Department of Computer & Mathematical Sciences

University of Toronto Scarborough

Term Test 2

MATC15H – Introduction to Number Theory

Examiner: J. Friedlander

Date: Feb. 24, 2018
Time: 1:00pm-2:30pm

FAMILY NAME: POON

GIVEN NAME(S): KEEGAN

STUDENT NUMBER: 1002423727

SIGNATURE:

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO

NOTES:

- There are 6 numbered pages in the test. It is your responsibility to ensure that, at the start of the exam and at the end of the exam, this booklet has all its pages.
- No calculators or other aids.
- Justify your answers.

FOR MARKERS ONLY	
Question	Marks
1	13 /13
2	9 /9
3	9 /18
TOTAL	31 **/40**

1. **[13 marks]** Give Euler's proof that there are infinitely many primes and state (but do not prove) the stronger version of his result.

Suppose there are finitely many primes $p_1, p_2, p_3, \dots, p_n$

then, $\left(\prod_{i=1}^{n} p_i\right) - 1$ cannot be a ~~mult~~ multiple of any of the given primes, since it is one difference from a multiple of each.

However, by the Fundemental Theorem of arithmetic, it can be decomposed into the product of primes. It is not the empty product since $\neq 1$, so it must factor into the given primes which do not divide it. Contradiction.

Suppose there are finitely many primes, then $\left(1+\frac{1}{p_1}+\frac{1}{p_1^2}+\dots+\frac{1}{p_1^r}\right)\left(1+\frac{1}{p_2}+\frac{1}{p_2^2}+\frac{1}{p_2^3}+\dots+\frac{1}{p_2^r}\right)\cdots\left(1+\frac{1}{p_n}+\frac{1}{p_n^2}+\dots+\frac{1}{p_n^r}\right)$ would generate $\frac{1}{n}$, $n \in \mathbb{N}$ for any n given large enough r (by FToA), but each is a finite geometric series $\sum_{i=0}^{n} \frac{1}{p^i}$, so for finite p, the infinite sum converges, so then $\sum_{i=1}^{\infty} \frac{1}{n} < \infty$, but this is the harmonic series, diverges Contradiction.

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Stronger result is that $\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges.

$$\frac{(p^{e+1}-1)(p+1)}{(p-1)(p+1)} = \frac{p^{e+2}+p^{e+1}-p-1}{p^2-1}$$

2. **[9 marks]** Show that every prime power $n = p^e$ is deficient, that is, satisfies $\sigma(n) < 2n$, and hence is not a perfect number.

$$\sigma(p^e) = \sum_{d \mid p^e} d = 1 + p + p^2 + \cdots + p^e$$

$$= \frac{p^{e+1}-1}{p-1}$$

$$\cancel{= \frac{p^{e+1}}{p-1}}$$

$$= \frac{p^{e+1}+p^{e+1}-p^{e+1}-1}{p-1}$$

$$= \frac{2p^{e+1}-p^{e+1}-1}{p-1}$$

$$= \frac{2p^{e+1}-(p^{e+1}+1)}{p-1}$$ since $p \geq 2$, $p^{e+1}+1 \geq 2p^e+1 \geq 2p^e$

$$< \frac{2p^{e+1}-2p^e}{p-1}$$

$$= \frac{2p^e(p-1)}{p-1}$$

$$= 2p^e \;//$$

$$2p^e = p^e + p^e = \frac{2p^e(p-1)}{(p-1)} = \frac{2p^{e+1}-2p^e}{p-1} = \frac{p^e(2p-2)}{p-1}$$

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Use but do **NOT** tear out.

3. **[18 marks]** Do each of the following, using the previous blank page if necessary.
Show, for which positive integers n,

a) $\tau(n)$ is an odd integer,

b) $\sigma(n)$ is an odd integer,

c) $\varphi(n)$ is an odd integer.

a) $\tau(n)$ odd iff n is the square of an integer ✓

This is because for all $d \mid n$, $\frac{n}{d} = k$ & $k \mid n$ so every divisor ~~has~~ gives another number that is also a divisor, if n not a square, d, k must be distinct, hence there are $2(j)$ divisors for nonsquares. But for squares $\exists\, r$, $r^2 = n$ so when summing divisors, r is counted once with no other secondary divisor, giving $\tau = 2j+1$.

b) $\sigma(n)$ odd for powers of 2 since $\frac{p^{k+1}-1}{p-1=1}$ = odd

OR?

as well as for ~~odd~~ even powers of an odd prime, extending multiplicatively to any number that is a product ~~s~~ of these.

This is since $1 + p + \cdots p^k \equiv k+1 \bmod (2)$ since p odd
$1 + 1 + \cdots + 1$

so for k even, $\sigma(p^k) \equiv k+1 \bmod (2)$
$\equiv 1 \Rightarrow$ odd

The only ~~other~~ other prime case is p^k p odd, k even, but evidently it must be $\equiv 0 \pmod 2$ by the previous justification so the only numbers s.t. $\sigma(n)$ odd are a product of the numbers given above.

c) $\varphi(n)$ odd iff $n \leq 2$, o/w given $n \neq 2$ & ~~and~~ $p^k \| n$ where p^k is the largest prime power

$\varphi(2) = 1 = 2^0(2-1)$ is odd.

$\varphi(n) = p^{k-1}(p-1) \cdot$ rest of product

So?

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Since ~~n≠2~~ $n \neq 2$ either $p = 2$ & $k > 1 \Rightarrow p^{k-1}$ even so product even

or p odd $\Rightarrow p-1$ even so product even.

or $n=1 \Rightarrow \varphi(n) = 0$ is even