On the Minimum Volume Covering Ellipsoid of Ellipsoids

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Abstract

We study the problem of computing a $(1+\epsilon)$-approximation to the minimum volume covering ellipsoid of a given set $\mathcal{S}$ of the convex hull of m full-dimensional ellipsoids in $\mathbb{R}^n$. We extend the first-order algorithm of Kumar and Yıldırım that computes an approximation to the minimum volume covering ellipsoid of a finite set of points in $\mathbb{R}^n$, which, in turn, is a modification of Khachiyan's algorithm. For fixed $\epsilon > 0$, we establish a polynomial-time complexity, which is linear in the number of ellipsoids m. In particular, the iteration complexity of our algorithm is identical to that for a set of m points. The main ingredient in our analysis is the extension of polynomial-time complexity of certain subroutines in the algorithm from a set of points to a set of ellipsoids. As a byproduct, our algorithm returns a finite "core" set $\mathcal{X} \subseteq \mathcal{S}$ with the property that the minimum volume covering ellipsoid of $\mathcal{X}$ provides a good approximation to that of $\mathcal{S}$. Furthermore, the size of $\mathcal{X}$ depends only on the dimension n and ϵ, but not on the number of ellipsoids m. We also discuss the extent to which our algorithm can be used to compute the minimum volume covering ellipsoid of the convex hull of other sets in $\mathbb{R}^n$. We adopt the real number model of computation in our analysis.

Keywords: Minimum volume covering ellipsoids, Löwner ellipsoid, core sets, approximation algorithms.

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1 Introduction

Given m full-dimensional ellipsoids $\mathcal{E}_1, \dots, \mathcal{E}_m$ in $\mathbb{R}^n$, let $\mathcal{S}$ denote their convex hull. In this paper, we are concerned with the problem of computing the minimum volume covering ellipsoid of $\mathcal{S}$, denoted by MVCE($\mathcal{S}$), also known as the Löwner ellipsoid of $\mathcal{S}$.

Ellipsoidal approximations of a compact convex set $\mathcal{S}$ with a nonempty interior play an important role in several applications. By the Löwner-John Theorem (see Theorem 2.1), the quality of an outside ellipsoidal approximation of such a set $\mathcal{S} \subset \mathbb{R}^n$ depends only on the dimension n. Therefore, MVCE($\mathcal{S}$) provides a good rounding of the set $\mathcal{S}$, which implies that certain characteristics of $\mathcal{S}$ can be approximated using an ellipsoidal rounding as long as MVCE($\mathcal{S}$) can be computed efficiently. For instance, an outside ellipsoidal approximation of $\mathcal{S}$ can be used to compute lower and upper bounds efficiently for a quadratic optimization problem over $\mathcal{S}$ (see Proposition 2.1).

The idea of approximating complicated objects using simpler ones is widely used in computational geometry and computer graphics. A common approach is to replace a complicated object by a simpler model covering it such as a minimum volume box or a sphere. More recently, ellipsoidal models have been proposed in the literature as they usually provide better approximations than bounding boxes or spheres (see, e.g., [23, 24, 14, 10]). The key idea is to construct a so-called bounding volume hierarchy [11], which is simply a tree of bounding volumes. The bounding volume at a given node encloses the bounding volumes of its children. The bounding volume of a leaf encloses a primitive. Such a data structure can be used for detection collision or ray tracing. For instance, if a ray misses the bounding volume of a particular node, then the ray will miss all of its children, and the children can be skipped. The ray casting algorithm traverses this hierarchy, usually in depth first order, and determines if the ray intersects an object. Therefore, if an ellipsoidal approximation is used, the construction of a bounding volume hierarchy requires the computation of the minimum volume covering ellipsoid of a union of ellipsoids at every node.

There is an extensive body of research on minimum volume covering ellipsoids of a finite set of points. We refer the reader to [15, 26, 17] and the references therein for a detailed account of numerous applications and several algorithms. In contrast, we are not aware of any specialized algorithms for the minimum volume covering ellipsoid of ellipsoids in the literature. It is known that the problem can be formulated as an instance of a convex determinant optimization problem with linear matrix inequalities [5, 2, 6] (see also Section 3), which is amenable to theoretically efficient algorithms proposed in [28, 27]. Our main objective in this paper is to establish that the problem of minimum volume covering ellipsoid of ellipsoids admits a sufficiently rich structure that enables us to extend the first-order algorithm of Kumar and Yıldırım [17], which, in turn, is a modification of Khachiyan's algorithm [15], that computes the minimum volume covering ellipsoid of a finite set of points in an almost verbatim fashion to a set of ellipsoids. The main ingredient in our analysis is the extension of polynomial-time complexity of certain subroutines in the algorithm of [17] from a set of points to a set of ellipsoids. We mainly rely on the complexity results of Porkolab and Khachiyan [20] on semidefinite optimization with a fixed number of constraints, which leads to the polynomial-time complexity of quadratic optimization over an ellipsoid – one of the subroutines in our algorithm (see Proposition 2.1). Throughout this paper, we adopt the real number model of computation [4], i.e., arithmetic operations with real numbers and comparisons can be done with unit cost.

Given $\epsilon > 0$ and a compact convex set $\mathcal{S}$, an ellipsoid $\mathcal{E}$ is said to be a $(1+\epsilon)$-approximation to MVCE($\mathcal{S}$) if

$$\mathcal{E} \supseteq \mathcal{S}, \quad \text{vol}\, \mathcal{E} \leq (1+\epsilon)\ \text{vol MVCE}(\mathcal{S}), \tag{1}$$

where vol $\mathcal{E}$ denotes the volume of $\mathcal{E}$. In this paper, we extend the first-order algorithm of [17] to compute a $(1+\epsilon)$-approximation to the minimum volume covering ellipsoid of ellipsoids. In particular, we establish that our extension has precisely the same iteration complexity as that of the algorithm of [17] (see Theorem 5.2). Furthermore, the overall complexity result is given by $O(mn^{O(1)}(\log n + [(1+\epsilon)^{2/n} - 1]^{-1}))$, which depends only linearly on the number of ellipsoids m (see Theorem 5.3). Here, $O(1)$ denotes a universal constant greater than four that does not depend on the particular instance. Therefore, our algorithm has a polynomial-time complexity for fixed ϵ and is especially well-suited for instances with $m \gg n$ and moderately small values of ϵ.

As a byproduct, our algorithm computes a finite set $\mathcal{X} \subseteq \mathcal{S}$ with the property that $\mathcal{X}$ provides a good approximation of $\mathcal{S}$. Moreover, the size of $\mathcal{X}$ depends only on the dimension n and the parameter ϵ but is independent of the number of ellipsoids m. In particular, $\mathcal{X}$ satisfies

$$\text{vol MVCE}(\mathcal{X}) \leq \text{vol MVCE}(\mathcal{S}) \leq \text{vol}\ \mathcal{E} \leq (1+\epsilon)\text{vol MVCE}(\mathcal{X}) \leq (1+\epsilon)\text{vol MVCE}(\mathcal{S}),$$

where $\mathcal{E}$ denotes the $(1+\epsilon)$-approximation computed by our algorithm, which implies that $\mathcal{E}$ is simultaneously a $(1+\epsilon)$-approximation to MVCE($\mathcal{X}$) and to MVCE($\mathcal{S}$) (see Proposition 5.2).

Following the literature, we refer to $\mathcal{X}$ as a "core set" [8, 7, 16, 17] since it provides a compact approximation to the input set $\mathcal{S}$. Recently, core sets have received significant attention and small core set results have been established for several geometric optimization problems such as the minimum enclosing ball problem and related clustering problems [16, 8, 7, 9, 1, 17]. Small core set results form a basis for developing practical algorithms for large-scale problems since many geometric optimization problems can be solved efficiently for small input sets.

The paper is organized as follows. We define our notation in the remainder of this section. In Section 2, we present some preliminary results and discuss the complexity of semidefinite feasibility and optimization. We then establish that the ellipsoid containment problem can be cast as a linear matrix inequality and can be solved in polynomial time. We present a convex optimization formulation of the MVCE problem in Section 3. Section 4 is devoted to a deterministic volume approximation algorithm that will serve as an initialization stage for our algorithm. In Section 5, we present and analyze a first-order algorithm for the MVCE problem. We discuss how to extend our algorithm to other input sets in Section 6. Section 7 concludes the paper with future research directions.

1.1 Notation

Vectors will be denoted by lower-case Roman letters. For a vector u, u_i denotes the ith component. Inequalities on vectors will apply to each component. e will be reserved for the vector of ones in the appropriate dimension, which will be clear from the context. e_j is the jth unit vector. Upper-case Roman letters will be reserved for matrices. $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices. The inner product in $\mathcal{S}^n$ is given by $U \bullet V := \text{trace}(UV) = \sum_{i,j} U_{ij} V_{ij}$ for any $U, V \in \mathcal{S}^n$. Note that $u^T A u = A \bullet uu^T$ for any $A \in \mathcal{S}^n$ and $u \in \mathbb{R}^n$. For $A \in \mathcal{S}^n$, $A \succ 0$ ($A \succeq 0$) indicates that A is positive definite (semidefinite) (i.e., the eigenvalues of A are strictly positive (nonnegative)). $\det(A)$ and $\text{rank}(A)$ denote the determinant and the rank of a square matrix A, respectively. The identity matrix will be denoted by I. For a finite set of vectors $\mathcal{V}$, $\text{span}(\mathcal{V})$ denotes the linear subspace spanned by $\mathcal{V}$. The convex hull of a set $\mathcal{S} \in \mathbb{R}^n$ is referred to as $\text{conv}(\mathcal{S})$. Superscripts will be used to refer to elements of a sequence of vectors or matrices. Lower-case Greek letters will represent scalars. i, j, and k will be reserved for indexing purposes and m and n will refer to the problem data. Upper-case script letters will be used for all other objects such as sets, operators, and ellipsoids. We also use the standard big-O and related notation.

