Demostraciones de Econometría

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Mínimos Cuadrados Ordinarios

1. El Estimador. $\hat{\beta}$

Derivación

Para el modelo

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

El problema a minimizar para obtener $\hat{\beta}_0$ y $\hat{\beta}_1$ es

$$\boxed{\min_{\{\beta_0,\beta_1\}} \sum_{i=1}^{T} u_i^2 = \sum_{i=1}^{T}(y_i - \beta_0 - \beta_1 x_i)^2}$$

Resolviendo para $\hat{\beta}_0$

$$\frac{\partial \sum_{i=1}^{T} u_i^2}{\partial \beta_0}|_{\beta=\hat{\beta}} = -2\sum_{i=1}^{T}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{T}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{T} y_i - \sum_{i=1}^{T}\hat{\beta}_0 - \sum_{i=1}^{T}\hat{\beta}_1 x_i = 0$$

$$\sum_{i=1}^{T} y_i - T * \hat{\beta}_0 - \hat{\beta}_1 \sum_{i=1}^{T} x_i = 0$$

$$T * \hat{\beta}_0 = \sum_{i=1}^{T} y_i - \hat{\beta}_1 \sum_{i=1}^{T} x_i$$

$$\hat{\beta}_o = \sum_{i=1}^{T}\frac{y_i}{T} - \hat{\beta}_1 \sum_{i=1}^{T}\frac{x_i}{T}$$

$$\boxed{\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}} \tag{1}$$

Resolviendo para $\hat{\beta}_1$

$$\frac{\partial \sum_{i=1}^{T} u_i^2}{\partial \beta_1}|_{\beta=\hat{\beta}} = -2\sum_{i=1}^{T}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)x_i = 0$$

$$\sum_{i=1}^{T} x_i(y_i - (\overline{y} - \hat{\beta}_1\overline{x}) - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{T} x_i[(y_i - \overline{y}) - \hat{\beta}_1(x_i - \overline{x})] = 0$$

$$\sum_{i=1}^{T} x_i(y_i - \overline{y}) - \hat{\beta}_1 \sum_{i=1}^{T} x_i(x_i - \overline{x})] = 0$$

$$\sum_{i=1}^{T}(y_i - \overline{y})(x_i - \overline{x}) = \hat{\beta}_1 \sum_{i=1}^{T}(x_i - \overline{x})^2$$

$$\boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{T}(y_i - \overline{y})(x_i - \overline{x})}{\sum_{i=1}^{T}(x_i - \overline{x})^2}} \tag{2}$$

Para un modelo con k variables y constante tendremos

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + ... + \beta_k x_{ki} + u_i$$

Matricialmente

$$Y_{T\times1} = X_{T\times(k+1)}\beta_{(k+1)\times1} + u_{T\times1}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}_{T\times1} = \begin{bmatrix} 1 & x_{1,1} & x_{1,2} & \cdots & x_{1,k} \\ 1 & x_{2,1} & x_{2,2} & \cdots & x_{2,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{T,1} & x_{T,2} & \cdots & x_{T,k} \end{bmatrix}_{T\times(k+1)} * \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}_{(k+1)\times1} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}_{T\times1}$$

El problema a minimizar para obtener el vector $\hat{\beta}$ es

$$\min_{\{\beta\}} \; u'u = (Y - X\beta)'(Y - X\beta)$$

$$u'u = Y'Y - \beta'X'Y - Y'X\beta + \beta X'X\beta$$

$$u'u = Y'Y - 2Y'X\beta + \beta X'X\beta$$

$$\frac{\partial u'u}{\partial \beta}|_{\beta=\hat{\beta}} = -2X'Y + 2X'X\hat{\beta} = 0$$

$$X'X\hat{\beta} = X'Y$$

$$\boxed{\hat{\beta} = (X'X)^{-1}(X'Y)}$$

2. Insesgamiento. $\mathbb{E}[\hat{\beta}]$

Derivación

Para el modelo

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

Tendremos

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{T}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{T}(x_i - \overline{x})^2}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{T}(x_i - \overline{x})[(\beta_0 + \beta_1 x_i + u_i) - (\beta_0 + \beta_1 \overline{x} + \overline{u})]}{\sum_{i=1}^{T}(x_i - \overline{x})^2}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{T}(x_i - \overline{x})(\beta_1(x_i - \overline{x}) + u_i)}{\sum_{i=1}^{T}(x_i - \overline{x})^2}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{T} \beta_1(x_i - \overline{x})^2 + (x_i - \overline{x})u_i}{\sum_{i=1}^{T}(x_i - \overline{x})^2}$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{T}(x_i - \overline{x})u_i}{\sum_{i=1}^{T}(x_i - \overline{x})^2}$$

