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Computation of Invariant Sets for Piecewise Affine Discrete Time Systems subject to Bounded Disturbances

S. V. Raković, P. Grieder, M. Kvasnica, D. Q. Mayne, M. Morari

***Abstract*— Piecewise affine (PWA) systems are useful models for describing non-linear and hybrid systems. One of the key problems in designing controllers for these systems is the inherent computational complexity of controller synthesis and analysis. These problems are amplified in the presence of state and input constraints and additive but bounded disturbances. In this paper we exploit *set invariance* and *parametric programming* to devise an efficient robust time optimal control scheme. Specifically, the state is driven into the maximal robust invariant set $\tilde{\Omega}_\infty$ in minimum time. We show how to compute $\tilde{\Omega}_\infty$ and derive conditions for finite time computation.**

Keywords: Piecewise Affine Dynamics, Set Invariance, Constrained Control, Robust Control.

I. Introduction

Piecewise affine (PWA) systems have attracted much interest in the research community since they can approximate non-linear systems [27] and because of their equivalence to many classes of hybrid systems [12]. Optimal control of constrained PWA systems is analyzed in [6], [18] while an explicit characterization of the optimal solution is obtained in [6], [15], [19]. The explicit solution takes the form of a PWA state feedback control law, i.e. the state space is partitioned into polyhedral sets (possibly overlapping) and in each of these sets the optimal control law is an affine function of the state. On-line implementation of the explicit control law reduces to a simple set-membership test.

It is general practice to provide guarantees on constraint satisfaction by adding a (robustly) controlled invariant terminal set constraint to the optimization problem [20]. Although computation of invariant sets has garnered great interest in the control community [5], [8], [14], [16], only few results for obtaining robustly positively invariant sets for PWA systems are reported; the most relevant work is an excellent thesis [14]. Our results are based on the results for linear systems in [16], [21] as well as recent extensions to PWA systems in [10], [15].

An algorithm for computing the maximal robustly positively invariant set $\tilde{\Omega}_\infty$ is described and sufficient conditions for finite termination of this algorithm are given. The set $\tilde{\Omega}_\infty$ is subsequently used to initialize an iterative computation scheme which converges to the maximal robustly attractive set $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$. A similar scheme is applied to obtain the maximal robustly control invariant set $\tilde{\mathcal{C}}_\infty$. We also introduce a set of efficient algorithms to obtain PWA feedback controllers that guarantee robust constraint satisfaction of the closed-loop system.

This paper is organised as follows. The preliminaries are given in Section II. Section III contains efficient algorithms for computational geometry needed in subsequent sections. Section IV presents a general framework for computing the maximal robustly positively invariant set $\tilde{\Omega}_\infty$ for a subset of the autonomous piecewise affine system. Section V gives an algorithm for computing moderate complexity controllers for perturbed PWA systems. Section VI provides two interesting examples demonstrating that our algorithms provide low/moderate complexity state feedback control. Section VII contains general conclusions.

All proofs as well as fuller exposition of the results of this paper can be found in [24].

Notation and Basic Definitions: Let $\mathbb{N} \triangleq \{0, 1, \ldots, \}$, $\mathbb{N}^+ \triangleq \{1, 2, \ldots, \}$, $\mathbb{N}_q \triangleq \{0, 1, 2, \ldots, q\}$ and $\mathbb{N}_q^+ \triangleq \{1, 2, \ldots, q\}$. We use convh to denote the convex hull. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces, a *polytope* is a (convex) closed and bounded *polyhedron* and a *P–collection* is a (possibly non-convex) union of a finite number of polyhedra. Given $\mathcal{A}$ and $\mathcal{B}$ two subsets of $\mathbb{R}^n$, the following are basic set operations: set complement $\mathcal{A}^c \triangleq \{x \in \mathbb{R}^n \mid x \notin \mathcal{A}\}$; set difference $\mathcal{A} \setminus \mathcal{B} \triangleq \{x \mid x \in \mathcal{A} \text{ and } x \notin \mathcal{B}\}$; symmetric set difference $\mathcal{A} \vartriangle \mathcal{B} \triangleq (\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A}) = \{x \mid (x \in \mathcal{A} \text{ and } x \notin \mathcal{B}) \text{ or } (x \in \mathcal{B} \text{ and } x \notin \mathcal{A})\}$; the Pontryagin (Minkowski set) difference $\mathcal{A} \ominus \mathcal{B} \triangleq \{x \in \mathbb{R}^n \mid x + b \in \mathcal{A},\ \forall b \in \mathcal{B}\}$; Minkowski set addition $\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A},\ b \in \mathcal{B}\}$.

