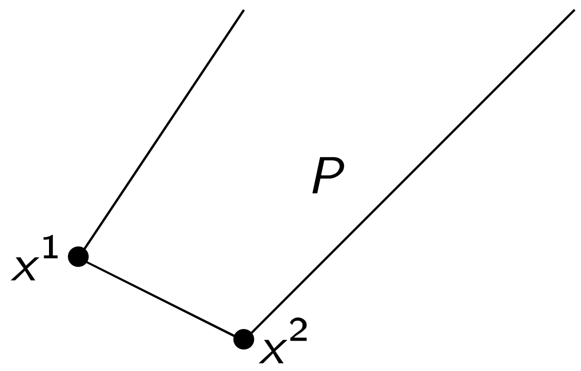
3.3 Basic feasible solutions and vertices of polyhedra

Due to the fundamental theorem of Linear Programming, to solve any LP it suffices to consider the vertices (finitely many) of the polyhedron P of the feasible solutions.



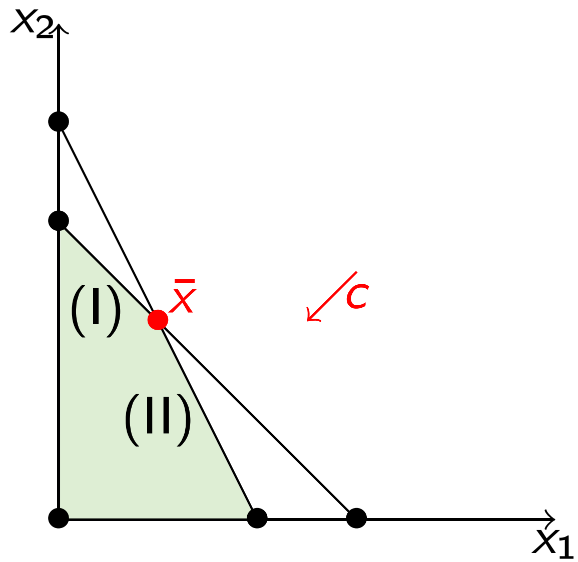
Since the geometrical definition of vertex cannot be exploited algorithmically, we need an algebraic characterization.

3.3 Basic feasible solutions and vertices of polyhedra

Which are the vertices of $P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ with only inequalities?

Example

$$\begin{array}{lrcl} \min & -x_1 - 3x_2 & & \\ \text{s. t.} & x_1 + x_2 & \leq & 6 \quad (\mathrm{I}) \\ & 2x_1 + x_2 & \leq & 8 \quad (\mathrm{II}) \\ & x_1, x_2 & \geq & 0 \end{array}$$



A vertex corresponds to the intersection of the hyperplanes associated to n inequalities.

Vertex $\bar{x}$ is the intersection of the hyperplanes of (I) and (II), i.e., the solution of equations

$$\begin{cases} x_1 + x_2 = 6 \\ 2x_1 + x_2 = 8 \end{cases}$$

3.3 Basic feasible solutions and vertices of polyhedra

What about the vertices of polyhedra expressed in standard form?

$$P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

We want to solve LPs in standard form. However, these are easier to describe if we start from

$$P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\},$$

transform it into standard form

$$P' = \{x \in \mathbb{R}^n : Ax + s = b, x, s \geq 0\}$$

and rename: $A^{'} := [A|I]$ and $x^{'} := (x^T|s^T)$, where A has m rows.

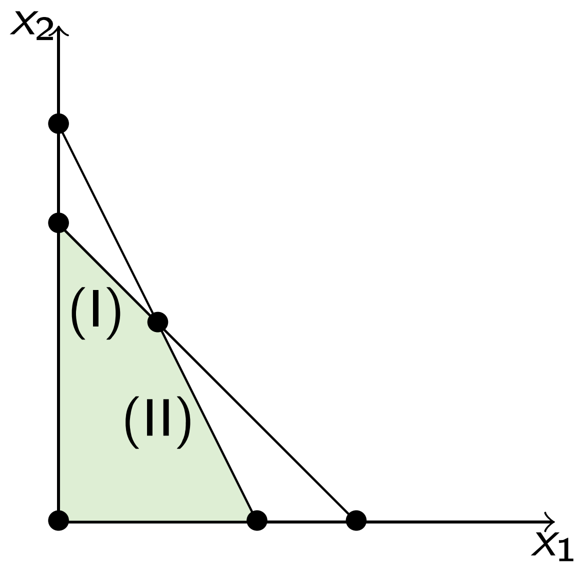
$Ax + s = b$ è lo stesso di fare $A'x' = b$

$$\overset{A}{\begin{bmatrix} A \end{bmatrix}}\overset{I}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}} \begin{bmatrix} x^T \\ s^T \end{bmatrix}$$

