

Certified Reasoning with Infinity

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Abstract. We demonstrate how infinities improve the expressivity, power, readability, conciseness, and compositionality of a program logic. We prove that adding infinities to Presburger arithmetic enables these improvements without sacrificing decidability. We develop Omega++, a Coq-certified decision procedure for Presburger arithmetic with infinity and benchmark its performance. Both the program and proof of Omega++ are parameterized over user-selected semantics for the indeterminate terms (such as $0 * \infty$).

1 Introduction

Formal software analysis and verification frameworks benefit from expressive, compositional, decidable, and readable specification mechanisms. Of course, these goals often conflict with each other: for example, it is easy to add expressivity if one is willing to give up decidability! Happily, we have found a free lunch: by adding the notion of “infinity” to the specification language we can usefully add to the expressivity, readability, and compositionality of our specifications while maintaining their decidability.

Specifically, we start from the well-established domains of separation logic [38] and Presburger arithmetic [37] and add two abstract/fictitious/ghost symbols ∞ and $-\infty$, for which we support a precise, well-defined semantics. Although a seeming-minor addition, these symbols add significantly to the expressivity and power of our logic.

For example, programmers in modern languages like Scala [1] are increasingly utilizing highly-abstract language features such as lazy evaluation and infinite data structures to code complicated computational tasks over large data structures in a more compact and maintainable way. Unfortunately, analyzing and verifying code using such features is not so easy: for example, while numerous specification mechanisms support induction, support for coinduction is rare indeed. In section 2.3 we will see that by setting the size of a data structure (*e.g.*, a list’s length) to ∞ , we can express and reason about its unbounded size without explicitly needing coinduction.

Infinities can help express many other useful ideas beyond infinite data structures. In section 2.4 we use infinities to increase the compositionality of our logic by showing that “lists” and “bounded lists” are equivalent when the bound is ∞ . Moreover, in section 2.5, we use ∞ to mix notions of partial and total correctness within a logic.

Infinities also add to our specification framework’s readability and conciseness. For example, we will see in section 2.2 that ∞ allows us to drop disjuncts in the specification for code that manipulates a sorted linked list.

Finally, infinities enable some interesting applications. In section 2.6 we apply the notion of quantifier elimination in Presburger arithmetic with infinities to infer pure (non-heap) properties of programs.

All of the previous gains are worthy in their own right, but our major technical advance is the development of Omega++, a sound and complete decision procedure for Presburger arithmetic with infinities (including arbitrary quantifier use). In other words, we do not sacrifice any of the computational advantage normally gained by restricting ourselves to Presburger arithmetic, despite the addition of infinities. We call our tool “Omega++” both to acknowledge the importance of the underlying Presburger solver Omega [17] and because we believe we have modestly incremented its utility.

Omega++ is written in Gallina, the specification language of Coq [2], allowing us to formally certify it (modulo the correctness of Omega itself, which we utilize as our backend). We extract our performance-tuned Gallina into OCaml and package it as a library, which we have benchmarked using the HIP/SLEEK verification toolset [9].

One notable technical feature of Omega++ is that it can handle several semantic variants of Presburger arithmetic with infinity. For example, Presburger arithmetic usually admits multiplication by a constant as a notational convenience, *e.g.* $3 \cdot x \stackrel{\text{def}}{=} x + x + x$. This obvious-seeming convenience becomes a little less obvious when one adds infinities: what is $0 \cdot \infty$? Mathematical sophisticates can—and *do*—disagree: some prefer 0 as a convention in certain contexts (including, reasonably, ours) [28], while others prefer the result to be undefined due to the indeterminate status of the corresponding limit forms [18]. When possible, Omega++ takes an agnostic approach to such disagreements by allowing the user to specify the semantics of some subtle cases. Omega++ is thus a certified compiler from a *set* of related source languages (Presburger arithmetics with infinities) to a fixed, well-understood target (vanilla Presburger).

The rest of this paper is structured as follows. In section 2 we demonstrate the value of adding infinities to Presburger arithmetic. In section 3 we develop a precise formal semantics for Presburger arithmetics with infinities. In section 4 we describe the algorithm for the decision procedure and state its key properties, all of which have been verified in Coq for our optimized implementation. In section 5 we discuss the implementation itself and in section 6 we benchmark its performance. Finally, in section 7 we describe some related work and conclude.

Omega++ is available for download and experimentation here:

<http://loris-7.ddns.comp.nus.edu.sg/~project/SLPAInf/>

2 Motivation

In this section, we highlight the benefits of augmenting a specification logic with infinities. For consistency we focus on separation logic [12,38] but other specification mechanisms which rely on Presburger arithmetic can enjoy similar benefits.

2.1 Orientation

Our flavor of separation logic is based on the HIP/SLEEK system [9], letting us run tests and benchmarks with a state-of-the-art verification toolchain. Methods are specified

with a pair of pre- and postcondition (Φ_{pr}, Φ_{po}), with the keyword `res` allowed in the Φ_{po} to refer to the return value. We have enhanced the logic to allow the symbols ∞ and $-\infty$ where it would normally require integers; we also allow quantification over infinities.

From a systems perspective, our setup is sketched in figure 1. First, entailment between separation logic formulae with infinities in HIP/SLEEK is reduced (*à la* Chin *et al.* [9]) to entailment between numeric formulae in Presburger arithmetic with infinities (PAInf). Our main contribution is the next phase, detailed in section 4, in which we translate PAInf to vanilla Presburger arithmetic (PA). Finally, we discharge PA proof obligations with Omega. There are other combinations of separation logic with extensions of PA (such as sets/multisets) that can be used to enhance the specification and verification process. We discuss them in section 7.2 with related work.

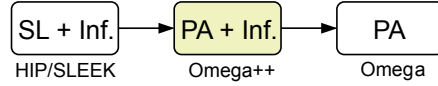


Fig. 1. Our setup: SL + Inf to PA

2.2 Infinities enable Concise Specifications

Let's start to see what infinities can buy us! Consider a simple program that inserts a new node into a sorted linked list, whose nodes are defined as follows:

```
data node {int val; node next; }
```

The data field `val` stores numerical information and the pointer field `next` points to the subsequent node in the structure. Consider next two alternative inductive predicates that characterize sortedness using only a single numeric parameter¹ that describes the list's minimum value:

Scenario 1 - no infinity enhancement:

```
sorted_ll⟨root, min⟩ ≡ root ↦ node⟨min, null⟩
  ∨ ∃ q, mtail · (root ↦ node⟨min, q⟩ * sorted_ll⟨q, mtail⟩ ∧ min ≤ mtail)
```

Scenario 2 - with infinity enhancement:

```
sorted_ll⟨root, min⟩ ≡ (root = null ∧ min = ∞)
  ∨ ∃ q, mtail · (root ↦ node⟨min, q⟩ * sorted_ll⟨q, mtail⟩ ∧ min ≤ mtail)
```

The base case of *Scenario 1* denotes a singleton, while its inductive case describes a linked list of length at least two. Though useable, this definition has a frustrating shortcoming: it cannot handle empty linked lists, since such lists do not have a finite minimum value. In contrast, *Scenario 2* handles the empty list gracefully since the minimum of an empty list can be defined to be just ∞ !

¹ Note that there are other ways of specifying sortedness, such as through the use of multi-set, that may also capture stronger properties, such as content preservation. However, they may require more complex provers in their reasoning.

The code for `insert` is in figure 2. Parameter `x` points to a sorted linked list, while `y` is the data node we wish to insert (preserving sortedness). Notice that the pre/post specifications in Scenario 1 require disjunctions to separate the cases when `x` is empty and nonempty, whereas Scenario 2 handles both cases uniformly. Infinities thus enable more *concise* and *readable* (easy to maintain) specifications.

<pre> node insert(node x, node y){ if (x == null) return y; else { if (y.val <= x.val){ y.next = x; return y; } else { x.next = insert(x.next, y); return x; } } } </pre>	<p><i>Scenario 1 :</i></p> $\Phi_{pr} : y \mapsto \text{node}\langle v, \text{null} \rangle \wedge x = \text{null}$ $\vee \text{sorted_ll}\langle x, a \rangle * y \mapsto \text{node}\langle v, \text{null} \rangle$ $\Phi_{po} : \text{sorted_ll}\langle \text{res}, b \rangle \wedge x = \text{null} \wedge b = v$ $\vee \text{sorted_ll}\langle \text{res}, b \rangle \wedge b = \min(a, v)$ <p><i>Scenario 2 :</i></p> $\Phi_{pr} : \text{sorted_ll}\langle x, a \rangle * y \mapsto \text{node}\langle v, \text{null} \rangle$ $\Phi_{po} : \text{sorted_ll}\langle \text{res}, b \rangle \wedge b = \min(a, v)$
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Fig. 2. Two pre-/post-specifications for insertion into a sorted linked list .

