Foundations of Operations Research

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3. Linear Programming (LP)

Definition 1

A Linear Programming (LP) problem is an optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s. t.} \quad & x \in X \subseteq \mathbb{R}^n \to \mathbb{R} \end{aligned} \tag{1}$$

vector of variables

where:

- the objective function $f : X \to \mathbb{R}$ is linear;
- the feasible region $X = \{x \in \mathbb{R}^n : g_i(x) r_i 0 \wedge i \in \{1, \ldots, m\}\}$ with $r_i \in \{=, \geq, \leq\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ linear functions, for $i = 1, \ldots, m$.

region in the n-space where the constraint (represented by the g function) are satisfied.

Definition 2

$x^* \in \mathbb{R}^n$ is an optimal solution of the LP (1) if $f(x^*) \leq f(x), \forall x \in X$.

A wide variety of decision-making problems can be formulated or approximated as LPs.
They often involve the optimal allocation of a given set of limited resources to different activities.

3. Linear Programming (LP)

General form:

$$\begin{array}{ll} \min & z = c_1x_1 + \cdots + c_nx_n \\ \text{s. t.} & a_{11}x_1 + \cdots + a_{1n}x_n (\geq, =, \leq) b_1 \\ & \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n (\geq, =, \leq) b_m \\ & x_1, \ldots, x_n \geq 0 \end{array}$$

Matrix notation

$$\begin{array}{ll} \min & z = c^T x \\ \text{s. t.} & Ax \geq b \\ & x \geq 0 \end{array} \quad , \qquad \begin{array}{ll} \min & z = [c_1 \ldots c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ \text{s. t.} & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ & \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq 0 \end{array}$$

3. Linear Programming (LP)

Historical sketch

Jean B. Joseph Fourier
(1786–1830)

L. V. Kantorovitch
(1912–1986)

G. B. Dantzig
(1914–2005)

- 1825/26: Fourier presents a method to solve systems of linear inequalities and discusses LPs with 2-3 variables.
- 1939: Kantorovitch lays the foundations of LP (Nobel prize, 1975).
- 1947: Dantzig independently proposes LP and invents the Simplex algorithm.

3. Linear Programming (LP)

Example 1: Diet problem

Given:

n aliments $j = 1, \ldots, n$

m nutrients (basic substances) $i = 1, \ldots, m$

a_{ij} amount of i-th nutrient contained in one unit of the j-th aliment

b_i daily requirement of the i-th nutrient

c_j cost of a unit of j-th aliment,

determine a diet that minimizes the total cost while satisfying all the daily requirements.

Decision variables:

- x_j = amount of j-th aliment in the diet, with $j = 1, \ldots, n$

$$\begin{aligned} \min \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{s. t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \ j = 1, \ldots, n \end{aligned}$$

3. Linear Programming (LP)

Example 2: Transportation problem (single product)

Given:

m production plants $i = 1, \ldots, m$

n clients $j = 1, \ldots, n$

c_{ij} unit transportation cost from plant i to client j

p_i maximum supply (production capacity) of plant i

d_j demand of client j

q_{ij} maximum amount transportable from plant i to client j

determine a transportation plan that minimizes the total costs while respecting plant capacities and client demands

Assumption: $\sum_{i=1}^{m} p_i \geq \sum_{j=1}^{n} d_j$

Decision variables:

- x_{ij} = amount of product transported from i to j, $i = 1, \ldots, m$, $j = 1, \ldots, n$

3. Linear Programming (LP)

$$
\begin{aligned}
\min \quad & z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s. t.} \quad & \sum_{j=1}^{n} x_{ij} \leq p_i, \quad i = 1, \dots, m && \text{(plant capacity)} \\
& \sum_{i=1}^{m} x_{ij} \geq d_j, \quad j = 1, \dots, n && \text{(client demand)} \\
& 0 \leq x_{ij} \leq q_{ij}, \ i = 1, \dots, m, \ j = 1, \dots, n && \text{(transportation capacity)}
\end{aligned}
$$

3. Linear Programming (LP)

Example 3: Production planning problem

Given:

n products $j = 1, \ldots, n$ which compete for resources
m resources $i = 1, \ldots, m$
c_j profit (selling price-cost) per unit of j-th product
a_{ij} amount of i-th resource needed to produce one unit of j-th product
b_i maximum available amount of i-th resource

determine a production plan that maximizes the total profit given the available resources.

