3.4 Simplex method

Given a LP in standard form:

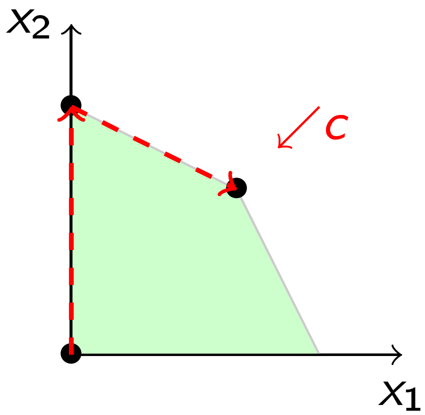
$$\begin{aligned} \min \quad & z = c^T x \\ \text{s. t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Examine a sequence of basic feasible solutions with non-increasing objective function values until an optimal solution is reached or the LP is found to be unbounded (G. Dantzig 1947).

George Dantzig
(1914-2005)

At each iteration, we move from a basic feasible solution to a "neighboring" basic feasible solution.

Generate a path (i.e., a sequence of adjacent vertices) along the edges of the polyhedron of the feasible solutions until an optimal vertex is reached.



3.4 Simplex method

Given the correspondence between the basic feasible solutions and the vertices, we need to describe how to :

- Find an initial vertex or establish that the LP is infeasible. (Achieved by applying the same method to another LP, see later.)
- Determine whether the current vertex is optimal.
- Move from a current vertex to a better adjacent vertex (in terms of objective function value) or establish that the LP is unbounded.

3.4 Simplex method

3.4.1 Optimality test

Given a LP $\min\{c^T x : Ax = b, x \geq 0\}$ and a feasible basis B of A, $Ax = b$ can be rewritten as

$$Bx_B + Nx_N = b \Rightarrow x_B = B^{-1}b - B^{-1}Nx_N$$

with $B^{-1}b \geq 0$.

Then, a basic feasible solution is such that $x_B = B^{-1}b$, $x_N = 0$.

By substitution we express the objective function in terms of only the non basic variables (for the current basis B):

$$\begin{aligned} c^T x &= (c_B^T \; c_N^T) \begin{bmatrix} x_B \\ x_N \end{bmatrix} = (c_B^T \; c_N^T) \begin{bmatrix} B^{-1}b - B^{-1}Nx_N \\ x_N \end{bmatrix} \\ &= c_B^T B^{-1}b - c_B^T B^{-1}Nx_N + c_N^T x_N \\ &= c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N)x_N \end{aligned}$$

- Here, $z_0 = c_B^T B^{-1}b$ is the cost of $x = (x_B \; x_N)$, where $x_B = B^{-1}b$, $x_N = 0$.
- The reduced costs of the non basic variables x_N are denoted by $\bar{c}_N^T := c_N^T - c_B^T B^{-1}N$.

3.4 Simplex method

Definition 17

$$\bar{c}^T = c^T - c_B^T B^{-1} A = [\overbrace{c_B^T - c_B^T B^{-1} B}^{0^T}, \overbrace{c_N^T - c_B^T B^{-1} N}^{\bar{c}_N^T}]$$

is the vector of reduced costs with respect to the basis B.

- The reduced costs are defined also for basic variables, but $\bar{c}_B = 0$.
- $\bar{c}_j$ represents the change in the objective function value if non basic x_j would be increased from 0 to 1 while keeping all other non basic variables to 0.
- The solution value changes by $\Delta z = \theta^* \bar{c}_j$.

3.4 Simplex method

Consider a LP $\min\{c^T x : Ax = b, x \geq 0\}$ and a feasible basis B.

Proposition 1

If $\bar{c}_N \geq 0$ then the basic feasible solution (x_B^T, x_N^T), where $x_B = B^{-1}b \geq 0$ and $x_N = 0$, of cost $c_B^T B^{-1} b$ is a global optimum.

Proof.

$\bar{c}^T \geq 0^T$ implies that

$$c^T x = c_B^T B^{-1} b + \bar{c}_N^T x_N \geq c_B^T B^{-1} b, \forall x \geq 0, Ax = b.$$

□

Note:

- For maximization problems we check whether $\bar{c}_N \leq 0$
- This optimality condition is sufficient but in general not necessary.

