Efficient Computation of Robust Positively Invariant Sets with Linear State-feedback Gain as a Variable of Optimization

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Abstract— **In this paper, we develop an algorithm for the efficient computation of Robust Positively Invariant sets for linear discrete-time systems subject to bounded additive disturbances and polytopic input constraints. The proposed algorithm simultaneously computes both the optimal invariant set and the corresponding state-feedback control law in one step by solving a single semidefinite program. Ellipsoidal as well as a polytopic characterization of the invariant sets is derived. In addition to the input constraints, the proposed method also allows for the incorporation of state constraints in a non-conservative manner. Furthermore, it is shown that for the case with a fixed control law, the proposed algorithm computes the optimal polytopic invariant set by solving a single Linear Program. The viability of the proposed scheme is demonstrated through numerical examples.**

Keywords— **Farkas Lemma, Linear Matrix Inequalities, Set invariance, S-procedure**.

I. Introduction

Robust Positively Invariant (RPI) set defines a bounded state-space region to which the system state can be confined, for all possible disturbances, through the application of a state-feedback control law K. RPI sets are important in the robustness analysis and synthesis of controllers for uncertain systems [1]. These sets also play a fundamental role in various robust model predictive control schemes [2], [3]. Furthermore, the RPI set is also a suitable target set in robust time-optimal control schemes [4]–[6].

The two invariant set structures most often considered in the literature are ellipsoidal and polytopic [7]. For these set structures, the problem of computing both the maximal and the minimal RPI set (or their suitable approximations) is significant.

Ellipsoidal sets are generally of the form $\mathcal{E}_Q := \{x \in \mathbb{R}^n : x^T Q x \leq 1\}$, where $Q = Q^T \geq 0$. Since the radii of the ellipse are the square root of the eigenvalues of Q^{-1}, therefore, the problem of optimizing the volume of this set generally considers $\det(Q^{-1})$ as an objective function [8]. This approach, however, has a few shortcomings. Firstly, the objective function $\det(Q^{-1})$ is non-convex unless monotonic transformations are performed. Secondly, the resulting algorithm generally involves Bilinear Matrix Inequalities (BMI) which means that computation of optimal RPI set requires iterative computations (e.g. a binary search). Finally, any non-symmetric input constraints can not be included in the formulation in a non-conservative manner. To avoid such issues, in this paper, we propose scaling the radii of the ellipse by a positive variable α while fixing the Q matrix i.e. we consider set of the form $\mathcal{E}_\alpha := \{x \in \mathbb{R}^n : (\frac{1}{\alpha^2}) x^T Q x \leq 1\}$, where $Q = Q^T \geq 0$ is fixed. In this case, since the volume of ellipse is proportional to the scaling, therefore, we consider α to be our *linear* objective function. Furthermore, we formulate the problem in a manner such that the optimal (smallest or largest volume-wise) set and the associated feedback law K can be computed simultaneously in *one step* through a semidefinite program which incorporates the non-symmetric input constraints without any extra conservatism.

Subsequently, we propose a polytopic RPI set characterization. Most existing methods for computing polytopic RPI sets do not allow for the linear feedback law K to be optimized, see e.g. [9] and [10], which in turn has an adverse effect on the size of the computed set. Moreover, in most schemes, computation of polytopic RPI set requires iterative set computations, see e.g. [11]–[14], which is often inefficient. We propose an algorithm to compute, exactly, the optimal (smallest or largest) polytopic RPI set and the associated controller K in *one step* through a semidefinite program. Furthermore, we show that for the case with a fixed K, the proposed algorithm computes the optimal polytopic set by solving a single Linear Program (LP).

This paper is organized as follows. Section II is concerned with the system description and definition of RPI sets. In Section III, an ellipsoidal characterization of the RPI sets is developed whereas, in Section IV, a polytopic characterization is derived. In Section V, we present numerical examples. Finally, we conclude in Section VI.