Decision variables:

- x_j = amount of j-th product, with $j = 1, \ldots, n$

$$
\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s. t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \; j = 1, \ldots, n
\end{aligned}
$$

3. Linear Programming (LP)

Assumptions of LP models

1. Linearity (proportionality and additivity) of the objective function and constraints.
   Proportionality: Contribution of each variable = constant $\times$ variable
   Drawback: does not account for economies of scale.
   Additivity: Contribution of all variables = sum of single contributions
   Drawback: Competing products $\Rightarrow$ profits are not independent.

2. Divisibility The variables can take fractional (rational) values.

3. Parameters are considered as constants which can be estimated with a sufficient degree of accuracy.
   More complex mathematical programs are needed to account for uncertainty in the parameter values.

LP "sensitivity analysis" allows to evaluate how sensitive an optimal solution is with respect to small changes in the parameter values (see end of Chapter 3).

3.1 Equivalent forms

General form:

$$
\begin{array}{lll}
\min(\max) & z = c^T x & \\
\text{s. t.} & A_1 x \geq b_1 & \text{inequality constraints} \\
 & A_2 x \leq b_2 & \text{inequality constraints} \\
 & A_3 x = b_3 & \text{equality constraints} \\
 & x_j \geq 0, \quad \text{for } j \in J \subseteq \{1, \ldots, n\} & \\
 & x_j \text{ free}, \quad \text{for } j \in \{1, \ldots, n\} \backslash J &
\end{array}
$$

Definition 3 (Standard form)

$$
\begin{array}{lll}
\min & z = c^T x & \\
\text{s. t.} & Ax = b & \text{only equality constraints and} \\
 & x \geq 0 & \text{all nonnegative variables}
\end{array}
$$

- The two forms are equivalent.
- Simple transformation rules allow to pass from one form to the other form.

Warning: The transformation may involve adding/deleting variables and/or constraints.

3.1 Equivalent forms

Transformation rules

- $\max c^T x = -\min(-c^T x)$

For transforming a maximization problem in a minimization one we will consider the minimization of the simmetric function and then multiplying it for -1

- $a^T x \leq b \Rightarrow \begin{cases} a^T x + s = b \\ s \geq 0 \end{cases}$, s a slack variable.

I need to add something to allow something smaller to be equal to something larger

- $a^T x \geq b \Rightarrow \begin{cases} a^T x - s = b \\ s \geq 0 \end{cases}$, s a surplus variable.

I need to add something to allow something larger to be equal to something smaller

- x_j unrestricted in sign $\Rightarrow \begin{cases} x_j = x_j^+ - x_j^- \\ x_j^+, x_j^- \geq 0 \end{cases}$. After substituting x_j with $x_j^+ - x_j^-$, we delete x_j from the problem.

any real number can be represented as the subtraction of two non negative numbers:
es: 16 = 16-0, here x^+=16 and x^-=0
-17 = 0 - 17, here x^+=0 and x^-=17

In this way we substitute the parameter xi (that can be negative) with two variables that are non-negative.

3.1 Equivalent forms

Example

General form:
$$\begin{aligned} \max \quad & f(x_1, x_2) = 2x_1 - 3x_2 \\ \text{s. t.} \quad & 4x_1 - 7x_2 \le 5 \\ & 6x_1 - 2x_2 \ge 4 \\ & x_1 \ge 0, \quad x_2 \in \mathbb{R} \end{aligned}$$

Visto che x2 non è non negativo

x2^+ x2^-

1. $x_2 = x_3 - x4,\ x_3, x_4 \ge 0$:

$$\begin{aligned} \max \quad & 2x_1 - 3x_3 + 3x_4 \\ \text{s. t.} \quad & 4x_1 - 7x_3 + 7x_4 \le 5 \\ & 6x_1 - 2x_3 + 2x_4 \ge 4 \\ & x_1, x_3, x_4 \ge 0 \end{aligned}$$

2. Introduce slack and surplus variables x_5 and x_6:

$$\begin{aligned} \max \quad & 2x_1 - 3x_3 + 3x_4 \\ \text{s. t.} \quad & 4x_1 - 7x_3 + 7x_4 + x_5 = 5 \\ & 6x_1 - 2x_3 + 2x_4 - x_6 = 4 \\ & x_1, x_3, x_4, x_5, x_6 \ge 0 \end{aligned}$$

