4.2 Cutting plane methods and Gomory fractional cuts

Consider a generic ILP problem:

$$(ILP) \quad \begin{array}{ll} \min & \underline{c}^T \underline{x} \\ \text{s.t.} & A\underline{x} \geq \underline{b} \\ & \underline{x} \geq \underline{0} \text{ integer} \end{array}$$

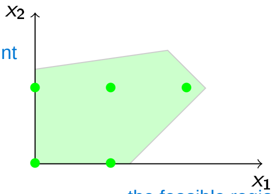
Feasible region:
$X = \{\underline{x} \in \mathbb{Z}^n : A\underline{x} \geq \underline{b}, \underline{x} \geq \underline{0}\}$

Assumption: a_{ij}, c_j and b_i integer. --> if are rational we can multiply each for sufficiently large numbers in order to have integer numbers

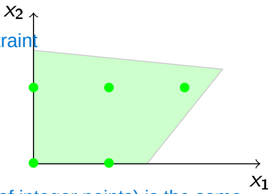
Note: The feasible region of an ILP can be described by different sets of constraints that may be weaker/tighter.

That is, infinite numbers of different formulations can lead to the same feasible region

ex: 3 constraint



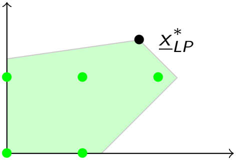
2 constraint

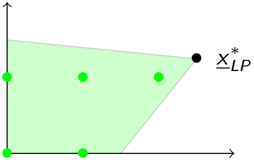


the feasible region (in terms of integer points) is the same

Infinitely many formulations!

Equivalent and ideal formulations





There are ∞ formulations

All formulations (with integrality constraints) are equivalent but the optimal solutions of the linear relaxations ($\underline{x}^*_{LP}$) can differ substantially.

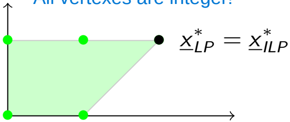
However, in terms of the LP relaxation, the feasible solution can differ a lot

Definition 4 Would be nice to get that LP feasible region that allows me to have vertex of the region that are integral points

The ideal formulation is that describing the convex hull of X, $conv(X)$, where $conv(X)$ is the smallest convex subset containing X.

All vertexes are integer!

Since all vertices have all integer coordinates, for any $\underline{c}$ we have $z^*_{LP} = z^*_{ILP}$ and the LP optimum is also the ILP optimum!



2 constraint + non-negativity constraint

Theorem 5

For any feasible region X of an ILP (bounded or unbounded), there exists an ideal formulation (a description of $conv(X)$ involving a finite number of linear constraints) but the number of constraints can be very large (exponential) with respect to the size of the original formulation.

In theory, the solution of any ILP can be reduced to that of a single LP!

However, the ideal formulation is often either very large or very difficult to determine.

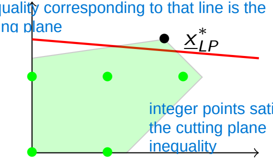
4.2.1 Cutting plane methods

A full description of $conv(X)$ is not required, we just need a good description in the neighborhood of the optimal solution.

the red line divides the value of the optimal solution of the relaxation by the all feasible solution of the correspondinf ILP. The inequality corresponding to that line is the cutting plane

Definition 6

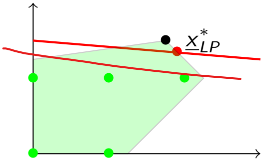
A cutting plane is an inequality $\underline{a}^T \underline{x} \leq b$ that is not satisfied by $\underline{x}^*_{LP}$ but is satisfied by all the feasible solutions of the ILP.



Idea:

Given an initial formulation, iteratively add cutting planes as long as the linear relaxation does not provide an optimal integer solution.

new cutting plain



1. Solve the relation
2. Check if that the solution has integer coordinates --> lucky case --> stop
3. if not, generate a cutting plane and start again

How to calculate the cutting plane?

