SOLUTIONS FOR TERM TEST, FEBRUARY 16, 2018
MATB43

(1) Given a real number a, prove that there exists a sequence of rational numbers converging to a.

Between any two real numbers there is a rational number. So for any $n \in \mathbb{N}$ there exists $a_n \in \mathbb{Q}$ such that

$$a - 1/n < a_n < a \ .$$

Given $\epsilon > 0$, there exists $N \in \mathbb{N}$, with $1/N < \epsilon$. Then for $n > N$,

$$|a - a_n| < 1/n < 1/N < \epsilon \ .$$

Therefore $\{a_n\}$ converges to a.

(2) Determine whether the following series converge or diverge.

(a) $\sum\limits_{k=2}^{\infty} 1/\log k$.

For all $k > 1$, $\log k < k$. Therefore $1/k < 1/\log k$. The harmonic series $\sum\limits_{k=2}^{\infty} 1/k$ diverges. Therefore by the comparison test, the series $\sum\limits_{k=2}^{\infty} 1/\log k$ diverges.

(b) $\sum\limits_{k=1}^{\infty}(1/3^k + 1/5^k)$. (Suggestion: $1/3^k + 1/5^k \leq (1/3 + 1/5)^k$)

By the binomial theorem, $1/3^k + 1/5^k \leq (1/3 + 1/5)^k = 8/15$. Since $8/15 < 1$, the geometric series $\sum\limits_{k=1}^{\infty}(8/15)^k$ converges. Therefore by the comparison test $\sum\limits_{k=1}^{\infty}(1/3^k + 1/5^k)$ converges.

(c) $\sum_{k=1}^{\infty}(-1)^{k+1}/\sqrt{k}$.

For $k > 0$, $1/\sqrt{k} \to 0$ as $k \to \infty$. and $1/\sqrt{k} > 0$. Therefore by the alternating series test, $\sum_{k=1}^{\infty}(-1)^{k+1}/\sqrt{k}$ converges.

(3) (a) Give a subsequence of the terms of the alternating harmonic series, whose sum does not converge.

Take the sequence $\{1/(2n-1)\}$, whose sum $\sum_{n=1}^{\infty} 1/(2n-1)$ is the sum of the positive terms in the alternating harmonic series. The sum of the positive terms in any conditionally convergent series diverges. Therefore this sum diverges.

(b) Show that if $\sum_{k=1}^{\infty} a_k$ converges conditionally then there is a subsequence of $\{a_k\}$, whose sum does not converge.

Take the subsequence of positive terms of $\{a_k\}$. As discussed in class, it must diverge since $\sum_{k=1}^{\infty} a_k$ converges conditionally.

(c) Prove that if $\sum_{k=1}^{\infty} a_k$ converges absolutely, and $\{b_l\}$ is any subsequence of $\{a_k\}$, then $\sum_{l=1}^{\infty} b_l$ converges absolutely.

Define a new sequence $\{c_k\}$ by replacing the terms in $\{a_k\}$ by 0 if they do not belong to the sequence $\{b_l\}$. It follows that for all k,

$$|c_k| \leq |a_k| \ .$$

The comparison test then shows that $\sum_{k=1}^{\infty} c_k$ converges absolutely. But

$$\sum_{k=1}^{\infty} c_k = \sum_{l=1}^{\infty} b_l \ .$$

So $\sum_{l=1}^{\infty} b_l$ converges absolutely.

(4) (a) Define functions f_n, for $n \in \mathbb{N}$, by

$$f_n(x) = \frac{x}{1+nx^2} \quad , x \in \mathbb{R} \ .$$

Let $f(x) = 0$, for all $x \in \mathbb{R}$. Show that $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$.

For fixed x, $\frac{x}{1+nx^2} \to 0$ as $n \to \infty$. Therefore $f_n(x) \to f(x)$.

(b) Determine the maximum and minimum of $f_n(x)$ on $\mathbb{R}$.

f_n is an odd function of x. The derivative of f_n is given by

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} \ .$$

So for $x \geq 0$,

$$f_n'(x) \begin{cases} > 0 & \text{for } x < 1/\sqrt{n} \\ = 0 & \text{for } x = 1/\sqrt{n} \\ < 0 & \text{for } x > 1/\sqrt{n} \end{cases}$$

Thus the maximum of f_n is $1/\sqrt{n}$ and the minimum is $-1/\sqrt{n}$.

(c) Show that $f_n \to f$ uniformly on $\mathbb{R}$.

Given $\epsilon > 0$, choose N so that $1/\sqrt{N} < \epsilon$. Then for all $n > N$,

$$|f_n(x) - f(x)| = |f_n(x)| < 1/\sqrt{n} < \epsilon \ ,$$

for all $x \in \mathbb{R}$. Therefore $f_n \to f$ uniformly on $\mathbb{R}$.

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Course MATB43
(e.g. ANTA01Y, PHL 202S)

Instructor JOHN SCHERK

Date of Examination FEB 16th 2018

Room Number HW 216 Seat Number

BOOK NO.

