt{\frac{t-1}{t}}$$

$$t = \infty, \quad then \sqrt{\frac{t-1}{t}} = 1$$

This implies that the series will change slowly as time increases.

part (e)

From the given series, the process can be transformed to

$$y_t = x_t - x_{t-1}$$

$$y_t = \delta\ t + \sum_{k\ =\ 1}^{t} \omega_t - (t-1)\delta + \sum_{k\ =\ 1}^{t-1} \omega_k$$

Thus,

$$y_t = \delta + w_t$$

$$E(y_t) = E(\delta + w_t)$$

$$E(y_t) = \delta$$

$$\gamma_y(s,t) = E[(y_s - \mu_s)(y_t - \mu_t)]$$

$$The\ \ case\ s = t$$

$$\gamma_y(s,t) = \sigma_w^2$$

$$The\ \ case\ s \neq t$$

$$\gamma_y(s,t) = 0$$

$$From\ \ y_t = \delta + w_t$$

The series seems to be stationary since this is a white noise process.

Problem 3

part (a)

Check causality

$$x_t + 1.6x_{t-1} + 0.64x_{t-2} = w_t$$

$$x_t = -1.6x_{t-1} - 0.64x_{t-2} + w_t$$

$$\emptyset(z) = 1 + 1.6z + 0.64z^2$$

```
z = c(1,1.6,0.64)
(a = polyroot(z))
```

```
## [1] -1.25-0i -1.25+0i
```

$$z_1 = -1.25\ \ and\ \ z_2 = -1.25\ \ are\ \ real\ \ and\ \ equal$$

```
z = c(1,1.6,0.64)
(a = polyroot(z))
```

```
## [1] -1.25-0i -1.25+0i
```

```
a[1]
```

```
## [1] -1.25-0i
```

```
Re(a[1])
```

```
## [1] -1.25
```

```r
Im(a[1])
```

```
## [1] -1.110223e-16
```

```r
z1 = sqrt(Re(a[1])^2+Im(a[1])^2)
sprintf('%s = %3.2f', '|z1|',z1)
```

```
## [1] "|z1| = 1.25"
```

```r
a[2]
```

```
## [1] -1.25+0i
```

```r
Re(a[2])
```

```
## [1] -1.25
```

```r
Im(a[2])
```

```
## [1] 1.110223e-16
```

```r
z2 = sqrt(Re(a[2])^2+Im(a[2])^2)
sprintf('%s = %3.2f', '|z2|',z2)
```

```
## [1] "|z2| = 1.25"
```

$$
\begin{aligned}
|z_1| = & \ 1.25 > 1 \\
|z_2| = & \ 1.25 > 1
\end{aligned}
$$

Therefore, this AR(2) model is causal.

Find the constant of the ACF using initial condition From

$$x_t = -1.6x_{t-1} - 0.64x_{t-2} + w_t$$

The result is

$$\gamma(h) = -1.6\gamma(h-1) - 0.64\gamma(h-2), \quad h = 0, 1, 2, ..$$

The difference equation for the ACF of the process is

$$\rho(h) = -1.6\rho(h-1) - 0.64\rho(h-2), \quad h = 0, 1, 2, ..$$

The initial conditions are

$$\rho(0) = 1 \ \ and \ \rho(1) = \frac{-1.6}{(1+0.64)}$$

```r
rho0 = 1
rho1 = -1.6/(1+0.64)
sprintf('%s = %3.6f', c('rho0','rho1'),c(rho0, rho1))
```

```
## [1] "rho0 = 1.000000"  "rho1 = -0.975610"
```