$$\mathbb{E}[\hat{\beta}_1] = \beta_1 + \mathbb{E}\left[\frac{\sum_{i=1}^{T}(x_i - \overline{x})u_i}{\sum_{i=1}^{T}(x_i - \overline{x})^2}\right]$$

$$\mathbb{E}[\hat{\beta}_1] = \beta_1 + \frac{\sum_{i=1}^{T}(x_i - \overline{x})\mathbb{E}(u_i)}{\sum_{i=1}^{T}(x_i - \overline{x})^2}$$

$$\boxed{\mathbb{E}[\hat{\beta}_1] = \beta_1} \tag{3}$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$$

$$\hat{\beta}_0 = \beta_0 + \beta_1 \overline{x} + \overline{u} - \hat{\beta}_1 \overline{x}$$

$$\mathbb{E}[\hat{\beta}_0] = \beta_0 + \beta_1 \overline{x} - \mathbb{E}[\hat{\beta}_1]\overline{x}$$

$$\mathbb{E}[\hat{\beta}_0] = \beta_0 + \beta_1 \overline{x} - \beta_1 \overline{x}$$

$$\boxed{\mathbb{E}[\hat{\beta}_0] = \beta_0} \tag{4}$$

Matricialmente para el modelo

$$Y = X\beta + u$$

Tendremos que

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + u)$$

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'u$$

$$\mathbb{E}[\hat{\beta}] = \beta + (X'X)^{-1}X'\mathbb{E}[u]$$

$$\boxed{\mathbb{E}[\hat{\beta}] = \beta}$$

3. La Eficiencia. $Var(\hat{\beta})$

Derivación

Para el modelo

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

Tendremos

$$Var(\hat{\beta}_1) = \mathbb{E}[(\hat{\beta}_1 - \mathbb{E}(\hat{\beta}_1))^2]$$

$$Var(\hat{\beta}_1) = \mathbb{E}\left[\frac{\sum_{i=1}^{T}(x_i - \overline{x})u_i}{\sum_{i=1}^{T}(x_i - \overline{x})^2}\right]^2$$

$$Var(\hat{\beta}_1) = \mathbb{E}\left[\frac{\sum_{i=1}^{T}(x_i - \overline{x})^2 u_i^2}{\left(\sum_{i=1}^{T}(x_i - \overline{x})^2\right)^2}\right]$$

$$Var(\hat{\beta}_1) = \frac{\sum_{i=1}^{T}(x_i - \overline{x})^2 \mathbb{E}(u_i^2)}{\left(\sum_{i=1}^{T}(x_i - \overline{x})^2\right)^2}$$

$$\boxed{Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{T}(x_i - \overline{x})^2}} \tag{5}$$

Matricialmente para el modelo

$$Y = X\beta + u$$

Tendremos que

$$V(\hat{\beta}) = \mathbb{E}\left[(\hat{\beta} - \mathbb{E}[\hat{\beta}])(\hat{\beta} - \mathbb{E}[\hat{\beta}])'\right]$$

$$V(\hat{\beta}) = \mathbb{E}\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right]$$

$$V(\hat{\beta}) = \mathbb{E}\left[((X'X)^{-1}X'u)((X'X)^{-1}X'u)'\right]$$

$$V(\hat{\beta}) = \mathbb{E}\left[(X'X)^{-1}X'uu'X(X'X)^{-1}\right]$$
$$V(\hat{\beta}) = (X'X)^{-1}X'\mathbb{E}[uu']X(X'X)^{-1}$$
$$V(\hat{\beta}) = \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$
$$\boxed{V(\hat{\beta}) = \sigma^2(X'X)^{-1}}$$

4. Logaritmos como una aproximación a cambios porcentuales.

Derivación

Podemos escribir la variación porcentual de una variable Y de la siguiente manera

$$\Delta \%Y = \frac{y_t - y_{t-1}}{y_{t-1}}$$

Paralelamente tenemos que

$$\Delta log(Y) = log(y_t) - log(y_{t-1})$$
$$\Delta log(Y) = log\left(\frac{y_t}{y_{t-1}}\right)$$
$$\Delta log(Y) = log\left(\frac{y_t}{y_{t-1}} + 1 - 1\right)$$
$$\Delta log(Y) = log\left(\frac{y_t}{y_{t-1}} + 1 - \frac{y_{t-1}}{y_{t-1}}\right)$$
$$\Delta log(Y) = log\left(1 + \frac{y_t - y_{t-1}}{y_{t-1}}\right)$$

Si tenemos que $\Delta \%Y = \frac{y_t - y_{t-1}}{y_{t-1}}$ es cercano a 0 entonces.

$$\boxed{\Delta log(Y) = log\left(1 + \frac{y_t - y_{t-1}}{y_{t-1}}\right) \approx \frac{y_t - y_{t-1}}{y_{t-1}}}$$