2 Preliminaries

A full-dimensional ellipsoid $\mathcal{E}_{Q,c}$ in $\mathbb{R}^n$ is specified by an $n \times n$ symmetric positive definite matrix Q and a center $c \in \mathbb{R}^n$ and is defined as

$$\mathcal{E} = \{x \in \mathbb{R}^n : (x - c)^T Q (x - c) \leq 1\}. \tag{2}$$

The matrix Q determines the shape and the orientation of $\mathcal{E}$. In particular, the axes of $\mathcal{E}$ are the eigenvectors $d^1, \ldots, d^n \in \mathbb{R}^n$ of Q and the length of each axis is given by ${\lambda_1}^{-1/2}, \ldots, {\lambda_n}^{-1/2}$, where $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues of Q. Therefore, the volume of $\mathcal{E}$, denoted by $\text{vol}\, \mathcal{E}$, is given by

$$\text{vol}\, \mathcal{E} = \eta \det Q^{-\frac{1}{2}} = \eta\, \Pi_{i=1}^n {\lambda_i}^{-1/2}, \tag{3}$$

where η is the volume of the unit ball in $\mathbb{R}^n$ [12]. Note that an ellipsoid $\mathcal{E}$ induces a norm on $\mathbb{R}^n$ via $\|x\|_{\mathcal{E}} := (x^T Q x)^{1/2}$. Therefore, every ellipsoid can be viewed as a translation of the unit ball in terms of the ellipsoidal norm induced by it.

We start with a classical result on the quality of the approximation of $\text{MVCE}(\mathcal{S})$ of a convex set $\mathcal{S} \subseteq \mathbb{R}^n$.

Theorem 2.1 (Löwner-John [13]) *Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a compact, convex set with a nonempty interior. Then, MVCE($\mathcal{S}$) exists and is unique and satisfies*

$$\frac{1}{n} MVCE(\mathcal{S}) \subseteq conv(\mathcal{S}) \subseteq MVCE(\mathcal{S}), \tag{4}$$

where the ellipsoid on the left-hand side is obtained by scaling MVCE($\mathcal{S}$) around its center by a factor of $1/n$. Furthermore, if $\mathcal{S}$ is symmetric around the origin, then the factor on the left-hand side of (4) can be replaced by $1/\sqrt{n}$.

We next state a well-known lemma that will be useful for our analysis.

Lemma 2.1 (Schur complement) *Let*

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$$

be a symmetric matrix with $B \in \mathcal{S}^\alpha$ *and* $D \in \mathcal{S}^\beta$. *Assume that* $D \succ 0$. *Then,* $A \succeq 0$ *if and only if* $B - CD^{-1}C^T \succeq 0$.

2.1 Complexity of Semidefinite Feasibility and Optimization

Consider the following feasibility problems:

1. **(PF)** Given $A_1, A_2, \ldots, A_\kappa \in \mathcal{S}^n$ and $\beta_1, \ldots, \beta_\kappa \in \mathbb{R}$, determine whether there exists a matrix $X \in \mathcal{S}^n$ such that

$$A_i \bullet X \leq \beta_i, \quad i = 1, \ldots, \kappa, \quad X \succeq 0.$$

2. **(DF)** Given $B_0, B_1, \ldots, B_\kappa \in \mathcal{S}^n$, determine whether there exists $y_1, \ldots, y_\kappa$ such that

$$B_0 + y_1 B_1 + y_2 B_2 + \ldots + y_\kappa B_\kappa \succeq 0.$$

The complexity of the problems **(PF)** and **(DF)** is still a fundamental open problem. In the real number model of computation, both problems are in NP since one can check in polynomial time whether a given symmetric matrix is positive semidefinite using Cholesky factorization. Ramana [21] proved that both problems belong to NP $\cap$ co-NP. Porkolab and Khachiyan [20] established the following complexity results, which, in turn, are mainly based on the first-order theory of the reals developed by Renegar [22].

Theorem 2.2 *Problems* **(PF)** *and* **(DF)** *can be solved in* $\kappa n^{O(\min\{\kappa, n^2\})}$ *and* $O(\kappa n^4) + n^{O(\min\{\kappa, n^2\})}$ *operations over the reals, respectively.*

In addition, let us consider the following optimization versions:

1. **(PO)** Given $D, A_1, A_2, \ldots, A_\kappa \in \mathcal{S}^n$ and $\beta_1, \ldots, \beta_\kappa \in \mathbb{R}$, solve

$$\alpha^* := \inf_{X \in \mathcal{S}^n} \{D \bullet X : A_i \bullet X \leq \beta_i, \quad i = 1, \ldots, \kappa, \quad X \succeq 0\}.$$

2. **(DO)** Given $B_0, B_1, \ldots, B_\kappa \in \mathcal{S}^n$ and $d \in \mathbb{R}^\kappa$, solve

$$\beta^* := \sup_{y_1, \ldots, y_\kappa \in \mathbb{R}} \{\sum_{i=1}^{\kappa} d_i y_i : B_0 + y_1 B_1 + y_2 B_2 + \ldots + y_\kappa B_\kappa \succeq 0\}.$$

The complexity results of Theorem 2.2 also extend to the optimization versions **(PO)** and **(DO)** [20].

Theorem 2.3 *For problems* **(PO)** *and* **(DO)***, each of the following can be solved in* $\kappa n^{O(\min\{\kappa,n^2\})}$ *and* $O(\kappa n^4)+n^{O(\min\{\kappa,n^2\})}$ *operations over the reals, respectively: (i) feasibility, (ii) boundedness, (iii) attainment of the optimal value, and (iv) computation of a least norm optimal solution.*

One important consequence of Theorems 2.2 and 2.3 is that semidefinite feasibility and semidefinite optimization can be solved in polynomial time if κ is fixed. We state this as a separate corollary.

Corollary 2.1 *Each of the four problems* **(PF)***,* **(DF)***,* **(PO)***, and* **(DO)** *can be solved in polynomial time for fixed* κ*.*

This result will play a key role in our algorithm as the semidefinite feasibility and semidefinite optimization problems we will encounter will always satisfy the condition of the corollary.

2.2 Ellipsoid Containment

In this section, we study the problem of deciding whether a given full dimensional ellipsoid $\mathcal{E}$ is contained in another full dimensional ellipsoid $\mathcal{E}^*$. Furthermore, we establish how to efficiently compute a point in $\mathcal{E}$ that is furthest from the center of $\mathcal{E}^*$ in terms of the ellipsoidal norm induced by $\mathcal{E}^*$.

We start with the following well-known result about polynomiality of quadratic optimization over an ellipsoid (see, e.g., [30]). Our treatment is closer to those of [25, 32], where we rely on the fact that the possibly nonconvex optimization problem admits a tight SDP relaxation, whose optimal solution can be used to compute an an optimal solution for the original problem.

Proposition 2.1 *Any quadratic function* $f : \mathbb{R}^n \to \mathbb{R}$ *can be maximized over a full-dimensional ellipsoid in* $O(n^{O(1)})$ *operations, where* $O(1)$ *is a universal constant greater than three.*

Proof. Let $f(x) := x^T Ax + 2b^T x + \gamma$, where $A \in \mathcal{S}^n$, $b \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$ and let $\mathcal{E} := \{x \in \mathbb{R}^n : (x-c)^T Q(x-c) \leq 1\}$ be a full-dimensional ellipsoid, where $Q \in \mathcal{S}^n$ is positive definite and $c \in \mathbb{R}^n$. We wish to solve

$$(P) \quad \max_{x \in \mathbb{R}^n} \{f(x) : x \in \mathcal{E}\}.$$

We consider the following SDP relaxation:

$$(SP) \quad \max_{X \in \mathcal{S}^{n+1}} \{F \bullet X : G \bullet X \leq 0, E_{n+1} \bullet X = 1, X \succeq 0\},$$

where

$$F := \begin{bmatrix} A & b \\ b^T & \gamma \end{bmatrix}, \quad G := \begin{bmatrix} Q & -Qc \\ -c^TQ & c^TQc - 1 \end{bmatrix}, \quad E_{n+1} = e_{n+1}e_{n+1}^T.$$

Note that (SP) is a relaxation of (P) since for any feasible solution $x \in \mathbb{R}^n$ of (P),

$$\begin{bmatrix} x \\ 1 \end{bmatrix} [x^T \quad 1] = \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

is a feasible solution of (SP) with the same objective function value. We claim that the relaxation is exact in the sense that the optimal values of (P) and (SP) coincide and an optimal solution of (SP) can be converted into an optimal solution of (P).

