3.5 Linear Programming duality

To any minimization (maximization) LP we can associate a closely related maximization (minimization) LP based on the same parameters.

Different spaces and objective functions but in general the optimal objective function values coincide.

Example The value of a maximum feasible flow is equal to the capacity of a cut (separating the source s and sink t) of minimum capacity.

3.5 Linear Programming duality

Motivation: estimate of the optimal value

Given

$$
\begin{array}{llrrrrll}
\max & z = & 4x_1 & +x_2 & +5x_3 & +3x_4 & & \\
\text{s. t.} & & x_1 & -x_2 & -x_3 & +3x_4 & \leq 1 & (1) \\
& & 5x_1 & +x_2 & +3x_3 & +8x_4 & \leq 55 & (2) \\
& & -x_1 & +2x_2 & +3x_3 & -5x_4 & \leq 3 & (3) \\
& & x_1, & x_2, & x_3, & x_4 & \geq 0 &
\end{array}
$$

find an estimate of the optimal value z*.

Any feasible solution provides a lower bound on z^*:

$$(0, 0, 1, 0) \Rightarrow z^* \geq 5$$

$$(2, 1, 1, 1/3) \Rightarrow z^* \geq 15$$

$$(3, 0, 2, 0) \Rightarrow z^* \geq 22$$

$$\ldots$$

Even if we are lucky, we are not sure it is the best lower bound!

Now we are interested in finding in a systematic way an estimate of z*

3.5 Linear Programming duality

Upper bounds:

- By multiplying constraint (2) by 5/3, we obtain an inequality that dominates the objective function: (the coefficients in the constraint we obtained are greater than the coefficients of the same variables on the objective function)

obj function <= $25/3\,x_1 + 5/3\,x_2 + 5x_3 + 40/3\,x_4 \leq 275/3 \quad \forall$ feasible solution

$$\Rightarrow z^* \leq 275/3$$

- By adding constraints (2) and (3), we obtain:

$$\overbrace{4x_1 + x_2 + 5x_3 + 3x_4}^{\text{objective function}} \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

$$\Rightarrow z^* \leq 58 \text{ (better upper bound)}$$

Linear combinations with nonnegative multipliers of inequality constraints yields valid upper bounds.

otherwise using negative coefficients I'm reversing the inequality

3.5 Linear Programming duality

General strategy:
Linearly combine the constraints with non negative multiplicative factors (i-th one multiplied by $y_i \geq 0$).

first case: $y_1 = 0, y_2 = 5/3, y_3 = 0$ --> we only considered the second constraint

second case: $y_1 = 0, y_2 = 1, y_3 = 1$

In general any such linear combination of (1), (2), (3) reads

$$y_1(x_1 - x_2 - x_3 + 3x_4) + y_2(5x_1 + x_2 + 3x_3 + 8x_4) + y_3(-x_1 + 2x_2 + 3x_3 - 5x_4)$$
$$\leq y_1 + 55y_2 + 3y_3$$

which is equivalent to:

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4$$
$$\leq y_1 + 55y_2 + 3y_3 \tag{2}$$

Now we should impose that the the coefficients are greater or equal than the coefficients of the objective function

Note: $y_i \geq 0$ so that the inequality direction is unchanged.

3.5 Linear Programming duality

To use the left hand side of (2) as upper bound on

$$z = 4x_1 + x_2 + 5x_3 + 3x_4$$

we must have

coefficient of x1:

coefficient of x2:

$$\begin{cases} y_1 & +5y_2 & -y_3 & \geq 4 \\ -y_1 & +y_2 & +2y_3 & \geq 1 \\ -y_1 & +3y_2 & +3y_3 & \geq 5 \\ 3y_1 & +8y_2 & -5y_3 & \geq 3 \\ y_1, & y_2, & y_3 & \geq 0 \end{cases}$$

In such a case, any feasible solution x satisfies

$$\overbrace{4x_1 + x_2 + 5x_3 + 3x_4}^{z} \leq \underbrace{y_1 + 55y_2 + 3y_3}_{\text{righten side of the combined linear equality}}$$

In particular: $z^* \leq y_1 + 55y_2 + 3y_3$

So we found an upper bound of the objective function

Since I want to find the best (lowest) upper bound, we need to minimize the righten size of the combined linear equality: this is the objective function of the dual

