University of Toronto at Scarborough
CSCD37—Analysis of Numerical Algorithms for Computational Mathematics, Winter 2018

Midterm examination

Duration: 90 minutes

Date and Time: Wednesday 14 March, 5:15–6:45 p.m.

Aids allowed: None; closed-book, no calculators.

Make sure that your examination booklet has 11 pages (including this one). Write your answers in the spaces provided. You will be rewarded for concise, well-thought-out answers, rather than long rambling ones. Please write legibly.

This exam was designed so that you have plenty of time to write it. Take a few minutes to read each question before you begin, and then start with the question you are most comfortable with.

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(Circle your family name.)

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YOU MUST SIGN THE FOLLOWING:

I declare that this exam was written by the person whose name and student # appear above.

Signature: ____________

Your grade

Wk 2 lecture 1.	5	/ 15	(a) 10 (b) 2 + 2 if converse (c) 1
2.	5	/ 5	
3.	15	/ 15	(a) 10 (b) 4 (c) 1
4.	8	/ 10	(a) 4 marks (b) 4 (c) 2
5.	7	/ 10	(a) 8 marks (b) 2 marks
Total	40	/ 55	

Graded by

Question 1

[15 marks]

Consider the overdetermined linear system $Ax = b, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, b \in \mathbb{R}^m, m > n$ and its least squares solution x^* yielding $\min_x \|Ax - b\|_2$.

a. Prove that x^* can be computed by solving the *Normal Equations* $A^T A x^* = A^T b$.

$$\|A^T A x^* - A^T b\|_2^2 = \cancel{(A^T A x^* - A^T b) \cdot (A^T A x^* - A^T b)^T}$$

if $\to$ $=0$ $\Rightarrow$ soln the same

$$= \cancel{(A^T A x^*)(A^T A x^*)^T - A^T A x^* (A^T b)^T}$$

$$= (A^T A x^* - A^T b)^T \cdot (A^T A x^* - A^T b)$$

$$= ((A^T A x^*)^T - (A^T b)^T) \cdot (A^T A x^* - A^T b)$$

$$= \|A^T A x^*\|_2^2 - (A^T A x^*)^T (A^T b) - (A^T b)^T (A^T A x^*) + \|A^T b\|_2^2$$

$$= \|A^T A x^*\|_2^2 - 2(A^T A x^*)^T (A^T b) + \|A^T b\|_2^2$$

See week #2 lecture

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b. What condition(s) are sufficient for x^* to be unique? Prove that x^* is unique if the condition(s) are satisfied. (Recall this is an "if and only if" proof.)

The matrix of A needs to have full column rank.

c. What is the major disadvantage of computing x^* by solving the Normal Equations?

The condition of the matrix heavily influences the error of x^* when computing it.

Question 2

[5 marks]

Recall that when computing the $PA = LU$ factorization of $A \in \mathcal{R}^{n\times n}$, the (possible) row interchange preceding the i-th stage of the triangularization is represented by a permutation matrix $P_i \in \mathcal{R}^{n\times n}$, where $P_i \equiv P_{ij}$ is simply the identity matrix with rows i and j interchanged, $j \geq i$.

Show that P_{ij} can be formulated as a Householder reflection

$$Q_{ij} = I - 2\frac{vv^T}{v^Tv}$$

with a suitable choice for $v \in \mathcal{R}^n$.

$$P_{ij} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0 \end{bmatrix} = I - 2\begin{bmatrix} 0 & & & \\ & \frac{1}{2} & -\frac{1}{2} & \\ & -\frac{1}{2} & \frac{1}{2} & \\ & & & 0 \end{bmatrix}$$

$\Rightarrow v = (e_i - e_j)$

then $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \end{bmatrix} = \begin{array}{c|c} & \begin{array}{cc} i & j \end{array} \\ \hline \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} 0 & & & \\ & 1 & \cdots & -1 \\ & -1 & \cdots & 1 \\ & & & 0 \end{bmatrix} \end{array}$

& $v^Tv = 2$

$= Q_{ij} = I - vv^T$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} 0 & & & & 0 \\ & 1 & 0 & -1 & \\ & 0 & 0 & 0 & \\ & -1 & 0 & 1 & \\ 0 & & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & & 0 \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ 0 & & & & 1 \end{bmatrix} = P_{ij}$$

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$$\begin{bmatrix} 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cancel{3} & \cancel{6} \\ \cancel{4} & \cancel{8} \\ \cancel{0} & \cancel{2} \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cancel{3} & \cancel{6} \\ \cancel{4} & \cancel{8} \\ \cancel{0} & \cancel{2} \end{bmatrix}$$

Question 3

[15 marks]

Consider the overdetermined linear system $Ax = b$ where

$$A = \begin{bmatrix} 3 & 6 \\ 4 & 8 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix}.$$

a. Compute the QR-factorization of A using Householder reflections. Show *all* intermediate calculations. For example, the first reflection is given by

$$Q_1 = I - 2\frac{vv^T}{v^Tv}, \quad v = a_1 - \|a_1\|_2 e_1,$$

where a_1 is the first column of A. Verify that $Q = Q_1^T Q_2^T$ is orthogonal.