Consider the following Lagrangian dual of (SP):

$$(SD) \quad \min_{\lambda,\beta} \{\beta : \ \lambda G + \beta E_{n+1} \succeq F, \lambda \geq 0\}.$$

We now make several observations about (SP) and (SD). Note that (SP) satisfies the Slater condition since

$$G \bullet \begin{bmatrix} cc^T & c \\ c^T & 1 \end{bmatrix} = -1 < 0,$$

which implies that strong duality holds between (SP) and (SD) and that the optimal value is attained in (SD). Furthermore, the feasible set of (SP) is bounded because the only solution to the system

$$G \bullet Y \leq 0, \quad E_{n+1} \bullet Y = 0, \quad Y \succeq 0, \quad Y \in \mathcal{S}^{n+1}$$

is $Y = 0$ since $Q \succ 0$. Therefore, the optimal value is also attained in (SP).

By Corollary 2.1, we can solve (SP) in $O(n^{O(1)})$ time (one can replace the equality constraint with two inequality constraints). Let X^* and (λ^*, β^*) denote optimal solutions of (SP) and (SD), respectively. It follows from optimality conditions that

$$X^* \bullet (\lambda^* G + \beta^* E_{n+1} - F) = 0, \quad \lambda^* G \bullet X^* = 0. \tag{5}$$

Since $G \bullet X^* \leq 0$, we can compute a rank-one decomposition of $X^* := \sum_{i=1}^{\rho} p^i (p^i)^T$, where $\rho := \text{rank}(X^*)$ and $p^i \in \mathbb{R}^{n+1}, p^i \neq 0, \ i = 1, \ldots, \rho$, in $O(n^3)$ operations such that $(p^i)^T G p^i \leq 0, \ i = 1, \ldots, \rho$ [25, Proposition 3]. We now construct a rank-one optimal solution of (SP) using this decomposition.

By (5), $\sum_{i=1}^{\rho} (p^i)^T (\lambda^* G + \beta^* E_{n+1} - F) p^i = 0$, which implies that

$$(p^i)^T (\lambda^* G + \beta^* E_{n+1} - F) p^i = 0, \quad i = 1, \ldots, \rho \tag{6}$$

by dual feasibility. Similarly, $\lambda^* G \bullet X^* = \lambda^* \sum_{i=1}^{\rho} (p^i)^T G p^i = 0$, which implies that

$$\lambda^* (p^i)^T G p^i = 0, \quad i = 1, \ldots, \rho \tag{7}$$

since $(p^i)^T G p^i \leq 0, \ i = 1, \ldots, \rho$ and $\lambda^* \geq 0$.

Let j be any index in $\{1, 2, \ldots, \rho\}$ and let us define

$$p^j = \begin{bmatrix} x \\ \alpha \end{bmatrix},$$

where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We claim that $\alpha \neq 0$. Otherwise, $0 \geq (p^j)^T G p^j = x^T Q x$, which implies that $x = 0$, contradicting the fact that $p^j \neq 0$. We now let $x^* := (1/\alpha)p_j$. Since $G \bullet x^*(x^*)^T \leq 0$ and $E_{n+1} \bullet x^*(x^*)^T = 1$, it follows from (6) and (7) that $x^*(x^*)^T$ is a rank-1 optimal solution of (SP), which implies that $(1/\alpha)x$ is an optimal solution of (P). (We remark that each of the indices in $\{1, 2, \ldots, \rho\}$ can be used to compute a different optimal solution of (P)). □

We now use Proposition 2.1 to give a simple proof to characterize ellipsoid containment.

Proposition 2.2 *Let $\mathcal{E} := \{x \in \mathbb{R}^n : (x - c)^T Q (x - c) \leq 1\}$ and $\mathcal{E}^* := \{x \in \mathbb{R}^n : (x - c^*)^T Q^* (x - c^*) \leq 1\}$ denote two full-dimensional ellipsoids. Then, $\mathcal{E} \subseteq \mathcal{E}^*$ if and only if there exists $\tau > 0$ such that*

$$\tau \begin{bmatrix} Q & -Qc \\ -c^T Q & c^T Q c - 1 \end{bmatrix} \succeq \begin{bmatrix} Q^* & -Q^* c^* \\ -c^{*T} Q & c^{*T} Q^* c^* - 1 \end{bmatrix}. \tag{8}$$

Proof. The statement follows directly from the $\mathcal{S}$-lemma [29] (see also [19] for a comprehensive treatment). However, we give a simple proof using standard duality arguments.

First, observe that (8) can be satisfied if and only if $\tau > 0$ since $Q^* \succ 0$. Consider

$$(P) \quad \max_{x \in \mathbb{R}^n} \; \{(x - c^*)^T Q^* (x - c^*) - 1 : \; (x - c)^T Q (x - c) - 1 \leq 0\}.$$

By a similar argument as in the proof of Proposition 2.1, it follows that

$$(SP) \quad \max_{X \in \mathcal{S}^{n+1}} \{F \bullet X : \; G \bullet X \leq 0, E_{n+1} \bullet X = 1, X \succeq 0\},$$

is a tight SDP relaxation of (P), where $F \in \mathcal{S}^{n+1}$ and $G \in \mathcal{S}^{n+1}$ are given by

$$F = \begin{bmatrix} Q^* & -Q^* c^* \\ -c^{*T} Q & c^{*T} Q c^* - 1 \end{bmatrix}, \quad G = \begin{bmatrix} Q & -Qc \\ -c^T Q & c^T Q c - 1 \end{bmatrix}.$$

The dual of (SP) is

$$(SD) \quad \min_{\lambda, \beta} \{\beta : \; \lambda G + \beta E_{n+1} \succeq F, \lambda \geq 0\}.$$

Let $v(P), v(SP)$, and $v(SD)$ denote the optimal values of (P), (SP), and (SD), respectively. It follows from the proof of Proposition 2.1 that

$$v(P) = v(SP) = v(SD). \tag{9}$$

Obviously, $\mathcal{E} \subseteq \mathcal{E}^*$ if and only if $v(P) \leq 0$. If (8) is feasible, then $(\lambda, \beta) = (\tau, 0)$ is a feasible solution of (SD), which implies that $v(P) = v(SD) \leq 0$. Conversely, if $v(P) \leq 0$, then let (λ^*, β^*) be an optimal solution of (SD) with optimal value $v(SD) = v(P)$. If $v(P) = 0$, then $\beta^* = 0$, which implies that λ^* is a feasible solution of (8). On the other hand, if $\beta^* = v(SD) < 0$, then

$$\lambda^* G \succeq \lambda^* G + \beta^* E_{n+1} \succeq F,$$

since $E_{n+1} \succeq 0$. This implies again that λ^* is a feasible solution of (8), which completes the proof. $\square$

We close this subsection by giving an equivalent characterization of (8), which will enable us to formulate the MVCE problem as a convex optimization problem in the next section.

Lemma 2.2 *Condition (8) is equivalent to*

$$\tau \begin{bmatrix} Q & -Qc & 0 \\ -c^T Q & c^T Qc - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq \begin{bmatrix} Q^* & -Q^* c^* & 0 \\ -c^{*T} Q^* & -1 & c^{*T} Q^* \\ 0 & Q^* c^* & -Q^* \end{bmatrix}. \tag{10}$$

Proof. We use the notation of Lemma 2.1. After rewriting (10) as a constraint of the form $A \succeq 0$, we let B denote the top left 2×2 block and define C and D, accordingly. The equivalence now simply follows from the Schur complement lemma since $D := Q^* \succ 0$. $\square$

We remark that condition (8) (or equivalently (10)) is a semidefinite constraint in a single variable. Therefore, it follows from Corollary 2.1 that the ellipsoid containment problem can be solved in polynomial time.

3 An Optimization Formulation

Let $\mathcal{E}_1, \ldots, \mathcal{E}_m \subset \mathbb{R}^n$ be m full-dimensional ellipsoids given by

$$\mathcal{E}_i := \{x \in \mathbb{R}^n : (x - c^i)^T Q^i (x - c^i) \leq 1\}, \ i = 1, \ldots, m, \tag{11}$$

where $Q^i \in \mathcal{S}^n$ are positive definite and $c^i \in \mathbb{R}^n, \ i = 1, \ldots, m$. In this section, we formulate the problem of computing the minimum volume ellipsoid covering $\mathcal{S} := \operatorname{conv}(\cup_{i=1}^m \mathcal{E}_i)$ as a convex optimization problem.

It follows from Lemma 2.2 that ellipsoid containment can be expressed as a semidefinite constraint. Clearly, an ellipsoid $\mathcal{E} := \{x \in \mathbb{R}^n : (x - c)^T Q (x - c) \leq 1\}$ covers $\mathcal{S}$ if and only if $\mathcal{E}_i \subseteq \mathcal{E}, \ i = 1, \ldots, m$. Since

$$\log \ \mathrm{vol}\ \mathcal{E} = \log \eta + \log \det Q^{-1/2} = \log \eta - \frac{1}{2} \log \det Q$$

by (3), the problem can be formulated as follows:

$$
\begin{array}{llrcl}
\text{(MCEE)} & \min_{Q,z} & -\frac{1}{2}\log\det Q & & \\
& \text{s.t.} & & & \\
& & \tau_i \begin{bmatrix} Q^i & -Q^i c^i & 0 \\ -{c^i}^T Q^i & {c^i}^T Q^i c^i - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \succeq & \begin{bmatrix} Q & -z & 0 \\ -z^T & -1 & z^T \\ 0 & z & -Q \end{bmatrix}, \quad i = 1, \ldots, m, \\
& & \tau_i & \geq & 0, \quad i = 1, \ldots, m, \\
& & Q & \succ & 0.
\end{array}
$$

(MCEE) is a convex optimization problem. If $(Q^*, z^*, \tau_1^*, \ldots, \tau_m^*)$ is an optimal solution of (MCEE), the minimum volume covering ellipsoid is given by $\mathcal{E}^* := \{x \in \mathbb{R}^n : (x - c^*)^T Q^* (x - c^*) \leq 1\}$, where $c^* := {Q^*}^{-1} z^*$.

