# University of Toronto at Scarborough
# Department of Computer & Mathematical Sciences

## MIDTERM TEST

## MATC27H3 – Introduction to Topology

Examiner: K. Smith

Date: October 19, 2018
Duration: 110 Minutes

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SIGNATURE:

**DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.**

**NOTES**:

- There are 7 numbered pages to the exam. It is your responsibility to ensure that, at the start of the exam, this booklet has all its pages.
- Answer all questions. Explain and fully justify all of your answers (unless instructed to do otherwise).
- **Show all your work**. Credit will not be given for answers if the work is not shown. If you need more space use the back of the page.
- No cell phones and any type of e-mail or instant messaging devices are allowed to be brought to the exam. Be sure that if you have any, that they are OFF and in your backpack away from you.

| FOR MARKERS ONLY | |
|---|---|
| Question | Marks |
| 1 | 12 /12 |
| 2 | 8 / 9 |
| 3 | 9 / 9 |
| 4 | 7 / 10 |
| 5 | 5 / 5 |
| 6 | 5 /6 |
| 7 | 3 / 4 |
| ~~8~~ | ~~/ 5~~ |
| BONUS | 5 / 5 |
| **TOTAL** | 54 /~~60~~ 55 |

well done!

1. **[12 marks] TRUE OR FALSE** Carefully read each statement. If the assertation must be true, the **circle T** (for true). Otherwise, **circle F** (for false). Justification is neither required nor rewarded, but a small workplace is given for your rough work. Each correct answer earns 2 point and each incorrect or blank answer earns 0 points.

(a) The collection $\{[a, b] \mid a, b \in \mathbb{R}\}$ is a basis for the standard topology (i.e., the Euclidean topology) on $\mathbb{R}$.

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**T or F**

(b) Let $X$ be a topological space. Arbitrary intersections and finite unions of closed sets of $X$ closed in $X$.

**T or F**

(c) On finite sets, topologies are by necessity also finite in cardinality.

**T or F**

(d) Consider $\mathbb{R}^2$ with the standard topology. The relative topology on $\mathbb{Z} \times \mathbb{Z}$ must be the discrete topology.

**T or F**

(e) Let $X$ be a non-empty set. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be basis of topologies $\tau$ and $\tau'$, respectively, on $X$. If $\tau$ is finer than $\tau'$, then $\mathfrak{B} \supseteq \mathfrak{B}'$.

**T or F**

$\mathcal{B} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$ $\mathcal{B}' = \{\{a\}, \{a, b\}, \{a, b, c, d, e\}\}$

(f) A collection $\mathcal{S}$ of subsets of a non-empty set $X$ is a subbasis for a topology $\tau$ on $X$ if and only if $X = \bigcup_{S \in \mathcal{S}} S$. In this case, the topology $\tau$ generated by $\mathcal{S}$ is

$$\tau = \{U \subseteq X \mid U \text{ is the union of finite intersections of sets from } \mathcal{S}\}.$$

**T or F**

2. **[9 marks]** Let $X$ be a infinite set and let $p \in X$ be an arbitrary point. Let $\tau_{F,p}$ be the collection of subsets of $X$ which either have finite complement or do not contain the point $p$. That is, the set $\tau_{F,p}$ is defined as

$$\tau_{F,p} = \{U \subseteq X \mid \text{ either } X - U \text{ is finite or } p \notin U\}.$$

Prove that $\tau_{F,p}$ is a topology on $X$.

1. $|X - X| = |\emptyset| = 0 < \infty$ so $X$ is in $\tau_{F,p}$
$p \notin \emptyset \Rightarrow \emptyset \in \tau_{F,p}$

2. Let $\mathcal{C}$ be a collection of open sets in $\tau_{F,p}$
The union $\bigcup_{c \in \mathcal{C}} c$ has two cases
① $\forall c \in \mathcal{C}$ $p \notin c$
$\Rightarrow p \notin \bigcup_{c \in \mathcal{C}} c$ as none of the elements include it.
② $\exists\, \mathcal{U} \in \mathcal{C}$ s.t. $X - \mathcal{U}$ is finite. (so $p \in \mathcal{U}$ for some $\mathcal{U}$)

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$\Rightarrow X \setminus \left(\bigcup_{c \in \mathcal{C}} c\right) \subseteq X \setminus \mathcal{U}$
$\Rightarrow \left|X \setminus \left(\bigcup_{c \in \mathcal{C}} c\right)\right| \leq \underbrace{|X \setminus \mathcal{U}|}_{\text{finite}}$
$\Rightarrow X \setminus \bigcup_{c \in \mathcal{C}} c$ is finite so $\in \tau$.

