University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB42H – Techniques of the Calculus of Several Variables II

Examiner: E. Moore

Date: February 26, 2018
Start time: 5:00pm
Duration: 110 minutes

FAMILY NAME: POON

GIVEN NAMES: KEEGAN

STUDENT NUMBER: 1002423727

TUTORIAL NUMBER: 5.

DAY AND TIME OF YOUR TUTORIAL: TUESDAY 9AM

CIRCLE THE NAME OF YOUR TUTORIAL LEADER:

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SIGNATURE: Keegan

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- NO AIDS.
- No electronic devices of any kind (e.g. calculators, smart phones, smart watches, tablets, computers, etc.) allowed.
- There are 12 numbered pages in the test. It is your responsibility to ensure that, at the start of the test, this booklet has all its pages.
- Answer all questions. EXPLAIN and JUSTIFY your answers.
- **Show all your work**. Credit will not be given for numerical answers if the work is not shown. If you need more space use the back of the page or the last page and indicate clearly the location of your continuing work.
- **Please note:** If you write in pencil you forfeit the right to any regrading.

question	1	2	3	4	5	6	7	8	9	10	11	total
	11	6	5	4.5	7	6	6	9	10	14.5	9	88
marks	15	6	5	5	10	6	14	9	10	20	10	110

Do not write above this line

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1. **[15 points]** Let $f(x)$ be defined by $f(x) = -x$, $0 \leq x < 2$, and extended from this with period 2 to the rest of $\mathbb{R}$.

8

(a) Find the Fourier series for $f(x)$.

$a_0 = \frac{2}{(2-0)} \int_0^2 f(x)\, dx$

$= -\int_0^2 x\, dx$

$= -\left[\frac{x^2}{2}\right]_0^2 = -4/2 = 2$

$a_k = \frac{2}{2} \int_0^2 f(x) \cos\left(k \frac{2\pi}{2} x\right) dx$

$= -\int_0^2 x\cos(k\pi x)\, dx$

Let $u = x$ $du = dx$, $dv = \cos(k\pi x)dx$ $v = \frac{\sin(k\pi x)}{k\pi}$

$= -\left(\left[\frac{x \sin(k\pi x)}{k\pi}\right]_0^2 - \int_0^2 \frac{\sin(k\pi x)}{k\pi} dx\right)$

$= \frac{1}{k\pi} \int_0^2 \sin(k\pi x) dx$

$= \frac{1}{k^2\pi^2} \left[\cos(k\pi x)\right]_0^2 = 0$

$b_k = \int_0^2 f(x) \sin\left(k \frac{2\pi}{2} x\right) dx$

$= -\int_0^2 x \sin(k\pi x) dx$

Let $u = x$, $du = dx$, $dv = \sin(k\pi x)dx$ $v = \frac{-\cos(k\pi x)}{k\pi}$

$= -\left(\left[\frac{-x\cos(k\pi x)}{k\pi}\right]_0^2 + \int_0^2 \frac{\cos(k\pi x)}{k\pi} dx\right)$

$= \frac{2}{k\pi} - \frac{1}{k\pi}\int_0^2 \cos(k\pi x) dx \quad = \frac{2}{k\pi} - \frac{1}{k\pi} \left[\frac{\sin(k\pi x)}{k\pi}\right]_0^2 \Rightarrow b_k = \frac{2}{k\pi}$

Therefore the Fourier series is

$F(x) = 1 + \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin(kx)$

(b) Find the energy of the first harmonic of the Fourier series in part (a).

0

$E(a_0) = \frac{1}{2} a_0^2 = 4/2 = 2$

A_1^2

Question 1. (cont'd)

(c) Determine whether the Fourier series in part (a) converges. If it does converge, what are the values to which it converges (on [0, 2])?

3

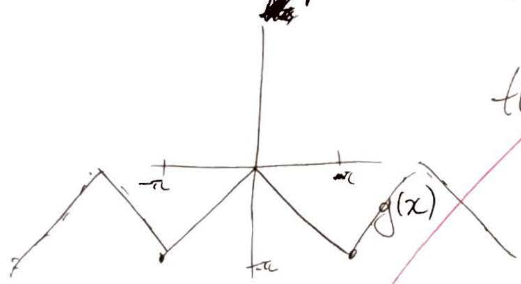
We know that the points within the interval $(0,2)$ are continuous on f (linear) so the Fourier series converges to the function $(-x)$ on the interior, by the fundamental theorem for Fourier series, however on the end points it converges to $\dfrac{\lim_{x\to 0^+} f(x) + \lim_{x\to 2} f(x)}{2} = -1$ on $x=0$ & 2

again by the theorem, so $F(x) \to \begin{cases} -1 & x=0,2 \\ -x & o/w \end{cases}$