II. Preliminaries

We consider the discrete time system defined by:

$$x^+ = f(x, u, w), \tag{1}$$

where x, u and w denote, respectively, the current state, control and disturbance and x^+ denotes the successor state. The function $f(\cdot)$ is piecewise affine in each of a finite number of polytopes $\{Q_i\},\ i \in \mathbb{N}_q^+$, with disjoint interiors

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S. V. Raković and D. Q. Mayne are with Department of Electrical and Electronic Engineering, Imperial College London, United Kingdom. E-mail: sasa.rakovic@imperial.ac.uk and d.mayne@imperial.ac.uk

P. Grieder, M. Kvasnica and M. Morari are with Automatic Control Laboratory, ETH Zurich, Switzerland. grieder@control.ee.ethz.ch, kvasnica@control.ee.ethz.ch and morari@control.ee.ethz.ch

that cover the region of state space of interest, i.e. $\mathbb{X} \subseteq \cup_{i \in \mathbb{N}_q^+} Q_i$. The system satisfies:

$$f(x,u,w) \triangleq A_i x + B_i u + c_i + w, \; x \in Q_i. \tag{2}$$

System (2) is subject to the following set of constraints:

$$(x,u,w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \tag{3}$$

The sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}$ are compact and polytopic and each set contains the origin in its interior. The following definitions are needed in the sequel:

Definition 1: A set $\Psi \subseteq \mathbb{X}$ is said to be a robustly controlled invariant set for the PWA system in (2) subject to the constraints in (3) if for every $x \in \Psi$ there exists a $u \in \mathbb{U}$ such that $f(x,u,w) \in \Psi$, $\forall w \in \mathbb{W}$.

Definition 2: The set $\tilde{\mathcal{C}}_\infty$ is said to be maximal robustly control invariant for the PWA system in (2) subject to the constraints in (3) if it contains all robustly control invariant sets.

Definition 3: For a robustly control invariant set Φ, the k-step robustly attractive set $\Gamma_k(\Phi)$ for the PWA system (2) subject to the constraints in (3) is defined by:

$$\Gamma_k \triangleq \tilde{\Theta}(\Gamma_{k-1}), \; k \in \mathbb{N}^+, \; \Gamma_0 = \Phi \text{ and}$$
$$\tilde{\Theta}(\Phi) \triangleq \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ s.t. } f(x,u,w) \in \Phi, \; \forall w \in \mathbb{W}\}$$

The maximal robustly attractive set $\tilde{\mathcal{K}}_\infty(\Phi)$ is defined by the union of all Γ_k, $k \in \mathbb{N}$, i.e. $\tilde{\mathcal{K}}_\infty(\Phi) \triangleq \cup_{k \in \mathbb{N}} \Gamma_k$.

III. Geometric Computations with P–collections

We now present some tools that are required for set computations with P–collections. For computation of the set difference of two polyhedra the reader is referred to [3]. The first two results show how the set difference of a P–collection and a polyhedron (or a P–collection) may be computed:

Proposition 1: Let $\mathcal{C} \triangleq \bigcup_{j=1}^p \mathcal{C}_j$ be a P–collection, where all the $\mathcal{C}_j$, $j \in \mathbb{N}_q^+$, are non-empty polyhedra. If $\mathcal{A}$ is a non-empty polyhedron, then $\mathcal{C} \setminus \mathcal{A} = \bigcup_{j=1}^p (\mathcal{C}_j \setminus \mathcal{A})$ is a P–collection.

Proposition 2: Let the sets $\mathcal{C} \triangleq \bigcup_{j=1}^p \mathcal{C}_j$ and $\mathcal{D} \triangleq \bigcup_{k=1}^q \mathcal{D}_k$ be P–collections, where all the $\mathcal{C}_j$, $j \in \mathbb{N}_p^+$, and $\mathcal{D}_k$, $k \in \mathbb{N}_q^+$, are non-empty polyhedra. If $\mathcal{E}_0 \triangleq \mathcal{C}$ and $\mathcal{E}_k \triangleq \mathcal{E}_{k-1} \setminus \mathcal{D}_k$, $k \in \mathbb{N}_q^+$ then $\mathcal{C} \setminus \mathcal{D} = \mathcal{E}_q$ is a P–collection. The reader is referred to [25] for proofs and comments on computational efficiency. If $\mathcal{C}$ and $\mathcal{D}$ are P–collections it follows that $\mathcal{C} \subseteq \mathcal{D}$ can be easily verified since $\mathcal{C} \subseteq \mathcal{D} \Leftrightarrow \mathcal{C} \setminus \mathcal{D} = \emptyset$, similarly $\mathcal{C} = \mathcal{D}$ is also easily verified since

$$\mathcal{C} = \mathcal{D} \Leftrightarrow (\mathcal{C} \setminus \mathcal{D} = \emptyset \text{ and } \mathcal{D} \setminus \mathcal{C} = \emptyset) \Leftrightarrow \mathcal{C} \triangle \mathcal{D} = \emptyset$$