The homogeneous difference equation of order two is

$$\emptyset(z) = 1 + 1.6z + 0.64z^2 = (1+0.8z)^2$$

So we have

$$z_1 = -1.25, \quad and \quad z_2 = -1.25$$

This is equal root case, then

$$\rho(h) = z_0^{-h}(c_1 + c_2 h)$$

By the initial conditions

$$\rho(0) = 1 \ \ and \ \rho(1) = \ -0.97561$$

We solve for

$$c_1 = 1$$

$$-0.8(c_2 + 1) = \rho(1)$$

$$-0.8(c_2 + 1) = -0.97561$$

```
c2 = (-0.97561 + 0.8)/-0.8
sprintf('%s = %3.6f', 'c2',c2)
```

```
## [1] "c2 = 0.219512"
```

Therefore, we can find the ACF by initial conditions using

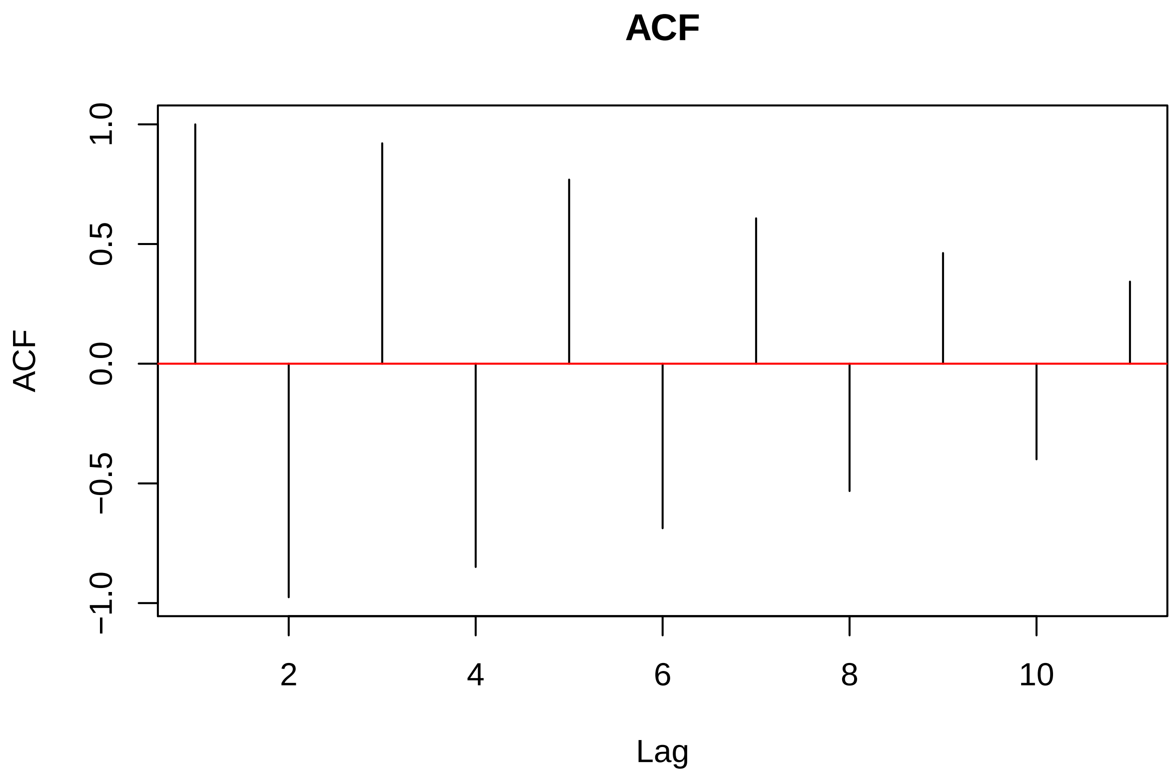
$$\rho(h) = -1.25^{-h}(1 + 0.219512h), \ h = 0, 1, 2, ..$$

Plot ACF values to lag 10

```
ACF = ARMAacf(ar = c(-1.6,-0.64), ma = 0, lag.max = 10)
ACF
```

```
##          0          1          2          3          4          5          6
##  1.0000000 -0.9756098  0.9209756 -0.8491707  0.7692488 -0.6873288  0.6074068
##          7          8          9         10
## -0.5319605  0.4623964 -0.3993796  0.3430736
```

```
plot(ACF, type = 'h', xlab = 'Lag', main = 'ACF')
abline(h=0, col ='red')
```



part (b)