We remark that the MVCE problem can be formulated in several different ways. Our formulation is similar to the one presented in [5] (see also [6]). Ben-Tal and Nemirovski [2] present a more general formulation that allows one to compute the MVCE of the union of ellipsoids under the weaker assumption that the convex hull of the union have nonempty interior (i.e., each ellipsoid $\mathcal{E}_i$ is allowed to be lower dimensional). However, since a lower dimensional ellipsoid can be approximated by a "thin" full-dimensional one, we continue to assume throughout this paper that each ellipsoid $\mathcal{E}_i$ is full-dimensional.

(MCEE) is an instance of a more general class of determinant maximization problems with linear matrix inequalities, for which efficient interior-point algorithms have been proposed by Vandenberghe et. al. [28] and Toh [27]. In addition, a negative power of the determinant of a symmetric positive definite matrix can be represented by a linear matrix inequality [2]. Therefore, (MCEE) can be cast as a larger linear SDP problem and can be solved to within any relative accuracy in polynomial time using primal-dual interior-point methods [18]. However, (MCEE) consists of m semidefinite constraints of size $(2n+1) \times (2n+1)$, which implies that interior-point algorithms quickly become computationally infeasible as the problem size increases. This is one of our motivations to develop a specialized algorithm for the MVCE problem.

4 Initial Volume Approximation

Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ denote m full-dimensional ellipsoids given by (11). We define $\mathcal{S} := \operatorname{conv}(\cup_{i=1}^m \mathcal{E}_i)$. In this section, we present a simple deterministic algorithm that identifies a finite subset $\mathcal{X}_0 \subseteq \mathcal{S}$ of size $2n$ such that vol MVCE$(\mathcal{X}_0)$ is a provable approximation to vol MVCE$(\mathcal{S})$.

Lemma 4.1 *Algorithm 4.1 terminates in $O(mn^3)$ time with a subset $\mathcal{X}_0 \subseteq \mathcal{S}$ with $|\mathcal{X}_0| = 2n$ such that*

$$\textit{vol } MVCE(\mathcal{S}) \leq n^{2n} \textit{vol } MVCE(\mathcal{X}_0). \tag{12}$$

Algorithm 4.1 Volume approximation algorithm

Require: Input set $\mathcal{E}_1, \ldots, \mathcal{E}_m \subseteq \mathbb{R}^n$

1: $\Psi \leftarrow \{0\}$, $\mathcal{X}_0 \leftarrow \emptyset$, $k \leftarrow 0$.
2: While $\mathbb{R}^n \setminus \Psi \neq \emptyset$ do
3: **loop**
4: $\quad k \leftarrow k+1$. Pick an arbitrary unit vector $b^k \in \mathbb{R}^n$ in the orthogonal complement of Ψ.
5: $\quad x^{2k-1} \leftarrow \arg\max_{i=1,\ldots,m}\{(b^k)^T x : x \in \mathcal{E}_i\}$, $\mathcal{X}_0 \leftarrow \mathcal{X}_0 \cup \{x^{2k-1}\}$.
6: $\quad x^{2k} \leftarrow \arg\min_{i=1,\ldots,m}\{(b^k)^T x : x \in \mathcal{E}_i\}$, $\mathcal{X}_0 \leftarrow \mathcal{X}_0 \cup \{x^{2k}\}$.
7: $\quad \Psi \leftarrow \mathrm{span}(\Psi, \{x^{2k-1} - x^{2k}\})$.
8: **end loop**

Proof. We first establish the running time of Algorithm 4.1. At step k, Ψ is given by the span of k linearly independent vectors since $\mathcal{S}$ is full-dimensional. Hence, upon termination, $\Psi = \mathbb{R}^n$. It follows that $|\mathcal{X}_0| = 2n$. At each step, we optimize a linear function over each of the m ellipsoids $\mathcal{E}_i$. Let $Q^i = (U^i)^T U^i$, $i = 1, \ldots, m$ denote the Cholesky factorization of Q^i, $i = 1, \ldots, m$, which can be computed in $O(mn^3)$ operations. Note that $\mathcal{E}_i = \{x \in \mathbb{R}^n : x = (U^i)^{-1}u + c^i, \|u\| \leq 1\}$, $i = 1, \ldots, m$. Therefore, at step k, each optimization problem has a closed form solution given by $\tilde{x}^{i,k} := c^i + (1/\|(U^i)^{-T} b^k\|)(U^i)^{-1}(U^i)^{-T} b^k$ with an optimal value of $(b^k)^T c^i \pm (1/\|(U^i)^{-T} b^k\|)(b^k)^T (U^i)^{-1}(U^i)^{-T} b^k$. For each ellipsoid $\mathcal{E}_i$, $\tilde{x}^{i,k}$ can be computed in $O(n^2)$ operations since U^i is upper triangular, which yields an overall computational cost of $O(mn^3)$ operations after n steps. Therefore, Algorithm 4.1 terminates after $O(mn^3)$ operations.

We now prove (12). It follows from the results of Betke and Henk [3] that vol conv$(\mathcal{S}) \leq n!$ vol conv$(\mathcal{X}_0)$. Combining this inequality with Theorem 2.1, we obtain

$$\frac{1}{n^n} \text{ vol MVCE}(\mathcal{S}) \leq \text{vol conv}(\mathcal{S}) \leq n! \text{ vol conv}(\mathcal{X}_0) \leq n! \text{ vol MVCE}(\mathcal{X}_0),$$

which implies that vol MVCE$(\mathcal{S}) \leq n! n^n$vol MVCE$(\mathcal{X}_0) \leq n^{2n}$ MVCE$(\mathcal{X}_0)$. □

5 A First-Order Algorithm

In this section, we present a first-order algorithm to compute a $(1+\epsilon)$-approximation to the minimum volume covering ellipsoid (MVCE) of the union of a set of full-dimensional ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_m \subseteq \mathbb{R}^n$ for $\epsilon > 0$. Our algorithm is a generalization of the first-order algorithm presented in [17] to compute the MVCE of a finite set of points, which, in turn, is obtained from a modification of Khachiyan's algorithm [15]. Our treatment closely follows the interpretation of Khachiyan's algorithm presented in [17].

As a byproduct, we establish the existence of a finite core set $\mathcal{X} \subseteq \mathcal{S}$ whose size depends only on the dimension n and the parameter ϵ, but is independent of the number of ellipsoids m.

Algorithm 5.1 A first-order algorithm

Require: Input set of ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_m \subset \mathbb{R}^n$ given by (11) and $\epsilon > 0$.

Run Algorithm 4.1 on $\mathcal{S} := \cup_{i=1}^m \mathcal{E}_i$ to obtain output $\mathcal{X}_0 := \{x^1, \ldots, x^{2n}\}$.
$u^0 \leftarrow (1/2n)e_n$
$w^0 \leftarrow \sum_{j=1}^{2n} x^j u_j^0$.
$(M^0)^{-1} \leftarrow n \sum_{j=1}^{2n} u_j^0 (x^j - w^0)(x^j - w^0)^T$.
$\mathcal{F}_0 \leftarrow \{x \in \mathbb{R}^n : (x - w^0)^T M^0 (x - w^0) \leq 1\}$.
$x^{2n+1} \leftarrow \arg\max_{i=1,\ldots,m} \{(x - w^0)^T M^0 (x - w^0) : x \in \mathcal{E}_i\}$.
$\epsilon_0 \leftarrow (x^{2n+1} - w^0)^T M^0 (x^{2n+1} - w^0) - 1$
$k \leftarrow 0$.
While $\epsilon_k > (1+\epsilon)^{2/n} - 1$
loop
    $\beta_k \leftarrow \frac{\epsilon_k}{n(1+\epsilon_k)-1}$.
    $k \leftarrow k + 1$.
    $u^k \leftarrow \begin{bmatrix} (1-\beta_{k-1})u^{k-1} \\ \beta_{k-1} \end{bmatrix}$.
    $w^k \leftarrow \sum_{j=1}^{2n+k} x^j u_j^k$.
    $(M^k)^{-1} \leftarrow n \sum_{j=1}^{2n+k} u_j^k (x^j - w^k)(x^j - w^k)^T$.
    $\mathcal{F}_k \leftarrow \{x \in \mathbb{R}^n : (x - w^k)^T M^k (x - w^k) \leq 1\}$.
    $\mathcal{X}_k \leftarrow \mathcal{X}_{k-1} \cup \{x^{2n+k}\}$.
    $x^{2n+k+1} \leftarrow \arg\max_{i=1,\ldots,m} \{(x - w^k)^T M^k (x - w^k) : x \in \mathcal{E}_i\}$.
    $\epsilon_k \leftarrow (x^{2n+k+1} - w^k)^T M^k (x^{2n+k+1} - w^k) - 1$.
end loop