3. Change the objective function sign:

$$\begin{aligned} \min \quad & -2x_1 + 3x_3 - 3x_4 \\ \text{s. t.} \quad & 4x_1 - 7x_3 + 7x_4 + x_5 = 5 \\ & 6x_1 - 2x_3 + 2x_4 - x_6 = 4 \\ & x_1, x_3, x_4, x_5, x_6 \ge 0 \end{aligned}$$

3.1 Equivalent forms

Other straightforward transformations

- $a^T x \leq b \Leftrightarrow -a^T x \geq -b$
- $a^T x \geq b \Leftrightarrow -a^T x \leq -b$
- $a^T x = b \Leftrightarrow \begin{cases} a^T x \geq b \\ a^T x \leq b \end{cases} \Leftrightarrow \begin{cases} a^T x \geq b \\ -a^T x \geq -b \end{cases}$

3.2 Geometry of Linear Programming

Example Capital budgeting

Capital of 10 000€ and two possible investments A and B with, respectively, 4% and 6% expected return.
Determine a portfolio that maximizes the total expected return, while respecting the diversification constraints:

- at most 75% of the capital is invested in A,
- at most 50% of the capital is invested in B.

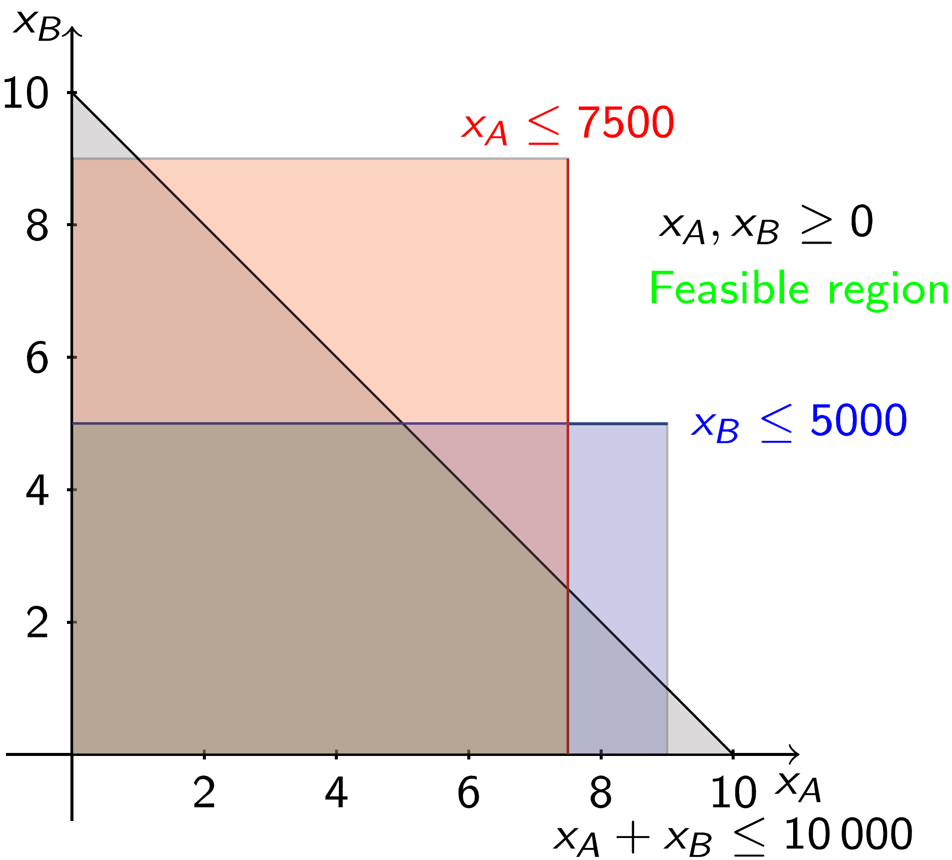
Model:

- x_A = amount invested in A
- x_B = amount invested in B

$$
\begin{array}{lrrcl}
\max & z = 0.04x_A & +0.06x_B & & \\
\text{s. t.} & x_A & +x_B & \leq & 10\,000 \\
 & x_A & & \leq & 0.75 \times 10\,000 \\
 & & x_B & \leq & 0.50 \times 10\,000 \\
 & x_A, & x_B & \geq & 0
\end{array}
$$