An efficient algorithm for computing the Pontryagin (Minkowski Set) difference of a P–collection and a polytope is discussed next. If $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathbb{R}^n$ it is known that (see for instance [14], [26]), $\mathcal{A} \ominus \mathcal{B} = [\mathcal{A}^c \oplus (-\mathcal{B})]^c$. The following algorithm implements the computation of the Pontryagin difference of a P–collection $\mathcal{C} \triangleq \cup_{j \in \mathbb{N}_p^+} \mathcal{C}_j$, where $\mathcal{C}_j, j \in \mathbb{N}_p^+$ are polytopes in $\mathbb{R}^n$, and a polytope $\mathcal{B} \subset \mathbb{R}^n$.

Algorithm 3.1 ($\mathcal{C} \ominus \mathcal{B}$):

1) Input: P–collection $\mathcal{C}$, polytope $\mathcal{B}$
2) $\mathcal{H} = \mathrm{convh}(\mathcal{C})$
3) $\mathcal{D} = \mathcal{H} \ominus \mathcal{B}$
4) $\mathcal{E} = \mathcal{H} \setminus \mathcal{C}$
5) $\mathcal{F} = \mathcal{E} \oplus (-\mathcal{B})$
6) $\mathcal{G} = \mathcal{D} \setminus \mathcal{F}$
7) Output: P–collection $\mathcal{G}$

Proposition 3: [24] Consider Algorithm 3.1, then $\mathcal{G} = \mathcal{C} \ominus \mathcal{B}$.

Algorithm 3.1 is illustrated on sample P–collections in Figures 1(a) to 1(f). It is important to note that in general $\mathcal{C} \ominus \mathcal{B} \neq \cup_{j \in \mathbb{N}_p^+} (\mathcal{C}_j \ominus \mathcal{B})$, but only $\cup_{j \in \mathbb{N}_p^+} (\mathcal{C}_j \ominus \mathcal{B}) \subseteq \mathcal{C} \ominus \mathcal{B}$ (*set equality* holds only in a very limited number of cases). Algorithm 3.1 for computation of the Pontryagin difference is conceptually similar to that proposed in [14], [15], [26]. However, computing the convex hull in the first step significantly reduces (in general) the number of sets obtained at step 3, which in turn results in fewer Minkowski set additions. Since computation of Minkowski set addition is expensive, a reasonable runtime improvement is expected. In principle, computation of the convex hull can be replaced by computation of any convex set containing the P–collection $\mathcal{C}$ – an easily computable alternative to the convex hull is $\mathcal{H} = \mathrm{env}(\mathcal{C})$, where env denotes the envelope [2]. Necessary computations can be efficiently implemented by using standard computational geometry software such as [9], [17], [28].

IV. Invariant Set Computation for PWA Systems

We first address the computation of a robustly positively invariant set for PWA systems around the origin. We assume that the origin is an equilibrium of system $x^+ = f(x,u,0)$ where $f(\cdot)$ is defined in (2), so that $c_i = 0$ for all $i \in \mathbb{N}_q^0$, where $\mathbb{N}_q^0 \subseteq \mathbb{N}_q^+$, is defined by:

$$\mathbb{N}_q^0 \triangleq \{i \in \mathbb{N}_q^+ \mid 0 \in Q_i\}$$

where 0 is the origin of the state space. Following [7], [10], [22], we assume that a stabilizing piecewise linear control law $\kappa(x)$ for the nominal system $x^+ = A_i x + B_i u$, $x \in Q_i$, $i \in \mathbb{N}_q^0$ along with the associated common quadratic Lyapunov function can be computed, so that

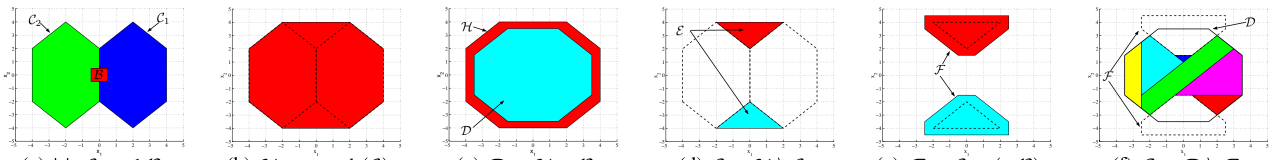
$$\kappa(x) = K_i x \text{ if } x \in Q_i^*, \; \forall i \in \mathbb{N}_q^0$$

