SOLUTIONS FOR TERM TEST
MARCH 23, 2018
MATB43

(1) Let $\sum_j a_j$ and $\sum_j b_j$ be series of positive terms. Suppose there exists a constant C such that

$$1/C \leq a_j/b_j \leq C \ ,$$

for all j. Prove that then both series converge or both series diverge.

Rewrite the inequalities as

$$b_j \leq C a_j \ , \quad a_j \leq C b_j \ ,$$

for all j.

Applying the comparison test to the first inequality, we get: if $\sum_j a_j$ converges, then $\sum_j b_j$ converges, and if $\sum_j b_j$ diverges, then $\sum_j a_j$ diverges.

Using the second inequality, we see that: if $\sum_j b_j$ converges, then $\sum_j a_j$ converges, and if $\sum_j a_j$ diverges, then $\sum_j b_j$ diverges.

Combining these statements gives us: both series converge or both series diverge.

(2) Let

$$\sinh x = \frac{e^x - e^{-x}}{2} \ , \qquad \cosh x = \frac{e^x + e^{-x}}{2} \ .$$

(a) Give power series expansions for $\sinh x$ and $\cosh x$.

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2} \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\end{aligned}$$

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2} \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\end{aligned}$$

(b) Prove that

$$\cosh^2 x - \sinh^2 x = 1 \ .$$

We have

$$\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x) \ .$$

Now

$$\begin{aligned}\cosh x - \sinh x &= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x} \\ \cosh x + \sinh x &= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^{x}\end{aligned}$$

Therefore

$$\cosh^2 x - \sinh^2 x = e^{-x} \cdot e^x = 1 \ .$$

(3) Give an example of non-empty closed sets $S_1 \subset S_2 \subset \cdots \subset S_j \subset \cdots$ such that $S = \cup_{j=1}^{\infty} S_j$ is open.

Let $S_j = [-1+1/j, 1-1/j]$, $j = 1, 2, \ldots$. For any $a \in (-1, 1)$, there exists $j > 0$ with

$$-1 + 1/j \le a \le 1 - 1/j \ .$$

Therefore $S \supseteq (-1, 1)$. On the other hand,

$$S_j \subset (-1, 1) \ ,$$

for all j. So $S \subseteq (-1,1)$. Thus $S = (-1,1)$, which is open.

(4) Find all accumulation points of the following sets. Explain your answers.

(a) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, 0 is an accumulation point. For any other $a \in \mathbb{R}$, there exists a neighbourhood which does not contain any other point in the set. Therefore 0 is the only accumulation point.

(b) $\{\frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N}\}$.

Since $\lim_{n\to\infty}(\frac{1}{m} + \frac{1}{n}) = \frac{1}{m}$, $\frac{1}{m}$ is an accumulation point for all $m \in \mathbb{N}$. As well, $\lim_{n\to\infty}(\frac{1}{n} + \frac{1}{n}) = \lim_{n\to\infty} \frac{1}{2n} = 0$ is an accumulation point For any other $a \in \mathbb{R}$, there exists a neighbourhood which does not contain any other point in the set. Therefore $\{\,\frac{1}{m} \mid m \in \mathbb{N}\,\} \cup \{0\}$ is the set of accumulation points.

(c) $\{(-1)^n\big(1 + \frac{1}{n}\big) \mid n \in \mathbb{N}\}$.

We have $\{(-1)^n\big(1 + \frac{1}{n}\big) \mid n \in \mathbb{N}\} = \{\big(1 + \frac{1}{2n}\big) \mid n \in \mathbb{N}\} \cup \{\big(-1 - \frac{1}{2n+1}\big) \mid n \in \mathbb{N}\}$. Since $\lim_{n\to\infty}(1 + \frac{1}{2n}) = 1$ and $\lim_{n\to\infty}(-1 - \frac{1}{2n+1}) = -1$, 1 and -1 are accumulation points. For any other $a \in \mathbb{R}$, there exists a neighbourhood which does not contain any other point in the set. Therefore these are the only accumulation points.