We now describe Algorithm 5.1. Given m full-dimensional ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_m \subset \mathbb{R}^n$ defined by (11), the algorithm calls Algorithm 4.1 on $\mathcal{S} := \mathrm{conv}\left(\cup_{i=1}^m \mathcal{E}_i\right)$ and computes a finite set $\mathcal{X}_0 \subset \mathcal{S}$ with $|\mathcal{X}_0| = 2n$. Next, a "trial ellipsoid" $\mathcal{F}_0$ is defined. Note that the center w^0 of $\mathcal{F}_0$ is simply the sample mean of $\mathcal{X}_0$ and M^0 is the inverse of the (scaled) sample covariance matrix of $\mathcal{X}_0$. ϵ_k measures the extent to which $\mathcal{F}_k$ should be enlarged around its center in order to cover $\mathcal{S}$. u^k can be viewed as a nonnegative weight vector whose components sum up to one. Note that the dimension of u^k increases at each iteration and is equal to $|\mathcal{X}_k|$. Unless the termination criterion is satisfied, the algorithm proceeds in an iterative manner as follows: At Step 13, u^k gets updated and is used to define w^k and M^k for the next trial ellipsoid $\mathcal{F}_k$. Observe that x^{2n+k} is precisely the farthest point in $\mathcal{S}$ from the center of the trial ellipsoid $\mathcal{F}_{k-1}$ in terms of its ellipsoidal norm. It is straightforward to verify that

$$w^k = (1-\beta_k)w^{k-1} + \beta_k x^{2n+k}, \quad k = 1, 2, \ldots \tag{13}$$

and

$$(M^k)^{-1} = (1-\beta_{k-1})(M^{k-1})^{-1} + n\beta_{k-1}(x^{2n+k} - w^{k-1})(x^{2n+k} - w^{k-1})^T, \quad k = 1, 2, \ldots. \tag{14}$$

It follows that the next trial ellipsoid $\mathcal{F}_k$ is obtained by shifting the center of $\mathcal{F}_{k-1}$ towards $\tilde{x}$ and its shape is determined by a convex combination of $(M^{k-1})^{-1}$ and a rank-one update. This update can be viewed as "enriching the eigenspace of $(M^{k-1})^{-1}$ in the direction $x^{2n+k} - w^{k-1}$". We refer the reader to [17, Section 4.1.2] for a related discussion. The parameter $\beta_k \in [0,1)$ solves the following line search problem as observed by Khachiyan [15]:

$$(LS(k)) \quad \max_{\beta \in [0,1]} \log\det\left[(1-\beta)(M^{k-1})^{-1} + n\beta(x^{2n+k} - w^{k-1})(x^{2n+k} - w^{k-1})^T)\right], \; k = 1, 2, \ldots.$$

Algorithm 5.1 terminates when the desired accuracy is achieved.

We next analyze the complexity of Algorithm 5.1. Our analysis resembles those of Khachiyan [15] and Kumar and Yıldırım [17]. The key ingredient in the complexity analysis is to demonstrate that Algorithm 5.1 produces a sequence $\{\mathcal{F}_k\}$ of trial ellipsoids with strictly increasing volumes. We utilize Lemma 4.1 to show that vol $\mathcal{F}_0$ is already a provable approximation to vol MVCE($\mathcal{S}$). The analysis will then be complete by establishing that each step of Algorithm 5.1 can be executed efficiently.

We start by proving that vol $\mathcal{F}_0$ is a provable approximation to vol MVCE($\mathcal{S}$).

Theorem 5.1 *Let $n \geq 2$. $\mathcal{F}_0$ defined in Algorithm 5.1 satisfies*

$$\log vol\ \mathcal{F}_0 \leq \log vol\ MVCE(\mathcal{S}) \leq \log vol\ \mathcal{F}_0 + 3n \log n. \tag{15}$$

Proof. We first establish that

$$\log \mathrm{vol}\ \mathcal{F}_0 \leq \log \mathrm{vol}\ \mathrm{MVCE}(\mathcal{X}_0), \tag{16}$$

where $\mathcal{X}_0 = \{x^1, \ldots, x^{2n}\}$ denotes the set of $2n$ points returned by Algorithm 4.1. Consider the following dual formulation to compute MVCE($\mathcal{X}_0$) (see, e.g., [15] or [26]):

$$\begin{array}{lll} (\mathbf{D}(\mathcal{X}_0)) & \max_u & \log\det \Pi(u) \\ & \text{s.t.} & e^T u = 1, \\ & & u \geq 0, \end{array}$$

where $u \in \mathbb{R}^{2n}$ is the decision variable and $\Pi : \mathbb{R}^{2n} \to \mathcal{S}^{n+1}$ is a linear operator given by

$$\Pi(u) := \sum_{j=1}^{2n} u_j \begin{bmatrix} x^j (x^j)^T & x^j \\ (x^j)^T & 1 \end{bmatrix}.$$

MVCE($\mathcal{X}_0$) can be recovered from an optimal solution u^* of ($\mathbf{D}(\mathcal{X}_0)$) [17, Lemma 2.1]. Furthermore, the optimal value of ($\mathbf{D}(\mathcal{X}_0)$) satisfies

$$\log \text{vol MVCE } (\mathcal{X}_0) = \log \eta + \frac{n}{2} \log n + \frac{1}{2} \log \det \Pi(u^*), \tag{18}$$

where η is the volume of the unit ball in $\mathbb{R}^n$.

Let us consider the feasible solution $\tilde{u} := (1/2n)e$ of ($\mathbf{D}(\mathcal{X}_0)$). We have

$$\Pi(\tilde{u}) = \begin{bmatrix} (1/2n) \sum_{j=1}^{2n} x^j (x^j)^T & w^0 \\ (w^0)^T & 1 \end{bmatrix} = \begin{bmatrix} I & w^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (1/n)(M^0)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ (w^0)^T & 1 \end{bmatrix}, \tag{19}$$

which implies that

$$\log \det \Pi(\tilde{u}) = -n \log n + \log \det (M^0)^{-1} = -n \log n + 2 \log \det (M^0)^{-1/2}. \tag{20}$$

However, $\log \text{vol } \mathcal{F}_0 = \log \eta + \log \det (M^0)^{-1/2}$. Combining this equality with (20), we obtain

$$\log \text{vol } \mathcal{F}_0 = \log \eta + \frac{n}{2} \log n + \frac{1}{2} \log \det \Pi(\tilde{u}).$$

Since $\tilde{u}$ is a feasible solution for the maximization problem ($\mathbf{D}(\mathcal{X}_0)$), it follows from (18) that $\log \text{vol } \mathcal{F}_0 \leq \log \text{vol MVCE}(\mathcal{X}_0)$.

Since $\mathcal{X}_0 \subseteq \mathcal{S}$, we clearly have $\log \text{vol MVCE } (\mathcal{X}_0) \leq \log \text{vol MVCE } (\mathcal{S})$, which proves the first inequality in (15). To prove the second inequality, we have

$$\begin{aligned} \max_{j=1,\ldots,2n} \ (x^j - w^0)^T M^0 (x^j - w^0) &\leq \sum_{j=1}^{2n} (x^j - w^0)^T M^0 (x^j - w^0), \\ &= M^0 \bullet \sum_{j=1}^{2n} (x^j - w^0)(x^j - w^0)^T, \\ &= 2M^0 \bullet (M^0)^{-1}, \\ &= 2n, \end{aligned}$$

which implies that the ellipsoid $\mathcal{G} := \{x \in \mathbb{R}^n : (x - w^0)^T (1/2n) M^0 (x - w^0) \leq 1\}$ covers $\mathcal{X}_0$. Therefore,

$$\begin{aligned} \log \text{vol MVCE}(\mathcal{X}_0) &\leq \log \text{vol } \mathcal{G}, \\ &= \log \eta + \frac{n}{2} \log n + \frac{n}{2} \log 2 + \log \det (M^0)^{-1/2}, \\ &\leq n \log n + \log \text{vol } \mathcal{F}_0, \end{aligned}$$

since $n \geq 2$, which implies that $\log \text{vol } \mathcal{F}_0 \geq \log \text{vol MVCE}(\mathcal{X}_0) - n \log n$. By Lemma 4.1, we have $\log \text{vol MVCE}(\mathcal{X}_0) \geq \log \text{vol MVCE}(\mathcal{S}) - 2n \log n$. Combining these inequalities, we obtain $\log \text{vol } \mathcal{F}_0 + 3n \log n \geq \log \text{vol MVCE}(\mathcal{S})$ as desired.

□

The next lemma relates $\log \text{vol } \mathcal{F}_k$ to $\log \text{vol MVCE}(\mathcal{S})$.

Lemma 5.1 *For any $k = 0, 1, 2, \ldots$, we have*

$$\log \mathit{vol}\ \mathcal{F}_k \leq \log \mathit{vol}\ \mathit{MVCE}(\mathcal{S}) \leq \log \mathit{vol}\ \mathcal{F}_k + \frac{n}{2} \log(1 + \epsilon_k). \tag{21}$$

Proof. By definition of ϵ_k, $\sqrt{1+\epsilon_k}\ \mathcal{F}_k \supseteq \mathcal{S}$, where $\sqrt{1+\epsilon_k}\ \mathcal{F}_k$ is given by expanding $\mathcal{F}_k$ around its center w^k by a factor of $\sqrt{1+\epsilon_k}$. Therefore, $\log \text{vol MVCE } (\mathcal{S}) \leq \log \text{vol}(\mathcal{F}_k) + \frac{n}{2} \log(1+\epsilon_k)$, which proves the second inequality in (21).