where

$$Q_i^* \triangleq \{x \mid x \in Q_i \cap \mathbb{X}, \; K_i x \in \mathbb{U}\}, \; i \in \mathbb{N}_q^0$$

and we define

$$\mathbb{X}_0 \triangleq \bigcup_{i \in \mathbb{N}_q^0} Q_i^* \tag{4}$$



(a) $\bigcup_j \mathcal{C}_j$ and $\mathcal{B}$. (b) $\mathcal{H} = \mathrm{convh}(\mathcal{C})$. (c) $\mathcal{D} = \mathcal{H} \ominus \mathcal{B}$. (d) $\mathcal{E} = \mathcal{H} \setminus \mathcal{C}$. (e) $\mathcal{F} = \mathcal{E} \oplus (-\mathcal{B})$. (f) $\mathcal{G} = \mathcal{D} \setminus \mathcal{F}$.

Fig. 1. Graphical Illustration of Algorithm 3.1.

Thus we consider the following autonomous system:

$$x^+ = f_a(x, w) \triangleq (A_i + B_i K_i)x + w, \ x \in Q_i^*, \ i \in \mathbb{N}_q^0. \quad (5)$$

and we will aim to compute a robustly positively invariant subset contained in $\mathbb{X}_0$ defined in (4). For any integer k, $\mathbf{w}_k$ denotes the sequence $\{w(0), w(1), \dots w(k-1)\}$, and $\phi(k; x_0, \mathbf{w}_k)$ denotes the solution of $x^+ = f_a(x, w)$ at time k if the initial state is x_0 and the disturbance sequence is $\mathbf{w}_k$. Given the non-empty set $\mathcal{A}$ and a function $f_a(\cdot)$ defined in (5) let

$$\tilde{\Psi}_k(\mathcal{A}) \triangleq \{\phi(k; x, \mathbf{w}_k) \mid x \in \mathcal{A}, \mathbf{w}_k \in \mathbb{W}^k\}$$

denote the k step reachable set and let

$$\Psi_k(\mathcal{A}) \triangleq \{\phi(k; x, \mathbf{0}) \mid x \in \mathcal{A}\}$$

denote the k step reachable set for the nominal system $x^+ = f_a(x, 0)$, where for any $k \in \mathbb{N}$, $\mathbb{W}^k \triangleq \mathbb{W} \times \mathbb{W} \times \dots \mathbb{W}$.

Definition 4: The maximal positively invariant set, Ω_∞, for the discrete time system $x^+ = f_a(x, 0)$, where $f_a(\cdot)$ is defined in (5) subject to the constraints in (4) is defined by

$$\Omega_\infty \triangleq \{x \in \mathbb{X}_0 \mid \phi(k; x, \mathbf{0}) \in \mathbb{X}_0, \forall k \in \mathbb{N}\}.$$

Definition 5: The maximal robustly positively invariant set, $\tilde{\Omega}_\infty$, for the discrete time system $x^+ = f_a(x, w)$, where $f_a(\cdot)$ is defined in (5), subject to the constraints in (4) is defined by:

$$\tilde{\Omega}_\infty \triangleq \{x \in \mathbb{X}_0 \mid \phi(k; x, \mathbf{w}_k) \in \mathbb{X}_0, \\ \forall \mathbf{w}_k \in \mathbb{W}^k, \ \forall k \in \mathbb{N}\}.$$

A. The Maximal Robustly Positively Invariant Set

Consider now the perturbed discrete-time system, defined in (5). The set of the states $\tilde{\Theta}_a(\Omega)$ that robustly evolve to $\Omega \subseteq \mathbb{X}_0$ for all $w \in \mathbb{W}$ in one step is:

$$\tilde{\Theta}_a(\Omega) \triangleq \{x \in \mathbb{X}_0 \mid f_a(x, w) \in \Omega, \ \forall w \in \mathbb{W}\}. \quad (6)$$

If $\Omega \subseteq \mathbb{X}_0$ is a P–collection, then the set $\tilde{\Theta}_a(\Omega)$ is also a P–collection by definition (6). Algorithm 4.1 provides a procedure for computing the maximal robustly positively invariant subset [1], [4], [8], [14]:

Algorithm 4.1 (Computation of $\tilde{\Omega}_\infty$):

1) $\Omega_0 = \mathbb{X}_0$
2) $\Omega_{k+1} = \tilde{\Theta}_a(\Omega_k)$
3) If $\Omega_{k+1} = \Omega_k$, return; Else, set $k = k + 1$ and goto 2.

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal robustly positively invariant set $\tilde{\Omega}_\infty$, otherwise $\tilde{\Omega}_\infty = \bigcap_{k \geq 0} \Omega_k$, however if $\Omega_k = \emptyset$ for some integer k then the simple conclusion is that $\tilde{\Omega}_\infty = \emptyset$.