4.2.2 Gomory fractional cuts

Let $\underline{x}^*_{LP}$ be an optimal solution of the linear relaxation of the current formulation $\min\{\underline{c}^T\underline{x} : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\}$ and $x^*_{B[r]}$ be a fractional basic variable. -> there must be at least one otherwise we have an integer solution so we finished

Ralph Gomory 1929-

$Bx_B + Nx_N = b \Rightarrow x_B + B^{-1}N x_N = B^{-1}b \Rightarrow x_B + \bar{N}x_N = \bar{b}$

The corresponding row of the optimal tableau is:

$\rightarrow x_B + \bar{N} x_N = \bar{b} \Rightarrow$

$\Downarrow$ per la riga r

$$\rightarrow x_{B[r]} + \underbrace{\sum_{j \in N}}_{x_j \text{ non basic}} \bar{a}_{rj} x_j = \underbrace{\bar{b}_r}_{fractional} \qquad (1)$$

	$x_1 \ldots$		$x \ldots$
$x_{B[r]}$ $\bar{b}_r$	1		$\bar{a}_{rj}$
	0		

Definition 7

The Gomory cut w.r.t. the fractional basic variable $x_{B[r]}$ is:

By considering X*LP:

$$\underbrace{\sum_{j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor)}_{\text{fractional part}} \underset{0}{\overset{\text{since is 0 in } x_N}{x_j}} \geq \underbrace{(\bar{b}_r - \lfloor \bar{b}_r \rfloor)}_{\text{fractional part}}$$

> 0 since $x^*_{B[r]} = \bar{b}$ and we know it's fractional.

$x_B + \bar{N} \cdot 0 = \bar{b} \Rightarrow x_B = \bar{b}$

$$x^*_{LP} = \begin{bmatrix} x^*_B \\ x^*_N \end{bmatrix} = \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix}$$

Is it a cutting plane? First, we need to verify that the optimal solution of the relaxation doesn't verify this inequality.

Let us verify that the inequality

$$\sum_{j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor) x_j \geq (\bar{b}_r - \lfloor \bar{b}_r \rfloor)$$

is a cutting plane with respect to $\underline{x}^*_{LP}$.

1) It is violated by the optimal fractional solution $\underline{x}^*_{LP}$ of the linear relaxation:

Obvious, since $(\bar{b}_r - \lfloor \bar{b}_r \rfloor) > 0$ and $x_j = 0$, $\forall j$ s.t. x_j non basic.

in fact it is fractional by assumption

2) It is satified by all integer feasible solutions:

For each feasible solution of the linear relaxation, we have

row of the canonical form:

$$x_{B[r]} + \sum_{j \in N} \lfloor \bar{a}_{rj} \rfloor x_j \leq x_{B[r]} + \sum_{j \in N} \bar{a}_{rj} x_j = \bar{b}_r \quad x_j \geq 0 \qquad (1)$$

and, in particular, for each integer feasible solution

I can round down because on the left now I have an integer

$$x_{B[r]} + \sum_{j \in N} \lfloor \bar{a}_{rj} \rfloor x_j \leq \lfloor \bar{b}_r \rfloor \quad x_j \text{ integer} \qquad (2)$$

this is called " integer form"

By subtracting (2) from (1), for each integer feasible solution we have:

$$\sum_{j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor) x_j \geq (\bar{b}_r - \lfloor \bar{b}_r \rfloor)$$

this is called "fractional form"

The "integer" form

$$x_{B[r]} + \sum_{j \in N} \lfloor \bar{a}_{rj} \rfloor x_j \leq \lfloor \bar{b}_r \rfloor$$

and the "fractional" form

$$\sum_{j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor) x_j \geq \bar{b}_r - \lfloor \bar{b}_r \rfloor$$

of the cutting plane are equivalent.