Notation and background material: The notation we use is fairly standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the space of n-dimensional (column) vectors whose entries are in $\mathbb{R}$, $\mathbb{R}^{n\times m}$ denotes the space of all $n \times m$ matrices whose entries are in $\mathbb{R}$ and $\mathbb{D}^{n\times n}$ denotes the space of diagonal matrices in $\mathbb{R}^{n\times n}$. For $A \in \mathbb{R}^{n\times m}$, A^T denotes the transpose of A. If $A \in \mathbb{R}^{n\times n}$ is symmetric, $\underline{\lambda}(A)$ denotes the smallest eigenvalue of A and we write $A \geq 0$ if $\underline{\lambda}(A) \geq 0$ and $A > 0$ if $\underline{\lambda}(A) > 0$. Analogous definitions apply to $\overline{\lambda}(A)$, $A \leq 0$ and $A < 0$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} := \{u + v | u \in \mathcal{U}, v \in \mathcal{V}\}$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^{b}$, we denote $\oplus_{i=a}^{b} \mathcal{U}_i := \mathcal{U}_a \oplus \cdots \oplus \mathcal{U}_b$. The identity matrix is denoted by I with the dimension inferred from the context. Let $z \in \mathbb{R}^n$ and denote the i-th element of z by z_i. Then $\mathrm{diag}(z)$ is the diagonal matrix whose (i, i) entry

IEEE Catalog Number: CFP10827-ART

ISBN: 978-1-4244-7314-4

is z_i. For square matrices $A_1, \ldots, A_m$, $\mathrm{diag}(A_1, \ldots, A_m)$ denotes the block diagonal matrix whose i-th diagonal block is A_i.

In the sequel, we mention a schur complement argument. This refers to the result that if $A = A^T$ and $C = C^T > 0$ then

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \geq 0 \Longleftrightarrow A - BC^{-1}B^T \geq 0.$$

We also use the following version of the Farkas Lemma (see e.g. [15]).

Lemma 1: Let $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Suppose there exists $\hat{y}$ such that $A\hat{y} < b$. Then the following two statements are equivalent:

1) $c^T y \leq d \; \forall y$ such that $Ay \leq b$.
2) $\exists \mu \in \mathbb{R}^m$ such that $\mu \geq 0$, $A^T \mu = c$ and $b^T \mu \leq d$.

Finally, we refer to the S-procedure. This is a family of procedures used to derive simple sufficient (occasionally necessary and sufficient) conditions, typically in the form of Linear Matrix Inequality (LMI) conditions, for the non-negativity or non-positivity of a quadratic function on a set described by quadratic inequality constraints [16]–[19].

II. Preliminaries

We consider the following linear discrete-time model:

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_w}$ are the state, input and bounded disturbance vectors at time k; A is the state transition matrix and B_u and B_w are the input and disturbance distribution matrices, respectively. We consider disturbance of the form

$$w_k \in W := \left\{ w \in \mathbb{R}^{n_w} : -v \leq w \leq v \right\}$$

where $v > 0$.

We also consider non-symmetric input constraints of the form:

$$u_k \in \mathcal{U} := \left\{ u \in \mathbb{R}^{n_u} : \underline{u} \leq u \leq \overline{u} \right\} \tag{2}$$

where $\overline{u} > 0$ and $\underline{u} < 0$.

Remark 1: Note that the inequalities given in the definition of sets W and $\mathcal{U}$ need to be satisfied element-wise.

We now recall the following definition [1]:

Definition 1 (RPI set): The set $Z \subset \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set of system (1) if, with the control law $u = -Kx$, it satisfies $(A - B_u K)Z \oplus B_w W \subseteq Z$.

Thus, if the initial state x_0 lies in the set Z, then all subsequent states will be kept within this set by the control law $u = -Kx$ [2]. Therefore, the set Z characterizes the evolution of the system (1) for all possible disturbances $w_k \in W$ [7].

III. Ellipsoidal Characterization

As mentioned in Section 1, we consider Z to be of the form:

$$Z := \mathcal{E}_\alpha = \left\{ x \in \mathbb{R}^n : \; x^T Q x \; \leq \; \alpha^2 \right\}$$

where $Q = Q^T \geq 0$ is known.