Check causality

$$x_t - 0.4x_{t-1} - 0.45x_{t-2} = w_t$$

$$x_t = 0.4x_{t-1} + 0.45x_{t-2} + w_t$$

$$\emptyset(z) = 1 - 0.4z - 0.45z^2$$

```
z = c(1,-0.4,-0.45)
(a = polyroot(z))
```

```
## [1]  1.111111-0i -2.000000+0i
```

$$z_1 = 1.11 \quad and \quad z_2 = -2 \quad are \quad real \quad and \quad distinct$$

```
z = c(1,-0.4,-0.45)
(a = polyroot(z))
```

```
## [1]  1.111111-0i -2.000000+0i
```

```
a[1]
```

```
## [1] 1.111111-0i
```

```
Re(a[1])
```

```
## [1] 1.111111
```

```r
Im(a[1])
```

```
## [1] -8.503417e-21
```

```r
z1 = sqrt(Re(a[1])^2+Im(a[1])^2)
sprintf('%s = %3.6f', '|z1|',z1)
```

```
## [1] "|z1| = 1.111111"
```

```r
a[2]
```

```
## [1] -2+0i
```

```r
Re(a[2])
```

```
## [1] -2
```

```r
Im(a[2])
```

```
## [1] 8.503417e-21
```

```r
z2 = sqrt(Re(a[2])^2+Im(a[2])^2)
sprintf('%s = %3.6f', '|z2|',z2)
```

```
## [1] "|z2| = 2.000000"
```

$$|z_1| = \ 1.111111 > 1$$
$$|z_2| = \ 2 > 1$$

Therefore, this AR(2) model is causal.

Find the constant of the ACF using initial condition From

$$x_t = 0.4x_{t-1} + 0.45x_{t-2} + w_t$$

The result is

$$\gamma(h) = 0.4\gamma(h-1) + 0.45\gamma(h-2), \quad h = 0, 1, 2, ..$$

The difference equation for the ACF of the process is

$$\rho(h) = 0.4\rho(h-1) + 0.45\rho(h-2), \quad h = 0, 1, 2, ..$$

The initial conditions are

$$\rho(0) = 1 \ \ and \ \rho(1) = \frac{0.4}{(1-0.45)}$$

```r
rho0 = 1
rho1 = 0.4/(1-0.45)
sprintf('%s = %3.6f', c('rho0','rho1'),c(rho0, rho1))
```

```
## [1] "rho0 = 1.000000" "rho1 = 0.727273"
```

The homogeneous difference equation of order two is

$$\emptyset(z) = 1 - 0.4z - 0.45z^2 = (1 - 0.9z)(1 + 0.5z)$$

So we have

$$z_1 = -1.11, \quad and \ \ z_2 = -2$$

This is distinct root case, then

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$$

By the initial conditions

$$\rho(0) = 1 \;\; and \; \rho(1) = \; 0.727273$$

We solve for

$$c_1 + c_2 = 1$$

$$0.9c_1 - 0.5c_2 = \rho(1)$$

$$0.9c_1 - 0.5c_2 = 0.727273$$

```
a <- rbind(c(1,1),
           c(0.9,-0.5))
b <- c(1,0.727273)
c <- solve(a, b)

sprintf('%s = %3.4f', c('c1', 'c2'), c(c[1],c[2]))
```

```
## [1] "c1 = 0.8766" "c2 = 0.1234"
```