Example

$$\begin{array}{llrrl} \max & z = & 8x_1 & +5x_2 & \\ \text{s. t.} & & x_1 & +x_2 & \leq 6 \\ & & 9x_1 & +5x_2 & \leq 45 \\ & & x_1, & x_2 & \geq 0 \text{ integer} \end{array}$$

Optimal tableau:

		x_1	x_2	s_1	s_2
$-z$	-41.25	0	0	-1.25	-0.75
x_1	3.75	1	0	-1.25	0.25
x_2	2.25	0	1	2.25	-0.25

with the fractional optimal basic solution $\underline{x}_B^* = (3.75, 2.25)^T$

The coordinates of the optimal solution are fractional so we can in principle perform 2 cuts

Select a row of the optimal tableau (a constraint) whose basic variable has a fractional value:

$$x_{B[r]} \rightarrow \quad x_1 - 1.25s_1 + 0.25s_2 = 3.75$$

Let's take the fractional parts of the aj : 0,75 for the -1,25 coefficient (floor is 2) and 0,25 for the 0,25 coefficient

Generate the corresponding Gomory cut: $0.75s_1 + 0.25s_2 \geq 0.75$.
(in fractional form)

3,75 − 3

Note: The integer and fractional parts of $a \in \mathbb{R}$ are

$$a = \lfloor a \rfloor + f, \quad \text{with } 0 \leq f < 1,$$

thus we have $-1.25 = -2 + 0.75$ and $0.25 = 0 + 0.25$.

To make the cut an equality we add a slack variable s3

Introduce the slack variable $s_3 \geq 0$ and add this cutting plane $-0.75s_1 - 0.25s_2 \leq -0.75$ to the tableau. $\rightarrow$ $-0{,}75s_1 - 0{,}25s_2 + s_3 = -0{,}75$

The new constraint "cuts" the optimal fractional solution $\underline{x}_B^* = (3.75, 2.25)^T$ of the linear relaxation of the ILP.

		x_1	x_2	s_1	s_2	s_3
$-z$	-41.25	0	0	-1.25	-0.75	0
x_1	3.75	1	0	-1.25	0.25	0
x_2	2.25	0	1	2.25	-0.25	0
s_3	-0.75	0	0	-0.75	-0.25	1

To efficiently reoptimize, we can apply a single iteration of the Dual simplex algorithm. --> appliable when we add just one constraint, thus reoptimize is very cheap

Optimal tableau:

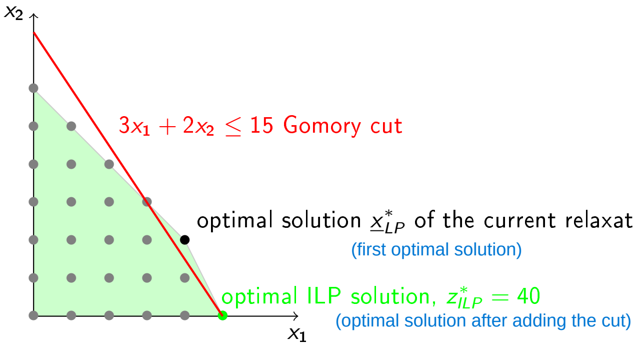
		x_1	x_2	s_1	s_2	s_3
$-z$	-40	0	0	0	-0.33	-1.67
x_1	5	1	0	0	0.67	-1.67
x_2	0	0	1	0	-1	3
s_1	1	0	0	1	0.33	-1.33

Since the optimal solution $\underline{x}^* = (5, 0, 1, 0, 0)^T$ ($z^* = 40$) of the linear relaxation of the new formulation is integer, $\underline{x}^*$ is also optimal for the original ILP.
No need to generate other Gomory cuts!

To express the Gomory cut $0.75s_1 + 0.25s_2 \geq 0.75$ in terms of the decision variables, we perform the substitution:

$$\begin{cases} s_1 = 6 - x_1 - x_2 \\ s_2 = 45 - 9x_1 - 5x_2 \end{cases} \Rightarrow \quad 3x_1 + 2x_2 \leq 15$$

Example



Very special case: original constraints + cut $\equiv$ ideal formulation!