3.5 Linear Programming duality

Since we look for the best possible upper bound on z^*:

(D)

$$
\begin{array}{llllll}
\min & y_1 & +55y_2 & +3y_3 & & \\
\text{s. t.} & y_1 & +5y_2 & -y_3 & \geq & 4 \\
 & -y_1 & +y_2 & +2y_3 & \geq & 1 \\
 & -y_1 & +3y_2 & +3y_3 & \geq & 5 \\
 & 3y_1 & +8y_2 & -5y_3 & \geq & 3 \\
 & y_1, & y_2, & y_3 & \geq & 0
\end{array}
$$

Original problem:

$$
\begin{array}{lllllllll}
\max & z = & 4x_1 & +x_2 & +5x_3 & +3x_4 & & \\
\text{s. t.} & y1 & x_1 & -x_2 & -x_3 & +3x_4 & \leq & 1 & + \\
 & y2 & 5x_1 & +x_2 & +3x_3 & +8x_4 & \leq & 55 & + \\
 & y3 & -x_1 & +2x_2 & +3x_3 & -5x_4 & \leq & 3 & + \\
 & & x_1, & x_2, & x_3, & x_4 & \geq & 0 &
\end{array}
$$

Definition 19

The problem (D) is the dual problem, while the original problem is the primal problem.

Which is the problem that allows me to find the lowest upper bound of the objective value of the original objective function. In case the original problem is minimization, it will allow us to maximize the lower bound

Inequalities are always in the "natural direction":
- in case of minimization problems the inequalities are >=
- in case of maximization, inequalities are <=

3.5 Linear Programming duality

Dual problem

$$
(P)\quad \begin{array}{ll} \max & z = c^T x \\ \text{s. t.} & Ax \leq b \\ & x \geq 0 \end{array}
\qquad\qquad
(D)\quad \begin{array}{ll} \min & w = b^T y \\ \text{s. t.} & A^T y \geq c \\ & y \geq 0 \end{array}
$$

What is the dual of a LP in standard form?

Same situation, the only difference is that we have twice as many constraint inequalities

$$
(P)\quad \begin{array}{ll} \min & z = c^T x \\ \text{s. t.} & Ax = b \\ & x \geq 0 \end{array}
\equiv
\begin{array}{ll} -\max & -z = -c^T x \\ \text{s. t.} & Ax \leq b \\ & -Ax \leq -b \\ & x \geq 0 \end{array}
\quad
\begin{array}{l} y\text{^}1 \\ y\text{^}2 \end{array}
$$

dual variables associated to the first and second set of inequalities

$$
\begin{array}{c} A \\ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \end{array}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
=
\begin{array}{l} y_1 (2x_1 + 3x_2) \\ y_2 (x_1 + 2x_2) \end{array}
\qquad
\begin{array}{c} A^T \\ \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \end{array}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
=
\begin{array}{l} 2y_1 + y_2 \\ 3y_1 + 2y_2 \end{array}
$$

By applying the derivation of the dual we get

$$
(D)\quad \begin{array}{ll} -\min & [b^T \quad -b^T] \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \\ \text{s. t.} & [A^T \quad -A^T] \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \geq -c \\ & y^1, y^2 \geq 0 \end{array}
\equiv
\begin{array}{ll} -\min & b^T (y^2 - y^1) \\ \text{s. t.} & -A^T (y^2 - y^1) \geq -c \\ & y^1, y^2 \geq 0 \end{array}
$$

3.5 Linear Programming duality

And letting $y = y^2 - y^1$ (unrestricted in sign),

$$\begin{array}{lll} -\min & b^T(y^2 - y^1) \\ \text{s. t.} & -A^T(y^2 - y^1) \geq -c \\ & y^1, y^2 \geq 0 \end{array} \equiv \begin{array}{ll} \max & w = b^T y \\ \text{s. t.} & A^T y \leq c \\ & y \in \mathbb{R}^m \end{array}$$

-> natural direction

(Because $y = y^2 - y^1$.)