2. **[6 points]** Give examples, including sketches, of periodic functions $g(x)$ and $h(x)$ each of which agrees with the function $f(x) = -x$, $0 \leq x \leq \pi$, but such that $g(x)$ has the property that all sine terms in its Fourier expansion are zero and $h(x)$ has the property that all cosine terms in its Fourier expansion are zero.

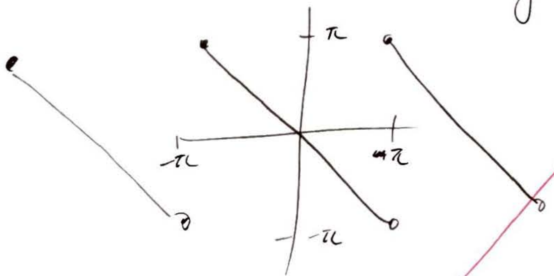
6

$g(x) = -|x|$, $-\pi \leq x \leq \pi$
extended with period 2π across $\mathbb{R}$



this is an even function, so all sin terms must be zero

$h(x) = -x$ $\quad -\pi \leq x \leq \pi$
extended like this again with period 2π across $\mathbb{R}$



this is an odd function, so its cosine terms must be zero.

3. **[5 points]** Find a Fourier series for $f(x) = \left(\cos(x) + \sin(2x)\right)^2$.

$$f(x) = \cos^2(x) + 2\cos(x)\sin(2x) + \sin^2(2x)$$

B3

$$= \cancel{\frac{1+\cos(2x)}{2}} \quad \frac{1+\cos(2x)}{2} + 2\cos(x)\sin(2x) + \frac{1-\cos(4x)}{2}$$

$$= \frac{\cos(2x)}{2} - \frac{\cos(4x)}{2} + 2\cos(x)\sin(2x) + 1$$

Expand...

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin 2x = \frac{e^{2ix}-e^{-2ix}}{2i}$$

$\cos x \sin 2x$

$$= \frac{1}{4i}\left(e^{ix}+e^{-ix}\right)\left(e^{2ix}-e^{-2ix}\right)$$

$$= \frac{1}{4i}\left(e^{3ix}-e^{-ix}+e^{ix}-e^{-3ix}\right)$$

$$= \frac{1}{2}\left(\frac{e^{3ix}-e^{-3ix}}{2i} + \frac{e^{ix}-e^{-ix}}{2i}\right)$$

$$= \frac{1}{2}\left(\sin(3x) + \sin(x)\right)$$

$$= \cancel{\tfrac{1}{2}\left(2\sin(x) + \cos(2x) + 2\sin(3x) + \cos(4x)\right)}$$

OK

$$= 1 + \sin(x) + \left(\tfrac{1}{2}\right)\cos(2x) + \sin(3x) - \left(\tfrac{1}{2}\right)\cos(4x)$$

5

4. **[5 points]** Determine the flow lines of the vector field $\mathbf{F}(x, y) = (y - 1,\ y + 1)$.

4.5

$$dx = y-1$$
$$dy = y+1$$

$$\frac{dy}{dx} = \frac{y+1}{y-1} = \cancel{\frac{y-1+2}{y-1}} = 1 + \frac{2}{y-1}$$

$$\cancel{\left(\frac{1}{1+\frac{2}{y-1}}\right) dy = 1\,dx}$$

$$\frac{y-1}{y+1}\,dy = 1\,dx$$

$$\frac{y+1-2}{y+1}\,dy = 1\,dx$$

$$1 - \frac{2}{y+1}\,dy = 1\,dx$$

$$\int 1 - \frac{2}{y+1}\,dy = \int 1\,dx$$

$$y - 2\log(y+1) = x + C$$

$(y \neq -1)$

The flow lines are of the form $y - 2\log(y+1) - x = C$

5. **[10 points]** Give the definition of each of the following:

(a) A *Fourier polynomial* of degree N.

