On photon splitting in LV QED

Konstantin Astapov[1], Dmitry Kirpichnikov[1], Petr Satunin[1*]

[1] *Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect, 7a, 117312 Moscow, Russia*

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Abstract

We calculate the width of the process of photon splitting to three photons in a simple model of quantum electrodynamics with broken Lorentz invariance. We show that this process may lead to a cut-off in very-high-energy part of astrophysical sources photon spectra. We obtain 95% CL bound on Lorentz violating mass scale for photons from the analysis of very-high-energy part Crab Nebula spectrum, obtained by HEGRA.

1 Introduction

Lorentz invariance (LI) is the one of fundamental laws of Nature. However, in several theoretical models it can be broken. These models are mostly motivated by different approaches to construct quantum theory of gravity (see review [1] and references therein). The common approach is to consider Lorentz invariance violation (LV) in matter sector is the framework of effective field theory (EFT) [**?**].

One of the differences between models with LV and LI ones is modification of particle reactions. In presence of hypothetical LV some processes forbidden in LI models may be allowed. The well-known examples are photon decay to an electron-positron pair $\gamma \to e^+e^-$ and vacuum Cerenkov radiation $e^- \to e^-\gamma$ [2]. Cross-sections of other processes may be modified as well. The absence of experimental observations of such effects constrain LV. Due to these processes a hypothesys of a certain type of LV may be experimentally tested. Experimental bounds are gathered in data tables [**?**].

2 The model.

The model is the standard QED with an addition of extra LV term, quartic on space derivatives, for a photon, quartic on space derivative for photon and suppressed by the second

***e-mail**: satunin@ms2.inr.ac.ru

power of LV mass scale M_{LV}.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \mp \frac{1}{2M_{LV}^2}F_{ij}\Delta^2 F^{ij} + i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi. \tag{1}$$

One may also consider similar LV terms in fermion Lagrangian. However, we will skip them for the following reason. The study [5] set the bound on LV mass scale for electron $M_{LV,e}$ at the level of 10^{16} GeV while the current limits [7] on this value are of the order of 10^{11} GeV. Thus, the photon dispersion relation is

$$E^2 = p^2 \pm \frac{p^4}{M_{LV}^2}. \tag{2}$$

In the paper we restrict ourselves only on superluminal photons — sign "+" in (2). Note that we keep CPT unbroken, so the dispersion relation (2) is the same for both photon polarizations.

Modified Feynman rules for the model (1) (and more general) are gathered in [6]. In particular, polarization sums for photons are

$$\varepsilon_\mu^*(k)\varepsilon_\nu(k) = -g_{\mu\nu} - \frac{k_0^2}{M_{LV}^2}u_\mu u_\nu, \tag{3}$$

where $u_\mu = (1, 0, 0, 0)$ is timelike 4-vector.

3 The bound from the absence of photon decay

In the presence of LV kinematics of some reactions modifies. The process of the photon decay $\gamma \to e^+e^-$, forbidden in the presence of LI, become allowed if the photon energy exceed a certain threshold. It can be simply illustrated in the language of "effective masses", defined as

$$m_{\gamma,eff}^2 \equiv E^2 - p^2 = \frac{p^4}{M_{LV}^2}\ . \tag{4}$$

In this notations photon decays if its effective mass exceed the double electron mass, $m_{\gamma,eff} \geq 2m_e$. The products of the reaction, electron and positron, bring approximately a half of the initial photon momentum. Once being allowed, the photon decay process is very rapid[1]. This process leads to an extremely sharp cutoff in photon spectrum from any astrophysical source. Thus, an observation of even single photon event of astrophysical origin with an energy E_γ gives the bound,

$$M_{LV} > \frac{E_\gamma^2}{2m_e}\ . \tag{5}$$

The recent analysis [8] using the highest-energy photons observed from the Crab nebula sets the constraint,

$$M_{LV,\gamma} > 2.8 \times 10^{12}\,\text{GeV}. \tag{6}$$

However, it is not only reaction which decrease VHE photon flux.

[1] When $m_{\gamma,eff} \gg 2m_e$ the decay width is given by $\Gamma_{\gamma\to e^+e^-} = (\alpha p_\gamma^3)/(3M_{LV,\gamma}^2)$, where α is the fine structure constant [6].

4 The photon splitting.

In the case $m_{\gamma,\text{eff}} < 2m_e$, the photon decay $\gamma \to e^+e^-$ is kinematically forbidden. However, a process of photon decay with several photons in the final state $\gamma \to \text{n}\gamma$, so-called photon splitting, is kinematically allowed whenever the photon dispersion relation is superluminal. In the context of LV, photon splitting was considered for QED with additional Chern-Simons term [9], and for photon with cubic dispersion relation [10]. A study for the case of quartic dispersion relation for a photon seems to be missed in the literature.