We follow a similar argument as in the proof of Theorem 5.1 to establish the first inequality (cf. (19), (20), and (18)). At step k of Algorithm 5.1, u^k is a feasible solution of the optimization problem $(\mathbf{D}(\mathcal{X}_\mathbf{k}))$. Therefore,

$$\begin{aligned} \log \text{vol } \mathcal{F}_k &= \log \eta + \frac{n}{2} \log n + \frac{1}{2} \log \det \Pi(u^k), \\ &\leq \log \eta + \frac{n}{2} \log n + \frac{1}{2} \log \det \Pi(u^k_*), \\ &= \log \text{vol MVCE}(\mathcal{X}_k), \end{aligned}$$

where u^k_* denotes the optimal solution of $(\mathbf{D}(\mathcal{X}_\mathbf{k}))$. Since $\mathcal{X}_k \subseteq \mathcal{S}$, the first inequality follows.
□

The following corollary immediately follows from Lemma 5.1.

Corollary 5.1 *For any $k = 0, 1, 2, \ldots$, $\epsilon_k \geq 0$. Furthermore, if Algorithm 5.1 does not terminate at step k, then $\epsilon_k > (1+\epsilon)^{2/n} - 1$.*

So far, we have established the following results: (i) vol $\mathcal{F}_0$ is a provable approximation to vol MVCE($\mathcal{S}$) and (ii) the sequence of ellipsoids $\mathcal{F}_k$ generated by Algorithm 5.1 yield lower bounds on vol MVCE($\mathcal{S}$). Our next goal is to demonstrate that $\{\text{vol } \mathcal{F}_k\}$ is a strictly increasing sequence, which implies that Algorithm 5.1 produces increasingly sharper lower bounds to vol MVCE($\mathcal{S}$). At this stage, it is worth noticing that the line search problem $LS(k)$ precisely computes the next trial ellipsoid which yields the largest increase in the volume for the particular updating scheme of Algorithm 5.1.

Proposition 5.1 *For any $k = 0, 1, 2, \ldots$,*

$$\log \mathit{vol}\ \mathcal{F}_{k+1} \geq \log \mathit{vol}\ \mathcal{F}_k + \begin{cases} \frac{1}{2}\log 2 - \frac{1}{4} > 0 & \mathit{if}\ \epsilon_k \geq 1, \\ \frac{1}{16}\epsilon_k^2 & \mathit{if}\ \epsilon_k < 1. \end{cases} \tag{22}$$

Proof. Our proof mimics Khachiyan's argument [15]. By definition of ϵ_k, we have $1+\epsilon_k = (x^{2n+k+1}-w^k)^T M^k (x^{2n+k+1}-w^k)$. It follows from (14) that

$$\begin{aligned}
\log\det(M^{k+1})^{-1} &= \log\det\left[(1-\beta_k)(M^k)^{-1} + n\beta_k(x^{2n+k+1}-w^k)(x^{2n+k+1}-w^k)^T\right], \\
&= \log\det\left[(M^k)^{-1}\left((1-\beta_k)I + n\beta_k M^k(x^{2n+k+1}-w^k)(x^{2n+k+1}-w^k)^T\right)\right], \\
&= \log\det(M^k)^{-1} + (n-1)\log(1-\beta_k) + \log\left[(1-\beta_k+n\beta_k(1+\epsilon_k)\right], \\
&= \log\det(M^k)^{-1} - (n-1)\log\left[1+\frac{\epsilon_k}{(n-1)(1+\epsilon_k)}\right] + \log(1+\epsilon_k), \\
&\geq \log\det(M^k)^{-1} + \log(1+\epsilon_k) - \frac{\epsilon_k}{1+\epsilon_k},
\end{aligned}$$

where we used the definition of β_k in the last equality and the inequality $\log(1+\kappa)\leq\kappa$ for $\kappa>-1$. Since $\log\mathrm{vol}\ \mathcal{F}_k = \log\eta + \log\det(M^k)^{-1/2} = \log\eta + (1/2)\log\det(M^k)^{-1}$, it follows that

$$\log\mathrm{vol}\ \mathcal{F}_{k+1} \geq \log\mathrm{vol}\ \mathcal{F}_k + \frac{1}{2}\log(1+\epsilon_k) - \frac{\epsilon_k}{2(1+\epsilon_k)}.$$

The assertion follows from the observation that $f(\kappa) := (1/2)\log(1+\kappa) - \kappa/[2(1+\kappa)]$ is a strictly increasing function for $\kappa\geq 0$ and $f(\kappa)\geq\kappa^2/16$ for $\kappa\in[0,1]$. □

We are now ready to analyze the iteration complexity of Algorithm 5.1. To this end, we define the following parameters:

$$\tau_\rho := \min\{k : \epsilon_k \leq 1/2^\rho\}, \quad \rho = 0,1,2,\ldots. \tag{23}$$

The next lemma establishes certain properties of τ_ρ.

Lemma 5.2 *τ_ρ satisfies the following relationships:*

$$\tau_0 = O(n\log n), \tag{24}$$
$$\tau_\rho - \tau_{\rho-1} \leq n2^{\rho+4}, \quad \rho = 1,2,\ldots. \tag{25}$$

Proof. By Theorem 5.1, $\log\mathrm{vol}\ \mathcal{F}_0 \leq \log\mathrm{vol}\ \mathrm{MVCE}(\mathcal{S}) \leq \log\mathrm{vol}\ \mathcal{F}_0 + 3n\log n$. At every iteration k with $\epsilon_k > 1$, we have $\log\mathrm{vol}\ \mathcal{F}_{k+1} - \log\mathrm{vol}\ \mathcal{F}_k \geq (1/2)\log 2 - 1/4$ by Proposition 5.1. Therefore, $\tau_0 = O(n\log n)$.

Let us now consider $\tau_\rho - \tau_{\rho-1}$, $\rho\geq 1$. For simplicity, let $\kappa := \tau_{\rho-1}$. By definition of $\tau_{\rho-1}$, it follows from Lemma 5.1 that $\log\mathrm{vol}\ \mathcal{F}_\kappa \leq \log\mathrm{vol}\ \mathrm{MCEE}\ (\mathcal{S}) \leq \log\mathrm{vol}\ \mathcal{F}_\kappa + (n/2)\log(1+2^{-(\rho-1)}) \leq \log\mathrm{vol}\ \mathcal{F}_\kappa + n2^{-\rho}$. By Proposition 5.1, $\log\mathrm{vol}\ \mathcal{F}_{k+1} - \log\mathrm{vol}\ \mathcal{F}_k \geq 2^{-(2\rho+4)}$ at every iteration k with $\epsilon_k > 2^{-\rho}$. Therefore, $\tau_\rho - \tau_{\rho-1} \leq n2^{-\rho}/2^{-(2\rho+4)} = n2^{\rho+4}$, which completes the proof. □

Lemma 5.2 enables us to establish the following result that will be useful.

Lemma 5.3 *Let $\mu\in(0,1)$. Then, Algorithm 5.1 computes an iterate with $\epsilon_k\leq\mu$ in $O\left(n(\log n + \mu^{-1})\right)$ iterations.*

Proof. Let σ be a positive integer such that $2^{-\sigma} \leq \mu \leq 2^{1-\sigma}$. Therefore, after $k = \tau_\sigma$ iterations, we already have $\epsilon_k \leq 2^{-\sigma} \leq \mu$. However,

$$\tau_\sigma = \tau_0 + \sum_{\rho=1}^{\sigma} (\tau_\rho - \tau_{\rho-1}) \leq \tau_0 + 32n \sum_{\rho=1}^{\sigma} 2^{\rho-1} \leq \tau_0 + 32n2^\sigma \leq O\left(n \log n + \frac{n}{\mu}\right),$$

where we used Lemma 5.2 and the inequality $2^\sigma \leq 2/\mu$. □

We are now in a position to establish the iteration complexity of Algorithm 5.1.

Theorem 5.2 *Algorithm 5.1 computes a* $(1+\epsilon)$*-approximation to MVCE(*$\mathcal{S}$*) after at most* $O(n(\log n + [(1+\epsilon)^{2/n} - 1]^{-1}))$ *iterations.*

Proof. We first establish that Algorithm 5.1 returns a $(1+\epsilon)$-approximation to MVCE($\mathcal{S}$) upon termination. Let κ denote the index of the final iterate. We have $\epsilon_\kappa \leq (1+\epsilon)^{2/n} - 1$. The trial ellipsoid $\mathcal{F}_\kappa$ satisfies $\mathcal{S} \subseteq \sqrt{1+\epsilon_\kappa}\, \mathcal{F}_\kappa$, which together with Lemma 5.1 implies that

$$\text{vol } \mathcal{F}_\kappa \leq \text{vol MVCE}(\mathcal{S}) \leq \text{vol} \sqrt{1+\epsilon_\kappa} \mathcal{F}_\kappa = (1+\epsilon_\kappa)^{n/2} \text{vol } \mathcal{F}_\kappa \leq (1+\epsilon) \text{vol } \mathcal{F}_\kappa.$$

Therefore, $\sqrt{1+\epsilon_\kappa}\, \mathcal{F}_\kappa$ is a $(1+\epsilon)$-approximation to MVCE($\mathcal{S}$).