In general we need to add a (very) large number of cuts.

4.2.3 Cutting plane method with fractional Gomory cuts

Algorithm 1: Cutting plane method with fractional Gomory cuts

```
Solve the linear relaxation min{c^T x : Ax = b, x ≥ 0}
Let x*_LP be an optimal basic feasible solution
while x*_LP has fractional components do
    Select a basic variable with a fractional value
    Generate the corresponding Gomory cut
    Add constraint to the optimal tableau of the linear relaxation
    Perform one iteration of the dual simplex algorithm
```

Solve the linear relaxation $\min\{\underline{c}^T\underline{x} : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\}$

Let $\underline{x}^*_{LP}$ be an optimal basic feasible solution

while $\underline{x}^*_{LP}$ *has fractional components* **do**

- Select a basic variable with a fractional value
- Generate the corresponding Gomory cut
- Add constraint to the optimal tableau of the linear relaxation
- Perform one iteration of the dual simplex algorithm

Theorem 8

If the ILP has a finite optimal solution, the cutting plane method finds one after adding a finite number of Gomory cuts.

(But often very large.)

In the worst case the number of needed cuts can grow exponentially wrt the size of the problem.

Example

$$\begin{array}{llrrl} \min & z = & & -x_2 & \\ \text{s. t.} & & 3x_1 & +2x_2 & \leq 6 \\ & & -3x_1 & +2x_2 & \leq 0 \\ & & x_1, & x_2 & \geq 0 \text{ integer} \end{array}$$

Solve the linear relaxation with the simplex algorithm:

		x_1	x_2	x_3	x_4
$-z$	0	0	-1	0	0
x_3	6	3	2	1	0
x_4	0	-3	2	0	1

$$x_3 = 6 - 3x_1 - 2x_2$$
$$x_4 = 3x_1 - 2x_2$$

		x_1	x_2	x_3	x_4
$-z$	0	$-3/2$	0	0	$1/2$
x_3	6	6	0	1	-1
x_2	0	$-3/2$	1	0	$1/2$

		x_1	x_2	x_3	x_4
$-z$	$3/2$	0	0	$1/4$	$1/4$
x_1	1	1	0	$1/6$	$-1/6$
x_2	$3/2$	0	1	$1/4$	$1/4$

The optimal solution $\underline{x}^* = (1, 3/2, 0, 0)^T$ has value $z^*_{LP} = -3/2$ (vertex A).

Generate the Gomory cut associated to the 2nd row: (the only fractional coordinate)

fractional form integer form

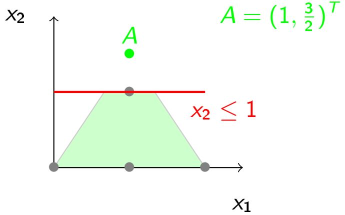
$$x_2 + x_3/4 + x_4/4 = 3/2 \Rightarrow x_2 + 0x_3 + 0x_4 \leq \lfloor 3/2 \rfloor,$$

namely the constraint $x_2 \leq 1$ (first cut).

(in canonical form)

Adding to the fractional form $1/4x_3 + 1/4x_4 \geq 1/2$ the surplus variable $x_5 \geq 0$, we obtain:

$$-1/4x_3 - 1/4x_4 + x_5 = -1/2.$$



Adding the corresponding row to the tableau:

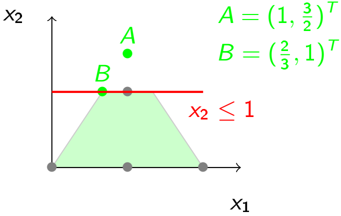
$$x_5 = -1/2 + x_3/4 + x_4/4 = 1 - x_2$$

		x_1	x_2	x_3	x_4	x_5
$-z$	$3/2$	0	0	$1/4$	$1/4$	0
x_1	1	1	0	$1/6$	$-1/6$	0
x_2	$3/2$	0	1	$1/4$	$1/4$	0
x_5	$-1/2$	0	0	$-1/4$	$-1/4$	1

To represent the cut in the original variables space, we proceed by substitution: the new surplus variable x_5 is expressed in terms of only x_1 and x_2.