3.2 Geometry of Linear Programming

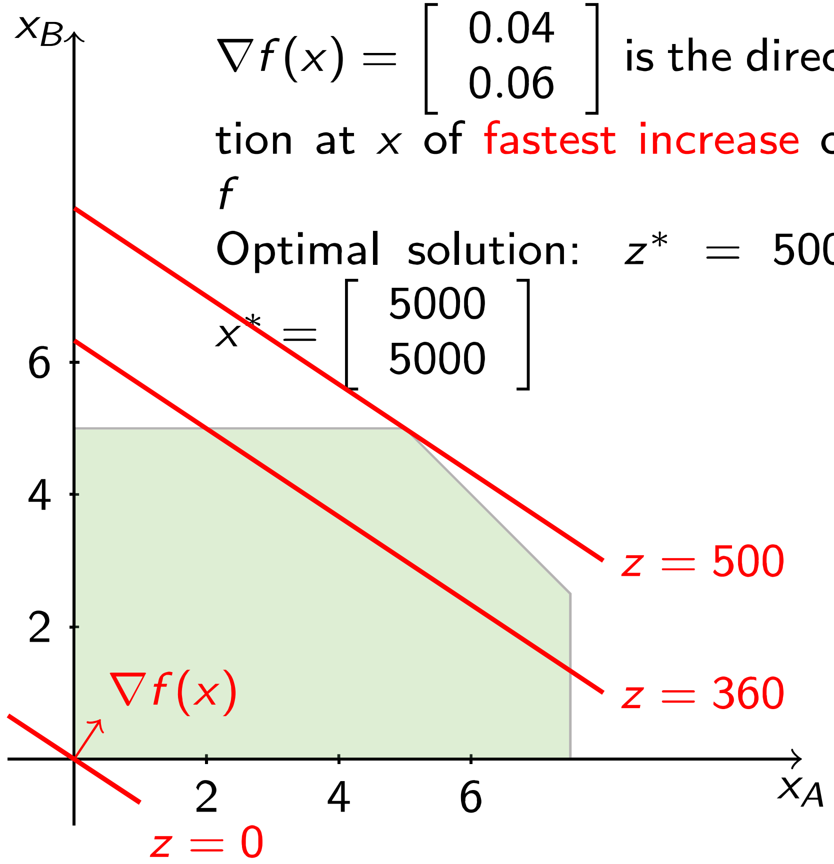
3.2.1 Graphical solution



derivata prima in zero di ciascuna delle due variabili

$\nabla f(x) = \begin{bmatrix} 0.04 \\ 0.06 \end{bmatrix}$ is the direction at x of fastest increase of f

Optimal solution: $z^* = 500$, $x^* = \begin{bmatrix} 5000 \\ 5000 \end{bmatrix}$



Definition 4

A level curve of value z of a function f is the set of points in $\mathbb{R}^n$ where f is constant and takes value z.

The level curves of a LP are lines: $0.04x_A + 0.06x_B = z$ (where z is a constant)

3.2 Geometry of Linear Programming

3.2.2 Vertices of the feasible region

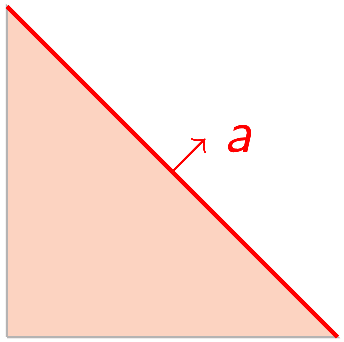
Consider a LP with inequality constraints (easier to visualize).

Definition 5

- $H = \{x \in \mathbb{R}^n : a^T x = b\}$ is a hyperplane;
- $H^- = \{x \in \mathbb{R}^n : a^T x \leq b\}$ is an affine half-space (half-plane in $\mathbb{R}^2$).

Each inequality constraint ($a^T x \leq b$) defines an affine half-space in the variable space.

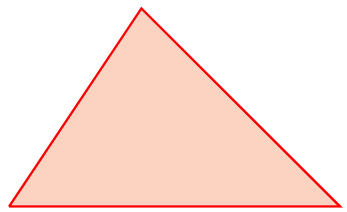
$$H = \{x \in \mathbb{R}^n : a^T x \leq b\}, \quad a \neq 0$$



3.2 Geometry of Linear Programming

Definition 6

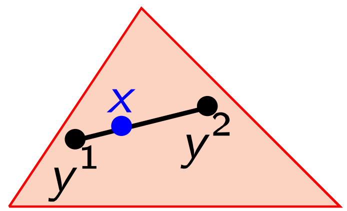
The feasible region X of any LP is a polyhedron P. (Intersection of a finite number of half-spaces.)