2.3 Infinities enable Infinite Data Structures

Consider the following two definitions corresponding to finite and infinite linked lists:

$$\text{ll}\langle \text{root}, n \rangle \equiv (\text{root} = \text{null} \wedge n = 0) \vee \exists q. (\text{root} \mapsto \text{node}\langle _, q \rangle * \text{ll}\langle q, n - 1 \rangle)$$

$$\text{ll_inf}\langle \text{root} \rangle \equiv \exists q. (\text{root} \mapsto \text{node}\langle _, q \rangle * \text{ll_inf}\langle q \rangle)$$

Here `n` captures the size of the list and `_` is used to denote an anonymous variable. While `ll` only requires inductive reasoning, traditionally `ll_inf` requires coinduction. We do not support coinduction directly, but we *can* relate both definitions using ∞ as follows:

$$\text{ll_inf}\langle \text{root} \rangle \leftrightarrow \text{ll}\langle \text{root}, \infty \rangle$$

We can prove this lemma via cyclic induction using our decision procedure for PAInf.

<pre> node take(node x, int k){ node y; if (k == 0) return null; else { y = new node(x.val, null); y.next = take(x.next, k - 1); return y; } } </pre>	<p>Original Specification :</p> $\Phi_{pr} : \text{ll}\langle x, n \rangle \wedge k \leq n$ $\Phi_{po} : \text{ll}\langle \text{res}, k \rangle * \text{ll}\langle x, n \rangle$ <p>Derived Specification :</p> $\Phi_{pr} : \text{ll_inf}\langle x \rangle$ $\Phi_{po} : \text{ll}\langle \text{res}, k \rangle * \text{ll_inf}\langle x \rangle$
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Fig. 3. Duplicating the first `k` elements of a linked list

Consider the code in figure 3, which takes a list `x` and duplicates its first `k` nodes, and for which the “Original Specification” is standard and unsurprising. More interestingly,

in a lazy context we can duplicate the first k nodes of an infinite list with the same code; we would like to verify the second specification. Using infinities and the above lemma, we can *derive the second specification from the first*, supporting some of coinduction's use cases indirectly and increasing the *expressivity* of our logic.

2.4 Infinities increase Compositionality

Consider this definition for an n -node linked list whose values are bounded by b :

$$\begin{aligned} \text{llB}\langle \text{root}, n, b \rangle &\equiv (\text{root} = \text{null} \wedge n = 0) \\ &\vee (\exists q, v. \text{root} \mapsto \text{node}\langle v, q \rangle * \text{llB}\langle q, n - 1, b \rangle \wedge v \leq b) \end{aligned}$$

Suppose we have a function f which uses this definition in its precondition:

$$\Phi_{pr} : \text{llB}\langle x, n, m \rangle * \dots$$

where x points to a linked list bounded by m . Now suppose we wish to call f from a program point at which we only have information about the shape and length of a linked list x but not any information about its bound, *e.g.* we satisfy $\text{ll}\langle x, n \rangle$ from section 2.3. With infinities this is easy: just instantiate m to ∞ since

$$\text{ll}\langle x, n \rangle \leftrightarrow \text{llB}\langle x, n, \infty \rangle$$

Without infinities, however, this is not so easy since we must first determine an appropriate bound for x 's values. Thus, infinities increase the *compositionality* of our logic, which in turn improves the reusability and conciseness of our specifications.

2.5 Infinities support Termination and Non-Termination Reasoning

Le *et al.* developed a technique to reason about termination and non-termination with a resource constraint $\text{RC}\langle \min, \max \rangle$ that tracks the minimum and maximum permitted execution steps [22]. Using Presburger arithmetic with infinity, terminating programs are modeled by $\text{RC}\langle _, \max \rangle \wedge \max < \infty$ and while non-terminating programs are captured by $\text{RC}\langle \infty, \infty \rangle$. Le *et al.* use Omega++ to discharge the associated proof obligations.

<pre> int length(node x){ if (x == null) return 0; else return (1 + length(x.next)); } </pre>	<p>Termination Spec :</p> $\begin{aligned} \Phi_{pr} &: \text{ls}\langle x, \text{null}, n \rangle * \text{RC}\langle _, M \rangle \wedge n < M \wedge M < \infty \\ \Phi_{po} &: \text{ls}\langle x, \text{null}, n \rangle * \text{RC}\langle _, M - (n + 1) \rangle \wedge \text{res} = n \end{aligned}$ <p>Non-Termination Spec :</p> $\begin{aligned} \Phi_{pr} &: \text{c11}\langle x, n \rangle * \text{RC}\langle \infty, \infty \rangle \\ \Phi_{po} &: \text{false} \end{aligned}$
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Fig. 4. Example 4: `length` terminates on proper lists and diverges on cyclic lists

Figure 4 demonstrates these resource constraints on a `length` function for linked lists. We show two specifications: the first shows that `length` terminates on finite lists `ls`, and the second shows that `length` diverges on circular lists `c11`.

$$\begin{aligned}
ls\langle root, p, n \rangle &\equiv (root = p \wedge n = 0) \\
&\vee \exists q \cdot (root \mapsto node\langle -, q \rangle * ls\langle q, p, n - 1 \rangle \wedge root \neq p) \\
c11\langle root, n \rangle &\equiv \exists q \cdot (root \mapsto node\langle -, q \rangle * ls\langle q, root, n - 1 \rangle)
\end{aligned}$$

2.6 Infinities support Analysis via Quantifier Elimination

Algorithmic quantifier elimination (QE) is a powerful technique for decision procedures in symbolic logic [15]. Kapur highlights the importance of geometric QE heuristics for the case of generating program invariants, distinguishing between octagonal and max-plus invariants [14]. While Kapur exploits the structure of verification conditions generated from numerical programs, we focus on generating inductive invariants for programs manipulating dynamically allocated data structures.

<pre> void append(node x, int a){ if (x.next == null) x.next = new node(a, null); else insert(x.next, a); } </pre>	<p>Shape Spec :</p> $\Phi_{pr} : ll\langle x, - \rangle \wedge x \neq null$ $\Phi_{po} : ll\langle x, - \rangle \wedge x \neq null$ <p>Spec with Inferred Pure :</p> $\Phi_{pr} : ll\langle x, n \rangle \wedge n > 0$ $\Phi_{po} : ll\langle x, n + 1 \rangle \wedge n > 0$
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Fig. 5. Pure Specification Inferred from PAInf QE

Consider the code in figure 5, which appends a node to the end of an acyclic linked list. The first specification only captures shape; it would be useful to infer size properties as well. We can do so by using PAInf-based QE to support inference of octagonal constraints with infinities in the presence of heap-based verification. Forward reasoning generates relational obligations which are then discharged by QE over PAInf, leading to the second specification with numeric properties.

3 Syntax and Parameterized Semantics

There are several benefits of adding the notion of infinity to a program logic. However, due to the presence of certain terms like $(\infty - \infty)$, it is an interesting problem to define the correct (or rather desired) semantics. We will now proceed to a formal discussion of Presburger arithmetic with infinity.

Our constraint language extends Presburger arithmetic with two abstract symbols designating positive (∞) infinity and negative $(-\infty)$ infinity. The language is detailed in figure 6. However, we would like to make some extra notes. First, we use a type based approach to distinguish between the domain of variables. The notation $w : \tau$ denotes that the variable w is of type τ ; thus it is a clear distinction between the domain of variables. Second, for performance reasons that are explained in section 5 we do not aim for a minimal input constraint language. That is the reason why the input language

also supports min and max constraints over expressions. The min and max constraints in the input language are translated to $\min_{=}$ and $\max_{=}$ (using $\pi \rightsquigarrow [v / \max(a_1, a_2)]\pi \wedge \max_{=}(v, a_1, a_2)$ and $\pi \rightsquigarrow [v / \min(a_1, a_2)]\pi \wedge \min_{=}(v, a_1, a_2)$).

$$\begin{aligned} \pi &::= \beta \mid \neg\pi \mid \pi_1 \wedge \pi_2 \mid \pi_1 \vee \pi_2 \mid \pi_1 \rightarrow \pi_2 \mid \exists(w : \tau) \cdot \pi \mid \forall(w : \tau) \cdot \pi \\ \beta &::= \mathbf{true} \mid \mathbf{false} \mid a_1 < a_2 \mid a_1 \leq a_2 \mid a_1 = a_2 \mid a_1 \neq a_2 \\ &\quad \mid a_1 \geq a_2 \mid a_1 > a_2 \\ a &::= k \mid v \mid c \times a \mid a_1 + a_2 \mid -a \mid a_1 - a_2 \mid \max(a_1, a_2) \mid \min(a_1, a_2) \\ k &::= c \mid \infty \mid -\infty \end{aligned}$$

where v, w are variable names; c is an integer constant

Fig. 6. PAInf: Input Constraint Language

Next, we present the parameterized semantic model for PAInf and establish theorems and lemmas that show the correctness of our decision procedure. All theorems and lemmas in this paper are machine checked in Coq. Parameters are introduced to adapt different possible ways of handling the tricky part of PAInf such as the terms $(\infty - \infty)$ and $(0 \times \infty)$. Since our semantics is parameterized, all procedures, theorems and lemmas based on the semantics are also parameterized. We start by defining an *environment* to map variables to values.