We now prove the iteration complexity. If $\epsilon \geq 2^{n/2} - 1$, then $(1+\epsilon)^{2/n} - 1 \geq 1$, which implies that $\tau_0 = O(n \log n)$ iterations already suffice. Otherwise, the result follows from Lemma 5.3. □

Remark 1 *The iteration complexity of Algorithm 5.1 is asymptotically identical to that of the algorithm of Kumar and Yıldırım [17] that computes the MVCE of a finite set of points.*

We now establish the overall complexity of Algorithm 5.1.

Theorem 5.3 *Algorithm 5.1 computes a* $(1+\epsilon)$*-approximation to MVCE(*$\mathcal{S}$*) in*

$$O\left(mn^{O(1)}(\log n + [(1+\epsilon)^{2/n} - 1]^{-1})\right)$$

operations, where $O(1)$ *denotes a universal constant greater than four.*

Proof. We already have the iteration complexity from Theorem 5.2. We need to analyze the computational cost of each iteration.

Let us start with the initialization stage. By Lemma 4.1, Algorithm 4.1 runs in $O(mn^3)$ operations. w^0 and $(M^0)^{-1}$ can be computed in $O(n^2)$ and $O(n^3)$ operations, respectively. The furthest point x^{2n+1} from the center of $\mathcal{F}_0$ can be determined by solving m separate quadratic optimization problems with a single ellipsoidal constraint. By Proposition 2.1, each optimization problem can be solved in $O(n^{O(1)} + n^3)$ operations. Finally, it takes $O(n^2)$ operations to compute ϵ_0. Therefore, the overall complexity of the initialization step is $O(m(n^{O(1)} + n^3))$ operations. Similarly, at iteration k, the major work is the computation of the furthest point x^{2n+k+1}, which can be performed in $O(m(n^{O(1)} + n^3))$ operations. Therefore, the overall running time of Algorithm 5.1 is given by $O(mn^{O(1)}(\log n + [(1 + \epsilon)^{2/n} - 1]^{-1}))$ operations. □

Remark 2 *The overall complexity of Algorithm 5.1 is linear in m, the number of ellipsoids. This suggests that, in theory, Algorithm 5.1 is especially well-suited for instances of the MVCE problem that satisfy $m \gg n$ and for moderate values of ϵ. In addition, if $\epsilon \in (0, 1)$, we have $(1+\epsilon)^{2/n} - 1 = \Theta(\epsilon/n)$, in which case, the running time of Algorithm 5.1 can be simplified to $O(mn^{O(1)}/\epsilon)$ operations, where $O(1)$ is a universal constant greater than five. Note that the running time of Algorithm 5.1 is polynomial for fixed ϵ.*

We close this section by establishing that the finite set of points collected by Algorithm 5.1 serves as a reasonably good approximation to $\mathcal{S}$ in the sense that their respective minimum volume covering ellipsoids are closely related.

Proposition 5.2 *Let κ denote the index of the final iterate of Algorithm 5.1. Then, $\mathcal{X}_\kappa$ satisfies*

$$\text{vol MVCE}(\mathcal{X}_\kappa) \le \text{vol MVCE}(\mathcal{S}) \le (1+\epsilon)\text{vol MVCE}(\mathcal{X}_\kappa). \tag{26}$$

In addition,

$$|\mathcal{X}_\kappa| = O(n(\log n + [(1+\epsilon)^{2/n} - 1]^{-1})). \tag{27}$$

Proof. We first prove (26). Note that the first inequality is obvious since $\mathcal{X}_\kappa \subseteq \mathcal{S}$. The second inequality follows from the relationships vol $\mathcal{F}_\kappa \le$ vol MVCE$(\mathcal{X}_\kappa) \le$ vol MVCE$(\mathcal{S})$ (see the proof of Lemma 5.1) and vol MVCE$(\mathcal{S}) \le (1+\epsilon)$vol $\mathcal{F}_\kappa$ (see the proof of Theorem 5.2).

Note that $|\mathcal{X}_\kappa|$ is simply given by $2n+\kappa$. Therefore, (27) simply follows from Theorem 5.2. □

Remark 3 *Proposition 5.2 establishes that Algorithm 5.1 computes a finite set of points $\mathcal{X}_\kappa \subseteq \mathcal{S}$ whose minimum volume covering ellipsoid is related to MVCE$(\mathcal{S})$ via (26). In addition, $|\mathcal{X}_\kappa|$ depends only on the dimension n and the approximation factor ϵ but is independent of the number of ellipsoids m. Furthermore, for $\epsilon \in (0, 1)$, we have $|\mathcal{X}_\kappa| = O(n^2/\epsilon)$. Therefore, $\mathcal{X}_\kappa$ serves as a finite core set for $\mathcal{S}$. Viewed from this perspective, Proposition 5.2 is an addition to the previous core set results for other geometric optimization problems [16, 8, 7, 9, 1, 17].*

Remark 4 *In [17], a similar core set result has been established for the MVCE problem for a finite set of points. It is remarkable that asymptotically the same result holds regardless of the underlying geometry of the input set. In particular, the main ingredient in [17] that leads to the improved complexity result over Khachiyan's algorithm [15] as well as the core set result is the initial volume approximation. In particular, the counterpart of this initialization stage (cf. Algorithm 4.1) enables us to extend the algorithm of Kumar and Yıldırım to a set of ellipsoids. Khachiyan's algorithm cannot be extended to a set of ellipsoids as it relies on the finiteness property of the input set. In the next section, we discuss how the framework of Algorithm 5.1 can be extended to compute the minimum volume covering ellipsoid of other sets.*

6 Minimum Volume Covering Ellipsoids of Other Sets

In this section, we discuss the extent to which Algorithm 5.1 can be used to compute the minimum volume covering ellipsoids of other input sets. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ denote m objects in $\mathbb{R}^n$ and let $\mathcal{S} := \mathrm{conv}\,(\cup_{i=1}^m \mathcal{T}_i)$. In order to extend Algorithm 5.1, we identify the following two subroutines that need to be implemented efficiently:

1. **Subroutine 1:** Optimizing a linear function over $\mathcal{S}$.

2. **Subroutine 2:** Maximizing a quadratic function over $\mathcal{S}$.

Note that Subroutine 1 is required by Algorithm 4.1. Similarly, the computation of the furthest point from the center of a trial ellipsoid (in its ellipsoidal norm) is equivalent to Subroutine 2. All of the other operations of Algorithm 5.1 can be performed efficiently for any input set $\mathcal{S}$. We now consider specific examples of input sets.

6.1 Set of Points

If $\mathcal{S}$ is given by the convex hull of m points in $\mathbb{R}^n$, both of the subroutines can be implemented in a straightforward way. In particular, Subroutine 1 can be performed in $O(mn^2)$ operations. Subroutine 2 reduces to evaluating a quadratic function at m points, which can be performed in $O(mn^2)$ operations, and then identifying the maximum in $O(m)$ steps. Consequently, Algorithm 5.1 can be used to compute the minimum volume covering ellipsoid of a set of points and reduces to the algorithm of Kumar and Yıldırım [17].

6.2 Set of Balls

Let $\mathcal{T}_i := \{x \in \mathbb{R}^n : \|x - c^i\| \leq r^i\},\ i = 1, \ldots, m$, where $c^i \in \mathbb{R}^n$ and $r^i \in \mathbb{R},\ i = 1, \ldots, m$. Since a full-dimensional ball is a special type of ellipsoid, Algorithm 5.1 can obviously be used for this input set. We mainly mention this input set to illustrate that Subroutine 1 admits a more efficient implementation in comparison with a set of ellipsoids. For any nonzero unit vector $d \in \mathbb{R}^n$, the linear function $d^T x$ is minimized and maximized over $\mathcal{T}_i,\ i = 1, \ldots, m$ at $x = c^i - r^i d$ and $x = c^i + r^i d$, respectively. Therefore, Subroutine 1 can be implemented in $O(mn^2)$ operations. However, Subroutine 2 still has the same complexity as in the case of ellipsoids. Consequently, Algorithm 5.1 can compute the minimum volume covering ellipsoid of a set of balls in asymptotically the same number of operations as in the case of ellipsoids.

6.3 Set of Half Ellipsoids

Let $\mathcal{T}_i := \{x \in \mathbb{R}^n : (x - c^i)^T Q^i (x - c^i) \leq 1, (d^i)^T x \leq \alpha^i\},\ i = 1, \ldots, m$, where $c^i \in \mathbb{R}^n$, $Q^i \in \mathcal{S}^n$ is positive definite, $d^i \in \mathbb{R}^n$ with $d^i \neq 0$, and $\alpha^i \in \mathbb{R}$. $\mathcal{T}_i$ is simply given by the intersection of a full-dimensional ellipsoid and a half space. We will call $\mathcal{T}_i$ a half-ellipsoid. To avoid trivialities, we assume that each $\mathcal{T}_i$ has a nonempty interior. It follows from the results of Sturm and Zhang [25] that the problem of optimizing any quadratic (hence linear) objective function over $\mathcal{T}_i$ can be cast as an equivalent SDP problem with a fixed number of constraints using a technique similar to that in the proof of Proposition 2.1. Since both Subroutines 1 and 2 naturally decompose into a linear and quadratic optimization problem over each $\mathcal{T}_i$, respectively, it follows from Corollary 2.1 that both of them can be implemented in polynomial time. Therefore, Algorithm 5.1 can compute the minimum volume covering ellipsoid of a set of half ellipsoids in polynomial time for any fixed $\epsilon > 0$.