Lemma 1: Let the set sequence $\{\Omega_k\}$ be generated by Algorithm 4.1. Then for any integer k the set Ω_k is a P–collection.

Some of the properties of Algorithm 4.1 may be established, see [1], [4], [8], [14] and also [13] where similar results are reported for unperturbed linear switched systems.

Lemma 2: Let the set Ω_j for some finite j be a fixed point (i.e., $\Omega_j = \Omega_{j+1}$) of Algorithm 4.1, then Ω_j is the maximal robustly positively invariant set.

B. Finite Termination of the Computation of the Maximal Robustly Positively Invariant Subset

We isolate a set of conditions that are sufficient to guarantee finite time termination of Algorithm 4.1. Our first step is to introduce the augmented system

$$x^+ = f_a^{aug}(x, w) \triangleq A^{aug} x + w \quad (7a)$$
$$A^{aug} \in \mathbb{A} \triangleq \{(A_i + B_i K_i), \ i \in \mathbb{N}_q^0\}. \quad (7b)$$

This augmented system $f_a^{aug}(x, w)$ corresponds to system (5) with $Q_i = \mathbb{R}^n$, i.e., any dynamic $i \in \mathbb{N}_q^0$ may be active at any time step. Let $\phi^{aug}(k, x_0, \mathbf{w_k})$ denote the set of states which is reachable from the initial state x_0 in k steps for $x^+ = f_a^{aug}(x, w)$ and a disturbance sequence $\mathbf{w_k}$. The k-step nominal and robust reachable sets for the augmented system are then given, respectively by:

$$\Psi_k^{aug}(\mathcal{A}) \triangleq \{\phi^{aug}(k, x_0, 0) | x_0 \in \mathcal{A}\}$$

and

$$\tilde{\Psi}_k^{aug}(\mathcal{A}) \triangleq \{\phi^{aug}(k, x_0, \mathbf{w_k}) | \mathbf{x_0} \in \mathcal{A}, \ \mathbf{w_k} \in \mathbb{W}^{\mathbf{k}}\}$$

Let the set $\mathcal{F}_k$ ($\mathcal{F}_k^{aug}$) be the k-step disturbance response for system defined in (5) ((7)) so that

$$\mathcal{F}_{k+1} \triangleq \tilde{\Psi}_1(\mathcal{F}_k), \quad \mathcal{F}_1 \triangleq \{0\},$$
$$\mathcal{F}_{k+1}^{aug} \triangleq \tilde{\Psi}_1^{aug}(\mathcal{F}_k^{aug}), \quad \mathcal{F}_1^{aug} \triangleq \{0\}.$$

It follows trivially, from definitions of the corresponding sets, that $\Psi_k^{aug}(\mathcal{A}) \supseteq \Psi_k(\mathcal{A})$, $\tilde{\Psi}_k^{aug}(\mathcal{A}) \supseteq \tilde{\Psi}_k(\mathcal{A})$, $\mathcal{F}_k^{aug} \supseteq \mathcal{F}_k$ for all $k \in \mathbb{N}$. As shown in [23], the set $\mathcal{F}_\infty^{aug} \triangleq \lim_{k \to \infty} \mathcal{F}_k^{aug}$ exists and is bounded by a compact robustly positively invariant set $\mathcal{F}$ satisfying $\mathcal{F}_k \subseteq \mathcal{F}_k^{aug} \subseteq \mathcal{F}_\infty^{aug} \subseteq \mathcal{F}$, $\forall k > 0$, if the nominal system $f_a^{aug}(x, 0)$ in (7) is absolutely asymptotically stable [11].

Theorem 1: [24] Suppose that there exists a compact robustly positively invariant set $\mathcal{F}$ satisfying $\mathcal{F}_\infty^{aug} \subseteq \mathcal{F} \subseteq \text{interior}(\mathbb{X}_0)$ and that the nominal augmented system (7) with $\mathbb{W} = \{0\}$ is absolutely asymptotically stable [11]; Then, Algorithm 4.1 terminates in finite time.

Remark 1: A detailed overview of the properties of the set sequences $\{\mathcal{F}_k\}$, $\{\mathcal{F}_k^{aug}\}$ and the set $\mathcal{F}_\infty^{aug} \triangleq \lim_{k\to\infty} \mathcal{F}_k^{aug}$ as well as algorithms to compute a robustly positively invariant set $\mathcal{F} \supseteq \mathcal{F}_\infty^{aug}$ for the augmented system (7) are discussed in more detail in [23].

Corollary 1: Suppose that the nominal system defined in (5) is asymptotically stable with $\mathbb{W} = \{0\}$ and $\mathbb{X}$ is a compact set that contains the origin in its interior. Then algorithm 4.1 computes the maximal positively invariant set Ω_∞ (see definition 4) in finite time.