Definition 1 *An environment for a universe τ of concrete values is a function $\phi_\tau : V \rightarrow \tau$ from the set of variables V to τ . For such a ϕ_τ , we denote by $\phi_\tau[x \mapsto a]$ the function which maps x to a and any other variable y to $\phi_\tau(y)$.*

We define the semantics of arithmetic operations and relations for PAInf formally in figure 7, denoted by $\llbracket \beta \rrbracket_{\mathbb{Z}_\infty}$. The subscript of $\llbracket \cdot \rrbracket$ denotes the domain of constants. \mathbb{Z}_∞ means $\mathbb{Z} \cup \{\infty, -\infty\}$. By analogy, $\llbracket \beta \rrbracket_{\mathbb{Z}}$ means the domain is \mathbb{Z} . With these definitions one can compute every atomic term into a truth value with respect to an environment ϕ_τ and domain of constants η as described in figure 9, and denoted by $\text{EVAL}_{\phi_\tau}^\eta$.

We define the satisfaction relation $\phi_\tau \models_\eta^{\text{sat}} \pi$ and dissatisfaction relation $\phi_\tau \models_\eta^{\text{dst}} \pi$ (in figure 10) for each logical formula π over the environment ϕ_τ and domain of constants η by structural induction on π . Sometimes, a formula π can neither be satisfied nor be dissatisfied. In that case, we say π is undetermined, which can be presented as $\phi_\tau \models_\eta^{\text{udt}} \pi$. We define two distinct relations for satisfaction and dissatisfaction as we support both two-valued and three-valued logic. In case of three-valued logic a formula can be neither satisfied nor dissatisfied (undetermined).

Much of the semantics for PAInf is “as you might expect”. For example, when all the values are finite, all of the operations and relations behave the same way they would in PA. On the other hand, any finite value plus ∞ equals ∞ and any finite value plus $-\infty$ equals $-\infty$. It is trickier to figure out what to do with the sum of ∞ and $-\infty$; we treat this as a meaningless value (much like the “value” of $\frac{0}{0}$ in the reals) denoted by “ \perp ”. If ∞ and $-\infty$ were actually inverses, we would need to admit the following whopper:

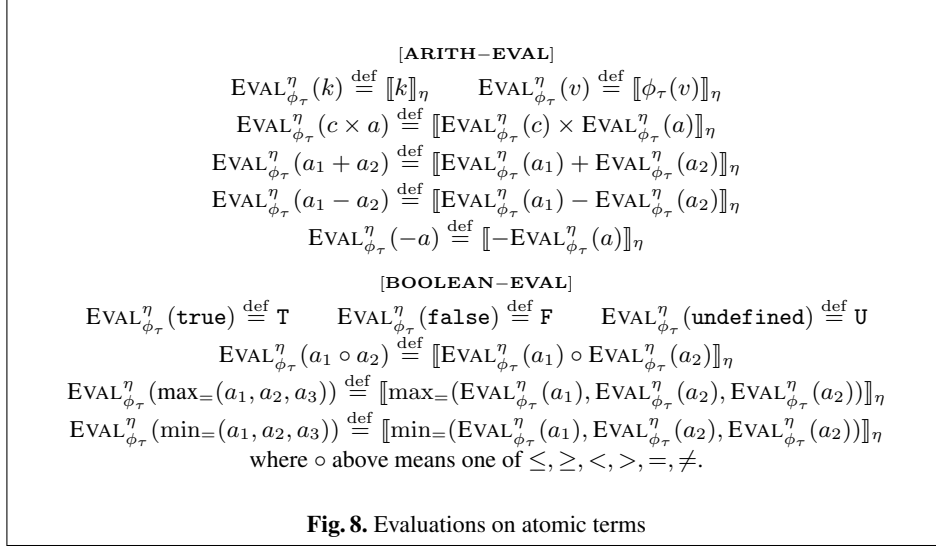
[ADDITION]			
$\llbracket k_1 + k_2 \rrbracket_{\mathbb{Z}_\infty} \stackrel{\text{def}}{=} \begin{cases} \perp & k_1 \text{ or } k_2 \text{ is } \perp \\ \perp & k_1 = \infty, k_2 = -\infty \\ \perp & k_1 = -\infty, k_2 = \infty \\ \infty & k_1 \text{ or } k_2 \text{ is } \infty, \text{ and neither is } -\infty \\ -\infty & k_2 \text{ or } k_2 \text{ is } -\infty, \text{ and neither is } \infty \\ \llbracket k_1 + k_2 \rrbracket_{\mathbb{Z}} & k_1 \text{ and } k_2 \text{ are finite} \end{cases}$			
[LESS-THAN-EQ]			
$\llbracket k_1 \leq k_2 \rrbracket_{\mathbb{Z}_\infty} \stackrel{\text{def}}{=} \begin{cases} \text{F/U} & k_1 \text{ or } k_2 \text{ is } \perp \\ \text{T} & k_2 = \infty \\ \text{T} & k_1 = -\infty \\ \text{T} & k_1 = k_2 = \infty \\ \text{T} & k_1 = k_2 = -\infty \\ \text{F} & k_1 = \infty, k_2 \neq \infty \\ \text{F} & k_1 \neq -\infty, k_2 = -\infty \\ \llbracket k_1 \leq k_2 \rrbracket_{\mathbb{Z}} & k_1 \text{ and } k_2 \text{ are finite} \end{cases}$			
		[IDENTITY]	
		$\llbracket k \rrbracket_{\mathbb{Z}_\infty} \stackrel{\text{def}}{=} k$	
		[NEGATION]	
		$\llbracket -k \rrbracket_{\mathbb{Z}_\infty} \stackrel{\text{def}}{=} \begin{cases} \perp & k = \perp \\ \infty & k = -\infty \\ -\infty & k = \infty \\ \llbracket -k \rrbracket_{\mathbb{Z}} & k \text{ is finite} \end{cases}$	
[OTHER-OPERATIONS-AND-RELATIONS]			
$\llbracket 0 \times k \rrbracket_{\mathbb{Z}_\infty} \stackrel{\text{def}}{=} \begin{cases} 0 & k \text{ is finite} \\ 0/\perp/k & k \text{ is not finite} \end{cases} \quad \llbracket c \times k \rrbracket_{\mathbb{Z}_\infty} \stackrel{\text{def}}{=} \begin{cases} \llbracket 0 \times k \rrbracket_{\mathbb{Z}_\infty} & c = 0 \\ k & c = 1 \\ \llbracket k + (c - 1) \times k \rrbracket_{\mathbb{Z}_\infty} & c > 1 \\ \llbracket -((-c) \times k) \rrbracket_{\mathbb{Z}_\infty} & c < 0 \end{cases}$			
$\begin{aligned} \llbracket k_1 \geq k_2 \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} \llbracket k_2 \leq k_1 \rrbracket_{\mathbb{Z}_\infty} & \llbracket k_1 > k_2 \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} \llbracket k_1 \geq k_2 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_1 \neq k_2 \rrbracket_{\mathbb{Z}_\infty} \\ \llbracket k_1 \neq k_2 \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} \neg \llbracket k_1 = k_2 \rrbracket_{\mathbb{Z}_\infty} & \llbracket k_1 = k_2 \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} \llbracket k_1 \leq k_2 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_2 \leq k_1 \rrbracket_{\mathbb{Z}_\infty} \\ \llbracket k_1 - k_2 \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} \llbracket k_1 + (-k_2) \rrbracket_{\mathbb{Z}_\infty} & \llbracket k_1 < k_2 \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} \llbracket k_1 \leq k_2 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_1 \neq k_2 \rrbracket_{\mathbb{Z}_\infty} \\ \llbracket \max_{=}(k_1, k_2, k_3) \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} (\llbracket k_1 = k_2 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_3 \leq k_2 \rrbracket_{\mathbb{Z}_\infty}) \vee (\llbracket k_1 = k_3 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_2 \leq k_3 \rrbracket_{\mathbb{Z}_\infty}) \\ \llbracket \min_{=}(k_1, k_2, k_3) \rrbracket_{\mathbb{Z}_\infty} &\stackrel{\text{def}}{=} (\llbracket k_1 = k_2 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_2 \leq k_3 \rrbracket_{\mathbb{Z}_\infty}) \vee (\llbracket k_1 = k_3 \rrbracket_{\mathbb{Z}_\infty} \wedge \llbracket k_3 \leq k_2 \rrbracket_{\mathbb{Z}_\infty}) \end{aligned}$			

Fig. 7. Operations and Relations in \mathbb{Z}_∞

$$0 = \infty + -\infty = \infty + (-\infty + 1) = (\infty + -\infty) + 1 = 1$$

In fact there is no perfect solution, since it is impossible to add a finite number of symbols to \mathbb{Z} while remaining a group. Lasaruk and Sturm [21] propose dodging part of this problem by using only a single value for both positive and negative infinity, which is both greater than *and* less than all finite values. This approach ensures that every sum is defined, although ∞ still does not have an inverse and you lose antisymmetry for \leq . We find the notion of a single infinity to be too restrictive as it prohibits us from expressing some of the motivating examples from section 2.