The leading order splitting process, photon splitting to two photons $\gamma \to 2\gamma$ do not occur if CPT is unbroken: fermion loop with odd number of propagators identically vanishes due to the Furry theorem (which violates if electrons are CPT-violated), see a discussion in [1].

Thus, the main splitting process is the photon decay to 3 photons $\gamma \to 3\gamma$. In order to estimate the width of the reaction, we follow the lines of [10], there the similar estimates have been done for the case of cubic photon dispersion relation.

Let us first follow the lines of [10] and obtain the similar estimasions for quartic dispersion relation (2) as was made for cubic dispersion relation in [10]. The authors made the following trick: they consider the effective mass for incoming photon via (4), make a boost into artificial "rest frame" of massive photon (there outcoming photons are considered to be massless), make estimations into the "rest frame", and make a boost back to laboratory frame.

For calculations of the matrix element the authors used Euler-Heisenberg Lagrangian

$$\mathcal{L}_{E\text{-}H} = \frac{2\alpha^2}{45m_e^4}\left[\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)^2 + 7\left(\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)^2\right] . \tag{7}$$

for calculation, which is applicable in the limit $m_{\gamma,eff} \ll m_e$ – far from the threshold of photon decay. The calculation [10] reads,

$$\Gamma\left(\gamma \to 3\gamma\right) \simeq \left(\frac{2\alpha^2}{45}\right)^2 \frac{1}{3!2^{11}\pi^9}\,\frac{m_{\gamma,eff}^{10}}{m_e^8\,E_\gamma} \times f \;\simeq\; 1.5 \times 10^{-20}\,\frac{E_\gamma^{19}}{m_e^8 M_{LV}^{10}} \times f. \tag{8}$$

Here f denotes dimensionless phase volume in the "rest frame" of the initial photon.

$$f = \frac{4\pi^4}{m_\gamma^9}\int \frac{d^3k_1\,d^3k_2\,d^3k_3}{E_1E_2E_3}\delta^4(p_\gamma - k_1 - k_2 - k_3)\,|\mathcal{M}|^2\,, \tag{9}$$

and $\mathcal{M}$ is the invariant matrix element obtained from Eq.(7) by omitting the global factor $2\alpha^2/(45m_e^4)$. By physical considerations, it should be of the order of unity. In the work [9] for different model direct calculations leads $f = 0.2$. Inverting the width (8), one obtains an estimation the mean free path for the photon.

$$L\left(\gamma \to 3\gamma\right) \simeq 0.5 \cdot f^{-1} \cdot \left(\frac{M_{LV}}{10^{13}\,\text{GeV}}\right)^{10}\left(\frac{E_\gamma}{40\,\text{TeV}}\right)^{-19}\ \text{kpc}. \tag{10}$$

We have normalized the photon energy to typical value of highest-energy detected photons – 40 TeV, and M_{LV} to the scale 10^{13} GeV. We see that for M_{LV} of this order high-energy photons even from galactic sources (distances of the order of several *kpc*) may be splitted during their propagation to Earth, so observations of 40 TeV and more energetic photons may set a bound on M_{LV} of the order of 10^{13} GeV. This value is several times larger than the bound from the absense of the photon decay (6) and so the splitting process is very relevant for setting actual constraint. However, estimation was done in [10] is very rude and should be verified by honest calculation in the laboratory frame.

5 Decay width

In comparison with culculations in Lorentz Invariant case, there are two differences. First, phase volume of the process becomes nonzero because of Lorentz non-invariant modifications of photon despersion relation. Second, the matrix element on the process has additionsl terms due to modified polarization sums.

5.1 Matrix element

To find the matrix element we apply the following technique. The Feynman rules for the relevant vertices are extracted from (7) in a way, the lowest order matrix element for a four-photon process can then be written as

$$M = M_{\mu\nu\rho\lambda}(k_1, k_2, k_3, k_4)\varepsilon_\mu(k_1)\varepsilon_\nu(k_2)\varepsilon_\rho(k_3)\varepsilon_\lambda(k_4) \tag{11}$$

where k_i are the photon four-momenta, and $\varepsilon(k)$ are the corresponding polarization vectors. For convenience of further calculations let us introduce tensors

$$T_{\mu\nu}(k, p) = 2(pk)g_{\mu\nu} - 2p_\mu k_\nu \tag{12}$$

and

$$T^{dual}_{\mu\nu}(k, p) = -4k^\rho p^\lambda \varepsilon^{\mu\nu\rho\lambda} \tag{13}$$