1

A Fourier ~~of~~ polynomial of degree n, with a period from a to b is defined ~~...~~ $\frac{1}{2}a_0 + \sum_{k=1}^{n} \{a_k \cos(k \frac{2\pi}{b-a} x) + b_k \sin(k \frac{2\pi}{b-a} x)$

for a periodic function f of period a to b

where a_0, a_k, b_k are the fourier coefficients, the definitions written in Q 1.

?

(b) A *(parametrized) path* in $\mathbb{R}^n$.

2

A parameterizal path γ in $\mathbb{R}^n$ is where $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{R}^n$

(c) A *Jordan curve* in $\mathbb{R}^2$.

0

A ~~J~~ Jordan curve is a closed, simple path ~~in~~

i.e. given $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$, $\gamma(a) = \gamma(b)$ & ~~its~~ γ never crosses over itself at any point.

(d) g is a *potential function* for the 1-form $\omega = F_1\, dx_1 + \cdots + F_n\, dx_n$.

2

g is a potential function $\Leftrightarrow$ $dg = \omega$, where $g : \mathbb{R}^n \to \mathbb{R}$ ~~of~~ of class C^2 in $\mathbb{R}^n$

Equivalently, for F, the corresponding function $F = (F_1, F_2, \ldots, F_n)$ to ω, $\nabla g = F$, so F is conservative.

(e) *Winding number*.

2

~~Given a~~ ~~closed~~ piecewise smooth ~~...~~ Jordan curve in $\mathbb{R}^2$, the winding number is defined as the integral $\frac{1}{2\pi}\int_\gamma \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$

or intuitively as the number of times the curve wraps around the origin.

6. **[6 points]** For the path $\gamma(t) = (e^{3t}, \sin(4t), t^{3/2})$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, calculate

(a) the speed $s(0)$

$\gamma'(t) = \left(3e^{3t}, 4\cos(4t), \frac{3}{2}t^{1/2}\right)$

$\gamma'(0) = (3e^{0}, 4\cos(0), 0)$

$= (3, 4, 0)$

$\|\gamma'(0)\| = \sqrt{3^2+4^2} = 5$

(b) the unit tangent vector $\boldsymbol{T}(0)$

tangent vector is $\gamma'(0) = (3,4,0)$

magnitude of said vector is 5

$\Rightarrow$ unit tangent vector $= (3/5, 4/5, 0)$

6

(c) the tangent line at $t = 0$.

where is γ @ 0?

$\gamma(0) = (e^{0}, \sin(0), 0)$

$= (1, 0, 0)$

So the tangent line is $\underline{(1,0,0) + k(3/5, 4/5, 0)}$

$k \in \mathbb{R}$

7. **[14 points]**

(a) Carefully state Green's Theorem.

Suppose γ a smooth Jordan curve in $\mathbb{R}^2$, going in the counterclockwise direction with R being the region enclosed by γ in $\mathbb{R}^2$. Let $\omega = F_1 dx + F_2 dy$ where $F_1, F_2 : [a,b] \subset \mathbb{R} \to \mathbb{R}$ are class C^1 across their domain, then

$$\int_\gamma \omega \, ds = \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

6

(b) State the corollary to Green's Theorem which gives the area of a region as a line integral.

Given γ a smooth (counterclockwise) Jordan curve in $\mathbb{R}^2$, R being the region enclosed by γ, Then the area of R

$$A(R) = \frac{1}{2}\int_\gamma -y \, dx + x \, dy$$

(c) Use a line integral to find the area enclosed by the closed curve $x^2 - 2xy + 3y^2 = 1$.

$$\frac{1}{2}\int_\gamma -y \, dx + x \, dy$$

- parametrize the curve

$x^2 - 2xy + 3y^2 = 1$

8. **[9 points]** Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$ be the path $\gamma(t) = \big(\sin(3t), \cos(3t), 4t\big)$.

(a) Find the arclength of the path γ.

$\int_\gamma 1 ds = \int_0^{2\pi} 1 \, (\|\gamma'(t)\|) \, ds$

$= \int_0^{2\pi} 5 \, ds$

$= 10\pi$

$\gamma'(t) = (3\cos(3t), -3\sin(3t), 4)$

$\|\gamma'(t)\| = \sqrt{(3\cos t)^2 + (3\sin t)^2 + 4^2}$

$= \sqrt{3^2(\cos^2 t + \sin^2 t) + 4^2}$

$= 5$

5

(b) Evaluate $\int_\gamma f \, ds$ where $f(x, y, z) = xy + 3y + z^2$.