In addition to the issues encountered by using a single infinity, handling comparisons with \perp is another challenge. A possible solution is treating all comparisons with \perp as false. This is reasonable but not perfect. For example, in this context, it is not the case that $x > y$ is equivalent to $\neg(x \leq y)$ when x or y are \perp . Interestingly this is the choice made by IEEE floating point standard [3]. Another possibility is to use a three-valued logic and treat any comparison with \perp as the “third unknown value”.



There are several three-valued logics studied in the literature [5]. We use Kleene's weak three-valued logic which interprets the unknown value as "Error" and propagates it to the entire formula. The truth tables for this three-valued logic are shown in figure 8. In three-valued logic, when x or y are \perp , $x > y$ and $\neg(x \leq y)$ are equivalent. In Omega++, user can choose between a two-valued or three-valued logic, which is indicated in [LESS-THAN-EQ] of figure 7. Note that in three-valued logic, according to the relation definition in figure 10, formulae like $\perp < 0$ are neither satisfied nor dissatisfied.

The definition of multiplication in the presence of infinities ($0 \times \infty$) can also be selected by the user as shown in figure 7. There are three possible choices for defining $0 \times \infty$: 0 , \perp and ∞ . For each of these options we can choose a two-valued or three-valued logic, thus Omega++ supports six different customized semantics in total. As described in section 6, for our experiments we use the semantics with three-valued logic and $0 \times \infty \stackrel{\text{def}}{=} 0$. However, in general any of the six customized semantics can be used as the decision procedure is parameterized over these choices and our certified proof guarantees that all choices are sound, complete and decidable.

In order to match the intuition of user, by design, most valid formulae in PA remain so in our semantics for PAInf, just as most invalid formulae in PA are still invalid in PAInf. Here are two short examples which are valid in both (if you drop the universe of quantification as you move from PAInf to PA):

$$\forall(x : \mathbb{Z}_\infty) \cdot \exists(y : \mathbb{Z}_\infty) \cdot x \leq y \qquad \forall(x : \mathbb{Z}_\infty) \cdot \forall(y : \mathbb{Z}_\infty) \cdot x + 1 = y + 1 \rightarrow x = y$$

However, there are differences. This formula is valid in PA but invalid in PAInf:

$$\forall(x : \mathbb{Z}_\infty) \cdot \exists(y : \mathbb{Z}_\infty) \cdot x + y = 0$$

$\phi_\tau \models_\eta^{\text{sat}} \beta$	iff $\text{EVAL}_{\phi_\tau}^\eta(\beta)$ is T.
$\phi_\tau \models_\eta^{\text{sat}} \neg\pi$	iff $\phi_\tau \models_\eta^{\text{dst}} \pi$ holds.
$\phi_\tau \models_\eta^{\text{sat}} \pi_1 \wedge \pi_2$	iff both $\phi_\tau \models_\eta^{\text{sat}} \pi_1$ and $\phi_\tau \models_\eta^{\text{sat}} \pi_2$ holds.
$\phi_\tau \models_\eta^{\text{sat}} \pi_1 \vee \pi_2$	iff both $\phi_\tau \models_\eta^{\text{sat}} \pi_1$ and $\phi_\tau \models_\eta^{\text{sat}} \pi_2$ holds, or both $\phi_\tau \models_\eta^{\text{dst}} \pi_1$ and $\phi_\tau \models_\eta^{\text{sat}} \pi_2$ holds, or both $\phi_\tau \models_\eta^{\text{sat}} \pi_1$ and $\phi_\tau \models_\eta^{\text{dst}} \pi_2$ holds.
$\phi_\tau \models_\eta^{\text{sat}} \pi_1 \rightarrow \pi_2$	iff both $\phi_\tau \models_\eta^{\text{sat}} \pi_1$ and $\phi_\tau \models_\eta^{\text{sat}} \pi_2$ holds, or both $\phi_\tau \models_\eta^{\text{dst}} \pi_1$ and $\phi_\tau \models_\eta^{\text{sat}} \pi_2$ holds, or both $\phi_\tau \models_\eta^{\text{dst}} \pi_1$ and $\phi_\tau \models_\eta^{\text{dst}} \pi_2$ holds.
$\phi_\tau \models_\eta^{\text{sat}} \exists(w : \tau) \cdot \pi$	iff $\phi_\tau[w \mapsto k] \models_\eta^{\text{sat}} \pi$ holds for some $k \in \tau$, and for all $k \in \tau$, either $\phi_\tau[w \mapsto k] \models_\eta^{\text{sat}} \pi$ or $\phi_\tau[w \mapsto k] \models_\eta^{\text{dst}} \pi$ holds.
$\phi_\tau \models_\eta^{\text{sat}} \forall(w : \tau) \cdot \pi$	iff $\phi_\tau[w \mapsto k] \models_\eta^{\text{sat}} \pi$ holds for all $k \in \tau$
$\phi_\tau \models_\eta^{\text{dst}} \beta$	iff $\text{EVAL}_{\phi_\tau}^\eta(\beta)$ is F.
$\phi_\tau \models_\eta^{\text{dst}} \neg\pi$	iff $\phi_\tau \models_\eta^{\text{sat}} \pi$ holds.
$\phi_\tau \models_\eta^{\text{dst}} \pi_1 \wedge \pi_2$	iff both $\phi_\tau \models_\eta^{\text{dst}} \pi_1$ and $\phi_\tau \models_\eta^{\text{dst}} \pi_2$ holds, or both $\phi_\tau \models_\eta^{\text{sat}} \pi_1$ and $\phi_\tau \models_\eta^{\text{dst}} \pi_2$ holds, or both $\phi_\tau \models_\eta^{\text{dst}} \pi_1$ and $\phi_\tau \models_\eta^{\text{sat}} \pi_2$ holds.
$\phi_\tau \models_\eta^{\text{dst}} \pi_1 \vee \pi_2$	iff both $\phi_\tau \models_\eta^{\text{dst}} \pi_1$ and $\phi_\tau \models_\eta^{\text{dst}} \pi_2$ holds.
$\phi_\tau \models_\eta^{\text{dst}} \pi_1 \rightarrow \pi_2$	iff both $\phi_\tau \models_\eta^{\text{sat}} \pi_1$ and $\phi_\tau \models_\eta^{\text{dst}} \pi_2$ holds.
$\phi_\tau \models_\eta^{\text{dst}} \exists(w : \tau) \cdot \pi$	iff $\phi_\tau[w \mapsto k] \models_\eta^{\text{dst}} \pi$ holds for all $k \in \tau$
$\phi_\tau \models_\eta^{\text{dst}} \forall(w : \tau) \cdot \pi$	iff $\phi_\tau[w \mapsto k] \models_\eta^{\text{dst}} \pi$ holds for some $k \in \tau$, and for all $k \in \tau$, either $\phi_\tau[w \mapsto k] \models_\eta^{\text{sat}} \pi$ or $\phi_\tau[w \mapsto k] \models_\eta^{\text{dst}} \pi$ holds.
$\phi_\tau \models_\eta^{\text{udt}} \pi$	iff neither $\phi_\tau \models_\eta^{\text{sat}} \pi$ or $\phi_\tau \models_\eta^{\text{dst}} \pi$ holds.

Fig. 9. Definition of satisfaction relation

The previous formula is false in PAInf when $x = \infty$. More generally, although \mathbb{Z}_∞ is not a group, it still has many useful algebraic properties, such as the following.

Lemma 1. + is Associative $\llbracket (a + b) + c \rrbracket_{\mathbb{Z}_\infty}$ and $\llbracket a + (b + c) \rrbracket_{\mathbb{Z}_\infty}$ are equal or both undefined.