6.4 Set of Intersections of A Pair of Ellipsoids

Let $\mathcal{T}_i := \{x \in \mathbb{R}^n : (x - c^i)^T Q^i (x - c^i) \leq 1, (x - d^i)^T Q^i (x - d^i) \leq 1\},\ i = 1, \ldots, m$, where $c^i \in \mathbb{R}^n$, $d^i \in \mathbb{R}^n$, and $Q^i \in \mathcal{S}^n$ is positive definite, $i = 1, \ldots, m$. Note that $\mathcal{T}_i$ is given by the intersection of two ellipsoids defined by the same matrix Q^i with different centers. Similarly to the previous case, Sturm and Zhang [25] establish that the problem of optimizing any quadratic (hence linear) objective function over $\mathcal{T}_i$ can be decomposed into two quadratic (linear) optimization problems over a half ellipsoid, each of which can be solved in polynomial time. Therefore, Algorithm 5.1 can compute the minimum volume covering ellipsoid of a set of intersections of a pair of ellipsoids in polynomial time for any fixed $\epsilon > 0$. We remark that the complexity of solving a general quadratic optimization problem over the intersection of two arbitrary ellipsoids is still an open problem.

6.5 Other Sets and Limitations

Based on the previous examples, it is clear that Algorithm 5.1 can be applied to any input set as long as Subroutines 1 and 2 admit efficient implementations. While Subroutine 1 can be performed efficiently for a rather large class of input sets (e.g., classes of convex sets that admit computable barrier functions [18]), Subroutine 2 can be efficiently implemented only in very special cases, some of which have been outlined in this section.

For instance, if $\mathcal{S}$ is given by the union of polytopes $\mathcal{T}_i := \{x \in \mathbb{R}^n : A^i x \leq b^i\},\ i = 1, \ldots, m$, where $A^i \in \mathbb{R}^{m \times n}$ and $b^i \in \mathbb{R}^m,\ i = 1, \ldots, m$, then Subroutine 1 reduces to linear programming, which can be solved efficiently using interior-point methods combined with a finite termination strategy [31]. However, maximizing a convex quadratic function over a polytope is in general an NP-hard problem. Therefore, even in the case of a single polytope defined by linear inequalities, the problem of computing the minimum volume covering ellipsoid is computationally intractable.

In summary, the extent to which Algorithm 5.1 can be applied to other input sets is largely determined by whether Subroutine 2 can be implemented efficiently. Since quadratic optimization over various feasible regions is an active area of research [25, 32], further progress in establishing polynomial complexity will widen the domain of input sets to which Algorithm 5.1 can be applied.

7 Concluding Remarks

In this paper, we established that the first-order algorithm of Kumar and Yıldırım [17] to compute the minimum volume enclosing ellipsoid (MVCE) of a finite set of points can be extended to compute the MVCE of a union of finitely many full-dimensional ellipsoids without compromising the polynomial complexity for a fixed approximation parameter $\epsilon > 0$. Moreover, the iteration complexity of our extension and the core set size remain asymptotically identical. We also discuss how the framework of our algorithm can be extended to compute the MVCE of other input sets in polynomial-time and present certain limitations. Our core set result is an addition to the recent thread of works on core sets for several geometric optimization problems [16, 8, 7, 9, 1, 17].

While our algorithm has a polynomial-time complexity in theory, it would be especially well-suited for instances of the MVCE problem with $m \gg n$ and moderately small values of ϵ. In particular, our algorithm would be applicable to the problem of constructing a bounding volume hierarchy as the objects lie in three-dimensional space (i.e., $n = 3$) and a fixed parameter ϵ usually suffices for practical applications. To the best of our knowledge, this is the first result in the literature towards approximating a union of ellipsoids by a finite subset whose size depends only on the dimension n and the parameter ϵ.

On the other hand, our algorithm would probably not be practical if a higher accuracy (i.e., a smaller ϵ) is required or if the dimension n is large. In addition, it is well-known that first-order algorithms in general suffer from slow convergence in practice, especially for smaller values of ϵ. Motivated by the core set result established in this paper and the encouraging computational results based on a similar core set result for the minimum enclosing ball problem [16], we intend to work on a cutting plane algorithm for the MVCE problem with an emphasis on establishing an upper bound on the number of added constraints to obtain a desired accuracy.

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References

[1] P. K. Aggarwal, R. Poreddy, K. R. Varadarajan, and H. Yu. Practical methods for shape fitting and kinetic data structures using core sets. In *Proceedings of the 20th Annual ACM Symposium on Computational Geometry*, 2004.

[2] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* MPS-SIAM Series on Optimization. SIAM Publications, Philadelphia, PA, 2001.

[3] U. Betke and M. Henk. Approximating the volume of convex bodies. *Discrete & Computational Geometry*, 10(1):15–21, 1993.

[4] L. Blum, M. Shub, and S. Smale. On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions, and universal machines. *Bulletin of the American Mathematical Society*, 21:1–46, 1989.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics. SIAM Publications, Philadelphia, PA, 1994.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[7] M. Bădoiu and K. L. Clarkson. Smaller core-sets for balls. In *Proceedings of the 14th ACM-SIAM Symposium on Discrete Algorithms*, pages 801–802, 2003.

[8] M. Bădoiu, S. Har-Peled, and P. Indyk. Approximate clustering via core-sets. In *Proceedings of the 34th Annual ACM Symposium on Theory of Computing*, pages 250–257, 2002.

[9] T. M. Chan. Faster core-set constructions and data stream algorithms in fixed dimensions. In *Proceedings of the 20th Annual ACM Symposium on Computational Geometry*, 2004.

[10] Y. Chien and J. Liu. Improvements to ellipsoidal fit based collision detection. Technical Report TR-IIS-03-001, Academia Sinica, Institute of Information Science, Taiwan, Taipei, R. O. C., 2003.

[11] J. H. Clark. Hierarchical geometric models for visible surface algorithms. *Communications of the ACM*, 19(10):547–554, 1976.

[12] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer, New York, 1988.

[13] F. John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays, presented to R. Courant on his 60th birthday January 8, 1948*, pages 187–204. Interscience, New York, 1948. Reprinted in: *Fritz John, Collected Papers Volume 2* (J. Moser, ed), Birkhäuser, Boston, 1985, pp. 543–560.

[14] M. Ju, J. Liu, S. Shiang, Y. Chien, K. Hwang, and W. Lee. Fast and accurate collision detection based on enclosed ellipsoids. *Robotica*, 19:381–394, 2001.

[15] L. G. Khachiyan. Rounding of polytopes in the real number model of computation. *Mathematics of Operations Research*, 21(2):307–320, 1996.

[16] P. Kumar, J. S. B. Mitchell, and E. A. Yıldırım. Approximate minimum enclosing balls in high dimensions using core-sets. *Journal of Experimental Algorithms*, 8, 2004. Article 1 (online).

[17] P. Kumar and E. A. Yıldırım. Minimum volume enclosing ellipsoids and core sets. *Journal of Optimization Theory and Applications*, 2005. To Appear.

[18] Yu. E. Nesterov and A. S. Nemirovskii. *Interior Point Polynomial Methods in Convex Programming*. SIAM Publications, Philadelphia, PA, 1994.

[19] I. Pólik and T. Terlaky. A comprehensive study of the $\mathcal{S}$-lemma. Technical report, McMaster University, Advanced Optimization Laboratory, 2004.

[20] L. Porkolab and L. Khachiyan. On the complexity of semidefinite programs. *Journal of Global Optimization*, 10(4):351–365, 1997.

[21] M. V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Mathematical Programming*, 77(2):129–162, 1997.

[22] J. Renegar. On the computational complexity and geometry of the first-order theory of the reals, Part I: Introduction; preliminaries; the geometry of the semi-algebraic sets; the decision problem for the existential theory of the reals. *Journal of Symbolic Computation*, 13(3):255–299, 1992.

[23] E. Rimon and S. Boyd. Obstacle collision detection using best ellipsoid fit. *Journal of Intelligent and Robotic Systems*, 18(2):105–126, 1997.

[24] S. Shiang, J. Liu, and Y. Chien. Estimate of minimum distance between convex polyhedra based on enclosed ellipsoids. In *Proceedings of the 2000 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 739–744, 2000.

[25] J. F. Sturm and S. Z. Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operations Research*, 28(2):246–267, 2003.

[26] P. Sun and R. M. Freund. Computation of minimum volume covering ellipsoids. *Operations Research*, 52(5):690–706, 2004.

[27] K. C. Toh. Primal-dual path-following algorithms for determinant maximization problems with linear matrix inequalities. *Computational Optimization and Applications*, 14(3):309–330, 1999.

[28] L. Vandenberghe, S. Boyd, and S. P. Wu. Determinant maximization with linear matrix inequality constraints. *SIAM Journal on Optimization*, 19(2):499–533, 1998.

[29] V. A. Yakubovich. $\mathcal{S}$-procedure in nonlinear control theory. *Vestnik Leningrad University*, 4:73–93, 1971. English translation from Russian original.

[30] Y. Ye. On affine scaling algorithms for nonconvex quadratic programming. *Mathematical Programming*, 56(3):285–300, 1992.

[31] Y. Ye. On the finite convergence of interior-point algorithms for linear programming. *Mathematical Programming*, 57(2):325–335, 1992.

[32] Y. Ye and S. Z. Zhang. New results on quadratic minimization. *SIAM Journal of Optimization*, 14(1):245–267, 2003.