3.4 Simplex method

Example

$$\begin{array}{llr} \min & -x_1 - x_2 & \\ \text{s. t.} & x_1 - x_2 + s_1 & = 1 \\ & x_1 + x_2 \quad\quad + s_2 & = 3 \\ & x_1,\ x_2,\ s_1,\ s_2 & \geq 0 \end{array}$$

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad c^T = [-1\ -1\ 0\ 0]$$

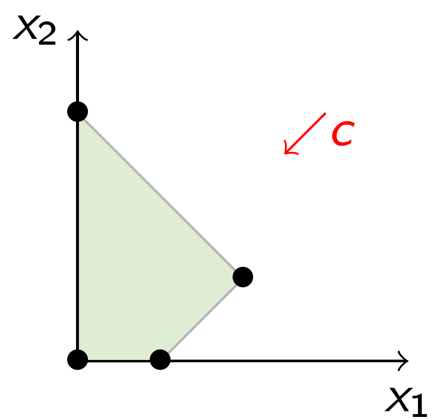
value of objective function : z = C^(T) * x

meaning this is the value of the objective function for the current basic feasible solution.

Consider $x_B = (x_1, s_2) = (1, 2)$, $x_N = (x_2, s_1)$, $z = -1$, so

non basis

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$



Therefore,

inverse of the matrix

$$c_B^T = c_N^T = [-1\ 0], \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and the reduced costs for x_2, s_1 are

The reduced cost for x2 is 2
The reduced cost for s1 is 0

$$\bar{c}_N^T = c_N^T - c_B^T B^{-1} N = [-2\ 0]$$

Increasing x_2 from 0 to 1 keeping $s_1 = 0$, we move along the edge to $(x_1, x_2) = (2, 1)$, and $z = -3$.

Since the reduced cost for x2 is negative

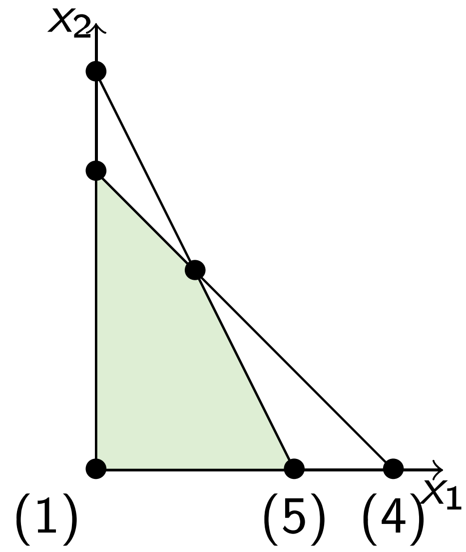
Since $\bar{c}_2 = -2 < 0$, increasing x_2 to 1 (keeping the other non-basic variables to 0) we improve the solution by -2.

Remember: In the context of the Simplex method, when we initially set up a basic feasible solution, we assign the non-basic variables to zero. In this example, the non-basic variables are x2x2 and s1s1, so they are set to 00 in the current basic feasible solution.

3.4 Simplex method

3.4.2 Move to an adjacent vertex (basic feasible solution)

$$\begin{array}{rcl} x_1+x_2+s_1 & & = 6 \\ 2x_1+x_2 & +s_2 & = 8 \\ x_1,\ x_2,\ s_1, & s_2 & \geq 0 \end{array}$$



Move from vertex (1) to vertex (5):

- In (1) $x_1 = 0, x_2 = 0 \Rightarrow s_1 = 6, s_2 = 8$ with $x_B = (s_1, s_2)$, $x_N = (x_1, x_2)$
- In (5) $x_2 = 0, s_2 = 0 \Rightarrow x_1 = 4, s_1 = 2$ with $x_B = (x_1, s_1)$, $x_N = (x_2, s_2)$

Thus, x_1 enters the basis B and s_2 exits the basis B.

Note: When moving from the current vertex to an adjacent vertex, we substitute one column of B (that of s_2) with one column of N (that of x_1).