3. ~~Ex~~ ~~Let~~
Induction: intersection of $n$ open sets is open
~~Case~~ Base, $n=2$ say $A, B$ open.
if ~~$p \notin A$~~ $p \notin A$ or $p \notin B \Rightarrow p \notin A \cap B$ so $A \cap B \in \tau$
else ~~$X \setminus A$~~ $X \setminus A$ finite & $X \setminus B$ finite
but $(A \cap B)^c = A^c \cup B^c$ & each term is finite, so their union must have finite elements $\Rightarrow A \cap B \in \tau$ ✓
Induction step
Sps holds for $n-1$, where $\mathcal{U}_i \in \tau$ .. for what $i$?
$\bigcap_{i=1}^{n} \mathcal{U}_i = \left(\bigcap_{i=1}^{n-1} \mathcal{U}_i\right) \cap \mathcal{U}_n$ [IH]
$= \mathcal{U}' \cap \mathcal{U}_n$ for some $\mathcal{U}' \in \tau$
$= \mathcal{U}''$ [by $n=2$] for some $\mathcal{U}'' \in \tau$.

3. Consider the set of rational numbers $\mathbb{Q}$ as a *subspace* of $\mathbb{R}$.

(a) **[4 marks]** Suppose that $\mathbb{R}$ is equipped with the discrete topology. Is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Make sure to fully justify your solutions.

Since it is the discrete topology, any possible subset of $\mathbb{R}$ must be open. Evidently $\mathbb{Q} \subseteq \mathbb{R} \Rightarrow \mathbb{Q} \in \mathcal{T}$.

It must also be closed as $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R} \Rightarrow \mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}$

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sorry

(b) **[5 marks]** Suppose that $\mathbb{R}$ is equipped with the finite complement topology $\tau_f$. Is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Make sure to fully justify your solutions.

*(Recall that $\tau_f = \{U \subseteq X \mid$ either $\cancel{X} - U$ is finite or $U = X\}$.)*

$\mathbb{Q}$ is not open, nor closed.

For it to be open, either $\mathbb{Q} = \mathbb{R}$ (false) or $|\mathbb{R} \setminus \mathbb{Q}| < \infty$. By density of $\mathbb{Q}^c$, we already know there infinitely many elements not in $\mathbb{Q}$, so it isn't open.

Now it is not closed either, $\mathbb{Q} \neq \emptyset \Rightarrow \mathbb{R} \setminus \mathbb{Q} \neq \mathbb{R}$ & since $\mathbb{Q}$ is also dense in $\mathbb{R}$, The compliment of $\mathbb{Q}^c$ is infinite as well.

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4. Let $X \neq \emptyset$ be a set. Provide a complete and accurate definition for each of the following.

(a) **[4 marks]** A *topology* $\tau$ on $X$.

1) $X, \emptyset \in \tau$ $\qquad \tau \subseteq P(X)$

2) For any arbitrary collection of sets $\mathcal{C} \subseteq \tau$, the union $\bigcup_{c \in \mathcal{C}} c$ is in $\tau$.

3 $\quad$ 3) For any ~~arbitrary~~ finite collection of sets $\mathcal{C} \subseteq \tau$, the intersection $\bigcap_{c \in \mathcal{C}} c$ is in $\tau$

(b) **[4 marks]** A *basis* $\mathfrak{B}$ for some topology on $X$.

1) $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B$ $\qquad \mathcal{B} \subseteq P(X)$

3 $\quad$ 2) if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ & $x \in X$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3$ & $B_3 \subseteq B_1 \cap B_2$.

(c) **[2 marks]** What it means for a basis $\mathfrak{B}$ to generate a topology $\tau_{\mathfrak{B}}$ on $X$.

$\forall u \in \tau_{\mathcal{B}}$ $\exists$ collection $\mathcal{B}' \subseteq \mathcal{B}$ s.t. $u = \bigcup_{b \in \mathcal{B}'} b$

4 $\quad$ & $\forall u \notin \tau_{\mathcal{B}}$ $\nexists$ " $\quad$ " $\quad$ "

5. **[5 marks]** Prove or disprove the following : The set $\mathfrak{B} = \{[a, b] \mid a, b \in \mathbb{R},\ a < b\}$ is a basis for some (not necessarily standard) topology on $\mathbb{R}$.

This is not a ~~topology~~ basis for a topology.

By 2) must have for $a < b < c \in \mathbb{R}$, $[a, b] \cap [b, c] = \{b\}$.

5 $\quad$ This means $\exists e, f$ s.t. $e < f$ & $b \in [e, f]$ (A) & $[e, f] \subseteq \{b\}$ (B)

but the only closed interval satisfying (A) & (B) is $[b, b]$

which does not satisfy $b < b$ ⓡ

6. **[6 marks]** Let $X_1, \ldots, X_n$ be a finite collection of topological spaces. Suppose that $C_i$ is an arbitrary closed subset of $X_i$, for $i = 1, \ldots, n$. Prove that $C_1 \times \cdots \times C_n$ is a closed in $X_1 \times \cdots \times X_n$.