So if we take the dual of an LP in standard form the only difference is that the dual variable are unrestricted in sign

Property 4

The dual of the dual problem coincides with the primal problem.

$$(P) \quad \begin{array}{ll} \max & z = c^T x \\ \text{s. t.} & Ax \leq b \\ & x \geq 0 \end{array} \qquad (D) \quad \begin{array}{ll} \min & w = b^T y \\ \text{s. t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

Note It does not matter which one is a maximum or minimum problem.

3.5 Linear Programming duality

General transformation rules

Primal (minimization)	Dual (maximization)
m constraints	m variables
n variables	n constraints
right handside	coefficients obj. fct
coefficients obj. fct.	right handside
A	A^T
equality constraints	unrestriced variables
unrestriced variables	equality constraints
inequality constraints $\geq$ $(\leq)$	variables ≥ 0 (≤ 0)
variables ≥ 0 (≤ 0)	inequality constraints $\leq$ $(\geq)$

like the example of dual of LP in standard form

in the natural direction

not in natural direction

Since the dual of the dual is the primal, I can read from right to left

non negative variables --> natural direction of inequality on constraints

3.5 Linear Programming duality

Example

$$(P) \quad \begin{array}{lllll} \max & x_1 & +x_2 & & \\ \text{s. t.} & x_1 & -x_2 & \leq & 2 \\ & 3x_1 & +2x_2 & \geq & 12 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lllll} \max & x_1 & +x_2 & & \\ \text{s. t.} & x_1 & -x_2 & \leq & 2 \\ & -3x_1 & -2x_2 & \leq & -12 \\ & x_1, & x_2 & \geq & 0 \end{array} \quad \overrightarrow{dual} \quad \begin{array}{lllll} \min & 2y_1 & -12y_2 & & \\ \text{s. t.} & y_1 & -3y_2 & \geq & 1 \\ & -y_1 & -2y_2 & \geq & 1 \\ & y_1, & y_2 & \geq & 0 \end{array}$$

since variables are
non negative,
inequalites will be in the
natural direction with
respect to the minimizaton -> >=

Example using the above rules

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s. t.} & x_1 - x_2 \leq 2 \\ & 3x_1 + 2x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{array} \quad \overrightarrow{dual} \quad \begin{array}{ll} \min & 2y_1 + 12y_2 \\ \text{s. t.} & y_1 + 3y_2 \geq 1 \\ & -y_1 + 2y_2 \geq 1 \\ & y_1 \geq 0, y_2 \leq 0 \end{array} \quad \overrightarrow{\bar{y}_2 = -y_2} \quad \begin{array}{ll} \min & 2y_1 - 12\bar{y}_2 \\ \text{s. t.} & y_1 - 3\bar{y}_2 \geq 1 \\ & -y_1 - 2\bar{y}_2 \geq 1 \\ & y_1, \bar{y}_2 \geq 0 \end{array}$$

3.5 Linear Programming duality

Example using the above rules

Write the dual of the following LP

$$
\begin{array}{lrrrl}
\min & 10x_1 & +20x_2 & +30x_3 & \\
\text{s. t.} & 2x_1 & -x_2 & & \geq 1 \\
 & & x_2 & +x_3 & \leq 2 \\
 & x_1 & & -x_3 & = 3 \\
 & x_1, & x_2 & & \leq 0
\end{array}
$$

3.5 Linear Programming duality

Theorem 20 (Weak duality theorem)

Given

$$(P) \quad \begin{array}{ll} \min & z = c^T x \\ s.\ t. & Ax \geq b \\ & x \geq 0 \end{array} \qquad (D) \quad \begin{array}{ll} \max & w = b^T y \\ s.\ t. & A^T y \leq c \\ & y \geq 0 \end{array}$$

set of feasible solution of the primal

set of feasible solutions of dual

with $X = \{x : Ax \geq b, x \geq 0\} \neq \emptyset$ *and* $Y = \{y : A^T y \leq c, y \geq 0\} \neq \emptyset$.
For every feasible solution $x \in X$ *of* (P) *and every feasible solution* $y \in Y$ *of* (D) *we have*

$$b^T y \leq c^T x.$$

N.B. Since the inequality is always satisfied, if I find a x* and y* for which the two objective function are equal, then tx and y* are optimal for the respective problems.

Proof.