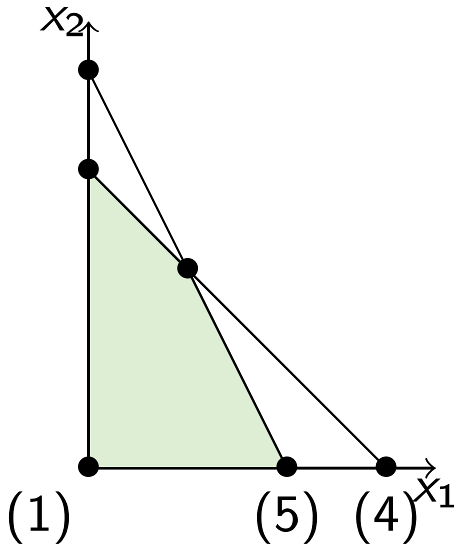
3.4 Simplex method

By expressing the basic variables in terms of the non basic variables, we obtain

$$s_1 = 6 - x_1 - x_2$$

$$s_2 = 8 - 2x_1 - x_2$$

Now we increase x_1, while keeping $x_2 = 0$.



Since $s_1 = 6 - x_1 \geq 0$ implies $x_1 \leq 6$ and $s_2 = 8 - 2x_1 \geq 0$ implies $x_1 \leq 8/2 = 4$, the upper limit on the increase of x_1 is: $x_1 \leq \min\{6, 4\} = 4$.

We move from vertex (1) to vertex (5) by letting x_1 enter the basis and s_2 exit the basis ($s_1 = 2$ and $s_2 = 0$).

Note: When $x_1 = 6$, we obtain the infeasible basic solution (4).

3.4 Simplex method

General case

Given a basis B, the system

$$Ax = b \Leftrightarrow \sum_{j=1}^{m} a_{ij} x_i = b_j, \text{ for } i = 1, \dots, m$$

can be expressed in canonical form

Identità

$$x_B + B^{-1}N x_N = B^{-1}b \Leftrightarrow x_B + \bar{N} x_N = \bar{b}$$

which emphasizes the basic feasible solution $(x_B, x_N) = (B^{-1}b, 0)$. This amounts to pre-multiply the system by B^{-1}:

$$\overbrace{B^{-1}B}^{=I} x_B + \underbrace{\overbrace{B^{-1}N}^{=\bar{N}}}_{(m \times m) * (m * (n-m)) = (m * (n-m))} x_N = \overbrace{B^{-1}b}^{=\bar{b}}$$

3.4 Simplex method

In the canonical form

$$x_{B_i} + \overbrace{\sum_{j=1}^{n-m} \bar{a}_{ij} x_{N_j}}^{\bar{N}} = \bar{b}_i, \text{ for } i = 1, \dots, m \quad (Ix_B + \bar{N}x_N = \bar{b})$$

basic variables are expressed in terms of non basic variables: $x_B = \bar{b} - \bar{N}x_N$.
If we increase the value of a non basic x_s (from value 0) while keeping all other non basic variables to 0, the system becomes

the others non basic variables are all 0s

$$x_{B_i} + \bar{a}_{is}x_s = \bar{b}_i \Leftrightarrow x_{B_i} = \bar{b}_i - \bar{a}_{is}x_s, \quad , \text{ for } i = 1, \dots, m$$

To guarantee $x_{B_i} \geq 0$ for each i, we need to satisfy

$$\bar{b}_i - \bar{a}_{is}x_s \geq 0 \Rightarrow x_s \leq \bar{b}_i / \bar{a}_{is}, \quad \text{for } \bar{a}_{is} \geq 0.$$

The value of x_s can be increased up to

$$\theta^* = \min_{i=1,\dots,m} \left\{ \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} \geq 0 \right\}$$

(If $\bar{a}_{is} \leq 0$ for any i, there is no limit to the increase of x_s.) The value of x_r where

$$r = \text{argmin}_{i=1,\dots,m} \left\{ \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} \geq 0 \right\}$$

decreases to 0 and exits the basis.

3.4 Simplex method

3.4.3 Change of basis (for minimization LP)

Let B be a feasible basis and x_s (in x_N) a non basic with reduced cost $\bar{c}_s < 0$.

- Increase x_s as much as possible (x_s "enters the basis") while keeping the other non basic variables equal to 0.
- The basic variable x_r (in x_B) such that $x_r \geq 0$ imposes the tightest upper bound θ^* on the increase of x_s (x_r leaves the basis).
- If $\theta^* > 0$, the new basic feasible solution has a better objective function value.
- The new basis differs w.r.t. the previous one by a single column (adjacent vertices).