Therefore, we can find the ACF by initial conditions using

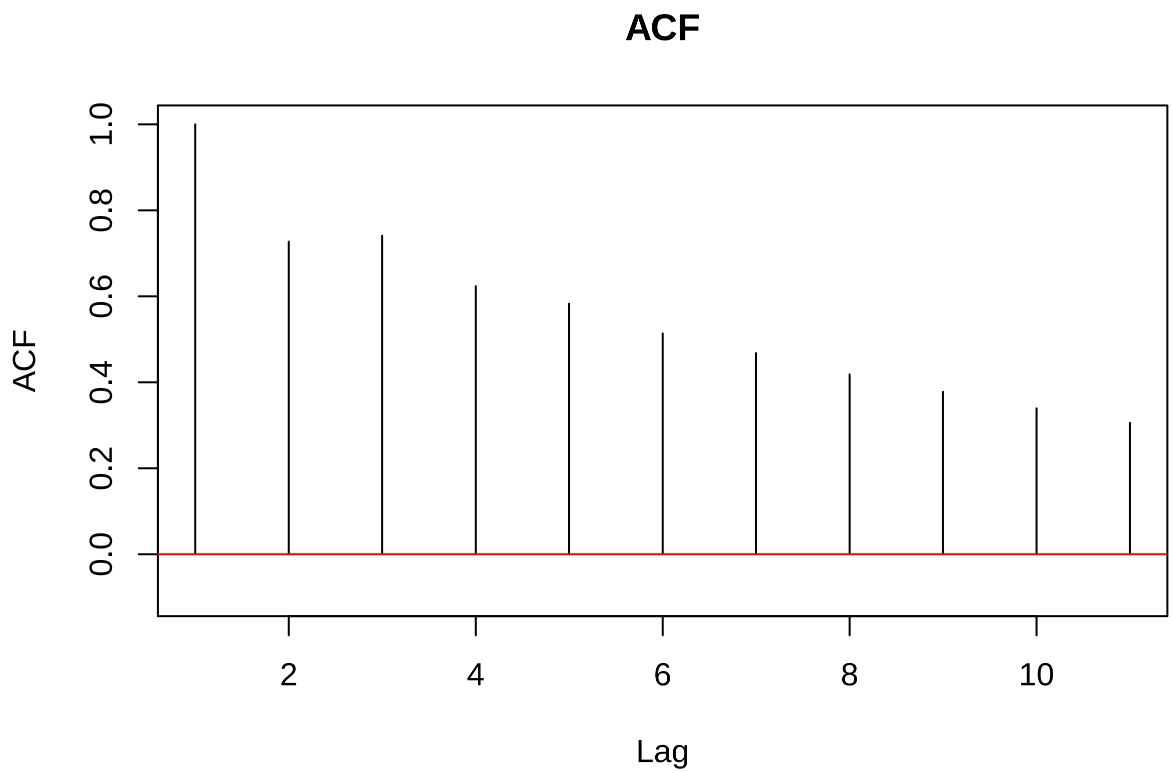
$$\rho(h) = 0.8766 \cdot 1.11^{-h} + 0.1234 \cdot (-2)^{-h}, \; h = 0, 1, 2, ..$$

Plot ACF values to lag 10

```
ACF = ARMAacf(ar = c(0.4,0.45), ma = 0, lag.max = 10)
ACF
```

```
##         0         1         2         3         4         5         6         7
## 1.0000000 0.7272727 0.7409091 0.6236364 0.5828636 0.5137818 0.4678014 0.4183224
##         8         9        10
## 0.3778396 0.3393809 0.3057802
```

```
plot(ACF, type = 'h', xlab = 'Lag', main = 'ACF', ylim = c(-0.1,1))
abline(h=0, col ='red')
```



part (c)