P can be empty or unbounded

Definition 7

A subset $S \subseteq \mathbb{R}^n$ is convex if for each pair $y^1, y^2 \in S$, S contains the whole segment connecting y^1 and y^2.

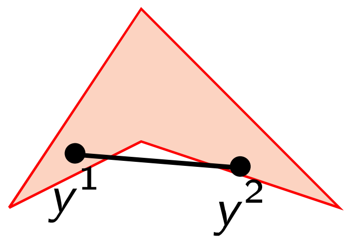


convex

Definition 8

The segment defined by $y^1, y^2 \in S$, defined by all the convex combinations of y^1 and y^2 is

$$[y^1, y^2] = \{x \in \mathbb{R}^n : x = \alpha y^1 + (1-\alpha) y^2 \wedge \alpha \in [0,1]\}$$



not convex

3.2 Geometry of Linear Programming

Property 1

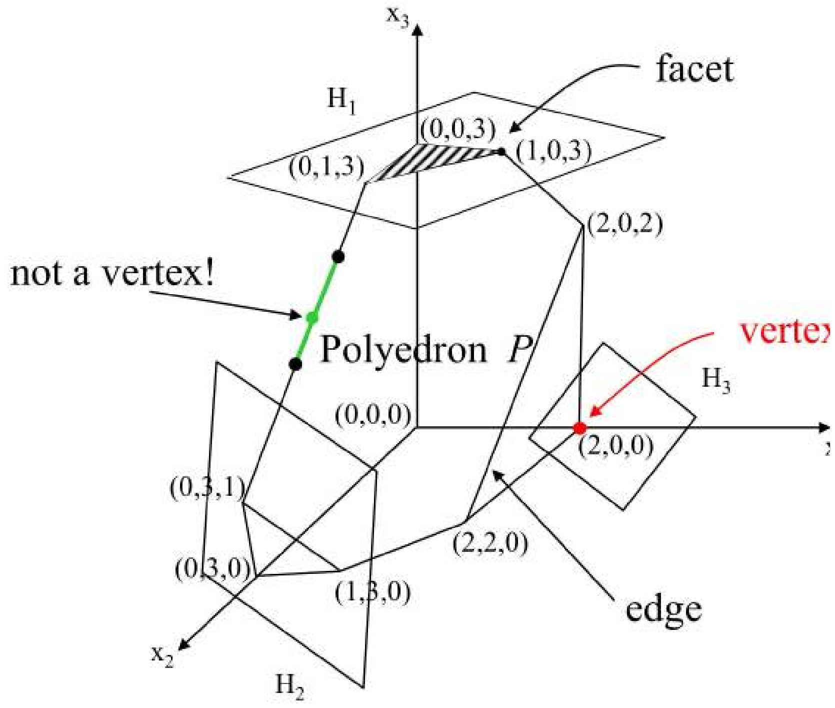
A polyhedron P is a convex set of $\mathbb{R}^n$.

Indeed:

- any half-space is convex;
- and the intersection of a finite number of convex sets is also a convex set.

Definition 9

A vertex of P is a point of P which cannot be expressed as a convex combination of two other distinct points of P.



Algebraically, x is a vertex of P iff:

$$x = \alpha y^1 + (1-\alpha) y^2,\ \alpha \in [0,1],\ y^1, y^2 \in P \Rightarrow x = y^1 \vee x = y^2$$

3.2 Geometry of Linear Programming

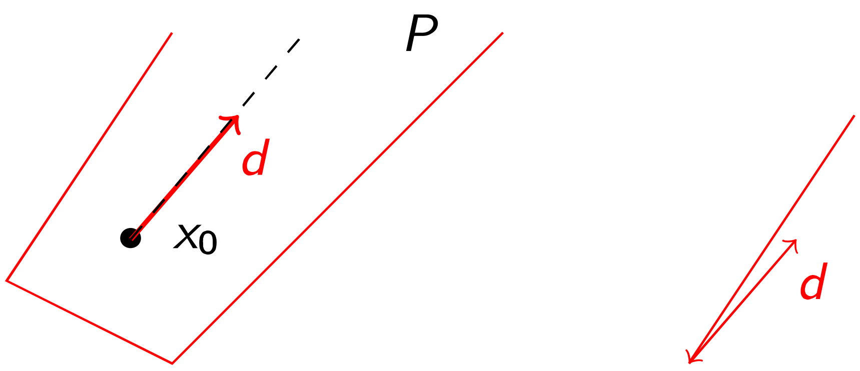
Property 2

A non-empty polyhedron $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ *(in standard form) or* $P = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$ *(in canonical form) has a finite number* (≥ 1) *of vertices.*