To go from the canonical form of the current basic feasible solution

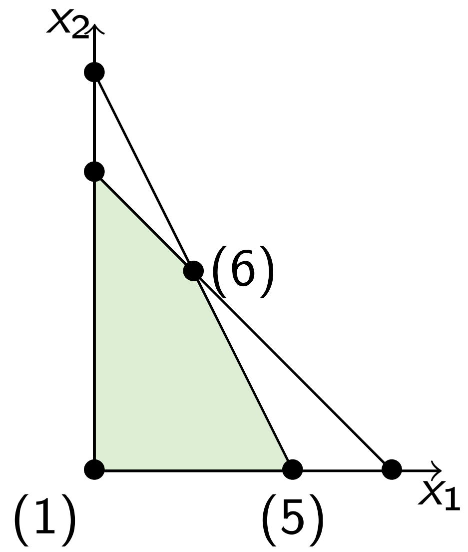
$$B^{-1}Bx_B + B^{-1}Nx_N = B^{-1}b$$

to that of an adjacent basic feasible solution, it is not necessary to compute B^{-1} from scratch.
B^{-1} of the new basis B can be obtained by applying to the inverse of the previous basis (which differs w.r.t a single column) a unique "pivoting" operation.

3.4 Simplex method

Example

$$\begin{array}{llr} \min & -x_1 - x_2 & \\ \text{s. t.} & x_1 + x_2 + s_1 & = 6 \\ & 2x_1 + x_2 + s_2 & = 8 \\ & x_1,\ x_2,\ s_1,\ s_2 & \geq 0 \end{array}$$



Move from vertex (1) to vertex (5).

System in canonical form w.r.t. the basis with s_1 and s_2 basic (vertex (1)):

pivot

$$\boxed{\begin{array}{lcl} s_1 & = & 6 - x_1 - x_2 \\ s_2 & = & 8 - 2x_1 - x_2 \end{array}} \Rightarrow \begin{array}{rrrrr} x_1 & +x_2 & +s_1 & & = 6 \\ 2x_1 & +x_2 & & +s_2 & = 8 \end{array}$$

columns of basic variables

x_1 enters the basis and s_2 exits

In the canonical form I want to express the basic variables in terms of non basic variales

System in canonical form w.r.t. the basis with x_1 and s_1 basic (vertex (5)):

I need the basic variable s1 to be expressed in terms of only non basic variables --> need to "eliminate" dependency from the other basic variable x1 by performing a substitution

$$\begin{array}{c} (4 - {}^1\!/_2 x_2 - {}^1\!/_2 s_2) + x_2 + s_1 = 6 \\ x_1 = 4 - {}^1\!/_2 x_2 - {}^3\!/_2 s_2 \end{array} \Rightarrow \begin{array}{rrrrr} & {}^1\!/_2 x_2 & +s_1 & -{}^1\!/_2 s_2 & = 2 \\ x_1 & +{}^1\!/_2 x_2 & & +{}^1\!/_2 s_2 & = 4 \end{array}$$

3.4 Simplex method

"Pivoting" operation

Same operations used in the Gaussian elimination method to solve systems of linear equations.

QUALE ESCE

QUALE ENTRA

Given $Ax = b$:

1. Select a coefficient $\bar{a}_{rs} \neq 0$ (the "pivot")
2. Divide the r-th row by $\bar{a}_{rs}$.
3. For each row i with $i \neq r$ and $\bar{a}_{is} \neq 0$, subtract the resulting r-th row multiplied by $\bar{a}_{is}$.

	s		A		b
	1	1	1	0	6
$r \rightarrow$	2	1	0	1	8

$\longrightarrow$

0	1/2	1	-1/2	2
1	1/2	0	1/2	4

$r/\bar{a}_{rs}$

$\frac{r}{2}$

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $(0 \;\; 1/2 \;\; 1 \;\; -\tfrac{1}{2}) - 1 \cdot (1 \;\; 1/2 \;\; 0 \;\; 1/2)$

$\bar{a}_{is}$

pivot

columns of basic variables

Note: This does not affect the feasible solutions.

3.4 Simplex method

Moving to an adjacent basic feasible solution (vertex)

Goals :

- improve the objective function value
- preserve feasibility

1. Which non basic variable enters the basis? (Ie, choice of pivot column s.)
   - Any one with reduced cost $\bar{c}_j < 0$.
   - One that yields the maximum Δz w.r.t. $z = c_B^T B^{-1} b$ (the actual decrement Δz also depends on θ^*).
   - Bland's rule: $s = \min\{j : \bar{c}_j < 0\}$. (Or $\bar{c}_j > 0$ for maximization problems.)