For every pair $x \in X$ and $y \in Y$, we have $Ax \geq b$, $x \geq 0$ and $A^T y \leq c$, $y \geq 0$ which imply that

$$b^T y \leq x^T A^T y \leq x^T c = c^T x$$

obj function of dual

$\square$

3.5 Linear Programming duality

Consequence:
If x is a feasible solution of (P) $(x \in X)$, y is a feasible solution of (D) $(y \in Y)$, and the values of the respective objective functions coincide, $c^T x = b^T y$, then

$$x \text{ is optimal for } (P) \text{ and } y \text{ is optimal for } (D).$$

Optimal solutions are denoted by x^* and y^*.

Remember this is the termination condition we have in the ford fulkerson algorithm (maximum network flow)

3.5 Linear Programming duality

N.B. It's also necessary that the problem has a finite optimal solution, otherwise we could have inf in the inequality

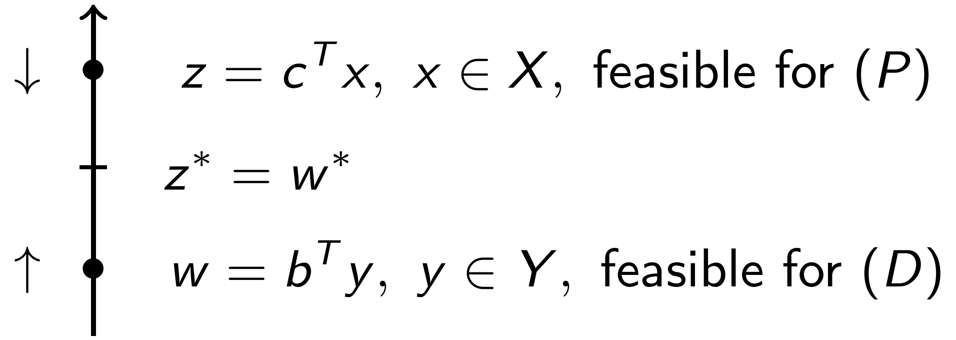
Theorem 21 (Strong duality theorem)

If $X = \{x : Ax \geq b, x \geq 0\} \neq \emptyset$ *and* $\min\{c^T x : x \in X\}$ *is finite, there exist* $x^* \in X$ *and* $y^* \in Y$ *such that* $c^T x^* = b^T y^*$*.*

N.B. the optimal solutions are different (they live in different spaces), but the values of the objective functions are the same!

That is,

$$\min\{c^T x : x \in X\} = \max\{b^T y : y \in Y\}$$



3.5 Linear Programming duality

Proof.

Derive an optimal solution of (D) from one of (P). Given

$$(P) \quad \begin{array}{ll} \min & z = c^T x \\ \text{s. t.} & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad (D) \quad \begin{array}{ll} \max & w = y^T b \\ \text{s. t.} & y^T A \leq c^T \\ & y \in \mathbb{R}^m \end{array}$$

and

$$x^* = \begin{bmatrix} x_B^* \\ x_N^* \end{bmatrix}, \quad \text{with } x_B^* = B^{-1}b, \ \ x_N^* = 0$$

an optimal feasible solution of (P), provided (after a finite # of iterations) by the Simplex algorithm with Bland's rule.

. . . $\square$

3.5 Linear Programming duality

Proof. Una generica y (?)

Consider $\bar{y}^T = c_B^T B^{-1}$. --> I can take those values it from the optimal table of the primal problem

- Verify that $\bar{y}$ is a feasible solution of (D):
  For the non-basic variables,

  Reduced cost

  $$\bar{c}_N^T = c_N^T - (c_B^T B^{-1})N = c_N^T - \bar{y}^T N \overbrace{\geq}^{\text{Since } x^* \text{ is optimal}} 0^T \Rightarrow \bar{y}^T N \leq c_N^T.$$

  (otherwise we could minimize further the value of z)

  For the basic variables

  Remember $\bar{c}^T = \begin{bmatrix} c_B^T - c_B^T B^{-1} B \\ c_N^T - c_B^T B^{-1} N \end{bmatrix} = \begin{bmatrix} 0^T \\ \bar{c}_N^T \end{bmatrix}$

  $$\bar{c}_B^T = c_B^T - (c_B^T B^{-1})B = c_B^T - \bar{y}^T B = 0^T \Rightarrow \bar{y}^T B \leq c_B^T.$$

  $\bar{y}^T A \leq c^T$ → CONSTRAINT OF DUAL ARE SATISFIED

  $A = [B \mid N]$

- According to weak duality, $\bar{y}$ is an optimal solution of (D):

  $$\bar{y}^T b = (c_B^T B^{-1})b = c_B^T (B^{-1} b) = c_B^T x_B^* = c^T x^*$$

  Hence, $\bar{y} = y^*$.