Then performing summation over all permutations of on-shell photons obtain following

$$M^1_{\lambda_1\lambda_2\lambda_3\lambda_4} = 8(T_{\lambda_1\lambda_2}(k_1, k_2)T_{\lambda_3\lambda_4}(k_3, k_4)+T_{\lambda_1\lambda_3}(k_1, k_3)T_{\lambda_2\lambda_4}(k_2, k_4)+T_{\lambda_1\lambda_4}(k_1, k_4)T_{\lambda_3\lambda_2}(k_3, k_2)) \tag{14}$$

and

$$M^{dual}_{\lambda_1\lambda_2\lambda_3\lambda_4} = 8(T^{dual}_{\lambda_1\lambda_2}(k_1, k_2)T^{dual}_{\lambda_3\lambda_4}(k_3, k_4)+T^{dual}_{\lambda_1\lambda_3}(k_1, k_3)T^{dual}_{\lambda_2\lambda_4}(k_2, k_4)+T^{dual}_{\lambda_1\lambda_4}(k_1, k_4)T^{dual}_{\lambda_3\lambda_2}(k_3, k_2)). \tag{15}$$

Unify both parts

$$M_{\lambda_1\lambda_2\lambda_3\lambda_4} = M^1_{\lambda_1\lambda_2\lambda_3\lambda_4} + \frac{7}{16}M^{dual}_{\lambda_1\lambda_2\lambda_3\lambda_4} \tag{16}$$

Finally for squared matrix element

$$M^*M = M^*_{\mu_1\nu_1\rho_1\lambda_1}M_{\mu\nu\rho\lambda}\ \varepsilon_{\mu 1}(k_1)\varepsilon_\mu(k_1)\ \varepsilon_{\nu 1}(k_2)\varepsilon_\nu(k_2)\ \varepsilon_{\rho 1}(k_3)\varepsilon_\rho(k_3)\ \varepsilon_{\lambda 1}(k_4)\varepsilon_\lambda(k_4), \tag{17}$$

where polarization sums $\varepsilon_\nu(k)\varepsilon_\mu(k)$ are given by eq.(3). The amplitude for for photon scattering $\gamma\gamma \to \gamma\gamma$ in LI theory agrees with the result in Schwartz book (p.717-718) (in Mandelstam variables)

5.2 Phase volume

If one outgoing photon carries away almost all of the initial photon energy, the emission angles tend to zero, as well as the phase volume. To obtain precise value for splitting width we carried out straightforward calculations in the laboratory frame.We consider the photon with 4-momentum k which splits to 3 photons with 4-momenta k_1, k_2, k_3. We divide 3-momenta of final photons to parallel $k_i^{\|}$ and perpendicular $k_i^{\perp}$ part (compared to initial photon 3-momenta). 3-momentum conservation gives: $\vec{k}_1^{\perp}+\vec{k}_2^{\perp}+\vec{k}_3^{\perp}=0$ and $k_1^{\|}+k_2^{\|}+k_3^{\|}=k$. Parallel momenta $k_i^{\|}$ may be reduced to dimensionless variables α_i:

$$k_i^{\|} = k\,\alpha_i, \qquad \alpha_1+\alpha_2+\alpha_3 = 1. \tag{18}$$

Perpendicular vectors are 2-vectors in a plane, perpendicular to the initial photon momentum $\vec{k}$. Each of them may be parametrized by module $k_i^{\perp}$ and polar angle φ_i; one of these angles (say, φ_3) is solved due to delta-function; another one decouples from the whole phase volume due to rotational symmetry; the only variable is φ_2. By simple geometric means, $k_3^{\perp}$ is written as follows:

$$(k_3^{\perp})^2 = (k_1^{\perp})^2 + (k_2^{\perp})^2 + 2k_1^{\perp}k_2^{\perp}\cos\varphi_2. \tag{19}$$

Let us describe the phase volume in the general way:

$$d\Phi = (2\pi)^4\delta(E-E_1-E_2-E_3)\delta^{(3)}\left(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3\right)\frac{1}{2E}\prod_{i=1,2,3}\frac{d\vec{k}_i}{(2\pi)^3(2E_i)}, \tag{20}$$

and consider aforementioned kinematic configuration. Keeping the highest order of magnitude of LV terms we neglect its contribution to denominator $E\,E_1\,E_2\,E_3 \approx |\vec{k}|\,|\vec{k}_1|\,|\vec{k}_2|\,|\vec{k}_3|$. We integrate out 3-momentum $\vec{k}_3$ eliminating one of delta-functions so the phase volume (20) goes to

$$d\Phi = \frac{1}{2^8\pi^5}\frac{1}{k}\frac{d\alpha_1\,d\alpha_2\,d^2k_1^{\perp}d^2k_2^{\perp}}{\alpha_1\alpha_2(1-\alpha_1-\alpha_2)}\delta(E-E_1-E_2-E_3). \tag{21}$$

In order to go further we make some simplifications. We say that $(k_i^{\perp}/k)^2$ and k^2/M^2 are of the same order of smallness, and introduce dimensionless variables β_1, β_2:

$$k_1^{\perp} = \frac{k^2}{M}\cdot\beta_1, \qquad k_2^{\perp} = \frac{k^2}{M}\cdot\beta_2. \tag{22}$$