Bland's rule for picking the variable entering the basis: if they have the same reduced cost, I will pick the one with the smallest index

2. Which basic variable leaves the basis?
   Min ratio test: index i with smallest $\bar{b}_i/\bar{a}_{is} = \theta^*$ (thightest upper bound on increase of x_s) among those with $\bar{a}_{is} > 0$ (otherwise no limit!).
   - Bland's rule: $r = \min\{i : \bar{b}_i/\bar{a}_{is} = \theta^*, \bar{a}_{is} > 0\}$
   - randomly. . .

Bland's rule for variable exiting the basis: can happen that when increasing a non basic variable up to the upper bound theta*, more than one basic variable goes to zero, in this case I will need to pick one who will enter the basis. That can be established by the bland's rule too. The variable with the smallest index will exit the basic

Unboundedness: If $\exists \bar{c}_j < 0$ with $\bar{a}_{ij} \le 0 \forall i$, no element of the j-th column can play the role of a pivot. $\Rightarrow$ The minimization problem is unbounded!

3.4 Simplex method

3.4.4 "Tableau" representation

z is considered as a variable. As initial state we consider z=0

$0 = c^T x - z$

System $\begin{cases} z = c^T x \\ Ax = b \end{cases}$ with (implicit) nonnegativity constraints

Initial tableau:

Column of the righten sight on the left

-z	$x_1 \; \cdots \; x_n$	z	
0	c^T	-1	$\rightarrow$ objective function
b	A	0 0	$\rightarrow$ m rows

The first column contains the:

- right hand side of the objective function
- right hand side vector

First column is associated t:
- the objective function value moved in the right side (first row)
- the right hand side vector b (second row)

3.4 Simplex method

Consider a basis B and a partition $A = [B\ N]$, with $0 = c^T x - z$, we have:

	$\overbrace{x_1 \ \cdots \ x_m}^{x_B}$	$\overbrace{x_{m+1} \ \cdots \ x_n}^{x_N}$	z
0	c_B^T	c_N^T	-1
$\overbrace{B^{-1}\, b}^{\bar b}$	$\overbrace{B^{-1}\, B}^{I}$	$\overbrace{B^{-1}\, N}^{\bar N}$	0 $\vdots$ 0

by "pivoting" operations (or pre-multiplying by B^{-1}) we put the tableau in canonical form with respect to B:

$z = c_{T_B} x_B + c_{T_N} x_N$ $\quad$ $x_B = \bar b - \bar N x_N$

$c^T x = z_0 + c_N^T x_N$

		$x_1 \ \cdots \ x_m$	$x_{m+1} \ \cdots \ x_n$
$-z$	$-z_0$	$0 \ \cdots \ 0$	$\bar c_N^T$
$x_{B[1]}$ $\vdots$ $x_{B[m]}$	$\underbrace{\bar b}_{B^{-1} b = x_B}$	I	$\bar N$

basic variables

$$z = c_B^T B^{-1} b + \bar c_N^T x_N = z_0 + \bar c_N^T x_N$$

$$\bar b = B^{-1} b$$

3.4 Simplex method

Example

First basics, trivial in this case

$$
\begin{array}{llllll}
\min & -x_1 & -x_2 & & & \\
\text{s. t.} & 6x_1 + 4x_2 & + x_3 & & = 24 \\
& 3x_1 - 2x_2 & & + x_4 & = 6 \\
& x_1, \quad x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

zero because initially $x_1, x_2 \in X_N \Rightarrow x_1 = x_2 = 0 \Rightarrow z(x_1, x_2) = 0$

Tableau w.r.t. the basis with columns 3, 4:

reduce costs

		x_1	x_2	x_3	x_4
$-z$	0	-1	-1	0	0
x_3	24	6	4	1	0
x_4	6	3	-2	0	1

basic variables

pivot

basic variables

x_1 enters in the basis and x_4 exits the basis

Pivot w.r.t. 3 amounts to deriving an expression for x_1 from the pivot row and substituting it in all other rows.

chosen with Bland's rule

$\theta^* = \min\left\{ \frac{24}{6}, \frac{6}{3} \right\} = \frac{6}{3}$ → PIVOT

RATIO FOR x_3 RATIO FOR x_4

Tableau w.r.t. the new basis with columns:

THIS ROW IS ALSO OBTAINED BY INVOLVING IT IN THE PIVOTING OPERATION, SINCE WE WANT IT TO BE INDIPENDENT FROM x_B, WE NEED TO SUBSTITUTE x_1 ($\in x_B$) IN IT

		x_1	x_2	x_3	x_4
$-z$	2	0	$-5/3$	0	$1/3$
x_3	12	0	8	1	-2
x_1	2	1	$-2/3$	0	$1/3$

this is x_B

since $\bar{b} = B^{-1}b = x_B$

Corresponding basic feasible solution: $x^T = (2, 0, 12, 0)$, with $z = -2$.

x_2 is the only non basic variable can "enter" the basis ($\bar{c}_2 = -5/3 < 0$)

x_3 is the only basic variable can "exit" the basis ($\bar{a}_{rs} = 8 > 0$)

$2/{-2/3} < 0 \Rightarrow x_1$ doesn't impose an upper bound on the variable entering the basis

3.4 Simplex method

		x_1	x_2	x_3	x_4
$-z$	2	0	$-5/3$	0	$1/3$
x_3	12	0	8	1	-2
x_1	2	1	$-2/3$	0	$1/3$

		x_1	x_2	x_3	x_4
$-z$	$9/2$	0	0	$5/24$	$-1/12$
x_2	$3/2$	0	1	$1/8$	$-1/4$
x_1	3	1	0	$1/12$	$1/6$

		x_1	x_2	x_3	x_4
$-z$	6	$1/2$	0	$1/4$	0
x_2	6	$3/2$	1	$1/4$	0
x_4	18	6	0	$1/2$	1

All reduced costs ≥ 0, then, the optimal basic (feasible) solution is:
${x^*}^T = (0, 6, 0, 18)$ with $z^* = -6$.

3.4 Simplex method

Algorithm 1: Simplex algorithm (LP with minimization)

Let $B[1], \ldots, B[m]$ be the column indices of the initial feasible basis B --> we need a first basis given to us

Construct the initial tableau $\bar{A} = \{\bar{a}[i,j] : 0 \leq i \leq m, 0 \leq j \leq n\}$ in canonical form w.r.t. B

unbounded $\leftarrow$ false --> if the problem is unbounded

optimal $\leftarrow$ false --> if the problem is optimal

while *optimal* = *false* ***and*** *unbounded* = *false* **do**

  if $\bar{a}[0,j] \geq 0 \forall j = 1, \ldots, m$ **then** *optimal* $\leftarrow$ true`/* for LP with min */`

  check wether all coefficient are greate or equal to zero, if so the optimality test is satisfied

  else

    select reduced cost associated to xs

    Select a non basic variable x_s with $\bar{a}[0,s] < 0$`/* negative reduced cost */`

    if ALL the ratios are negative, then the problem is unbounded.

    if $\bar{a}[i,s] \leq 0 \forall i = 1, \ldots, m$ **then** *unbounded* $\leftarrow$ true

    else othewise we can pick the minimum from the non negative ones

      $r \leftarrow \text{argmin}\left\{\frac{\bar{a}[i,0]}{\bar{a}[i,s]} \text{ with } 1 \leq i \leq m \text{ and } \bar{a}[i,s] > 0\right\}$

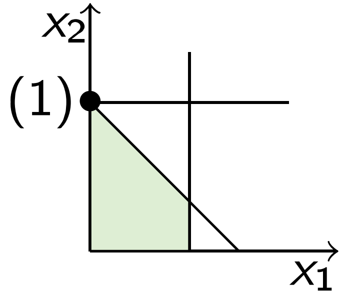
      pivot(r,s)`/* update tableau */` pick the pivot based on which is the ration chosen

      $B[r] \leftarrow s$

3.4 Simplex method

3.4.5 Degenerate basic feasible solutions and convergence

$$
\begin{array}{lllllll}
x_1 & & +s_1 & & & = 2 \\
 & +x_2 & & +s_2 & & = 3 \\
x_1 & +x_2 & & & +s_3 & = 3 \\
x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$



Point (1) corresponds to the basic solution
$x = (0, 3, 2, 0, 0)$

Definition 18

A basic feasible solution x is degenerate if it contains at least one basic variable $x_j = 0$.

that means not only the non basic variables are 0, but also at least a basic variable

A solution x with more than $n - m$ zeros corresponds to several distinct bases!