$f(\gamma(t)) = f(\sin(3t), \cos(3t), 4t) = \sin(3t)\cos(3t) + 3\cos(3t) + 16t^2$

$= \frac{1}{2}\sin(6t) + 3\cos(3t) + 16t^2$

$\int_\gamma f ds = \int_0^{2\pi} f(\gamma(t)) \|\gamma'(t)\| dt$

$= 5\left[\frac{1}{2}\int_0^{2\pi} \sin 6t \, dt + 3\int_0^{2\pi} \cos(3t) \, dt + 16\int_0^{2\pi} t^2 dt\right]$

$= (5 \cdot 16)\left[\frac{t^3}{3}\right]_0^{2\pi}$

$= \frac{5 \cdot 2^7 \pi^3}{3}$

4

9. **[10 points]** For each of the following differential forms ω determine if ω is exact. If ω is exact, use the algorithm given in class to find the potential function g.

(a) $\omega = (2xyz^2 + 2)\,dx + (x^2z^2 + ze^{yz})\,dy + (2x^2yz + e^{yz})\,dz$.

$\frac{\partial}{\partial y} F_1 = 2xz^2 = \frac{\partial}{\partial x} F_2 = 2xz^2$ $\quad \frac{\partial}{\partial x} F_3 = 4xyz$

$\frac{\partial}{\partial z} F_1 = 4xyz \quad \frac{\partial}{\partial z} F_2 = 2xz^2 + e^{yz} + yze^{yz} \neq \frac{\partial}{\partial y} F_3 = 2x^2z + ze^{yz}$

The domain of ω is simply connected ~~by~~

However, ~~b~~ it is not closed, so it cannot be exact.

2

(b) $\omega = (2x\sin z + ye^x)\,dx + (e^x + z)\,dy + (x^2\cos z + y + 2z)\,dz$.

$\frac{\partial}{\partial y} F_1 = e^x = \frac{\partial F_2}{\partial x} = e^x \quad \frac{\partial F_3}{\partial x} = 2x\cos z$

$\frac{\partial}{\partial z} F_1 = 2x\cos z \quad \frac{\partial}{\partial z} F_2 = 1 = \frac{\partial F_3}{\partial y} = 1$

The domain of ω is $\mathbb{R}^3$ which is simply connected

& ~~as~~ ω is closed, so it is ~~ex~~ exact

$\Rightarrow g = \int F_1\,dx + h(y,z)$

$= \int 2x\sin z + ye^x\,dx + h(y,z)$

$= x^2\sin z + ye^x + h(y,z)$

$\frac{\partial g}{\partial y} = F_2$

$e^x + h'(y,z) = e^x + z$

$\Rightarrow h(y,z) = yz + k(z)$

$\frac{\partial g}{\partial z} = F_3$

$x^2\cos z + y + k'(z) = x^2\cos z + y + 2z$

$\Rightarrow k(z) = z^2$

So $g = x^2\sin z + ye^x + yz + z^2 + C$, $C \in \mathbb{R}$

8

10. **[20 points]** Evaluate the following line integrals. ~~$F_1 dx =$~~ ~~$x^2 \sin(y^2) - xy^2$~~

(a) $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F}(x, y) = (x \sin(y^2) - y^2,\ x^2 y \cos(y^2) + 3x)$, and γ is a counterclockwise parametrization of the boundary curve of the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$ and $(0, 2)$.

F_1 F_2

~~$F_1 dx$~~ $\frac{d}{dy}\ x\sin(y^2) - y^2$

$= 2yx\cos(y^2) - 2y$

$\frac{d}{dx}\ x^2 y\cos(y^2) + 3x$

$= 2xy\cos(y^2) + 3$

$(1,-1) - (0,-2)$

$= (1, 1)$

$\gamma_1'(t) = (1,1)$ $\quad \gamma_1(t) = (t, t-2)$ $\quad 0 \le t \le 1$

$2 = (0,t)$ $\quad \gamma_2(t) = (1, t-1)$ $0 \le t \le 2$ $\quad (1,1) - (1,-1)$