Definition 10

Given a polyhedron P, a vector $d \in \mathbb{R}^n$ with $d \neq 0$ is an unbounded feasible direction of P if, for every point $x_0 \in P$, the "ray" $\{x \in \mathbb{R}^n : x = x_0 + \lambda d \wedge \lambda \geq 0\}$ is contained in P.

Identificata una direzione, se prendo un qualsiasi punto nel poliedro e lo sparo lungo la direzione scelta non uscirò mai dal poliedro.
In questo caso tale direzione è detta "unbounded feasible direction" del poliedro.



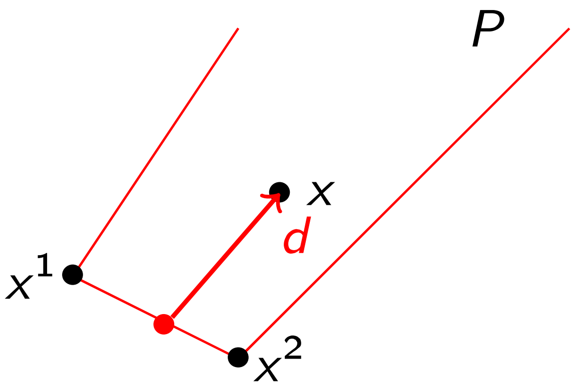
3.2 Geometry of Linear Programming

Theorem 11 (representation of polyhedra – Weyl-Minkowski)

Every point x of a polyhedron P can be expressed as a convex combination of its vertices $x^1, \ldots, x^k$ plus (if needed) an unbounded feasible direction d of P:

$$x = \alpha_1 x^1 + \cdots + \alpha_k x^k + d$$

where the multipliers $\alpha_i \geq 0$ satisfy $\alpha_1 + \cdots + \alpha_k = 1$.



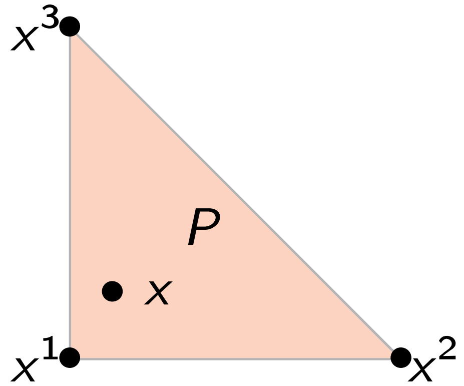
$\forall x \in [x^1, x^2], d = 0$

3.2 Geometry of Linear Programming

Definition 12

A polytope is a bounded polyhedron, that is, it has the only unbounded feasible direction $d = 0$.

Consequence Every point x of a polytope P can be expressed as a convex combination of its vertices.



Then, $x = \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3$, with $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$ $(d = 0)$.

3.2 Geometry of Linear Programming

Theorem 13 (Fundamental theorem of Linear Programming)

Consider a LP $\min\{c^T x : x \in P\}$*, where* $P \subseteq \mathbb{R}^n$ *is a non-empty polyhedron of the feasible solutions (in standard or canonical form). Then either there exists (at least) one optimal vertex or the value of the objective function is unbounded below on* P*.*

Proof.

- Case 1: P has an unbounded feasible direction d such that $c^T d < 0$.
  P is unbounded and the values $z = c^T x$ tend to $-\infty$ along the direction d.
- Case 2: P has no unbounded feasible direction d such that $c^T d < 0$, that is, for all of them we have $c^T d \geq 0$.
  Any point of P can be expressed as:
  $$x = \sum_{i=1}^{k} \alpha_i x^i + d,$$
  where $x^1, \ldots, x^k$ are the vertices of P, $\alpha_i \geq 0$ with $\alpha_1 + \cdots + \alpha_k = 1$, and $d = 0$, or d is an unbounded feasible direction.

$(\cdots)$

□

3.2 Geometry of Linear Programming

Proof.