We would have many basic feasible solutions that correspond to the same vertex, since by making the basic variable that equals to zero exit and making a non basic variable entering, the vertex is the same.
If a basic variable xi in Xb is zero, than is ratio will be zero so the upper bound on the entering variable is zero. That is, the exiting variable is zero and the entering variable is 0. Different bases (since different variables are involved) but same vertex.

Same vertex:

- More than n constraints (the m of $Ax = b$ and more than $n - m$ among the n of $x \geq 0$) are satisfied with equality ("active").

In this case we have a mapping basis -> vertex which is not one to one, so the simplex method, which only is based on changing of the basis, doesn't know the vertex is always the same, and can get stuck in a cycle

3.4.5 Degenerate basic feasible solutions and convergence

In the presence of degenerate basic feasible solutions (BFSs), a basis change may not decrease the objective function value:

The simplex method changes the bases because it sees that the reduced cost of a variable is good for upgrading the objective function, but then the upper bound of the increasing of the variable is zero, then there is no real increase of objective function

- if the current BFS is degenerate, one can have $\theta^* = 0$ and hence the new BFS is identical (same vertex).

Note that a degenerate BFS can arise from a non degenerate one: even if $\theta^* > 0$, several basic variables may go to 0 when x_s is increased to θ^*.

- One can cycle through a sequence of "degenerate" basis associated to the same vertex.

When I am making a non basic variable entering the basis by increasing is value up to theta* >0, it can be that that theta* is imposed at the same time by two basis variables (the ratio of the two variables is the same). Usually, it happens that the variable that imposes the upper bound becomes zero after the increase of the entering value, but in this case two basic variables go to zero. So we need to choose which one will exit the base, and the other one will stay in the base and impose the upper bound for the next entering variable to 0. That is, we reached a degenerate BFS since there is a basic variable which is =0.

3.4 Simplex method

Several "anticycling" rules have been proposed for the choice of the variables that enter and exit the bases (indices r and s).

Bland's rule: Among all candidate variables to enter/exit the basis (x_s/x_r) always select the one with smallest index.

Robert Bland

Proposition 2

The Simplex algorithm with Bland's rule terminates after $\leq \binom{n}{m}$ *iterations.*

Note: The number of pivots is finite.

- In some "pathological" cases (see e.g. Klee & Minty 72), the number of iterations grows exponentially w.r.t. n or m. -> so in theory it is not polynomial, in the practical application it is very efficient
- However, the Simplex algorithm is overall very efficient and extensive experimental campaigns showed a number of iterations that grows linearly w.r.t. m ($m \leq \ldots \leq 3m$) and very slowly ($\approx$ logarithmically) w.r.t. n.

m is usually very smaller than n

3.4 Simplex method

3.4.6 Two-phase simplex method

The overall simplex method is called a two-phase method:
- First phase to get the first basis if it is not straightforward
- Second phase we apply the iterations we saw in the previous examples

Phase 1: Determine an initial basic feasible solution.

Example:

$$\begin{array}{llll} \min & x_1 & & +x_3 \\ \text{s. t.} & x_1 + 2x_2 & & \leq 5 \\ & & x_2 + 2x_3 & = 6 \\ & x_1, & x_2, & x_3 \geq 0 \end{array}$$

There is no sub-matrix $I_{2\times 2}$ of A! $\Rightarrow$

$$\begin{array}{lllll} \min & x_1 & & +x_3 & \\ \text{s. t.} & x_1 + 2x_2 & & +x_4 & = 5 \\ & & x_2 + 2x_3 & & = 6 \\ & x_1, & x_2, & x_3, & x_4 \geq 0 \end{array}$$

If not so I can multiply both sides by -1

Given the problem (P): $\begin{array}{ll} \min & z = c^T \\ \text{s. t.} & Ax = b \\ & x \geq 0 \end{array}$, with $b \geq 0$, an auxiliary LP with artificial variables y_i, $i = 1, \ldots, m$, is written:

$$(P_A) \quad \begin{array}{ll} \min & v = \sum_{i=1}^{m} y_i \\ \text{s. t.} & Ax + Iy = b \\ & x \geq 0, \; y \geq 0 \end{array}$$

--> we forget the original obj function, we only focus on constraint (auxiliary obj funct)

There is an obvious initial basic feasible solution $y = b \geq 0$ and $x = 0$.