TOTAL NUMBER OF BOOKS USED

TUT 3

TERM

EXAMINER'S REPORT

1	25
2	25
3	25
4	20
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11	
12	
TOTAL	95

INSTRUCTIONS

Write the information sought in the spaces above.

Write the answers on the RULED SIDE ONLY; all calculations or rough drafts of answers should be shown, preferably on the unruled side.

Clearly identify the question to which each answer applies; whenever the answer to a question is divided, note at the end of the first section "see also work on page ".

If a page is left blank write on it "see work on page ".

If more than one book is used, indicate the total number on the cover of each. At the conclusion of the examination, place all other books inside Book No. 1.

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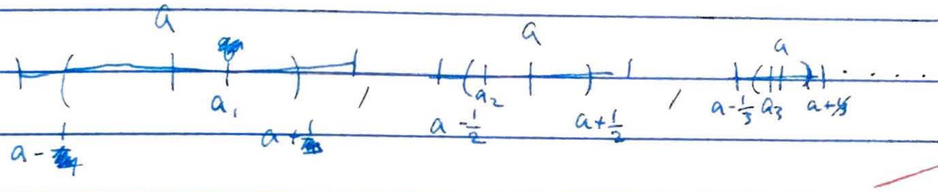
1. ~~Consider the set $S=\{q\in\mathbb{Q}, q<a\}$~~
~~Since the rationals are countable, and~~

~~Given $a\in\mathbb{R}$, Let $m=\lfloor a\rfloor$, $n=\lceil a\rceil$~~
~~Then consider the function $f(n)=\begin{cases}\frac{m+n}{2}, & n=0\end{cases}$~~

Consider the interval $I_n=[a-\frac{1}{n}, a+\frac{1}{n}]$ around a, where $n\in\mathbb{N}$.
Due to the fact that $\mathbb{Q}$ dense in $\mathbb{R}$, $\exists q\in I_n$ $\forall n\in\mathbb{N}$.
This means that we ~~take any arbitrary rational~~ can form a sequence $\{a_1, a_2, ..\}$ by having a_1 be an arbitrary rational in I_1 & a_2 being in I_2 and so forth. This is a valid sequence mapping $(n\in\mathbb{N})\to\mathbb{R}$. Now, for any $\varepsilon>0$, we can have our $N=\frac{1}{\varepsilon}$, then for any element $i_n\in I_n$, $|i_n-a|<|a+\frac{1}{n}-a|=|\frac{1}{n}|<|\frac{1}{(\frac{1}{\varepsilon})}|=\varepsilon$ so any element in this interval has the property of being less than ε. $\therefore$ $\forall\varepsilon>0, \exists N$ s.t. $n>N\Rightarrow|a_n-a|<\varepsilon$ and this is independent of the specific element we choose in I_n, as all of them share this property.



Graphical representation of the elements in the sequence.

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2 a) $\sum_{k=2}^{\infty} \frac{1}{\log(k)}$ given $a_j = \frac{1}{\log(j)}$, for all $j \geq 2$, $0 < \frac{1}{j\log(j)} < \frac{1}{\log(j)}$

So by comparison test, if $\sum \frac{1}{j\log(j)}$ diverges, so does $\sum \frac{1}{\log(j)}$

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Using the integral test, (Since $\frac{1}{j\log(j)}$ cont & integrable over $[2, \infty)$)

$$\int_2^{\infty} \frac{1}{k\log(k)}\,dk \overset{u=\log(k),\ du=\frac{1}{k}}{=} \int_{\log(2)}^{\infty} \frac{1}{u}\,du = \Big[\log(u)\Big]_{\log(2)}^{\infty} = \infty - \log(\log(2))$$

$\Rightarrow$ it diverges, $\therefore$ by C.T. so does $\sum_{k=2}^{\infty} \frac{1}{\log k}$

(Using binomial thm, there other positive elements other than $1/3^k$ & $1/5^k$)

b) $\sum_{k=1}^{\infty} \left(\frac{1}{3^k} + \frac{1}{5^k}\right)$ $0 < \left(\frac{1}{3^k} + \frac{1}{5^k}\right) \leq (1/3 + 1/5)^k$, so by C.T. if $(1/3 + 1/5)^k$ converges, so does $\left(\frac{1}{3^k} + \frac{1}{5^k}\right)$

Now, $\sum_{k=1}^{\infty} (\frac{1}{3} + \frac{1}{5})^k = \sum_{k=1}^{\infty} \left(\frac{8}{15}\right)^k$, but this is a geometric series with $8/15 < 1$ so we know that it must converge. Therefore by C.T. $\sum_{k=1}^{\infty}\left(\frac{1}{3^k} + \frac{1}{5^k}\right)$ must also converge.

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c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ Using the alternating series test, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so it converges.

& $\frac{1}{\sqrt{n}}$ decreasing

$\left(\frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{-1}{2\sqrt{x^3}} < 0\ \forall x\right)$

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$\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

3a) A possible subsequence is simply the positive elements of the harmonic series. ($\sum_{k=1}^{\infty} a_k^+$ of $\sum_{k=1}^{\infty} a_k$). This is divergent, since we know that for any divergent series, both the sum of its positive elements & the sum of its negative elements must diverge ~~or~~ or else the sum absolutely converges or diverges.