$\square$

3.5 Linear Programming duality

Corollary 22

For any pair of primal-dual problems (P) and (D), only four cases can arise:

$(P)\backslash(D)$	$\exists$ optimal solution	unbounded LP	infeasible LP
$\exists$ optimal solution	*1.* (and the objective function values coincide)	*1.*	*1.*
unbounded LP	*1.*	*2.*	*2.*
infeasible LP	*1.*	*3.*	*4.*

- Strong duality theorem $\Rightarrow$ 1.
- Weak duality theorem $\Rightarrow$ 2. and 3.
- 4. can arise. For instance, both problems below are infeasible:

$$(P)\quad \begin{array}{lrrl} \min & -4x_1 & -2x_2 & \\ \text{s. t.} & -x_1 & +x_2 & \geq 2 \\ & x_1 & -x_2 & \geq 1 \\ & x_1, & x_2 & \leq 0 \end{array} \qquad (D)\quad \begin{array}{lrrl} \max & 2y_1 & +y_2 & \\ \text{s. t.} & -y_1 & +y_2 & \leq -4 \\ & y_1 & -y_2 & \leq -2 \\ & y_1, & y_2 & \leq 0 \end{array}$$

3.5 Linear Programming duality

Economic interpretation

The primal and dual problems correspond to two complementary point of views on the same "market".

Diet problem:

n aliments $j = 1, \ldots, n$

m nutrients $i = 1, \ldots, m$ (vitamines, ...)

a_{ij} quantity of i-th nutrient in one unit of j-th aliment

b_i requirement of i-th nutrient

c_j cost of one unit of j-th aliment

Point of view of the manager of canteen:

minimize the cost

Interpretation: we have variable yi associated to each nutrient. The point of view of the dual is the person who is producing the pills of the nutrients. Variables is the price

total nutrient satisfying the daily requirement

$$(P) \quad \begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s. t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

$$(D) \quad \begin{aligned} \max \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s. t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \quad j = 1, \ldots, n \\ & y_i \geq 0, \quad i = 1, \ldots, m \end{aligned}$$

this requires that

3.5 Linear Programming duality

Interpretation of the dual problem:
A company that produces pills of the m nutrients needs to decide the nutrient unit prices y_i so as to maximize income.

es: if he needs 10 proteins, he will buy 10 units of nutrient i (Protein)

- If the costumer buys nutrient pills, he will buy b_i units for each i, $i = 1 \dots m$.
- The price of the nutrient pills must be competitive:

if a meal has 10 proteins (aij), then the price of buying 10 proteins pills (10* y_protein) must cost ess than the meal cost cj

$$\sum_{i=1}^{m} a_{ij} y_i \leq c_j, \quad j = 1, \dots, n$$

(cost of the pills that are equivalent to 1 unit of j-th aliment)

If both programs (P) and (D) admit a feasible solution, the strong duality theorem implies that

$$z^* = w^*.$$

An "equilibrium" exists (two alternatives with the same cost).

Note: Strong connection with Game theory (zero-sum games).

3.5 Linear Programming duality

Optimality conditions

Given

$$(P)\quad \begin{array}{ll} \min & z = c^T x \\ \text{s. t.} & Ax \geq b \\ & x \geq 0 \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & w = b^T y \\ \text{s. t.} & y^T A \leq c^T \\ & y \geq 0 \end{array}$$

Two feasible solutions $x^* \in X$ and $y^* \in Y$, with $X = \{Ax \geq b \wedge x \geq 0\}$ and $Y = \{y^T A \leq c^T \wedge y \geq 0\}$, are optimal iff

$$y^{*T} b = c^T x^*$$

If x_j and y_i are unknown, this is a single equation in $n + m$ unknowns!