The new optimal tableau: since the last solution is not feasible since the value of x5 is <=0. We need to compute the next tableu either with two face simplex (which allows us to find a feasible solution in order to then apply the simplex method), or the dual simplex method (not explained)

		x_1	x_2	x_3	x_4	x_5
$-z$	1	0	0	0	0	1
x_1	$2/3$	1	0	0	$-1/3$	$2/3$
x_2	1	0	1	0	0	1
x_3	2	0	0	1	1	-4



Optimal solution $\underline{x}^* = (2/3, 1, 2, 0, 0)^T$ is still fractional (vertex B).

The integer form of the Gomory cut associated to the 1st row is

$$x_1 - x_4 \leq \lfloor 2/3 \rfloor = 0,$$

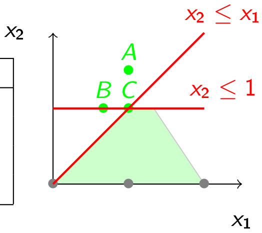
which, by replacing x_4 with $x_4 = 3x_1 - 2x_2$, is equivalent to $-2x_1 + 2x_2 \leq 0$ (second cut).

The fractional form of the cut is $2/3x_4 + 2/3x_5 \geq 2/3$, so it suffices to include the surplus variable $x_6 \geq 0$ and add the corresponding row to the "extended" tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	1	0	0	0	0	0	1
x_1	2/3	1	0	0	$-1/3$	2/3	0
x_2	1	0	1	0	0	1	0
x_3	2	0	0	1	1	-4	0
x_6	$-2/3$	0	0	0	$-2/3$	$-2/3$	1

We obtain the optimal tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	1	0	0	0	0	1	0
x_1	1	1	0	0	0	1	$-1/2$
x_2	1	0	1	0	0	1	0
x_3	1	0	0	1	0	-5	$3/2$
x_4	1	0	0	0	1	1	$-3/2$



$A = (1, 3/2)^T$
$B = (2/3, 1)^T$
$C = (1, 1)^T$

The optimal solution of the linear relaxation $\underline{x}^* = (1, 1, 1, 1, 0, 0)^T$ corresponds to the vertex C whose components are all integer.

Note:

The formulation is not ideal (the polytope has still a fractional vertex), the constraint $x_1 + x_2 \leq 2$ that is needed to describe $conv(X)$ is not required for this objective function.

Gomery cuts cut a lot at the beginning but then they start to cut very litte to almost nothing

4.2.4 Generic and specific cutting planes

There exist other types of generic cutting planes (different from the fractional Gomory cuts) and a large number of classes of cutting planes for specific problems.

The "deepest" cuts are the "facets" of $conv(X)$!

The thorough study of the combinatorial structure of various problems (e.g., TSP, set covering, set packing, ...) led to:

- characterization of entire classes of facets,
- efficient procedures for generating them.

Generate cuts --> shrink the feasible region then I stop cutting since there is the tailing effect and I start branching. I will have 2 regions, then I start again generating cuts and then branching

4.2.5 Idea of Branch-and-Cut

Branch and bound + cutting planes

The "combined" Branch-and-Cut approach aims at overcoming the disadvantages of pure Branch-and-Bound (B&B) and pure cutting plane methods.

For each subproblem (node) of B&B, several cutting planes are generated to improve the bound and try to find an optimal integer solution.

Whenever the cutting planes become less effective, cut generation is stopped and a branching operation is performed.

Very efficient approach: state of the art

Advantages:

The cuts tend to strengthen the formulation (linear relaxation) of the various subproblems.

The long series of cuts without sensible improvement are interrupted by branching operations.