- Case 2 (cont.):
  For any $x \in P$, we have $d = 0$ or $c^T d \geq 0$ and hence

$$c^T x = c^T \left(\sum_{i=1}^{k} \alpha_i x^i + d \right) = \sum_{i=1}^{k} \alpha_i c^T x^i + c^T d \geq \min_{i=1,\dots,n} \{c^T x^i\}$$

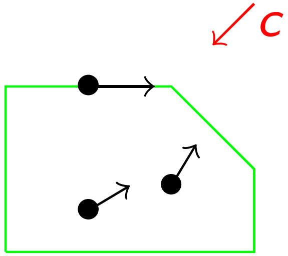
  since $\alpha_i \geq 0$, for any i, and $\alpha_1 + \cdots + \alpha_k = 1$.

$\square$

3.2 Geometry of Linear Programming

Geometrically

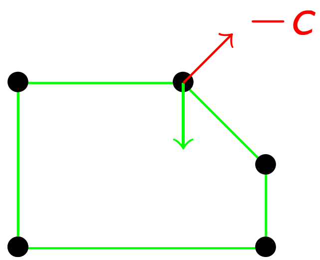
An interior point $x \in P$ cannot be an optimal solution:



c = direction of fastest increase in z (constant gradient)

$\exists$ always an improving direction

In an optimal vertex all feasible directions (for a sufficiently small step) are "worsening" directions:



• vertices
improving directions
feasible directions

The theorem implies that, although the variables can take fractional values, LPs can be viewed as combinatorial problems. Thus:

> *We "only" need to examine the vertices of the polyhedron of the feasible solutions!*

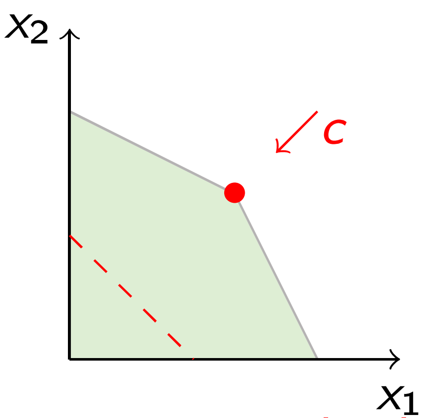
However:

- these are finite but often exponential;
- and the graphical method is only applicable for $n \leq 3$.

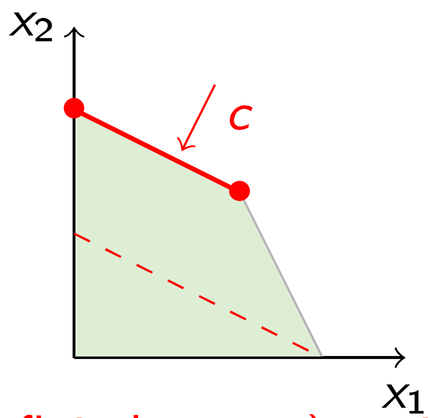
3.2 Geometry of Linear Programming

3.2.3 Four types of Linear Programs

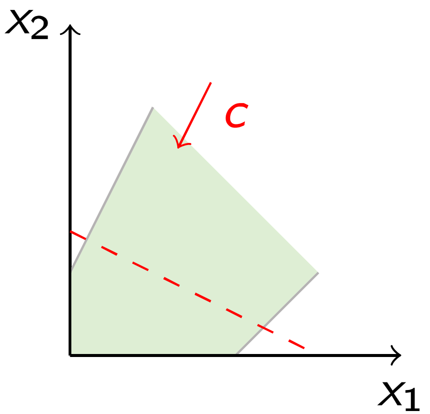
Note: Since we want to min $c^T x$, better solutions are found by moving along $-c$.



A unique optimal solution



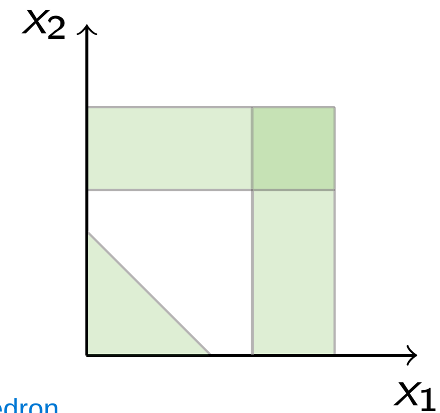
Multiple (infinitely many) optimal solutions



Unbounded polyhedron and unlimited objective function value

Unbounded LP

If we have an unbounded polyhedron we could still have a bounded solution (if c comes from a bound)



Empty polyhedron (no feasible solution)

Infeasible LP