~~Suppose ... the open compliment of ... the closed subset $C_i$~~

Given $U_i = X_i \setminus C_i$, know that $X_1 \times \cdots \times U_i \times \cdots \times X_n$ is open

and so $X_1 \times X_2 \times \cdots \times X_n \setminus X_1 \times \cdots \times U_i \times \cdots X_n = (X_1 \times \cdots \times C_i \times \cdots \times X_n) \cup (\emptyset \times \cdots \times U_i \times \cdots)$

$= X_1 \times \cdots \times C_i \times \cdots \times X_n$ is closed

So let $S_i = X_1 \times \cdots \times C_i \times \cdots \times X_n$,

The arbitrary intersection of closed sets are closed $\Rightarrow$ $\bigcap_{i=1}^{n} S_i$ is closed

$= C_1 \times \cdots \times C_n$

5

7. **[4 marks]** Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces. Suppose that for each $\alpha \in J$, $U_\alpha$ is an arbitrary open subset of $X_\alpha$. Is $\prod_{\alpha \in J} U_\alpha$ open in $\prod_{\alpha \in J} X_\alpha$? Make sure to cite appropriate definitions or theorems in your justification. Provide an explicit counter-example in your justification if your answer is 'no'.

It is not open in $\prod_{\alpha \in J} X_\alpha$.

if we take $\prod_{n \in \mathbb{N}} \mathbb{R}$ as the product of topological spaces &

$(0,1) = U_\alpha \ \forall \alpha$, by the product topology definition we have

that $\prod_{\alpha \in J} (0,1) = \bigcup_{\beta \in L} \bigcap_{i=1}^{k} \pi^{-1}(U_\beta)$ $k \in \mathbb{N}$, $L$ arbitrary index set

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but for any preimage $\pi^{-1}(U_\beta)$ there are an infinite # of projections to $\mathbb{R}$, even with a finite intersection, finite projections to non $\mathbb{R}$ open sets will not change that. The infinite union will not take away elements either so each projection cannot decrease in size.

But $\prod_{\alpha \in J} (0,1)$ does not have infinite projections to $\mathbb{R}$ ↯

not included

8. **[5 marks]** Let $X = \mathbb{R}$ and let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. Suppose that the following families are bases for a topology on $X$:

- $\mathfrak{B} = \{(a, b) \mid a, b \in \mathbb{R},\ a < b\}$
- $\mathfrak{B}' = \{(a, b) \mid a, b \in \mathbb{R},\ a < b\} \cup \{(a, b) - K \mid a, b \in \mathbb{R},\ a < b\}$.

Prove that the topologies, $\tau$ and $\tau'$, generated by $\mathfrak{B}$ and $\mathfrak{B}'$ respectively, are not comparable.

Suppose $\tau \supseteq \tau'$ then $\exists \mathcal{B}_1 \subset \mathcal{B}$ s.t. $\bigcup_{b \in \mathcal{B}_1} b = (-1,1) - K$
$= (-1,0] \cup (0,1) - K$

Now $(0,1) - K \in \tau$ since can take union of $\bigcup_{i=1}^{\infty} (\frac{1}{i+1}, \frac{1}{i})$,
but $(-1,0]$ has closed interval $\Rightarrow$ cannot be in $\tau$
& $(-1,k)$ where $k > 0$ $\notin (-1,1) - K$ as $\exists i$ s.t. $\frac{1}{i}$ arbitrary close to $0$ #

Suppose ~~...~~ $\tau \subseteq \tau'$

BONUS **[5 marks]** Let $X = \{a, b, c, d, e\}$. Suppose that $\tau_X$ is the discrete topology. Find a subbase $\mathcal{S}$ for (i.e., that generates) $\tau_X$ which does not contain any singleton sets. Make sure to fully justify why your $\mathcal{S}$ satisfies all of the specified requirements.

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$S = \{\{a,b\}, \{c,d\}, \{b,c\}, \{d,e\}, \{a,e\}\}$

each single ton set can be generated through finite intersection

$\{a\} = \{a,e\} \cap \{a,b\}$
$\{b\} = \{a,b\} \cap \{b,c\}$
$\{c\} = \{b,c\} \cap \{c,d\}$
$\{d\} = \{c,d\} \cap \{d,e\}$
$\{e\} = \{d,e\} \cap \{e,a\}$

this means that the arbitrary unions of finite intersections will include everything, as the single tons can be arranged to give any subset.


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$\mathbb{R}\times\{1\}$

$(\mathbb{R}\times\mathbb{R}\setminus\{1\}) \cup \{\emptyset\times\{1\}\}$

1 ✓ 2 ✓ 3 ✓ 4 ✓ 5 ✓ 6 ✓ 7 ✓ 8 B ✓

if it was comparable

$\{a, b, c, d, e\}$ $\forall x\in X\ \exists B\in\mathcal{B}$ s.t. $x\in B \wedge B'\supseteq B$

a ,

$\forall B'\in\mathcal{B}', \forall x\in B', \exists B\in\mathcal{B}$ s.t. $x\in B$ & $B\subseteq B'$

$K = 1, 1/2, 1/3, 1/4, 1/5, \dots$

Definition of Product topology

$S_i = \{\pi_i^{-1}(U_i) \mid U_i\in\tau_i\}$

Subbasis $= S = \bigcup_{i\in I} S_i$