(**Hint**: You may find the result in the previous question useful when computing this factorization.)

$$V_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \sqrt{3^2+4^2+0}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$$

$$\frac{VV^T}{V^TV} = \frac{\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}[-2\ 4\ 0]}{[-2\ 4\ 0]\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}} = \begin{bmatrix} 4 & -8 & 0 \\ -8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \div [4+16+0]$$

$$= \begin{bmatrix} 4 & -8 & 0 \\ -8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \div 20$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{20}\right)\begin{bmatrix} 4 & -8 & 0 \\ -8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4/10 & -8/10 & 0 \\ -8/10 & 16/10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6/10 & 8/10 & 0 \\ 8/10 & -6/10 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_1 A = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 4 & 8 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9/5 + 16/5 & 18/5 + 32/5 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Use permutation $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$Q_2 Q_1 A = \boxed{\begin{bmatrix} 5 & 10 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = R}$$

[Continue your answer on the next page ...]

[. . . continue your answer to (a) here.]

$$v_2 = \begin{bmatrix}2\\0\end{bmatrix} \qquad \cancel{\frac{vv^T}{v^Tv}} \quad \cancel{\frac{\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}2&0\end{bmatrix}}{\begin{bmatrix}2&0\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}} = \begin{bmatrix}4&0\\0&0\end{bmatrix}/4} = \begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$\cancel{Q_3 = I - 2\frac{v_2v_2^T}{v_2^Tv_2}}$$

$$\cancel{= \begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}2&0\\0&0\end{bmatrix}}$$

$$\cancel{= \begin{bmatrix}-1&0\\0&1\end{bmatrix}}$$

Rectangular uppertriangular

$$Q_2Q_1A = \begin{bmatrix}5&10\\0&2\\0&0\end{bmatrix}$$

$$\boxed{Q = Q_1^TQ_2^T = \begin{bmatrix}-3/5&4/5&0\\4/5&-3/5&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = \begin{bmatrix}-3/5&0&4/5\\4/5&0&-3/5\\0&1&0\end{bmatrix}}$$

$$Q^T = \begin{bmatrix}-3/5&4/5&0\\0&0&1\\4/5&-3/5&0\end{bmatrix} \qquad QQ^T = \begin{bmatrix}-3/5&0&4/5\\4/5&0&-3/5\\0&1&0\end{bmatrix}\begin{bmatrix}-3/5&4/5&0\\0&0&1\\4/5&-3/5&0\end{bmatrix}$$

$$= \begin{bmatrix}16/25+9/25&0&0\\0&16/25+9/25&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I$$

$\Rightarrow Q$ orthogonal.

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b. Use the QR-factorization computed in (a) to solve the least squares problem $\min_x \|Ax - b\|_2$.

$$\min_x \|Ax-b\|_2$$
$$= \min_x \|QRx-b\|_2$$
$$= \min_x \|Q^TQRx-Q^Tb\|_2$$
$$= \min_x \|Rx-Q^Tb\|_2$$

multiplying by orthogonal matrix doesn't affect the norm

$$Q^Tb = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & -3/5 & 0 \end{bmatrix}\begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 27/5 + 48/5 = 75/5 = 15 \\ 4 \\ \frac{36}{5} - \frac{36}{5} = 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 15 \\ 4 \\ 0 \end{bmatrix}$$

$$2x_2 = 4$$
$$x_2 = 2$$
$$5x_1 + 10x_2 = 15$$
$$5x_1 = -5$$
$$x_1 = -1$$
$$\Rightarrow \boxed{x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

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c. What is the norm of the minimum residual given by your solution in (b), and what does this tell you about the row-rank of the original system?

The norm is $\left\| \begin{bmatrix} 3 & 6 \\ 4 & 8 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix} - \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix} \right\| = 0$

$\Rightarrow$ it ~~is of~~ full rank since the solution is 0.