If the original system, defined in (1) is piecewise linear, i.e. $\mathbb{N}_q^0 = \mathbb{N}_q^+$, then Algorithm 4.1 computes the maximal robustly positively invariant set. In general, $\mathbb{N}_q^0 \neq \mathbb{N}_q^+$ so that non-existence of $\tilde{\Omega}_\infty$ for system (5) does not imply non-existence of this set for the original system in (1). However, considering the discrete time system defined in (1) the issue of finite time termination of Algorithm 4.1 becomes significantly more complicated.

V. Maximal robustly controlled invariant set

This section shows how computation of $\tilde{\Omega}_\infty$ permits the computation of $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$. A similar procedure can be employed for the computation of $\tilde{\mathcal{C}}_\infty$. If the computation scheme is applied using multi-parametric programming [3], a robust minimum-time controller is automatically computed by the proposed scheme. The proposed procedure for computation of the state feedback controller is based on the results in [10], [15].

The Algorithm below describes the computation scheme for $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$, $\tilde{\mathcal{C}}_\infty$ or a minimum time state-feedback controller for generic PWA systems, depending on the specific implementation:

Algorithm 5.1 (Computation $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$ or $\tilde{\mathcal{C}}_\infty$):

1) Define a target set $\mathcal{S}_0$ and set $\mathcal{S}_0 = \mathcal{S}_0 \ominus \mathbb{W}$, $k = 0$.
2) Compute $\mathcal{S}_{k+1} = \Theta(\mathcal{S}_k)$, where
$$\Theta(\mathcal{S}_k) \triangleq \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ s.t. } f(x,u,0) \in \mathcal{S}_k\}, \tag{8}$$
If $\mathcal{S}_{k+1}$ is computed by solving a multi-parametric program [10], this will automatically yield a PWA control law which drives all states $x \in \mathcal{S}_{k+1}$ into the set $\mathcal{S}_k$ in one time step.
3) If $\mathcal{S}_{k+1} = \mathcal{S}_k$, return; Else, set $k = k+1$, $\mathcal{S}_k = \mathcal{S}_k \ominus \mathbb{W}$ and goto step 2.

The sets $\mathcal{S}_k$ are, in general, P–collections, making Algorithm 5.1 computationally demanding.

The following two results are standard and well–known results in set invariance or viability theory:

Theorem 2: Suppose that $\mathcal{S}_0 = \mathbb{X}$ and that there exists a $k^* \in \mathbb{N}$ such that $\mathcal{S}_{k^*} = \mathcal{S}_{k^*+1}$. Then, Algorithm 5.1 terminates and $\tilde{\mathcal{C}}_\infty = \mathcal{S}_{k^*}$.

Theorem 3: Suppose that $\mathcal{S}_0 = \tilde{\Omega}_\infty$ and that there exists a $k^* \in \mathbb{N}$ such that $\mathcal{S}_{k^*} = \mathcal{S}_{k^*+1}$. Then, Algorithm 5.1 terminates and $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty) = \mathcal{S}_{k^*}$.

It is clear that at each time k, the effective target set is a P–collection, i.e. $\mathcal{S}_k = \bigcup_{l\in L_k} \mathcal{S}_k^l$ where L_k has a finite cardinality, that changes with time k, so that
$$\Theta(\mathcal{S}_k) = \bigcup_{l\in L_k} \Theta(\mathcal{S}_k^l). \tag{9}$$
where $\mathcal{S}_k^l$ is a polytopic set. For each time k and for each $(i,l) \in \mathbb{N}_q^+ \times L_k$ let
$$\begin{aligned}\mathcal{Z}_k^{(i,l)} \triangleq \{(x,u) \mid \quad & (x,u) \in (Q_i \cap \mathbb{X}) \times \mathbb{U}), \\ & A_i x + B_i u + c_i \in \mathcal{S}_k^l\}\end{aligned} \tag{10}$$
and
$$\mathcal{X}_k^{(i,l)} \triangleq \text{Proj}_X \mathcal{Z}_k^{(i,l)} \tag{11}$$
It follows from the definition of $\Theta(\cdot)$ that $\Theta(\mathcal{S}_k^l) = \cup_{i\in\mathbb{N}_q^+} \mathcal{X}_k^{(i,l)}$ and $\Theta(\mathcal{S}_k) = \cup_{(i,l)\in(\mathbb{N}_q^+\times L_j)} \mathcal{X}_k^{(i,l)}$. Let
$$V(x,u,i) \triangleq u'Ru + (x_i^+)'Q(x_i^+)$$
where $x_i^+ \triangleq A_i x + B_i u + c_i$ and $Q = Q' \succeq 0,\ R = R' \succ 0$. For each time k and for each $(i,l) \in \mathbb{N}_q^+ \times L_j$ let the problem $\mathcal{P}_k^{(i,l)}(x)$ be defined as:
$$V_k^o(x,i,l) = \min_u \{V(x,u,i) \mid (x,u) \in \mathcal{Z}_k^{(i,l)}\} \tag{12}$$
and let
$$\mu_k^o(x,i,l) = \arg\min_u \{V(x,u,i) \mid (x,u) \in \mathcal{Z}_k^{(i,l)}\} \tag{13}$$
denote the argument of $V_k^o(x,i,l)$. Since the set $\mathcal{Z}_k^{(i,l)}$ is a polytopic set and since $V(x,u,i)$ is quadratic in (x,u) it follows that the problem $\mathcal{P}_k^{(i,l)}(x)$ is a quadratic program for each k and each $(i,l) \in \mathbb{N}_q^+ \times L_j$ and therefore can be solved as multi parametric quadratic program. In order to obtain a controller which drives every feasible state x into a target set $\tilde{\Omega}_\infty$ in minimum time, it is therefore necessary to compute $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$ by solving a sequence of multi-parametric programs in Step 2. of Algorithm 5.1.