Check causality

$$x_t - 1.2x_{t-1} + 0.85x_{t-2}$$

$$x_t = 1.2x_{t-1} - 0.85x_{t-2} + w_t$$

$$\emptyset(z) = 1 - 1.2z + 0.85z^2$$

```
z = c(1,-1.2,0.85)
(a = polyroot(z))
```

```
## [1] 0.7058824+0.8235294i 0.7058824-0.8235294i
```

$$z_1 = 0.7058824 + 0.8235294 \quad and \quad z_2 = 0.7058824 - 0.8235294 \quad are \quad the \quad complex \quad roots$$

```
z = c(1,-1.2,0.85)
(a = polyroot(z))
```

```
## [1] 0.7058824+0.8235294i 0.7058824-0.8235294i
```

```
a[1]
```

```
## [1] 0.7058824+0.8235294i
```

```
Re(a[1])
```

```
## [1] 0.7058824
```

```r
Im(a[1])
```

```
## [1] 0.8235294
```

```r
z1 = sqrt(Re(a[1])^2+Im(a[1])^2)
sprintf('%s = %3.6f', '|z1|',z1)
```

```
## [1] "|z1| = 1.084652"
```

```r
a[2]
```

```
## [1] 0.7058824-0.8235294i
```

```r
Re(a[2])
```

```
## [1] 0.7058824
```

```r
Im(a[2])
```

```
## [1] -0.8235294
```

```r
z2 = sqrt(Re(a[2])^2+Im(a[2])^2)
sprintf('%s = %3.6f', '|z2|',z2)
```

```
## [1] "|z2| = 1.084652"
```

$$|z_1| = \ 1.084652 > 1$$
$$|z_2| = \ 1.084652 > 1$$

Therefore, this AR(2) model is causal.

Find the constant of the ACF using initial condition From

$$x_t = 1.2x_{t-1} - 0.85x_{t-2} + w_t$$

The result is

$$\gamma(h) = 1.2\gamma(h-1) - 0.85\gamma(h-2), \quad h = 0, 1, 2, ..$$

The difference equation for the ACF of the process is

$$\rho(h) = 1.2\rho(h-1) - 0.85\rho(h-2), \quad h = 0, 1, 2, ..$$

The initial conditions are

$$\rho(0) = 1 \ \ and \ \rho(1) = \frac{1.2}{(1+0.85)}$$

```r
rho0 = 1
rho1 = 1.2/(1+0.85)
sprintf('%s = %3.6f', c('rho0','rho1'),c(rho0, rho1))
```

```
## [1] "rho0 = 1.000000" "rho1 = 0.648649"
```

The homogeneous difference equation of order two is

$$\emptyset(z) = 1 - 1.2z + 0.85z^2$$

So we have

$$z_1 = 0.7058824 + 0.8235294 \ \ and \ \ z_2 = 0.7058824 - 0.8235294$$

This is complex roots case, then

$$\rho(h) = a\,|z_1|^{-h}\,cos(h\theta + b)$$

By the initial conditions

$$\rho(0) = 1 \ \ and \ \rho(1) = \ 0.648649$$

Find

$$\theta$$

```
z = c(1,-1.2,0.85)
(a = polyroot(z))
```

```
## [1] 0.7058824+0.8235294i 0.7058824-0.8235294i
```

```
arg = Arg(a[1])
sprintf('%s = %3.6f', 'arg',arg)
```

```
## [1] "arg = 0.862170"
```

We solve for

$$a\cos(b) = 1$$

$$a \cdot 1.084652^{-1} cos(0.86217(1) + b) = \rho(1)$$

$$a \cdot 1.084652^{-1} cos(0.86217 + b) = 0.648649$$

$$\frac{1}{cos(b)} 1.084652^{-1} cos(0.86217 + b) = 0.648649$$

$$b = tan^{-1}(\frac{(0.648649 \ \cdot \ 1.084652) - cos(0.86217)}{-sin(0.86217)})$$

```
x = 0.648649*1.084652
y = cos(0.86217)
z = -sin(0.86217)
b = atan((x-y)/z)
a = 1/cos(b)
sprintf('%s = %3.4f', c('a','b'), c(a,b))
```

```
## [1] "a = 1.0024"  "b = -0.0694"
```