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1	25
2	25
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INSTRUCTIONS

Write the information sought in the spaces above.

Write the answers on the RULED SIDE ONLY; all calculations or rough drafts of answers should be shown, preferably on the unruled side.

Clearly identify the question to which each answer applies; whenever the answer to a question is divided, note at the end of the first section "see also work on page ".

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1) $\frac{1}{C} \leq a_j / b_j \leq C$

$\Rightarrow b_j/C \leq a_j \leq b_j C$

$0 \leq \frac{1}{C} \sum_{j=1}^{\infty} b_j \leq \sum_{j=1}^{\infty} a_j \leq C \sum_{j=1}^{\infty} b_j$

So if b_j converges/diverges, we can immediately do a comparison test since if b_j diverges $\Rightarrow \frac{1}{C} \sum b_j$ diverges & since

$\frac{1}{C} \sum_{j=1}^{n} b_j \leq \sum_{j=1}^{n} a_j$ $\frac{1}{C} |b_j| \leq |a_j| \Rightarrow a_j$ diverges as well since partials of b_j diverge.

Similarly if b_j converges, $0 \leq a_j \leq C b_j$ & $C \sum b_j$ converges so CT $\Rightarrow a_j$ converges again.

Since all terms are positive, $\Rightarrow C$ also positive & $1/C$ also positive

$\Rightarrow \left(\frac{1}{C}\right)^{-1} \geq \left(\frac{a_j}{b_j}\right)^{-1} \geq (C)^{-1}$ as x^{-1} decreasing function & continuous on $(0, \infty)$

$\Rightarrow C \geq b_j / a_j \geq 1/C$

so $1/C \leq b_j / a_j \leq C$

therefore, b_j also converges/diverges with a_j, symmetrically to how a_j was proven.

2a) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all $x \in (-\infty, \infty)$ so $\forall x$, $\boxed{\frac{e^x - e^{-x}}{2}} = \frac{\sum_{k=0}^{\infty} \frac{x^k}{k!} - \sum \frac{(-x)^k}{k!}}{2}$

$= \frac{1}{2} \sum_{k=0}^{\infty} \left[\frac{x^k}{k!} - \frac{(-x)^k}{k!}\right] \longrightarrow$ ~~one~~ on even k $= \frac{x^{2k}}{(2k)!} - \frac{a(-x)^{2k}}{(2k)!} = \frac{x^{2k}}{(2k)!} - \frac{x^{2k}}{(2k)!} = 0$

on odd k $= \frac{x^{2k+1}}{(2k)!} - \frac{-(x)^{2k+1}}{(2k)!} = 2\frac{x^{2k+1}}{(2k+1)!}$

$\Rightarrow \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ with the same radius of convergence

While for $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}}{2}$ 15

$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x^k}{k!} + \frac{(-x)^k}{k!}\right) \longrightarrow$ on even k $= \frac{x^{2k}}{(2k)!} + \frac{(-x)^{2k}}{(2k)!}$

$= 2\frac{x^{2k}}{(2k)!}$

$\Rightarrow \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ on odd k $= \frac{x^{2k+1}}{(2k+1)!} + \frac{-x^{2k+1}}{(2k+1)!}$

With the same radius of convergence. $= \emptyset$

b) $\cosh^2 x - \sinh^2 x$

$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + 2e^{x-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^{x-x} + e^{-2x}}{4}$ 10

$= \frac{4e^0}{4} = 1$ //

3) $S_n = [-1+1/n, 1-1/n]$ $\quad -1+1/_{n+1} < -1+1/n$ & $1-1/_{n+1} > 1-1/n \Rightarrow S_n \subset S_{n+1}$

Then $S = \bigcup_{i=1}^{\infty} S_i = (-1, 1)$ is open

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Pf/ $0 \in S_n \ \forall n$

Let $0 < |a| < 1$, then $1 - |a| = \varepsilon > 0$

$\Rightarrow |a| = 1 - \varepsilon$

But $\forall \varepsilon > 0 \ \exists N$ s.t. $n > N \Rightarrow |\frac{1}{n}| < \varepsilon$ (since $\lim_{n\to\infty} \frac{1}{n} = 0$)