Lemma 2. + is Commutative $\llbracket a + b \rrbracket_{\mathbb{Z}_\infty}$ and $\llbracket b + a \rrbracket_{\mathbb{Z}_\infty}$ are equal or both undefined.

Lemma 3. 0 is the Additive Identity $\llbracket a + 0 \rrbracket_{\mathbb{Z}_\infty}$ and a are equal for all defined a .

Lemma 4. + is Monotonic If $\llbracket a \leq b \rrbracket_{\mathbb{Z}_\infty}$ is T and if both $\llbracket a + c \rrbracket_{\mathbb{Z}_\infty}$ and $\llbracket b + c \rrbracket_{\mathbb{Z}_\infty}$ are defined, then $\llbracket a + c \leq b + c \rrbracket_{\mathbb{Z}_\infty}$ is also T.

<table><tr><th>P</th><th>$\neg P$</th></tr><tr><td>true</td><td>false</td></tr><tr><td>false</td><td>true</td></tr><tr><td>undefined</td><td>undefined</td></tr></table>			P	$\neg P$	true	false	false	true	undefined	undefined																																																				
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Fig. 10. Truth Tables for Three-valued Logic

4 Reasoning with Infinity

For the following discussion we assume the existence of a solver for Presburger arithmetic (such as Omega [17]). Our focus is to automate the reasoning of ghost infinities by leveraging on existing solvers.

Note that $v \in \mathbb{Z}_\infty$, is the same as, $v \in \mathbb{Z} \vee v = \infty \vee v = -\infty$. This fact can be used to give a quantifier elimination procedure for PAInf as shown in figure 11. However, using this approach naively leads to an explosion in the size of formulae to be checked. As an example, consider the following formula,

$$\forall x, y, z \cdot (z = \infty \wedge y = x + z \wedge x < \infty)$$

Using the [FORALL-INF] rule to eliminate the three quantified variables (x , y and z), leads to $3^3 (= 27)$ constraints. To avoid this problem, we support both kinds of quantifiers ($\exists(w : \mathbb{Z})$ and $\exists(w : \mathbb{Z}_\infty)$) in the implementation. This allows for a more efficient quantifier elimination as variables with finite domain do not give rise to new disjunctions in formulae. Since, infinity is added as a ghost constant only in the specification logic, all program variables are still in finite domain. Supporting two kinds of quantifiers matches nicely with the distinction between the domain of specification variables (\mathbb{Z}_∞) and program variables (\mathbb{Z}). In section 6 we compare our system with an implementation of PAI from [21] and demonstrate the effectiveness of using our procedure for quantifier elimination.

For checking satisfiability in the PAInf we use the algorithm shown in figure 12. We denote the procedure for satisfiability checking as $SAT(\pi)$. The algorithm has four steps: (i) first we eliminate the quantifiers starting with the innermost quantifier, (ii) next we apply a normalization which detects tautologies and contradictions in constraints

$\begin{array}{c} \text{[EXISTS-INF]} \\ \exists(w : \mathbb{Z}_\infty) \cdot \pi \rightsquigarrow \exists(w : \mathbb{Z}) \cdot \pi \\ \vee[\infty/w] \pi \\ \vee[-\infty/w] \pi \end{array}$	$\begin{array}{c} \text{[FORALL-INF]} \\ \forall(w : \mathbb{Z}_\infty) \cdot \pi \rightsquigarrow \forall(w : \mathbb{Z}) \cdot \pi \\ \wedge[\infty/w] \pi \\ \wedge[-\infty/w] \pi \end{array}$
--	--

Fig. 11. PAInf: Quantifier Elimination (INF-TRANS)

using infinity, (iii) then we eliminate min-max and constant constraints and (iv) finally we solve the resulting formula using an existing PA solver Omega.

$SAT(\pi)$	$\pi_F = \text{INF-TRANS}(\pi)$	(1) <i>Quantifier Elimination</i>
$\implies SAT(\pi_F)$	$\pi_N = \text{INT-TRANS}(\pi_F)$	(2) <i>Normalization</i>
$\implies SAT(\pi_N)$	$\pi_G = \text{SIMP}(\pi_N)$	(3) <i>Simplification</i>
$\implies SAT(\pi_G)$		(4) <i>Omega</i>

Fig. 12. PAInf: SAT Checking

At a high level the intuition behind the SAT checking algorithm is as follows: after quantifier elimination, the π_F formula has quantifiers only on the finite domain variables. The normalization, detects tautologies and contradictions in constraints using infinity and rewrites the formula to π_G . The normalization eliminates all the infinite constants from the formula. The resulting formula (π_G) is in PA and its satisfiability can be checked using Omega. Next we describe the steps in the SAT checking algorithm in detail.

$\begin{array}{c} \text{[EVAL-FIN]} \\ v \rightsquigarrow Z \\ c \rightsquigarrow Z \\ -Z \rightsquigarrow Z \\ Z + Z \rightsquigarrow Z \\ Z - Z \rightsquigarrow Z \\ c \times Z \rightsquigarrow Z \end{array}$	$\begin{array}{c} \text{[EVAL-INF]} \\ \infty + \infty \rightsquigarrow \infty \\ -\infty + (-\infty) \rightsquigarrow -\infty \\ -\infty + Z \rightsquigarrow -\infty \\ Z + (-\infty) \rightsquigarrow -\infty \\ \infty + Z \rightsquigarrow \infty \\ Z + \infty \rightsquigarrow \infty \end{array}$	$\begin{array}{c} \text{[EVAL-BOT]} \\ \infty + (-\infty) \rightsquigarrow \perp \\ -\infty + \infty \rightsquigarrow \perp \\ \perp + Z \rightsquigarrow \perp \\ Z + \perp \rightsquigarrow \perp \\ \perp + \perp \rightsquigarrow \perp \\ -\perp \rightsquigarrow \perp \end{array}$
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Fig. 13. PAInf: Evaluation Check

4.1 Normalization and Simplification

We define a set of rewriting rules based on the semantics of formulae in PAInf. We work only with closed-form formulae, thus after applying the quantifier elimination given in figure 11, all the remaining variables are in the finite domain (\mathbb{Z}). It is possible to compare the variables with infinities by evaluating their values (as they are all finite) using the semantics given in the section 3. This is performed by the `Evaluation Check` function in figure 13 which reduces each expression to a finite value Z . Thus for the normalization rules in figure 14 we only need to consider the integer values (Z) and the infinity constants. Note that, the `Evaluation Check` is only applied as part of the normalization process, with the purpose of checking the finiteness and eliminating infinity. In particular, the actual formula is only transformed to a form without infinity constants; it is not evaluated to a value.

The normalization process uses the rewriting rules given in figure 14 (rules for \neq , \geq , $<$ and $\min=$ are similar and omitted for brevity). These rules detect the tautologies and contradictions in the usage of ∞ and $-\infty$, thus all the constraints involving ∞ and $-\infty$ are eliminated. After the application of these rules the given formula is reduced to a form which can be solved by existing PA solvers like Omega.

We also proved the following theorems and lemmas about quantifier elimination INF-TRANS and normalization INT-TRANS. These theorems and lemmas hold for both two-valued/three-valued logics and all choices of $(0 \times \infty)$. Hence, the Coq certified proof of these theorems and lemmas is also parameterized. Note that for quantifier elimination the universe of environment τ and the domain of constants η are both instantiated to \mathbb{Z}_∞ .

Lemma 5. Quantifier Elimination $\phi_{\mathbb{Z}_\infty} \models_{\mathbb{Z}_\infty}^{\text{sat}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}_\infty}^{\text{sat}} \text{INF-TRANS}(\pi)$,
 $\phi_{\mathbb{Z}_\infty} \models_{\mathbb{Z}_\infty}^{\text{dst}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}_\infty}^{\text{dst}} \text{INF-TRANS}(\pi)$,

For infinity elimination τ is \mathbb{Z}_∞ and η is \mathbb{Z} . This is due to the fact that after quantifier elimination the domain of all the variables is finite.

Lemma 6. Infinity Elimination $\phi_{\mathbb{Z}} \models_{\mathbb{Z}_\infty}^{\text{sat}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{sat}} \text{INT-TRANS}(\pi)$,
 $\phi_{\mathbb{Z}} \models_{\mathbb{Z}_\infty}^{\text{dst}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{dst}} \text{INT-TRANS}(\pi)$.

So for the total transformation $\text{TRANS}(\pi) = \text{INT-TRANS}(\text{INF-TRANS}(\pi))$ used in satisfiability checking, we have the following theorem:

Theorem 1. Satisfiability Checking $\phi_{\mathbb{Z}_\infty} \models_{\mathbb{Z}_\infty}^{\text{sat}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{sat}} \text{TRANS}(\pi)$,
 $\phi_{\mathbb{Z}_\infty} \models_{\mathbb{Z}_\infty}^{\text{dst}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{dst}} \text{TRANS}(\pi)$,

Gallina, the internal functional language of Coq is strongly normalizing. Thus, all functions written in Coq must terminate.

