# Advanced Agent-Based Modeling HW 2

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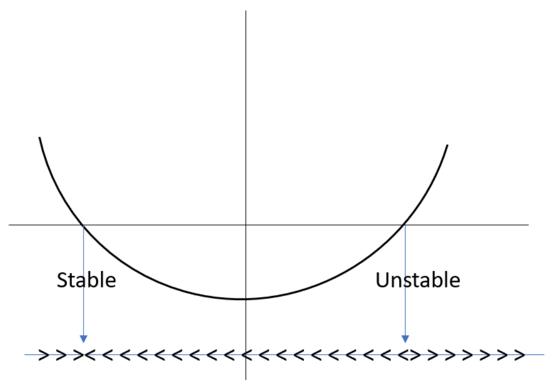
**Problem 1:** In class, we studied the dynamical system below. It is "well-behaved" because no equilibrium is semi-stable, and neither is an endpoint. Can you make an argument for the following claim: For any well-behaved G, Successive equilibria must alternate in stability.

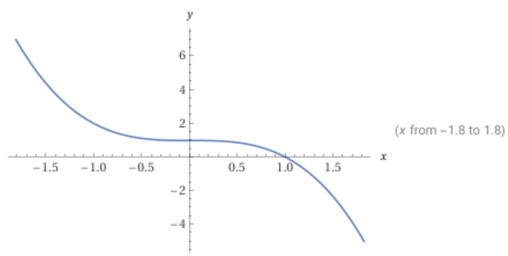



**Answer 1:**

- We can begin by recognizing the equilibria occurs when $G(x) = 0$ and that the stability of a point is dependent upon the eigenvalues at that critical point $x^*$ such that:
  1. $G(x^*) < 0 \implies$ Stable (decay)
  2. $G(x^*) > 0 \implies$ Unstable (growth)
  3. $G(x^*) = 0 \implies$ Semistable (saddle point)
     (a) When the critical point has an eigenvalue with a complex component, the absolute value of the coefficient of the imaginary component determines the radius of the oscillation.
- Then, we can consider successive equilibria by recognizing that the first-derivative of the growth function $\frac{dG(x)}{dx}$ which determines the concavity of the growth function, whose sign determines whether the parabola is open upwards or downwards. Furthermore, since the first derivative is

continuous over a closed and bounded interval, continuously differentiable on the open cover corresponding to the interval, and the function is equal at the endpoints of the interval, by Rolle's theorem, there must exist a critical point of the first derivative. Thus, the concavity of the function must change between successive equilibria and thus must alternate in stability (assuming $G(x)$ to be well-defined).

**Problem 2:** G(x) = 1 − x^3. Identify and classify all equilibria. Verify algebraically that x* = 1.




**Answer 2:**

- To find the equilibria of the growth function $G(x)$, we must find the points $x^*$, such that $G(x^*) = 0$. We can find these critical points by setting $G(x) = 0$, and to verify the given critical point, we can make the substitution $x = x^* = 1$ and see if $G(x) = 0$, such that:

$$
\begin{aligned}
G(x^*) := 1 - (x^*)^3 = 0 & \qquad (G(x) \text{ Definition}) \\
= 1 - (1)^3 & \qquad (\text{Substitution}) \\
= 1 - 1 & \qquad (\text{Simplify}) \\
= 0 & \qquad (\text{Simplify}) \\
\implies x^* = 1 \text{ is a valid critical point.} &
\end{aligned}
$$

- Since the growth function is a cubic, there are three roots, but since $x$ is cubed, there must be a multiplicity of three associated with the equilibria at $x^* = 1$. Furthermore, since the plot shows the growth function as decreasing as it crosses the axis, we can assert that the equilibria is stable.

**Problem 3:** Consider the dynamical system with growth rate $G(x) = x^2 + b$. If we start with b > 0 what is the equilibrium structure? Now if we reduce b, something dramatic happens when it crosses the x axis (going negative.) What happens?