3.3 Basic feasible solutions and vertices of polyhedra

Example

$$(P)\quad \begin{array}{rcl} x_1 + x_2 & \leq & 6 \\ 2x_1 + x_2 & \leq & 8 \\ x_1, x_2 & \geq & 0 \end{array} \Rightarrow (P')\quad \begin{array}{rcl} x_1 + x_2 + s_1 & = & 6 \\ 2x_1 + x_2 + s_2 & = & 8 \\ x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$



Taking the intersection of the lines associated to (I) and (II) in P, amounts in P' to let $s_1 = s_2 = 0$.

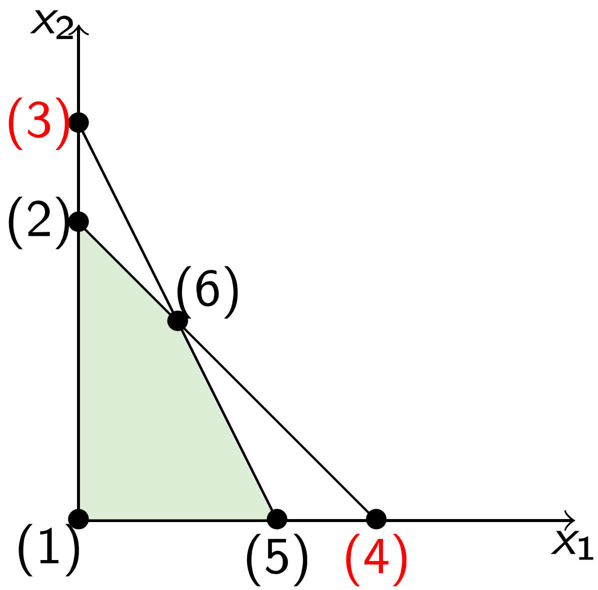
Notes:

- Every constraint in P corresponds to a slack variable in P'. When the slack variable is set to 0 the constraint is satisfied with "=".
  For example, $s_1 = 0 \to x_1 + x_2 = 6$.
- Any vertex of P is the intersection of the hyperplanes associated to n inequalities. This is equivalent to setting the corresponding variables in P' to 0.

3.3 Basic feasible solutions and vertices of polyhedra

Example (continued)
Compute all the intersections

$$\begin{aligned} x_1 + x_2 + s_1 &= 6 \\ 2x_1 + x_2 + s_2 &= 8 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$



(1) $x_1 = 0, x_2 = 0 \Rightarrow s_1 = 6, s_2 = 8$

(2) $x_1 = 0, s_1 = 0 \Rightarrow x_2 = 6, s_2 = 2$

(3) $x_1 = 0, s_2 = 0 \Rightarrow x_2 = 8, s_1 = -2$

(4) $x_2 = 0, s_1 = 0 \Rightarrow x_1 = 6, s_2 = -4$

(5) $x_2 = 0, s_2 = 0 \Rightarrow x_1 = 4, s_1 = 2$

(6) $s_1 = 0, s_2 = 0 \Rightarrow x_1 = 2, x_2 = 4$

The intersections where some x_j or s_i are < 0 yield infeasible solutions.

3.3 Basic feasible solutions and vertices of polyhedra

Which are the vertices of a polyhedron in standard form?

Example

$$P = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1, x \geq 0\}$$

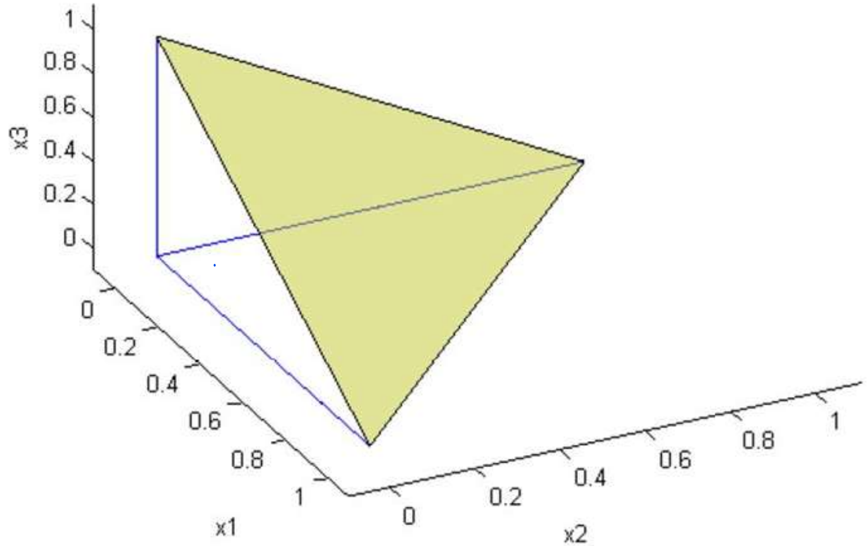
n. variabili

with $n = 3$ and $m = 1$.