$\Rightarrow |a| = 1 - \varepsilon < 1 - \frac{1}{n}$

$\Rightarrow |a| < 1 - 1/n$ so $a \in (-1+1/n, 1-1/n)$ for some $n \in \mathbb{N}$

$\therefore a \in S$

But $n > 0 \Rightarrow -1+1/n > -1 \ \forall n$ & $1-1/n < 1 \ \forall n$

so $S = (-1, 1)$

Now $(-1,1)$ is clearly open since for any $x \in (-1,1)$ $\varepsilon = |x| - 1 > 0$ so $(x-\varepsilon, x+\varepsilon)$ is a neighbourhood within S around x.

4a) $0 \notin \{1/n \mid n \in \mathbb{N}\}$ is the only possible accumulation point. Since the sequence $\{1/n\}$ converges to 0, there are infinitely many points within $(0-\varepsilon, 0+\varepsilon)$ that belong in the sequence. Also know that $\{1/n \mid n \in \mathbb{N}\} \subset (0, 1]$ so no accumulation points outside that range. Given any $1/x$, $x \in \mathbb{R}^+$, there is at most 1 element of the set in $(1/x - \delta, 1/x + \delta)$, as the distance between $1/n$, $1/n+1$ surrounding $x > \delta$.

where $\delta = \frac{\frac{1}{\lfloor x \rfloor + 1} - \frac{1}{\lfloor x \rfloor + 2}}{2}$ $\lfloor x \rfloor$ being the integer part of x.

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b) Every point of the previous set $\{1/n \mid n \in \mathbb{N}\}$ is an accumulation point in this set. Given the justification used earlier, $1/m$ gets arbitrarily close to zero $\Rightarrow$ infinitely many points near $1/n$, formally

Since $\forall \varepsilon > 0, \exists M > 0$ s.t. $m > M \Rightarrow |1/m| < \varepsilon$

$\Rightarrow 1/n < |1/m| + 1/n < \varepsilon + 1/n$ $\quad 1/n > 0$

$\Rightarrow 1/n < |1/m| + |1/n| < \varepsilon + 1/n$ $\quad$ since $1/m + 1/n > 0$

$1/n < |1/m + 1/n| < \varepsilon + 1/n \Rightarrow \frac{1}{m} + \frac{1}{n} \in (1/n - \varepsilon, 1/n + \varepsilon)$

$\exists$ infinitely $m > M$, so for ε precision to $1/n$, $\exists$ infinitely many points.

0 is also an accumulation point for similar reasons to the first, $a/m = n$

$\{\frac{a}{n}\} \to 0$ as $n \to \infty$

c) The accumulation points are $\{1, -1\}$

first examine $\{|1 + 1/n| \mid n \in \mathbb{N}\}$ $\quad \lim_{n \to \infty} \frac{1}{n} \to 0$ so $1 + 1/n \to 1$ as $n \to \infty$.

So again there must be accumulation points at 1 or -1

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if we take the subsets $E = \{(1 + 1/2n) \mid n \in \mathbb{N}\}$ and $\{-1 - 1/(2n+1) \mid n \in \mathbb{N}\} = O$ we can that the even n generate a sequence whose limit is 1, so there is an infinite sequence of points converging to 1 $\Rightarrow$ infinitely many points around 1 exactly the same as done in b). Similarly for O, $-1/(2n+1)$ converges to 0, so the sequence generated by ordering each element by its value of n, converges to -1, giving another accumulation point.

Also know that for any $x \neq 1, -1$, Since $|s| \in S\{(-1)^n (1 + 1/n) \mid n \in \mathbb{N}\}$ within ε of ± 1 for any sufficiently large N, $\Rightarrow$ for sufficiently large n, $\exists$ neighborhood around x such that no elements of $\{(-1)^n (1 + 1/n) \mid n \in \mathbb{N}, n > N\}$ are in it $\Rightarrow$ not accumulation point.