Therefore, we can find the ACF by initial conditions using

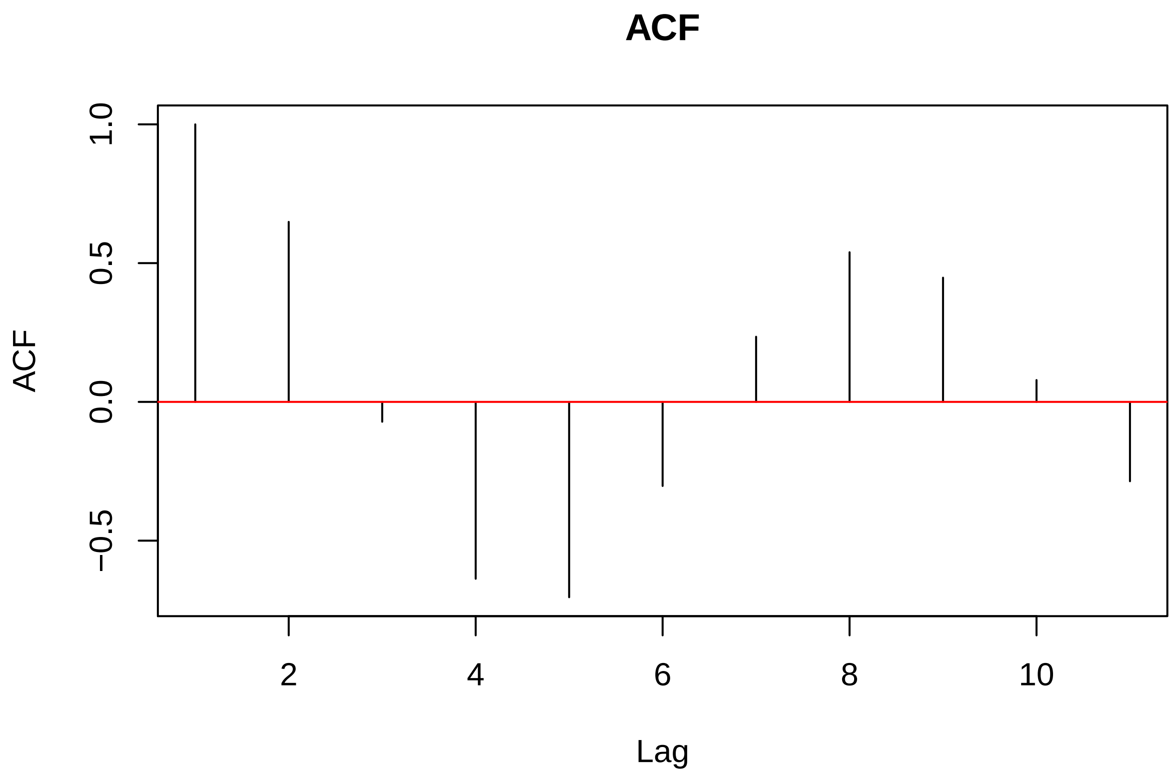
$$\rho(h) = 1.0024 \cdot 1.084652^{-h} cos(0.86217h - 0.0694), \ \ h = 0, 1, 2, ..$$

Plot ACF values to lag 10

```
ACF = ARMAacf(ar = c(1.2,-0.85), ma = 0, lag.max = 10)
ACF
```

```
##           0           1           2           3           4           5
##  1.00000000  0.64864865 -0.07162162 -0.63729730 -0.70387838 -0.30295135
##           6           7           8           9          10
##  0.23475500  0.53921465  0.44751583  0.07868654 -0.28596460
```

```
plot(ACF, type = 'h', xlab = 'Lag', main = 'ACF')
abline(h=0, col ='red')
```