n. equazioni

$A : m \times n$

$x : n \times 1$



Property 3

For any polyhedron $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$:

1. *the facets (edges in $\mathbb{R}^2$) are obtained by setting one variable to 0,*
2. *the vertices are obtained by setting n-m variables to 0.*

In the above example: $3 - 1 = 2$ variables set to 0 for vertices.

3.3 Basic feasible solutions and vertices of polyhedra

Algebraic characterization of the vertices

Consider any $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ in standard form.

Assumption:
$A \in \mathbb{R}^{m \times n}$ is such that $m \leq n$ of rank m (A is of full rank). This is equivalent to assume that there are no "redundant" constraints.

Example

$$
\begin{array}{cccll}
x_1 & +x_2 & +x_3 & = & 2 \text{ (I)} \\
x_1 & +x_2 & & = & 1 \text{ (II)} \\
x_1 & & +x_3 & = & 1 \text{ (III)} \\
x_1, & x_2, & x_3 & \geq & 0
\end{array}
$$

Since (I) = (II) + (III), then (I) can be dropped.

- If $m = n$, there is a unique solution of $Ax = b$. ($x = A^{-1}b$)
- If $m < n$, there are ∞ solutions of $Ax = b$: the system has $n - m$ degrees of freedom ($n - m$ variables can be fixed arbitrarily). By fixing them to 0, we get a vertex.

3.3 Basic feasible solutions and vertices of polyhedra

$$P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

with n variables, m constraints, and $A \in \mathbb{R}^{m \times n}$.

Definition 14

A basis of such a matrix A is a subset of m columns of A that are linearly independent and form an $m \times m$ non singular matrix B.

$$A = [\ \overbrace{B}^{m}\ |\ \overbrace{N}^{n-m}\ ]$$

First permute the columns of A, then partition A into $[B|N]$

3.3 Basic feasible solutions and vertices of polyhedra

Let $x^T = [\ \overbrace{x_B^T}^{m \text{ components}} \mid \overbrace{x_N^T}^{n-m \text{ components}}\ ]$, then any system $Ax = b$ can be written as $Bx_B + Nx_N = b$ and for any set of values for x_N, if B is non singular, we have

$$x_B = B^{-1}b - B^{-1}Nx_N$$

Definition 15

- A basic solution is a solution obtained by setting $x_N = 0$ and, consequently, letting $x_B = B^{-1}b$.
- A basic solution with $x_B \geq 0$ is a basic feasible solution.
- The variables in x_B are the basic variables and those in x_N are then non basic variables.

Note: By construction (x_B^T, x_N^T) together satisfy $Ax = b$.

Theorem 16

$x \in \mathbb{R}^n$ is a basic feasible solution iff x is a vertex of $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$.

3.3 Basic feasible solutions and vertices of polyhedra

Example

$$
\begin{array}{lllllllll}
\min & 2x_1 + x_2 + 5x_3 & & & & & & & \\
\text{s. t.} & x_1 + x_2 & +x_3 & +x_4 & & & & = 4 \\
& x_1 & & & +x_5 & & & = 2 \\
& & x_3 & & & +x_6 & & = 3 \\
& 3x_2 & +x_3 & & & & +x_7 & = 6 \\
& x_1, \; x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq 0
\end{array}
\quad
A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}
\quad
b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}
$$

- Choosing columns 4, 5, 6, 7, we have $B = I = B^{-1}$, so $x_B = B^{-1}b = b \geq 0$, thus we obtain a basic feasible solution.
- Choosing columns 2, 5, 6, 7, we have

$$
B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}, \; B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}, \; x_B = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -6 \end{bmatrix}
$$

and thus we obtain an infeasible basic solution.

3.3 Basic feasible solutions and vertices of polyhedra

Number of basic feasible solutions

At most one basic feasible solution for each choice of the $n - m$ non basic variables (to be set to zero) out of the n variables:

$$\#\text{ basic feasible solutions } \leq \binom{n}{n-m} = \frac{n!}{(n-m)!(n-(n-m))!} = \binom{n}{m}.$$