$= (0, 2)$

$3 = (-1, 1)$ $\quad \gamma_3(t) = (1-t, 2+t)$ $\quad 0 \le t \le 1$

$(0,2) - (1,1)$

$4 = (0,-1)$ $\quad \gamma_4(t) = (0, 2-t)$ $0 \le t \le 4$ $= (-1, 1)$

$(0,-2) - (0,2)$

$= (0, -4)$

2

$$\int_\gamma F ds = \int_0^1 \left(t\sin((t-2)^2) - (t-2)^2\right) + \left(t^2(t-2)\cos((t-2)^2) + 3(t)\right) dt$$

$$+ \int_0^2 F(\gamma_2) \cdot \gamma_2'(t)\, dt + \int_0^1 F(\gamma_3) \cdot \gamma_3'(t)\, dt + \int_0^4 F(\gamma_4) \cdot \gamma_4'(t)\, dt$$

$= \omega\ ds$

(b) $\int_\gamma \underbrace{(y-2z)}_{F_1} dx + \underbrace{(x+3z)}_{F_2} dy + \underbrace{(-2x+3y+2z)}_{F_3} dz$, where $\gamma(t) = (2\cos 2t, \sin^3 t, \cos^2 t)$, $0 \le t \le \frac{\pi}{2}$.

$\frac{d}{dy} F_1 = 1$ $\quad \frac{d}{dx} F_2 = 1$ $\quad \frac{d}{dx} F_3 = -2$

$\frac{d}{dz} F_1 = -2$ $\quad \frac{d}{dz} F_2 = 3$ $\quad \frac{d}{dy} F_3 = 3$ $\quad \gamma(0) = (2, 0, 1)$

ω is exact with $g =$ $xy - 2xz + 3yz + z^2$

$\gamma(\frac{\pi}{2}) = (2\cos\pi, \sin^3\frac{\pi}{2}, \cos^2\frac{\pi}{2})$

$= (-2, 1, 0)$

By Generalized FTOC, $\int_\gamma \omega\, ds = g(\gamma(\frac{\pi}{2})) - g(\gamma(0))$

$= g(-2, 1, 0) - g(2, 0, 1)$

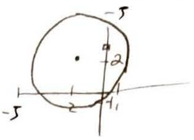
~~$= g(-4, 1, -1)$~~

3.5

Question 10. (cont'd)

(c) $\int_\gamma \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$, where $\gamma(t)$ is a parametrization of the circle $(x+2)^2+(y-2)^2 = 9$ oriented in the clockwise direction.

$(0+2)^2+(0-2)^2 = 4+4 = 8 < 9$



γ is a ~~...~~ piecewise smooth Jordan curve & ω is of the form of the winding number integral, so ~~...~~ the result is simply $-$(times around origin)$\cdot 2\pi$ since γ is clockwise

$$\Rightarrow \int_\gamma \omega\, ds = -1(2\pi) = -2\pi$$

5

(d) $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x,y,z) = (\sin x,\ \cos y,\ xz)$ and $\gamma(t) = (t^3, -t^2, t)$, $0 \le t \le 1$.

$F(\gamma(t)) = F(t^3, -t^2, t)$

$= (\sin t^3, \cos -t^2, t^4)$

$= (\sin t^3, \cos t^2, t^4)$

$\gamma'(t) = (3t^2, -2t, 1)$

$\int_\gamma F \cdot ds = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt$

$= \int_0^1 3t^2 \sin t^3 + \cos t^2 (-2t) + t^4\, dt$

$= \left[-\cos(t^3) - \sin(t^2) + \frac{t^5}{5}\right]_0^1$

$= -\cos(1) + \cos(0) - \sin(1) + \sin(0) + (1/5)$

$= 4/5 - (\sin(1) + \cos(1))$

$\frac{6}{5}$

4/5

11. [10 points]

(a) Give a parametrization of the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y - 1$.

Parametrize the cylinder as $(x,y) = (\cos\theta, \sin\theta)$ (typical ... formula)

then set $z = \cos\theta - 1$ subbing in y

So the ~~cylinder~~ curve γ is $\theta \mapsto (\cos\theta, \sin\theta, \cos\theta - 1)$

range ?

4

(b) Give a parametrization of the piece of the plane $z = 2x + y + 3$ which lies over the unit disk $x^2 + y^2 \leq 1$.

Parametrize the surface of the plane,

To describe the disk, use $\Phi(r,\theta)$, so $(x,y) = (r\cos\theta, r\sin\theta)$

describing the disk. $0 \leq r \leq 1$ $0 \leq \theta \leq 2\pi$

To parametrize the surface, take the x, y & plug into plane equation

giving $z = 2r\cos\theta + r\sin\theta + 3$

$\Rightarrow$ ~~$\Phi(x,y)$~~ $\underline{\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 2r\cos\theta + r\sin\theta + 3)}$

5