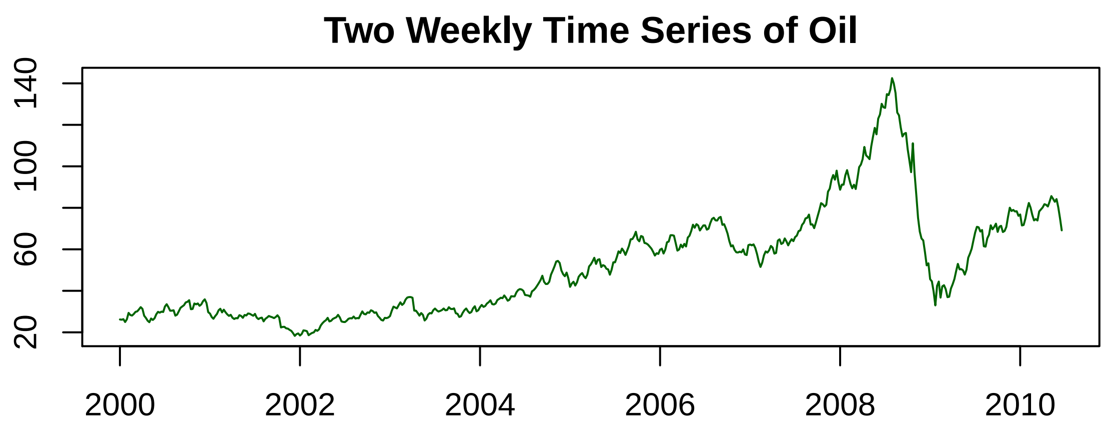
Problem 4

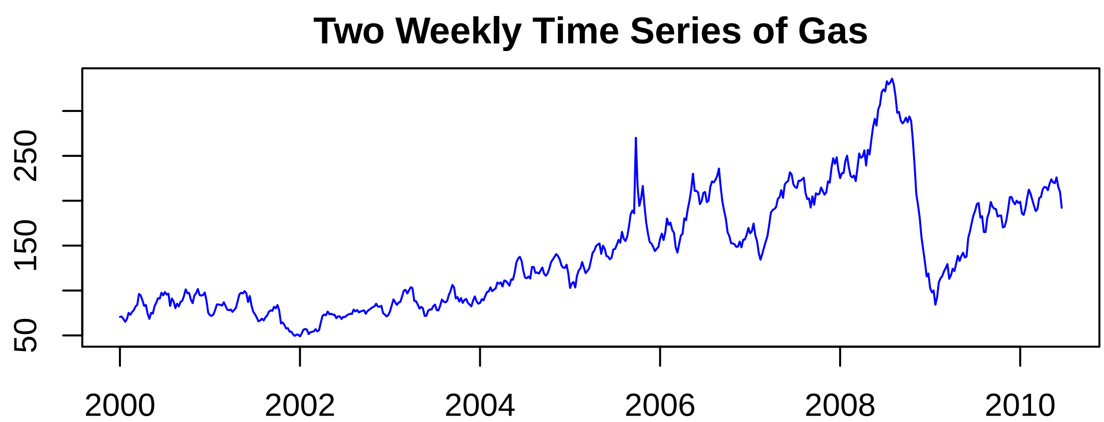
part (a)

```
library(astsa)
```

```
## Warning: package 'astsa' was built under R version 4.0.3
```

```
par(mfrow=c(2,1),mar=c(2,2,2,4))
plot(oil, col="darkgreen", main="Two Weekly Time Series of Oil", ylab="Dollars per Barrel")
plot(gas, col="blue", main="Two Weekly Time Series of Gas", ylab="Cents per Gallon")
```