Theorem 2. Termination *Satisfiability checking in PAInf (figure 12) terminates.*

The quantifier elimination with infinity expands the logical formula π and the normalization introduces many logical constants. We introduce a simplification function SIMP which recursively eliminates logical constants according to the rules in figure 15

[NORM-INF-EQ]	[NORM-INF-LEQ]	[NORM-INF-LT]
$\perp = _ \rightsquigarrow \text{error}$	$\perp \leq _ \rightsquigarrow \text{error}$	$\perp < _ \rightsquigarrow \text{error}$
$_ = \perp \rightsquigarrow \text{error}$	$_ \leq \perp \rightsquigarrow \text{error}$	$_ < \perp \rightsquigarrow \text{error}$
$Z = \infty \rightsquigarrow \text{false}$	$Z \leq \infty \rightsquigarrow \text{true}$	$Z < \infty \rightsquigarrow \text{true}$
$\infty = \infty \rightsquigarrow \text{true}$	$\infty \leq \infty \rightsquigarrow \text{true}$	$\infty < \infty \rightsquigarrow \text{false}$
$-\infty = \infty \rightsquigarrow \text{false}$	$-\infty \leq \infty \rightsquigarrow \text{true}$	$-\infty < \infty \rightsquigarrow \text{true}$
$-\infty = Z \rightsquigarrow \text{false}$	$-\infty \leq Z \rightsquigarrow \text{true}$	$-\infty < Z \rightsquigarrow \text{true}$
$-\infty = -\infty \rightsquigarrow \text{true}$	$-\infty \leq -\infty \rightsquigarrow \text{true}$	$-\infty < -\infty \rightsquigarrow \text{false}$
$\infty = Z \rightsquigarrow \text{false}$	$\infty \leq Z \rightsquigarrow \text{false}$	$\infty < Z \rightsquigarrow \text{false}$
$\infty = -\infty \rightsquigarrow \text{false}$	$\infty \leq -\infty \rightsquigarrow \text{false}$	$\infty < -\infty \rightsquigarrow \text{false}$
$Z = -\infty \rightsquigarrow \text{false}$	$Z \leq -\infty \rightsquigarrow \text{false}$	$Z < -\infty \rightsquigarrow \text{false}$
[NORM-EQ-MAX]		
$\text{max}_=(\infty, \infty, \infty) \rightsquigarrow \text{true}$	$\text{max}_=(-\infty, Z, Z) \rightsquigarrow \text{false}$	$\text{max}_=(_, _, \perp) \rightsquigarrow \text{error}$
$\text{max}_=(-\infty, -\infty, -\infty) \rightsquigarrow \text{true}$	$\text{max}_=(\infty, Z, -\infty) \rightsquigarrow \text{false}$	$\text{max}_=(\infty, Z, Z) \rightsquigarrow \text{false}$
$\text{max}_=(-\infty, Z, -\infty) \rightsquigarrow \text{false}$	$\text{max}_=(\infty, -\infty, \infty) \rightsquigarrow \text{true}$	$\text{max}_=(\infty, \infty, Z) \rightsquigarrow \text{true}$
$\text{max}_=(-\infty, \infty, -\infty) \rightsquigarrow \text{false}$	$\text{max}_=(-\infty, \infty, Z) \rightsquigarrow \text{false}$	$\text{max}_=(Z, \infty, Z) \rightsquigarrow \text{false}$
$\text{max}_=(\infty, -\infty, -\infty) \rightsquigarrow \text{false}$	$\text{max}_=(\infty, -\infty, Z) \rightsquigarrow \text{false}$	$\text{max}_=(_, \perp, _) \rightsquigarrow \text{error}$
$\text{max}_=(-\infty, -\infty, Z) \rightsquigarrow \text{false}$	$\text{max}_=(Z, \infty, -\infty) \rightsquigarrow \text{false}$	$\text{max}_=(\infty, Z, \infty) \rightsquigarrow \text{true}$
$\text{max}_=(\infty, \infty, -\infty) \rightsquigarrow \text{true}$	$\text{max}_=(-\infty, Z, \infty) \rightsquigarrow \text{false}$	$\text{max}_=(\perp, _, _) \rightsquigarrow \text{error}$
$\text{max}_=(Z, -\infty, -\infty) \rightsquigarrow \text{false}$	$\text{max}_=(Z, -\infty, \infty) \rightsquigarrow \text{false}$	$\text{max}_=(Z, Z, \infty) \rightsquigarrow \text{false}$
$\text{max}_=(-\infty, -\infty, \infty) \rightsquigarrow \text{false}$	$\text{max}_=(-\infty, \infty, \infty) \rightsquigarrow \text{false}$	$\text{max}_=(Z, \infty, \infty) \rightsquigarrow \text{false}$
[NORM-INF-ERR]		
$\text{error} \rightsquigarrow \text{false}$ (two-valued logic)		
$\text{error} \rightsquigarrow \text{undefined}$ (three-valued logic)		

Fig. 14. PAInf: Normalization (INT-TRANS)

in order to reduce the length of a formula. As Omega doesn't support $\text{max}_=$ or $\text{min}_=$ we also include the elimination of $\text{max}_=$ and $\text{min}_=$ in SIMP. Note that for three-valued logic, logical constants contains a third value: **undefined** which is not supported by Omega. Our SIMP function actually propagates **undefined** to the whole formula such that we know if a formula is undetermined before calling Omega due of the following theorem:

Theorem 3. Decide Undetermined $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{udt}} \pi$ if and only if $\text{SIMP}(\pi) = \text{undefined}$

Thus, we do not need to extend Omega to support **undefined**. SIMP also preserves the validity of formulae:

Theorem 4. Simplification $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{sat}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{sat}} \text{SIMP}(\pi)$, $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{dst}} \pi$ if and only if $\phi_{\mathbb{Z}} \models_{\mathbb{Z}}^{\text{dst}} \text{SIMP}(\pi)$.

[ELIM]	
$\text{max}_=(a_1, a_2, a_3) \rightsquigarrow (a_1 = a_2 \wedge a_3 \leq a_2) \vee (a_1 = a_3 \wedge a_2 \leq a_3)$	
$\text{min}_=(a_1, a_2, a_3) \rightsquigarrow (a_1 = a_2 \wedge a_2 \leq a_3) \vee (a_1 = a_3 \wedge a_3 \leq a_2)$	
[SIMP]	
$\beta \rightsquigarrow \text{ELIM}(\beta)$	
$\text{undefined} \wedge \pi \rightsquigarrow \text{undefined}$	$\pi \wedge \text{undefined} \rightsquigarrow \text{undefined}$
$\text{true} \wedge \pi \rightsquigarrow \pi$	$\pi \wedge \text{true} \rightsquigarrow \pi$
$\text{false} \wedge \pi \rightsquigarrow \text{false}$	$\pi \wedge \text{false} \rightsquigarrow \text{false}$
$\text{undefined} \vee \pi \rightsquigarrow \text{undefined}$	$\pi \vee \text{undefined} \rightsquigarrow \text{undefined}$
$\text{true} \vee \pi \rightsquigarrow \text{true}$	$\pi \vee \text{true} \rightsquigarrow \text{true}$
$\text{false} \vee \pi \rightsquigarrow \pi$	$\pi \vee \text{false} \rightsquigarrow \pi$
$\text{undefined} \rightarrow \pi \rightsquigarrow \text{undefined}$	$\pi \rightarrow \text{undefined} \rightsquigarrow \text{undefined}$
$\text{false} \rightarrow \pi \rightsquigarrow \text{true}$	$\pi \rightarrow \text{true} \rightsquigarrow \text{true}$
$\text{true} \rightarrow \pi \rightsquigarrow \pi$	$\pi \rightarrow \text{false} \rightsquigarrow \neg \pi$
$\neg \text{true} \rightsquigarrow \text{false}$	$\neg \text{false} \rightsquigarrow \text{true}$
$\neg \text{undefined} \rightsquigarrow \text{undefined}$	
$\forall(w : \tau) \cdot \text{undefined} \rightsquigarrow \text{undefined}$	$\exists(w : \tau) \cdot \text{undefined} \rightsquigarrow \text{undefined}$
$\forall(w : \tau) \cdot \text{true} \rightsquigarrow \text{true}$	$\exists(w : \tau) \cdot \text{true} \rightsquigarrow \text{true}$
$\forall(w : \tau) \cdot \text{false} \rightsquigarrow \text{false}$	$\exists(w : \tau) \cdot \text{false} \rightsquigarrow \text{false}$