As already remarked, each resulting controller covers a convex set and their union is equal to $\Theta(\mathcal{S}_k)$. In on-line application, several controller regions may be associated to any given state, since the different controllers will overlap. In order to obtain closed loop trajectories which enter the initial target set $\tilde{\Omega}_\infty$ in minimum time, the controller region associated to the lowest iteration must be selected, i.e.
$$k^o = \min_{k,\ l} \{k \in \mathbb{N} \mid x \in \mathcal{X}_k^{(i,l)}\}.$$
where $\mathcal{X}_k^{(i,l)}$ is defined in (11). The controller partition associated to $\mathcal{X}_{k^o}^{(i,l)}$ is then activated and $\mu_{k^o}^o(x,i,l)$ applied in a standard multi-parametric manner [3]. Note that a state may be contained in several partitions with equal iteration number k^o. If this is the case, any of the controllers associated to k^o may be applied.

Remark 2: If a controller partition computed at iteration k_1 is covered by controller regions of equal or lower iteration number $k_2 \leq k_1$, the respective partition may be discarded since it will never be applied on-line. This may lead to drastic decreases in controller complexity. Efficient methods to check whether a polytope is covered by a P–collection are given in [10].

The multi-parametric min-max optimal control scheme in [15] requires each region partition to be maintained from iteration to iteration. Algorithm 5.1 only requires the representation of the feasible set $\mathcal{S}_k$ to be stored which is expected to mitigate the combinatorial explosion. It is therefore reasonable to assume that the minimum-time algorithm proposed here will yield controllers of relatively low complexity versus min-max controllers. This is also indicated by results for nominal systems in [10].

Remark 3: Note that $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$ may not be finitely determined. In order to obtain a controller, it may therefore be necessary to abort Algorithm 5.1 after a predefined maximum number of iterations or after the state space of interest has been covered. The resulting minimum time controller will then only cover a subset of $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$.

VI. Numerical Examples

In order to illustrate the proposed procedure we consider two second order PWA systems.

Our first example is the following 2-dimensional problem adopted from [18]:

$$x^+ = A_i x + B_i u + c_i + w \tag{14}$$

where $i = 1$ if $x_1 \leq 1$ and $i = 2$ if $x_1 \geq 1$ and

$$A_1 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

and the additive disturbance w is bounded:

$$w \in \{w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.1\}. \tag{15}$$

The system is subject to constraints $-x_1 + x_2 \leq 15$, $-3x_1 - x_2 \leq 25$, $0.2x_1 + x_2 \leq 9$, $x_1 \geq -6$, $x_1 \leq 8$, and $-1 \leq u \leq 1$, whereas weight matrices for the optimization problem are $Q = I$ and $R = 1$. We applied Algorithm 5.1 to the PWA system (14) and obtained a controller that guarantees robust constraints satisfaction for all admissible disturbances (15). The terminal set was obtained by computing the maximal robustly positively invariant set [16] for $x^+ = (A_1 + B_1 K_1)x + w$, where K_1 is the Riccati LQR feedback controller. Using the MPT toolbox [17] the algorithm converged after 75 seconds at iteration 15. The resulting PWA control law is defined over a polyhedral partition consisting of 417 regions and is depicted in Figure 2.