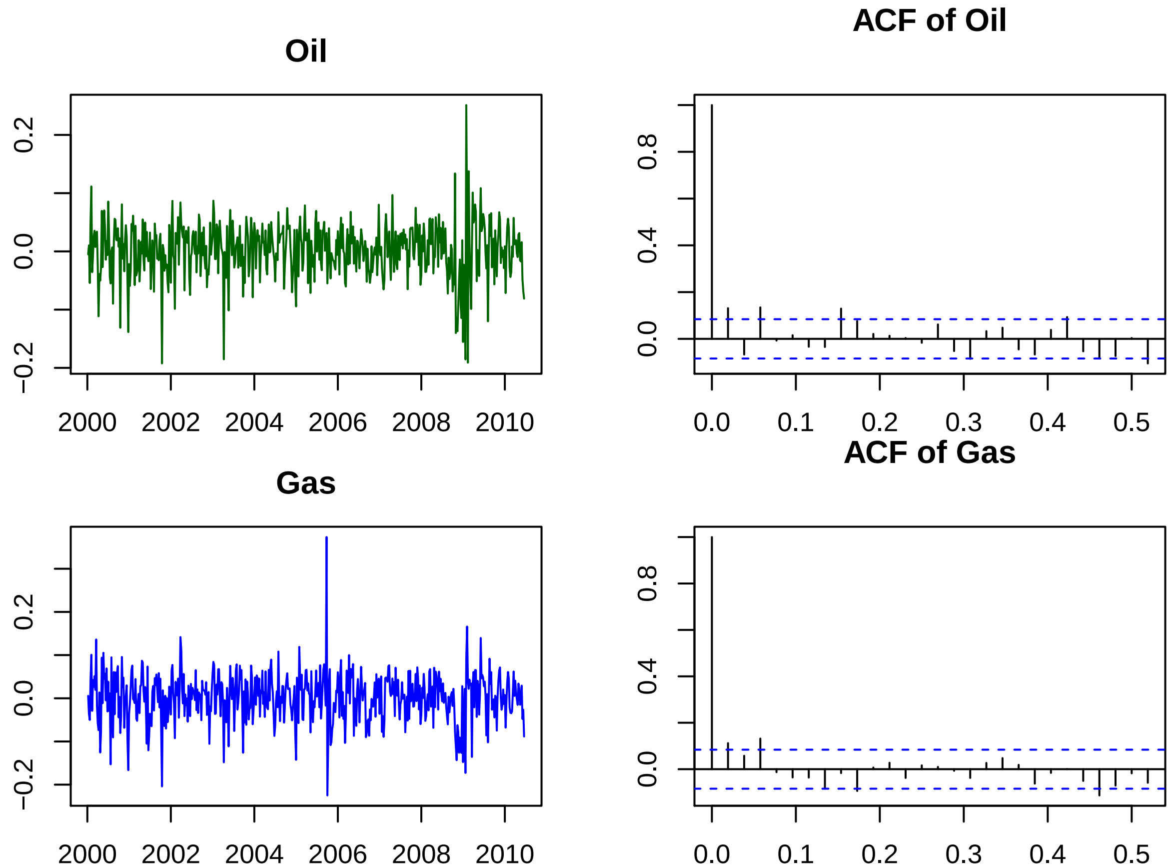




As we know that a stationary time series is one whose properties do not depend on time, time series with trends, or with seasonality, are not stationary since the trend and seasonality will affect the value of the time series at difference times. As a result, it can be observed from the plots of two weekly time series oil and gas that the mean of oil and gas prices increased constantly until 2008. However, the mean price fell and then increased from 2008 to 2010. It can be concluded that the mean of oil and gas prices are not constant over time. Therefore, the series are not stationary.

part (b)

```
par(mfrow=c(2,2),mar=c(2,2.8,2.8,2))
plot(diff(log(oil)), main="Oil", col="darkgreen", ylab = "Oil")
acf(diff(log(oil)), main="ACF of Oil")
plot(diff(log(gas)), main="Gas", col="blue", ylab= "Gas")
acf(diff(log(gas)), main="ACF of Gas")
```



From the plots above, it can be concluded that the transformed data seems to be stationary except the gas price near year 2006 and the oil price near year 2009. Since the ACF plots indicate exponential decay and white noise process, these indicate the series are stationary.