2

1

Question 4

[10 marks]

The simple interpolation problem involves finding a polynomial $p \in \mathcal{P}_n$ which interpolates the $n+1$ data points $\{(x_i, y_i), i = 0, \ldots, n\}$, x_i unique, y_i not necessarily unique. In C37 we proved such a polynomial exists, and discussed three algorithms for constructing the polynomial. We also introduced the error formula for simple polynomial interpolation, $E(x) = y(x) - p(x)$, and explained how to bound it. This formula, of course, requires that $y_i, i = 0, \ldots, n$ come from an underlying function y. In D37, we derived the error formula and discussed it in detail.

a) $E(x) = y(x) - p(x)$
$= \frac{y^{(n+1)}}{(n+1)!}(\psi) \prod_{i=0}^{n} (x - x_i)$
$\psi \in \text{Span}\{x, x_0, \ldots, x_n\}$

a. Write the error formula for simple polynomial interpolation. Clearly define each of its components.

$$\frac{f^{(n+1)}(\psi)}{(n+1)!} \overbrace{(x - x_0)(x - x_1)\cdots(x - x_n)}^{= W(x)}$$

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x_i are the interpolations points given,
f is the underlying function being interpolated (y)
ψ is some point in the span of $\{x_0, x_1, \ldots, x_n, x\}$

b. Give two examples of a function y, one for which, when the formula in (a) is bounded, the bound approaches (and reaches) 0 as $n \to \infty$, and the other for which the bound $\to \infty$ as $n \to \infty$.

bounded $\to 0$
$y = x$, polynomials are interpolated exactly

$\to \infty$
$y = e^{x^2}$

2

b) $y = x^2$
$y = 1/(1+25x^2)$

[Continue your answer on the next page ...]

[...continue your answer to (b) here.]

c. For your second function in (b), can anything be done to damp the bound (reduce its growth) as $n \to \infty$? Explain.

Can use Chebyshev points to reduce the error caused by $\prod_{i=0}^{n}(x - x_i)$, Since any possible monic polynomial will be less than the polynomial generated with Chebyshev points as proven in lecture.

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Question 5

[10 marks]

Consider Simpson's Rule for estimating the definite integral $I(f) = \int_a^b f(x)\,dx$:

$$S(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_a^b \frac{f^{(3)}(\xi_x)}{3!} \underbrace{(x-a)\left(x-\frac{a+b}{2}\right)(x-b)}_{g(x)}$$

pull out $\left(x-\frac{a+b}{2}\right)$ with f

or split the integral $a \to \frac{a+b}{2}$ $\frac{a+b}{2} \to b$

When deriving the error formula for Simpson's Rule we used a trick to allow us to separate the integrand with MVT. This trick involved introducing an artificial cubic interpolant; the details are rather technical and will not be reproduced here.

During the discussion, at least two other approaches were suggested that would allow us to separate the integrand with MVT without introducing an artificial cubic interpolant. These approaches, interestingly enough, were far less technical.

a. Using either one of the alternate approaches discussed, derive an error bound for Simpson's Rule. Your bound **must**, in some way, separate f from the integrand.

The interpolation points for Simpson's rule are a, b & $\frac{a+b}{2}$ so the ~~[illegible]~~ error formula looks like

Splitting the integral

$$\int_a^b \frac{f^{(3)}(x)}{3!}(x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx$$

$$= \frac{1}{3!}\int_a^b f^{(3)}(x)(x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx$$

$$= \frac{1}{3!}\left[\int_a^{\frac{a+b}{2}} f^{(3)}(x)(x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx + \int_{\frac{a+b}{2}}^b f^{(3)}(x)(x-a)\left(x-\frac{a+b}{2}\right)(x-b)\right]$$

[MVT] since doesn't change sign on $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$

$$= \frac{1}{3!}\left[f^{(3)}(\xi_1)\int_a^{\frac{a+b}{2}}(x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx + f^{(3)}(\xi_2)\int_{\frac{a+b}{2}}^b (x-a)\left(x-\frac{a+b}{2}\right)(x-b)\right]$$

where $\xi_1 \in (a, \frac{a+b}{2})$, $\xi_2 \in (\frac{a+b}{2}, b)$

[Continue your answer on the next page ...]

[. . . continue your answer to (a) here.]

Let ξ = maximum of ξ_1, ξ_2, maximizing $|f^{(3)}(\xi)|$

$$\leq \left| \frac{1}{3!} f^{(3)}(\xi) \int_a^b (x-a)(x-\tfrac{a+b}{2})(x-b)\,dx \right|$$

$=$

Needed to finish the integration

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b. Is your bound derived in (a) preferable to the bound derived in lecture? Explain.

No, the coefficient on the lecture bound ~~was~~ was much lower, but since the derivative is lower, it may handle cases with blowing up derivatives slightly better. ~~[illegible]~~

It does not ~~[illegible]~~ however, only rely on one point in the span, may ~~[illegible]~~ be ~~[illegible]~~ loose due ~~[illegible]~~ to picking a maximum of 2 elements.

1

No, ~~[illegible]~~

END OF EXAM