Our second example is the following 2-dimensional PWA system with 4 dynamics [10]:

$$x^+ = A_i x + B_i u + w \tag{16}$$

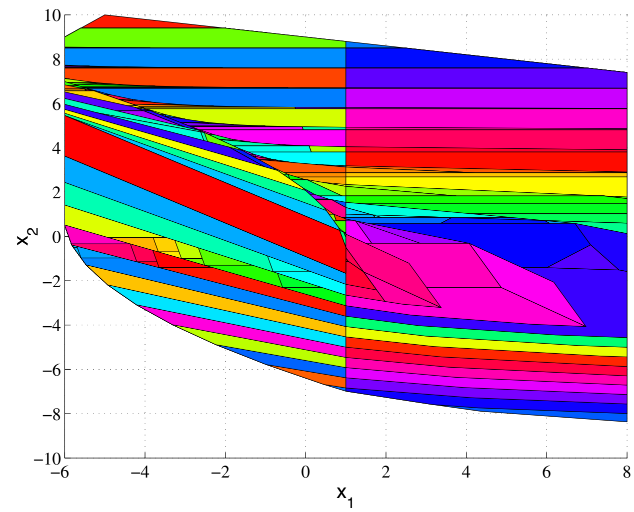


Fig. 2. Final robust controller for Example 1.

where

$$i = \begin{cases} 1, & \text{if } x_1 \geq 0 \ \& \ x_2 \geq 0 \\ 2, & \text{if } x_1 \leq 0 \ \& \ x_2 \leq 0 \\ 3, & \text{if } x_1 \leq 0 \ \& \ x_2 \geq 0 \\ 4, & \text{if } x_1 \geq 0 \ \& \ x_2 \leq 0 \end{cases}$$
$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}.$$

with the following bounds on the disturbance:

$$w \in \{w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.2\}. \tag{17}$$

One can observe, that the system is a perturbed double integrator in the discrete time domain, with different orientation of the vector field. The output and input constraints, respectively, are: $-5 \leq x_1 \leq 5$, $-5 \leq x_2 \leq 5$, $-1 \leq u \leq 1$, whereas weight matrices for the optimization problem are $Q = I$ and $R = 1$.

By solving the SDP in [10], the following stabilizing feedback controllers are obtained: $K_1 = [-0.5897 \quad -0.9347]$, $K_2 = [0.5897 \quad 0.9347]$, $K_3 = [0.5897 \quad -0.9347]$ and $K_4 = [-0.5897 \quad 0.9347]$. Subsequently $\tilde{\Omega}_\infty$ is computed according to Algorithm 4.1 and used as a target set for the robust minimum-time Algorithm 5.1. This technique yields a controller that guarantees robust constraint satisfaction for all possible realizations of the disturbance bounded by (17). Using the MPT toolboxa controller partition with 508 regions is obtained at iteration 17 after 107 seconds of computation time (see Figure 3).

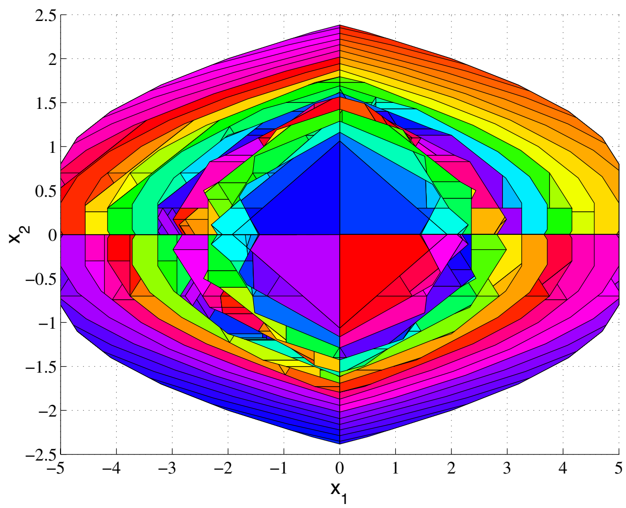


Fig. 3. Final robust controller for Example 2.

VII. Conclusions

We have introduced efficient techniques for computations with P–collections, which can be combined with set invariance theory to compute non-convex robustly positively invariant sets for piecewise-affine (PWA) systems. Specifically we have presented methods to compute the maximal robustly positively invariant subset $\tilde{\Omega}_\infty$, the maximal robustly controllable set $\tilde{\mathcal{C}}_\infty$ and the maximal robustly attractive set $\tilde{\mathcal{K}}_\infty(\tilde{\Omega}_\infty)$ for PWA systems. In addition, sufficient conditions for finite time determination of $\tilde{\Omega}_\infty$ are given. We have furthermore shown how these methods may be used to obtain robust state feedback controllers for PWA systems if combined with multi-parametric programming techniques. The proposed controller robustly drives the state into the target set $\tilde{\Omega}_\infty$ in minimum time.

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