**Answer 3:**

- Since the equilibria is dependent upon where the growth function $G(x) = 0$, we can recognize the fact that $G(x)$ is a quadratic polynomial that opens upwards and that the value of $b$ determines the y-intercept. Since $b > 0$, the parabola will never pass through the $x$-axis, as the quadratic is strictly non-negative, which means we have a strictly positive value plus a non-negative value, which will never go to zero. If we change $b < 0$, then we have a strictly negative value plus a strictly non-negative value, which must eventually cross the $x$-axis.

**Problem 4:** What is meant by the term “Agentization?”

**Answer 4:**

- "Agentization" is the process of taking a dynamical system, which relies on the mean-field hypothesis, and converting it to an agent-based model. To do this, instead of treating individuals in the dynamical system as identical, we initialize agents in a space-time domain to enable more realistic agent-interactions, where the agents may potentially have heterogenous initial traits. To address the transition rates between compartments, we randomly select float value less than one and compare it to the transition rate to determine the agent's behavior.

**Problem 5:** Open the 2D Congestion Code. Run it and estimate the equilibrium from the plot. In line 10, reduce the city-radius to 1.5. Estimate this equilibrium from the Plot. Explain clearly why these differ.

**Answer 5:**

- The equilibria when $r = 3.5$ occurs at when $G(t) = 0$. When $t = 0$, the simulation is just starting and the stability at that point is unstable, as we traverse along the flow going towards the next equilibria. The next equilibria is reached when the city reaches maximum capacity and cannot fit anymore agents and represents a stable point, as $G(x)$ will continue to be zero. When $r = 1.5$, $G(x)$ still maintains the same equilibria behavior, but reaches it's stable equilibria sooner. In both cases, the second equilibria for $t > 0$ represents the city reaching maximum capacity.

**Problem 6:** Open the ZOMBIE Model.

1. True or False. If the zombie congestion penalty c is raised to 0.04, then the number of zombies and humans are equal in equilibrium.
2. Notice that c = a in this case. Can you prove analytically that if c = a, then H = Z in equilibrium? [Hint: set H* = Z* and solve]

**Answer 6:**

We begin by recognizing the general form of the difference equations governing the human and zombie populations:

$$H_{t+1} = H_t - aH_tZ_t + bH_t \qquad \text{(Human Population)}$$

$$Z_{t+1} = Z_t + aH_tZ_t - cZ_t^2 \qquad \text{(Zombie Population)}$$

- $H_t$: Human population at discrete time $t \in \mathbb{Z}_{\geq 0}$
- $Z_t$: Zombie population at discrete time $t \in \mathbb{Z}_{\geq 0}$
- $a$: Human–zombie interaction (infection) rate
- $b$: Human reproduction rate
- $c$: Zombie congestion (intraspecific competition) coefficient

1. True. We can analytically show this by converting our difference equations to continuous time, such that:

$$\dot{H} = -aHZ + bH \tag{Human Population}$$
$$\dot{Z} = aHZ - cZ^2 \tag{Zombie Population}$$

2. To solve this system analytically for it's non-trivial equilibria, let $\dot{H} = \dot{Z} = 0$, so that we can solve for , such that:

$$\dot{H} = \dot{Z} = 0 \tag{Equilibria Def}$$
$$-aHZ + bH = aHZ - cZ^2 \tag{Substitution}$$
$$H(-aZ + b) = 0 \implies Z^* = \frac{b}{a} \tag{Zombie Equilibria}$$
$$Z(aH + cZ) = 0 \implies H^* = \frac{c}{a}Z^* \tag{Human Equilibria}$$

3. Now that we have our equilibria of the Human and Zombie populations, we can recognize that when $c = a = 0.04$, $\dot{H} = \frac{c}{a}\dot{Z} = \dot{Z} \implies \dot{H} = \dot{Z}$, thus we have shown analytically that the number of zombies and humans are equal in equilibrium.