Por ejemplo, si hay un cambio porcentual de 3 %

$$ln(1 + 0{,}03) \approx 0{,}03$$
$$ln(1{,}03) \approx 0{,}03$$
$$0{,}0295588 \approx 0{,}03$$

Una aproximación bastante razonable

Por lo tanto, si queremos estimar el cambio porcentual de la variable Y ante cambios porcentuales de la variable X, podemos estimar lo siguiente

$$log(y_t) = \beta_0 + \beta_1 log(x_t) + u_t$$

¿Por qué?

$$\beta_1 = \frac{\Delta log(y_t)}{\Delta log(x_t)} \approx \frac{\frac{y_t - y_{t-1}}{y_{t-1}}}{\frac{x_t - x_{t-1}}{x_{t-1}}}$$

Nota: Típicamente se usa el logaritmo natural (logaritmo en base e) para estimar regresiones dado que este tiene propiedades que facilitan mucho la vida al hacer operaciones.

5. Sesgo por variable omitida

Derivación

Si el modelo real es:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + u_i \tag{6}$$

Pero nosotros estimamos

$$y_i = \gamma_0 + \gamma_1 x_{1,i} + e_i \tag{7}$$

Entonces llegaremos a que:

$$\hat{\gamma_1} = \frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{T}(x_{1,i} - \overline{x})^2}$$

Reemplazamos y_i y el $\overline{y}$ del modelo real.

$$\hat{\gamma_1} = \frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})[(\beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + u_i) - (\beta_0 + \beta_1 \overline{x_1} + \beta_2 \overline{x_2} + \overline{u})]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}$$

$$\hat{\gamma_1} = \frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})[\beta_1 x_{1,i} - \beta_1 \overline{x_1} + \beta_2 x_{2,i} - \beta_2 \overline{x_2} + u_i - \overline{u}]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}$$

$$\hat{\gamma_1} = \frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})[\beta_1(x_{1,i} - \overline{x_1}) + \beta_2(x_{2,i} - \overline{x_2}) + u_i]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}$$

$$\hat{\gamma_1} = \frac{\beta_1 \sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2 + \beta_2 \sum_{i=1}^{T}(x_{1,i} - \overline{x_1})(x_{2,i} - \overline{x_2}) + \sum_{i=1}^{T}(x_{1,i} - \overline{x_1})u_i]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}$$

$$\hat{\gamma_1} = \beta_1 + \frac{\beta_2 \sum_{i=1}^{T}(x_{1,i} - \overline{x_1})(x_{2,i} - \overline{x_2})}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2} + \frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})u_i]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}$$

Aplicamos esperanza

$$\mathbb{E}[\hat{\gamma}_1] = \mathbb{E}[\beta_1] + \mathbb{E}\left[\frac{\beta_2 \sum_{i=1}^{T}(x_{1,i} - \overline{x_1})(x_{2,i} - \overline{x_2})}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}\right] + \mathbb{E}\left[\frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})u_i]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}\right]$$

$$\mathbb{E}[\hat{\gamma}_1] = \beta_1 + \frac{\beta_2 \sum_{i=1}^{T}(x_{1,i} - \overline{x_1})(x_{2,i} - \overline{x_2})}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2} + \frac{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})\mathbb{E}[u_i]}{\sum_{i=1}^{T}(x_{1,i} - \overline{x_1})^2}$$

$$\boxed{\mathbb{E}[\hat{\gamma}_1] = \beta_1 + \underbrace{\beta_2 \frac{Cov(X_1, X_2)}{Var(X_1)}}_{Sesgo}}$$

La dirección del sesgo depende del signo que tomen β_2 y $Cov(X_1, X_2)$. Se dice que el estimador $\hat{\gamma}_1$ está "sobrevalorado" si el sesgo es positivo, y está "subvalorado" si el sesgo es negativo. Esto se resume en la siguiente tabla:

	$Cov(X_1, X_2) > 0$	$Cov(X_1, X_2) < 0$
$\beta_2 > 0$	Sobrevalorado	Subvalorado
$\beta_2 < 0$	Subvalorado	Sobrevalorado

Apéndice

Apéndice N°1: Sea $\{x_i, y_i : i = 1, \ldots, n\}$ un conjunto de n par de observaciones, tendremos de que:

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0$$

Demostración

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = \sum_{i=1}^{n} x_i - \bar{x} \underbrace{\sum_{i=1}^{n} 1}_{=n}$$

$$\boxed{\therefore \quad \sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0}$$

Apéndice N°2: Sea $\{x_i, y_i : i = 1, \ldots, n\}$ un conjunto de n par de observaciones, tendremos de que:

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i \cdot (x_i - \bar{x})$$

Demostración

$$\begin{aligned}
\sum_{i=1}^{n} (x_i - \bar{x})^2 &= \sum_{i=1}^{n} \left(x_i^2 - 2 \cdot x_i \cdot \bar{x} + \bar{x}^2\right) \\
&= \sum_{i=1}^{n} x_i^2 - 2 \cdot \bar{x} \cdot \sum_{i=1}^{n} x_i + \bar{x}^2 \cdot \sum_{i=1}^{n} 1 \quad / \quad \sum_{i=1}^{n} x_i = n \cdot \bar{x} \\
&= \sum_{i=1}^{n} x_i^2 - 2 \cdot n \cdot \bar{x}^2 + n \cdot \bar{x}^2 \\
&= \sum_{i=1}^{n} x_i^2 - n \cdot \bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \cdot \bar{x} = \sum_{i=1}^{n} \left(x_i^2 - x_i \cdot \bar{x}\right)
\end{aligned}$$

$$\boxed{\therefore \quad \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i \cdot (x_i - \bar{x})}$$

Apéndice N°3: Sea $\{x_i, y_i : i = 1, \ldots, n\}$ un conjunto de n par de observaciones, tendremos de que:

$$\sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y}) = \sum_{i=1}^{n} y_i \cdot (x_i - \bar{x}) = \sum_{i=1}^{n} x_i \cdot (y_i - \bar{y})$$

Demostración

$$\begin{aligned}
\sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y}) &= \sum_{i=1}^{n} (x_i \cdot y_i - x_i \cdot \bar{y} - y_i \cdot \bar{x} + \bar{x} \cdot \bar{y}) \quad / \quad \sum_{i=1}^{n} z_i = n \cdot \bar{z} \\
&= \sum_{i=1}^{n} x_i \cdot y_i - n \cdot \bar{x} \cdot \bar{y} - n \cdot \bar{y} \cdot \bar{x} + \underbrace{\sum_{i=1}^{n} \bar{x} \cdot \bar{y}}_{=n \cdot \bar{x} \cdot \bar{y}} \\
&= \sum_{i=1}^{n} x_i \cdot y_i - n \cdot \bar{x} \cdot \bar{y} = \sum_{i=1}^{n} x_i \cdot y_i - \sum_{i=1}^{n} \bar{x} \cdot y_i = \sum_{i=1}^{n} (x_i \cdot y_i - \bar{x} \cdot y_i)
\end{aligned}$$

$$\boxed{\therefore \quad \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y}) = \sum_{i=1}^{n} y_i \cdot (x_i - \bar{x}) = \sum_{i=1}^{n} x_i \cdot (y_i - \bar{y})}$$

Apéndice N°4:

- Introduciremos el concepto de esperanza iterada, la cual dice que si $\mathbb{E}|y| < \infty$ entonces para cualquier vector aleatorio de x, se cumple que:

$$\mathbb{E}(\mathbb{E}(y|x)) = \mathbb{E}(y) \tag{8}$$

A esta se le conoce como la Ley simple de las expectativas iteradas, la cual nos dice, en resumidas cuentas, que el promedio de los promedios condicionales es el promedio incondicional.

Por ejemplo, consideremos la siguiete tabla:

Y	X
1	1
2	1
3	2
4	2

En este caso tendremos que:

$$E[Y] = \frac{1+2+3+4}{4} = 2{,}5$$

$$E[Y/X = 1] = \frac{1+2}{2} = 1{,}5$$

$$E[Y/X = 2] = \frac{3+4}{2} = 3{,}5$$

$$E[E[Y/X]] = \frac{1{,}5 + 3{,}5}{2} = 2{,}5$$

Por lo que es fácil de ver que

$$\boxed{E[E[Y/X]] = E[Y]}$$

Cuando x es discreta:

$$\mathbb{E}(\mathbb{E}(y|x)) = \sum_{j=1}^{\infty} \mathbb{E}(y|x_j) Pr(x = x_j)$$

Cuando x es continua:

$$\mathbb{E}(\mathbb{E}(y|x)) = \int_{\mathbb{R}^k} \mathbb{E}(y|x) f_x(x) dx$$

La Ley general de las esperanzas iteradas permite dos condiciones para las variables establecidas, la Ley de la experanza iterada nos dice que:
Si $\mathbb{E}|y| < \infty$ entonces para cualquier vectores x_1 y x_2 aleatorios, se cumple de que:

$$\mathbb{E}(\mathbb{E}(y|x_1, x_2)|x_1) = \mathbb{E}(y|x_1)$$

Esta nos dice, en resumidas cuentas que la variable que contenga la mayor cantidad de información es la que termina ganando.

Apéndice N°5: Teniendo de que $\mu_x = \mathbb{E}(x)$, la varianza se puede expresar como:

$$V(x) = \mathbb{E}\left((x - \mathbb{E}(x))^2\right) = \mathbb{E}(x^2) - \left(\mathbb{E}(x^2)\right)^2$$