Cart Pole

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1 Modeling

In this report, a model of cart pole is found in details. It should be mentioned that most of the material is driven from MIT Underacuted Robotics. Also, SharpNEAT has a reach explanation of the problem In Fig. 1, the overall model of the cart pole system with its dimensions and states are presented.

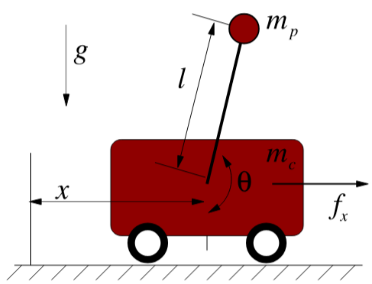

Figure 1: Cart Pole System (see MIT Underacuted Robotics).

The equations of the motion are as follows:

$$\ddot{x} = \frac{f + m_p \sin\theta(l\dot{\theta}^2 + g\cos\theta)}{m_c + m_p \sin^2\theta} \tag{1}$$

$$\ddot{\theta} = \frac{-f\cos\theta - m_p l\dot{\theta}^2\cos\theta\sin\theta - (m_p + m_c)g\sin\theta}{l(m_c + m_p\sin^2\theta)} \tag{2}$$

where m is the mass of the pendulum's point mass, M is the combined mass of pendulum and the cart, l is the length of the pendulum rod, g is the gravitational acceleration, and f is an external force applied to the cart.

By considering the following state variables

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{3}$$

The state space representation of the system when the mass is concentrated at a fixed distance l from the axis of the rotation is as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{f+m_p \sin x_3 (l x_4^2 + g \cos x_3)}{m_c + m_p \sin^2 x_3} \\ x_4 \\ \frac{-f \cos x_3 - m_p l x_4^2 \cos x_3 \sin x_3 - (m_p + m_c) g \sin x_3}{l(m_c + m_p \sin^2 x_3)} \end{bmatrix} \tag{4}$$

In other words,

$$\dot{x} = g(x, f) \tag{5}$$

2 Friction

There are 3 sources of friction

- Friction between the cart and the track
- Friction in the pendulum's pivot joint
- Air resistance

2.1 Cart-Track Friction

A simplified friction derived from Coulomb's law is considered as

$$F_f = -\mu_c \dot{x} \tag{6}$$

3 Model Predictive Control (MPC)

The goal is to swing up the pole from downward position and erect it upward. This goal could be translated as maximizing the potential energy of the system and minimizing the kinetic energy at the same time. The potential an kinetic energy at each time step is defined as

$$\begin{aligned} U_i =& - m_p g l cos x_{3,i} \\ K_i =& \frac{1}{2}(m_c + m_p) x_{2,i}^2 + m_p x_{2,i} x_{4,i} l cos x_{3,i} + \frac{1}{2} m_p l^2 x_{4,i}^2 \end{aligned} \tag{7}$$

Therefore, the problem could be formulated as

$$\begin{aligned} &\min_{x_{0:N}, u_{0:N-1}} \sum_{i=0}^{N} U_i - K_i \\ subject to :& x_0 = x_0 \\ & x_{i+1} = g(x_i, f_i) \\ & f_{lb} \leq f_i \leq f_{ub} \end{aligned} \tag{8}$$

where N is the MPC horizon, f_{lb}, and f_{ub} are the lower and upper bounds of the excerted force.