Fig. 15. Definition of Simplification

5 Implementation

The Omega++ decision procedure (along with the proofs of all the associated lemmas and theorems) is implemented in Coq. Since Omega++ supports six different customized semantics our implementation of the transformation and proofs is modular and composable. The customized parts—two- or three-valued logic, value of $(0 \times \infty)$ are abstracted as different module types composed by parameters and axioms. Each concrete choice instantiates the module type via definitions and lemmas. For example, part of our module type for “value” looks like this:

```
Module Type SEM_VAL.
  Parameter Val : Set.
  Parameter truth_and : Val -> Val -> Val.
  Parameter truth_or : Val -> Val -> Val.
  Parameter truth_not : Val -> Val.
  Axiom truth_and_comm : forall v1 v2, truth_and v1 v2
    = truth_and v2 v1.
  ...
End SEM_VAL.
```

where `truth_and`, `truth_or` and `truth_not` are truth tables for conjunction, disjunction and negation. Given this module type our module implementation for a value in three-valued logic looks like this:

```

Module Three_Val_NoneError <: SEM_VAL.
  Inductive Val_Impl := VTrue | VFalse | VError.
  Definition Val := Val_Impl.
  Definition truth_and (v1 v2 : Val) := ...
  Definition truth_or (v1 v2 : Val) := ...
  Definition truth_not v := ...
  Lemma truth_and_comm : forall v1 v2, truth_and v1 v2
    = truth_and v2 v1.
  Proof. intros; destruct v1, v2; simpl; trivial. Qed.
  ...

```

We put our definition of semantics in another module which is parameterized by these module types. This method enables us to define transformations and prove theorems in a highly modular and compositional way, regardless of the concrete values of those parameter modules.

The following table presents some statistics for our Coq implementation of Omega++. The first column shows the file name, while the second and third columns are the number of lines in the file taken by the program and its soundness proof, respectively. Our total development is a modest 3,988 lines and the ratio of proof to program is a reasonable 2.35. All of the source files in this table are bundled with our submission.

<i>Coq File</i>	<i>Program</i>	<i>Proof</i>	<i>Time (s)</i>	<i>Description</i>
Theory.v	585	737	20.68	<i>Syntax and Semantics; SIMP</i>
Transformation.v	350	1,203	31.07	<i>INF-TRANS, INT-TRANS</i>
Simplification.v	0	856	338.96	<i>Tactics/lemmas for SIMP</i>
Extraction.v	257	0	1.27	<i>Module to extract OCaml code</i>
	1,192	2,796	391.98	<i>Total Coq</i>

The fourth column gives the time taken by Coq to verify the file (*i.e.*, proof/type checking), using a 2.6 GHz Intel Core i7 with 16 GB of DDR3 RAM. Note that type checking times have very little to do with file length. For example, `Transformation.v` has 1,553 lines (combined program and proof), but takes less than 32 seconds to verify. On the other hand, verifying the 44 lines of the SIMP procedure, whose code is contained in `Theory.v`, takes more than five minutes!

We gain a number of benefits in exchange for implementing Omega++ in Coq. We get proof of termination for free since Gallina (the extractable pure functional language of Coq) is strongly normalizing. More importantly, we get full machine-checked formal correctness proofs for our source code with respect to a well defined semantics for Presburger arithmetic with infinity. Coq's extraction facility then transforms the Gallina program into OCaml (or Haskell or Scheme), which we then compile and run as normal. A very simple handwritten interface in OCaml (`omegapp.ml`, 162 lines) hides the natural ugliness of auto-generated code from other OCaml modules and enables a useful optimization within the generated Coq code as detailed below.

Although extraction seems straightforward, there are a surprising number of pitfalls. We will next highlight the key optimizations we used to get good performance and discuss how the program affected the proof—and vice versa.

In the implementation we directly handle all of the logical operators and min-max constraints of the constraint language (figure 6), even though the “obvious” strategy would be to desugar aggressively. Unfortunately, sugar-free formulae are actually quite a bit larger than their svelter sugared cousins, resulting in a significant performance penalty. Working with fully-sugared formulae has a significant impact on the proofs because we must handle more cases than would otherwise be necessary.

Similarly, we allow the input formulae to specify, for each quantifier, whether the domain of quantification is over \mathbb{Z} or over \mathbb{Z}_∞ . Quantifier elimination is expensive, and our “user”—the HIP/SLEEK verification toolset—often knows when a variable must be finite: in particular, program variables must be finite, whereas specification variables need not be. Communicating this fact to Omega++ resulted in significant performance gains, but again increased the proof effort due to the necessity of handling more cases.

To enable min/max, reduce the length of the output, eliminate redundant clauses, and propagate the undefined value, we implemented some basic simplifications (figure 15). The SIMP procedure was easy to implement but very painful to verify due to the vast number of cases we need to consider. In the end we wrote some custom proof tactics in Ltac (Coq’s proof tactic language) which crunched through the tedium while we ate lunch.

The previous examples all trade one-time verification effort for a better-performing algorithm. On the other hand, sometimes the proof improves the program. Before we started on our Coq implementation, we did a OCaml prototype for the quantifier-free fragment of the problem. That prototype’s version of normalization did additional case analysis. Due to our careful treatment of quantifier elimination we were able to prove that much of this case analysis was unnecessary in our Coq tool. Moreover, the Coq development identified a soundness bug in the OCaml prototype, which allowed the invalid transformation $x \geq y \leadsto x+1 > y$, which is false when $x = y = \infty$.

We also used one engineering trick to boost the performance of the extracted code. The code uses strings to represent both variables and (arbitrary-sized) integers, but Coq’s encoding of strings is less efficient than OCaml’s. We therefore usually treat strings as an abstract type within Coq and manipulate them via an interface to OCaml’s string functions, passed in using a functor. This interface takes only a few lines of `omegapp.ml` and results in a noticeable performance gain.

Overall, Omega++ is far better than our previous OCaml prototype. Consider:

<i>Tool</i>	<i>Sound</i>	<i>Complete</i>	<i>Termination</i>	<i>Semantics</i>	<i>Verified</i>
<i>OCaml Prototype</i>	<i>No</i>	<i>No</i>	<i>Unclear</i>	<i>Unclear</i>	<i>No</i>
<i>Omega++</i>	<i>Yes</i>	<i>Yes</i>	<i>Guaranteed</i>	<i>Precise</i>	<i>in Coq</i>

Of course, our OCaml prototype is a bit of a straw man, but we have been quite convinced that the substantial effort that it took to write Omega++ in Coq was well-rewarded. Moreover, as we will soon see, Omega++ has comparable performance to our OCaml prototype, despite solving a trickier problem in a much more through way.

6 Experiments

To benchmark Omega++ we integrated it into the HIP/SLEEK verification toolset [9] and developed a suite of tests (mostly searching and sorting programs) whose specifications use ∞ in interesting ways. The source code for each of these programs can be investigated in detail and tested with Omega++ [44] on our web site. In all the experiments we selected three-valued logic in Omega++ and used $0 \cdot \infty \stackrel{\text{def}}{=} 0$ as these are the appropriate choices for program verification.

We used a 3.20GHz Intel Core i7-960 processor with 16GB memory running Ubuntu Linux 10.04 for our benchmarks, the first set of which are detailed in the table below. The first column lists the test name and the second gives its lines of code.

<i>Benchmark</i>	<i>LOC</i>	<i>Disjuncts (\mathbb{Z})</i>	<i>Time (Ω)</i>	<i>Disjuncts (\mathbb{Z}_∞)</i>	<i>Time ($\Omega++$)</i>
<i>Insertion Sort</i>	30	4	0.14	2	0.15
<i>Selection Sort</i>	69	14	0.36	7	0.35
<i>Binary Search Tree</i>	105	12	0.43	6	0.35
<i>Bubble Sort</i>	110	12	0.29	9	0.50
<i>Merge Sort</i>	91	6	0.32	4	1.81
<i>Priority Queue</i>	207	16	0.84	10	2.73
<i>Infinite List</i>	41	–	–	4	0.36
<i>Infinite Tree</i>	39	–	–	2	0.54
<i>Stream</i>	28	–	–	2	0.68
<i>Total Correctness</i>	21	–	–	2	0.21
<i>Sorting with Min and Max</i>	79	–	–	7	1.82

The third and fifth columns show that \mathbb{Z}_∞ enables more readable and concise specifications. Specifically, the third column gives the number of disjunctions required to express the test’s specifications using \mathbb{Z} , whereas the fifth column expresses the same properties using \mathbb{Z}_∞ . For each test in the first group (top six), \mathbb{Z}_∞ requires fewer disjunctions. We do need to be a bit careful: although the specifications are informally for the same property (e.g., “sortedness”), typically the specifications in \mathbb{Z}_∞ are formally stronger since the embedded quantification occurs over larger sets. Note that we do not claim that Omega++ eliminates the disjunctions from reasoning since the quantifiers over infinities hide the disjunctions inside them. However, using infinities provides a useful abstraction to express the same property as the given specification is more concise.