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You should try to show it!

b) Consider the positive sequence of elements of a_k, $\{a_k^+\}$
& likewise, the negative elements $\{a_k^-\}$

Now, Suppose that none of them diverged, then $\sum_{k=1} a_k^+ + \sum_{k=1} a_k^-$ ~~conv~~
We would have that $\underbrace{\sum_{k=1}^{\infty} a_k^+}_{\text{convergent}} - \underbrace{\sum_{k=1}^{\infty} a_k^-}_{\text{convergent}}$

it might be empty...

would sum to some finite $c < \infty$. But this expression is simply terms! a rearrangement of the absolute value sum of each element ~~ax~~ of a_k.
Since absolute convergence sums independently of order, ~~thus~~ this sum is absolutely convergent, but that is a contradiction $\therefore$ either $\{a_k\}$, $\{b_k\}$ or both must diverge. Just take a_k^+ & show why!

OK! 10

c) Let A_n be the partial sums of $\sum_{k=1}^{n} |a_k|$, & similarly for B_n.
Since $\sum a_k$ absolutely converges, A_n must converge ~~and hence have the Cauchy property~~. ~~So $\forall \varepsilon > 0, \exists N_a$ s.t. $m, n > N \Rightarrow |A_m - A_n| < \varepsilon$~~
~~If B_n has this property as well, $\sum b_k$ must be absolutely convergent.~~
~~Then for B_n, choose $N_b = N_a \; \forall \varepsilon$~~
~~Since the subsequence of $\{b_k\}_{k \geq 8}$~~

Suppose A_n converges to c, then the sequence of partial sums A_n is bounded above by c (since there are no negative elements in ~~the absolute~~ sum of absolute terms)
It is also bounded below by zero, again since every term ≥ 0.
But B_n must also be bounded by above by c since B_n is one of the partial sums A_k without finitely many of its terms. We also know $B_n \geq 0$ since $|b_k| \geq 0$. Therefore B_n ~~B_n~~ is bounded, since $b_k \geq 0$, it is also increasing $\therefore$ B_n is Bounded & monotone, by BMCT, B_n converges $\Rightarrow \sum_{k=1} |b_k|$ converges) i.e. $\sum b_k$ absolutely converges.

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OK!

need absolute value

4a) ~~Sin~~ Given, ε.

For pointwise convergence to f, pick $N = \frac{1}{\varepsilon x} - \frac{1}{x^2}$,

so any $n \geq N$ ~~⇔~~

$\Rightarrow \left|\frac{x}{1+nx^2}\right| < \left|\frac{x}{1+(\frac{1}{\varepsilon x}-\frac{1}{x^2})x^2}\right|$ X

$\left|\frac{x}{1+(\frac{x}{\varepsilon}-1)}\right|$

$= \left|\frac{x}{(\frac{x}{\varepsilon})}\right|$

$= \varepsilon \Rightarrow \left|\frac{x}{1+nx^2}\right| < \varepsilon$

so it converges pointwise to 0 for all x.

$\frac{x}{1+nx^2} < \varepsilon$

$x < \varepsilon(1+nx^2)$

$x < \varepsilon + \varepsilon n x^2$

$-\varepsilon n x^2 < \varepsilon - x$

$\varepsilon n x^2 > x - \varepsilon$

$n x^2 > \frac{x}{\varepsilon} - 1$

$n > \frac{1}{\varepsilon x} - \frac{1}{x^2}$

$x = 0$?

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$\frac{x}{1+nx^2} <$

b) $\frac{d}{dx}\frac{x}{1+nx^2} = \frac{d}{dx}(x)(1+nx^2)^{-1}$

$= (1+nx^2)^{-1} + -(x)(1+nx^2)^{-2}(2xn)$

$= \frac{(1+nx^2) - 2x^2n}{(1+nx^2)^2}$

$= \frac{1-nx^2}{(1+nx^2)^2}$ So the min & max are $\frac{-1}{\sqrt{n}}$ & $\frac{1}{\sqrt{n}}$ respectively

Since f_n does not diverge as $x \to -\infty$ or $x \to \infty$, local min/max = global min/max.

? proof is min/max ? (2)

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c) For uniform convergence to f, we know Given ε, n, $f_n(x) \in [\frac{-1}{\sqrt{n}}, \frac{1}{\sqrt{n}}]$

so pick $N = \frac{1}{\varepsilon^2} \Rightarrow |f_n(x)| < \frac{1}{\sqrt{n}}$

$\pm\frac{1}{2\sqrt{n}}$

$< \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}}$

$= \frac{1}{(\frac{1}{\varepsilon})}$

$= \varepsilon$

Therefore, $\forall \varepsilon > 0, \exists N$ s.t. $|f_n(x) - 0| < \varepsilon \ \forall x \in \mathbb{R}$

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