Since $y^{*T} b \leq y^{*T} A x^* \leq c^T x^*$, we have

=

$$y^{*T} b = y^{*T} A x^* \text{ and } y^{*T} A x^* = c^T x^*$$

and therefore

$$y^{*T}(Ax^* - b) = 0 \text{ and } (c^T - y^{*T} A) x^* = 0$$

These are $m + n$ equations in $n + m$ unknowns!

Necessary and sufficient optimality conditions!

3.5 Linear Programming duality

Complementary slackness conditions

$x^* \in X$ and $y^* \in Y$ are optimal solutions of, respectively, (P) and (D) iff

$$y_i^* \overbrace{(a_i^T x^* - b_i)}^{\text{slack } s_j \text{ of } j\text{-th constraint of (D)}} = 0, \quad i = 1, \ldots, m$$

--> (corresponding to the optimal solution of the primal)

$$\underbrace{(c_j^T - {y^*}^T A_j)}_{\text{slack } s_i \text{ of } i\text{-th constraint of (P)}} x_j^* = 0, \quad j = 1, \ldots, n$$

-> --> (corresponding to the optimal solution of the dual)

where a_i denotes the i-th row of A and A_j denotes the j-th column of A.

At optimality, the product of each variable with the corresponding slack variable of the constraint of the relative dual is $= 0$.

3.5 Linear Programming duality

We never use the optimality condition for solving linear problems, but it is important to see the link between P and D problems

Example:

$$(P) \quad \begin{array}{lrrrl} \min & 13x_1 & +10x_2 & +6x_3 & \\ \text{s. t.} & 5x_1 & +x_2 & +3x_3 & = 8 \\ & 3x_1 & +x_2 & & = 3 \\ & x1, & x2, & x3 & \geq 0 \end{array} \qquad (D) \quad \begin{array}{lrrl} \max & 8y_1 & +3xy2 & \\ \text{s. t.} & 5y_1 & +3y_2 & \leq 13 \\ & y_1 & +y_2 & \leq 10 \\ & 3y_1 & & \leq 6 \end{array}$$

| x1
| x2
| x3

since in P there are equality constraints, the slacks are set to zero, thus the first equation on the previous slide doesn't apply since it is always satisfied

Verify that the feasible $x^* = (1, 0, 1)$ is an optimal, non degenerate solution of (P).

Suppose this is true and derive, via the complementary slackness conditions, the corresponding optimal solution of (D).

Since (P) is in standard form, the conditions $y_i^*(a_i^T x^* - b_i) = 0$ hold for $i = 1, 2$.

Now we focus on the second set of conditions:

- Condition $(c_j^T - {y^*}^T A_j)x_j^* = 0$ is satisfied for $j = 2$ because $x_2^* = 0$.
- Since $x_1^*, x_3^* > 0$, we obtain the conditions:

$$5y_1 + 3y_2 = 13, \quad 3y_1 = 6 \quad \Rightarrow \quad y_1 = 2, \quad y_2 = 1$$

Thus, $y^* = (2, 1)$ is the optimal solution of (D) and $b^T y^* = 19 = c^T x^*$.

That is, by solving the complementary slackness conditions we can find the optimal solution for both problems.

3.5 Linear Programming duality

Economic interpretation for the diet problem

- $\displaystyle\sum_{j=1}^{n} a_{ij}x_j^* > b_i \Rightarrow y_i^* = 0$

  If the optimal diet includes an excess of i-th nutrient, the costumer is not willing to pay $y_i^* > 0$.

- $\displaystyle y_i^* > 0 \Rightarrow \sum_{j=1}^{n} a_{ij}x_j^* = b_i$

  If the company selects a price $y_i^* > 0$, the costumer must not have an excess of i-th nutrient.

- $\displaystyle\sum_{i=1}^{m} y_i^* a_{ij} < c_j \Rightarrow x_j^* = 0$ (not convenient for the costumer to buy aliment)

  price of the pills equivalent to the nutrients contained in one unit of j-th aliment is lower than the unit price of the aliment.

- $\displaystyle x_j^* > 0 \Rightarrow \sum_{i=1}^{m} y_i^* a_{ij} = c_j$

  If the costumer includes the j-th aliment in the optimal diet, the company must have selected competitive prices y_i^* (price of the nutrients in pills contained in a unit of the j-th aliment is not lower than c_j).