Thus,

$$E_i = k\left(\alpha_i + \frac{k^2}{2M^2}\alpha_i^3 + \frac{k^2}{2M^2}\frac{\beta_i^2}{\alpha_i}\right), i=1,2 \tag{23}$$

$$E_3 = k\,(1-\alpha_1-\alpha_2) + \frac{1}{2k}\frac{\left(k_3^{\perp}\right)^2}{1-\alpha_1-\alpha_2} + \frac{k^3(1-\alpha_1-\alpha_2)^3}{2\cdot M^2}, \tag{24}$$

These expressions may be substituted to the delta-function. The inner part of delta-function $(E_1 + E_2 + E_3 - E)$ become

$$\frac{k^3}{2M^2}\left(3(\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2+\alpha_1\alpha_2)+\frac{1}{1-\alpha_1-\alpha_2}\left(\frac{1-\alpha_2}{\alpha_1}\beta_1^2+\frac{1-\alpha_1}{\alpha_2}\beta_2^2+2\beta_1\beta_2\cos\varphi_2\right)\right) \tag{25}$$

The phase volume (21) becomes

$$d\Phi = \frac{1}{2^8\pi^5}\frac{2\pi}{k}\left(\frac{k^2}{M}\right)^4\frac{d\alpha_1\, d\alpha_2\, d\varphi_2 d(\beta_1^2)d(\beta_2^2)}{\alpha_1\alpha_2(1-\alpha_1-\alpha_2)}\delta\left[...\right], \tag{26}$$

here additional 2π factor is the result of integration over φ_1. Integration over φ_2 solves delta-function. We obtain $\frac{k^3}{2M^2}\frac{2\beta_1\beta_2}{1-\alpha_1\alpha_2}\sin\varphi_2$ in the denominator, φ_2 is expressed as

$$\cos\varphi_2 = \frac{3(\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2)(1-\alpha_1-\alpha_2+\alpha_1\alpha_2)-\frac{1-\alpha_2}{\alpha_1}\beta_1^2-\frac{1-\alpha_1}{\alpha_2}\beta_2^2}{2\beta_1\beta_2} \tag{27}$$

Phase volume:

$$d\Phi = \frac{1}{2^8\pi^5}2\pi\frac{k^4}{M^2}\frac{d\alpha_1\, d\alpha_2\, d\beta_1 d\beta_2}{\alpha_1\alpha_2}\frac{1}{\sin\varphi_2|_{\varphi_2=\varphi_2(\alpha_1,\alpha_2,\beta_1,\beta_2)}}. \tag{28}$$

The area of integration over $(\beta_1,\ \beta_2)$ is determined by the condition:

$$-1 < \cos\varphi_2 < 1.$$

It is an area between two ellipses (see Fig.1) depending on α_1, α_2.

Integration in (28) may be done first over $(\beta_1,\ \beta_2)$ and after over α_1, α_2. The area of integration over α_1, α_2 is a triangle: $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$.

First, we integrate over $(\beta_1,\ \beta_2)$ for different values of fixed α_1, α_2 (see file calcs3.nb). Phase volume is maximal at the symmetric configuration $\alpha_1 = \alpha_2 = 0.33$.

The next step is inegration over longitudial momenta. Integration over α_2 with fixed α_1 (or vice versa, it must be the same) gives us energy losses. Integral over both α_1, α_2 gives the splitting width.

$$\Gamma = 4\times 10^4 (\frac{2\alpha^2}{45})^2\frac{k^{20}}{2^7\pi^4 m_e^8 M^{10}} \tag{29}$$

6 Scalar Products

$$(k\, k_i) = \frac{k^4}{2M^2}\left(\alpha_i + \alpha_i^3 + \beta_i^2/\alpha_i\right),\ i = 1, 2 \tag{30}$$

$$(k_1\, k_2) = \frac{k^4}{2M^2}\left(\alpha_1\alpha_2(\alpha_1^2+\alpha_2^2) - 3(\alpha_1+\alpha_2)(1-\alpha_1-\alpha_2)(1-\alpha_1-\alpha_2+\alpha_1\alpha_2) + \frac{\beta_1^2}{\alpha_1} + \frac{\beta_2^2}{\alpha_2}\right) \tag{31}$$

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Figure 1: The area of integration over (β_1, β_2) is between two ellipses for the symmetric case $\alpha_1 = \alpha_2 = 0.33$

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Figure 2: The phase volume integrated over perpendicular momenta $(\beta_1,\ \beta_2)$, as the function of longitudial momenta (α_1, α_2). The peak is on the symmetric configuration $\alpha_1 = \alpha_2 = 0.33$

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