The difference in formal strength is the fundamental reason why the times given in columns four and six differ. Column four gives the time (including all of HIP/SLEEK) using Omega, whereas column six gives the time using Omega++. For the first four examples Omega++ is comparable to Omega, but in the final two of the first group of tests we believe the difference in the domain of quantification results in a significantly harder theorem in \mathbb{Z}_∞ , and thus, a noticeably longer runtime.

The second group of tests (bottom five) shows that \mathbb{Z}_∞ is more expressive: the specifications for each of these tests is not expressible using only \mathbb{Z} . The runtimes we get using (HIP/SLEEK and) Omega++ are encouragingly modest.

Comparison with similar tools. Lasaruk and Sturm [21] also propose extending Presburger arithmetic with infinity. Their work differs from ours in several respects. First, they only add a single infinity value, thus dodging any thorny—but in our view, important—semantic issues involving $\infty - \infty$. More importantly, Lasaruk and Sturm describe an algorithm but do not provide an implementation. For benchmarking purposes, we implemented their algorithm and tested it using the constraints generated from our test suite. We also compared our previous OCaml prototype. Our results are as follows:

<i>Benchmark</i>	<i>Calls</i>	<i>Time (PAI)</i>	<i>Time (Proto)</i>	<i>Time ($\Omega++$)</i>
<i>Insertion Sort</i>	100	4.58	0.78	0.39
<i>Selection Sort</i>	245	>600.00	0.62	0.78
<i>Binary Search Tree</i>	116	150.00	0.48	0.50
<i>Bubble Sort</i>	336	>600.00	1.25	1.34
<i>Merge Sort</i>	155	>600.00	1.05	1.92
<i>Priority Queue</i>	778	>600.00	FAIL	1.20
<i>Infinite List</i>	121	66.88	0.35	0.24
<i>Infinite Tree</i>	94	38.80	0.54	0.73
<i>Stream</i>	126	>600.00	0.27	0.21
<i>Total Correctness</i>	120	>600.00	0.31	0.16
<i>Sorting with Min and Max</i>	376	>600.00	0.29	0.19
<i>Entailment Examples</i>	124	1.89	FAIL	1.42
<i>Lemma Examples</i>	35	1.88	1.27	1.65
<i>Total (except PQ and EE)</i>	1,824	>3,862.14	7.21	8.11

The second column gives the number of times the associated decision procedure was called for each test. The third column gives the times for Lasaruk and Sturm’s “PAI” algorithm; many of the tests timed out after 10 minutes. The fourth column gives the times for our OCaml prototype “Proto”; notice that for two of the tests Proto failed (completeness holes). The fifth column gives the times for Omega++.

It is obvious that PAI, at least when implemented directly as given by Lasaruk and Sturm [21], is uncompetitive. Thus, Omega++ is always faster than PAI. When comparing Proto to Omega++, recall that Proto is only trying to solve the simpler problem of quantifier-free formulae. Despite this, for many of our tests the tools perform similarly. For a few tests, some of Proto’s heuristics result in appreciably better times; we plan to study these tests in more detail in the future to try to improve Omega++. Overall, Omega++’s performance is competitive.

Inference. As described in section 2.6, quantifier elimination in Presburger arithmetic with infinity can help with invariant generation of octagonal constraints. The table below benchmarks using Omega++ for this analysis technique. The first column gives the test name. The second and third columns give the user-provided spatial pre- and postconditions in separation logic. The fourth column gives the inferred pure specification, while the last column gives the time used by Omega++. The final test is noteworthy because the inferred invariant uses min/max constraints.

<i>Method</i>	<i>Pre</i>	<i>Post</i>	<i>Inferred</i>	<i>Time (Omega++)</i>
<i>Create</i>	true	$ll\langle res, m \rangle$	$m=n$	0.13
<i>Delete</i>	$ll\langle x, n \rangle$	$ll\langle res, m \rangle$	$n-1 \leq m$	0.17
<i>Insert</i>	$ll\langle x, n \rangle \wedge x \neq null$	$ll\langle x, m \rangle$	$n=m-1$	0.13
<i>Copy</i>	$ll\langle x, n \rangle * ll\langle res, m \rangle$	$ll\langle x, m \rangle$	$m=n$	0.16
<i>Remove</i>	$ll\langle x, n \rangle \wedge x \neq null$	$ll\langle x, m \rangle$	$n-1 \leq m \wedge m \leq n$	0.19
<i>Return</i>	$ll\langle x, n \rangle$	$ll\langle x, m \rangle$	$m=n \wedge 0 \leq m$	0.07
<i>Traverse</i>	$ll\langle x, n \rangle$	$ll\langle x, m \rangle$	$m=n$	0.12
<i>Get</i>	$ll\langle x, n \rangle \wedge x \neq null$	$ll\langle res, m \rangle$	$m=n-2 \wedge 2 \leq n$	0.11
<i>Head</i>	$ll\langle x, n \rangle * ll\langle y, m \rangle$	$ll\langle res, n+m-1 \rangle$	$1=\min(n, m)$	0.21

7 Related Work

7.1 Ghost Variables

Reynolds demonstrated that *ghost variables* [39] were useful for verifying sequential programs. Their importance is highlighted when proving program, object or loop invariants [27], refining between two transition systems [26] or when considering program's security aspects [24]. In a concurrent setting, whether adopting a Hoare [39] or a VDM-style [13] program logic, specification formulae are generally extended with ghost variables in order to explicitly record the information of interest between communicating processes. Our work enriches specifications by extending the domain of ghost values with the mathematical concepts of positive and negative infinity.

7.2 Decision Procedures

Presburger arithmetic [37] is one of the canonical examples of an important decidable problem. Kuncak *et al.* [19,20] presented a decision procedure for a quantifier-free fragment of Boolean Algebra with Presburger arithmetic which can be used to prove a mixed set-based constraint with symbolic cardinality and linear arithmetic. QFBAPA was later extended to the more challenging case of multisets [35] and proved to be NP-complete [36]. The VCDryad [34] framework combines separation logic with decision procedures for sets and multi sets to verify programs with natural proofs. The combination of set/multi-sets with PA even though quite useful cannot be used to specify some examples like the infinite data structure example we presented in section 2.3.

Another line of work is focussed on not only on proving decidability of certain logic fragments [8,4] but also on both the applicability of exhibited decision procedures [27,11] as well as their efficiency [7]. In particular, arithmetic decision procedures [32,29,33] are a fundamental part of interactive systems such as: Isabelle/HOL [31], Coq [6] and ACL2 [16]. Considerable attention is also paid to the problem of exploiting methods which combine decision procedures from different domains [10,25], resulting in a decision of a union of theories the majority being based on the foundations built by Nelson-Oppen[30] or Shostak [40].

7.3 Extending Decision Procedures

Lasaruk and Sturm [21] were the first to tackle the problem of extending PA with infinity, proving completeness and decidability. As discussed in section 6, our work differs from theirs by providing distinct positive and negative infinities as well as by providing an implementation. In this paper, we build an implementation on top of Omega calculator [17], and certify it in Coq [2]. The general problem of adding infinities to the set of reals was addressed by Weispfenning [42]. This was later extended to mixed real and integer quantifier elimination in [43]. Another interesting extension of decision procedures for real arithmetic is the addition of infinitesimals. The proof assistant Isabelle/HOL [31] supports the use of such infinitesimals. Loos and Weispfenning [23] first proposed a virtual substitution approach for quantifier elimination of infinitesimals. In Omega++, we use a similar virtual substitution to eliminate infinities as part of the decision procedure. For the case of linear formulas, the use of substitution for elimination of quantifiers was first proposed in [41].

8 Conclusion.

We presented Omega++, a decision procedure for Presburger arithmetic with infinity \mathbb{Z}_∞ . Infinity is a useful abstraction, increasing a program logic’s ability to model infinite data structures, reason about termination, and compose more elegantly. Moreover, specifications with infinity are often more concise. Finally, quantifier elimination for infinities enables an extension to an existing analysis technique.

Omega++ itself is a sound and complete decision procedure for \mathbb{Z}_∞ , and has been Coq-certified to respect a precise formal semantics for \mathbb{Z}_∞ . We integrated Omega++ into an existing verifier and evaluated it on a benchmark of small programs, demonstrating that it can perform well in practice. Omega++ demonstrates that we can develop useful, efficient, and certified programs for program verification and analysis.

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