3.4 Simplex method

A partire dalla soluzione banale y=b e x=0, possiamo applicare il simplex method, facendo entrare x nella base e facendo uscire y dalla base. Se riusciamo ad arrivare alla soluzione ottimale, in cui y*=[0] (tutte le y fuori dalla base) e in cui tutte le x sono dentro la base x*, vuol dire che i constraint Ax*=b sono rispettati e quindi quella base formata solo da x è una basic feasible solution per il problema originale, che possiamo usare come base di partenza per applicare il simplex method.

1. If $v^* > 0$, then (P) is infeasible. --> v* è l'objective function con y*=[0], x* base
2. If $v^* = 0$, clearly $y^* = 0$ and x^* is a basic feasible solution of (P).
   - If y_i is non basic $\forall i$, with $1 \leq i \leq m$, delete the corresponding columns and obtain a tableau in canonical form w.r.t. a basis; the row of z must be determined by substitution.
   - If there is a basic y_i (the basic feasible solution is degenerate), we perform a "pivot" operation w.r.t. a coefficient $\neq 0$ of the row of y_i so as to "exchange" y_i with any non basic variable x_j.

cf. example

SIAMO NEL CASO IN CUI $y^* = 0$ MA UNA DELLE y_i È NELLA BASE, QUINDI ABBIAMO UNA BASE DEGENERE. POSSIAMO COMUNQUE RICONDURCI AD UNA BASE PER IL PROB ORIGINALE

3.4 Simplex method

(P)

$$\begin{array}{llr} \min & x_1+x_2+10x_3 & \\ \text{s. t.} & x_2 + 4x_3 = 2 & \\ & -2x_1+x_2 -6x_3 = 2 & \\ & x_1,\ x_2,\ x_3 \geq 0 & \end{array}$$

(P_A)

$$\begin{array}{lll} \min & v = & y_1+y_2 \\ \text{s. t.} & & x_2+4x_3+y_1 = 2 \\ & & -2x_1+x_2-6x_3 + y_2 = 2 \\ & & x_1,\ x_2,\ x_3,\ y_1,\ y_2 \geq 0 \end{array}$$

We write $v = y_1 + y_2$ in canonical form by substituting the expression of y_1 and y_2 in terms of x_1, x_2 and x_3.

		x_1	x_2	x_3	y_1	y_2
$-v$	-4	2	-2	2	0	0
y_1	2	0	1	4	1	0
y_2	2	-2	1	-6	0	1

BASIS

		x_1	x_2	x_3	y_1	y_2
$-v$	0	2	0	10	2	0
x_2	2	0	1	4	1	0
y_2	0	-2	0	-10	-1	1

SINCE WE ARE IN CASE 2 OF PREVIOUS SLIDE (y_2 is BASIC)

By swaping the roles of y_2 and x_1 or x_3, we identify an optimal solution of the Phase I problem: $x^* = (0, 2, 0)$, $y^* = (0, 0)$, with optimal value $v^* = 0$.

WE NEED TO PIVOT AS DESCRIBED IN PREVIOUS SLIDE

3.4 Simplex method

By selecting as "pivot" the coefficient -2 of the row of y_2, we obtain:

		x_1	x_2	x_3	y_1	y_2
$-v$	0	0	0	0	1	1
x_2	2	0	1	4	1	0
x_1	0	1	0	5	$1/2$	$-1/2$

NEW BASIS

Therefore, this is an optimal basic feasible solution of (P_A) and $x = (0, 2, 0)$ is a basic feasible solution of (P).

With respect to the original LP,

$$z = x_1 + x_2 + 10x_3$$

We should express the objective function with respect to only the non basic variable, in our case x3

is not in the canonical form, because x_3 is a non basic variable, but substituting:

$$\begin{cases} x_2 = 2 - 4x_3 \\ x_1 = -5x_3 \end{cases} \Rightarrow z = 2 + x_3$$

we obtain the tableau corresponding to the initial basic feasible solution of (P):

		x_1	x_2	x_3
$-z$	-2	0	0	1
x_2	2	0	1	4
x_1	0	1	0	5

Since the basic feasible solution found is (already) optimal, there is no need for Phase II!

3.4 Simplex method

3.4.7 Polynomial-time algorithms for LP

- Ellipsoid method (L. Khachiyan 1979)
  Theoretically important.
- Interior point methods (N. Karmarkar 1984, . . .)
  Very efficient variants (e.g. barrier methods) for some types of instances (e.g. sparse and large-scale).

Leonid Khachiyan (1952-2005)

Narendra Karmarkar (1957-)