Our problem is to find the optimal (smallest or largest) α, call it α_o, and corresponding controller K such that

$$\begin{aligned} \alpha_o =: \quad & \min / \max \quad \alpha \\ & \underline{u} \leq -K\mathcal{E}_\alpha \leq \overline{u} \\ & (A - B_u K)\mathcal{E}_\alpha \oplus B_w W \subseteq \mathcal{E}_\alpha \end{aligned} \tag{3}$$

Remark 2: The constraint $(A - B_u K)\mathcal{E}_\alpha \oplus B_w W \subseteq \mathcal{E}_\alpha$ ensures that the set $\mathcal{E}_\alpha$ is invariant by construction. Furthermore, for optimality, we seek to minimize (or maximize) the volume of the ellipse by considering α to be the objective function.

The optimization problem in (3) is generally NP-hard. Therefore, our approach is to use an S-Procedure to derive necessary and sufficient conditions - on α and K - for the satisfaction of the input constraints and, subsequently, sufficient conditions for the invariance constraint in (3). This will allow us to obtain an upper/lower bound on α_o.

A. Necessary and Sufficient Conditions for the Input Constraints.

The input constraints in (3) can be written as:

$$-e_{uj}^T K x \; \leq \; e_{uj}^T \overline{u} \quad , \qquad \forall x \in \mathcal{E}_\alpha \tag{4}$$

$$-e_{uj}^T K x \; \geq \; e_{uj}^T \underline{u} \quad , \qquad \forall x \in \mathcal{E}_\alpha \tag{5}$$

where $j = 1, 2, \ldots, n_u$ and e_{uj} denotes the jth column of the $n_u \times n_u$ identity matrix.

Remark 3: The problems (4) and (5) belong to a class of Trust Region Subproblems (TRS) for which exact semidefinite relaxation exists [20], [21]. Therefore, the conditions we derive in this subsection are both necessary and sufficient. This ensures that the non-symmetric input constraints are incorporated into the algorithm in a non-conservative manner.

Using S-procedure, it can be verified that:

$$-e_{uj}^T (Kx + \overline{u}) = -\overline{\mu}_j (\alpha^2 - x^T Q x) - s^T L_{\overline{u}}^j (\overline{\mu}_j, K, \alpha) s \; \leq \; 0$$

where $0 \leq \overline{\mu}_j \in \mathbb{R}$, $s = [\; x^T \quad 1 \;]^T$ and

$$L_{\overline{u}}^j (\overline{\mu}_j, K, \alpha) := \begin{bmatrix} \overline{\mu}_j Q & \frac{1}{2} K^T e_{uj} \\ \star & e_{uj}^T \overline{u} - \overline{\mu}_j \alpha^2 \end{bmatrix}$$

where $\star$ denotes the term readily inferred from symmetry.

The matrix $L_{\overline{u}}^j (\overline{\mu}_j, K, \alpha)$ is non-linear due to the (2,2) entry. Pre- and post-multiplying by $\mathrm{diag}(\alpha, 1)$ and redefining

IEEE Catalog Number: CFP10827-ART

ISBN: 978-1-4244-7314-4

$\overline{\mu}_j := \alpha^2\overline{\mu}_j$ and $\hat{K} := \alpha K$ yields the following necessary and sufficient conditions for the input constraint in (4):

$$\overline{\mu}_j \ \geq \ 0, \qquad \mathcal{L}^j_{\overline{u}}(\overline{\mu}_j, K, \alpha) := \begin{bmatrix} \overline{\mu}_j Q & \frac{1}{2}\hat{K}^T e_{uj} \\ \star & e^T_{uj}\overline{u} - \overline{\mu}_j \end{bmatrix} \geq 0. \tag{6}$$

Analogous to the above procedure, it can be shown that the following conditions are both necessary and sufficient for the satisfaction of input constraint in (5):

$$\underline{\mu}_j \ \geq \ 0, \qquad \mathcal{L}^j_{\underline{u}}(\underline{\mu}_j, K, \alpha) := \begin{bmatrix} \underline{\mu}_j Q & -\frac{1}{2}\hat{K}^T e_{uj} \\ \star & -e^T_{uj}\underline{u} - \underline{\mu}_j \end{bmatrix} \geq 0. \tag{7}$$

for all $j = 1, 2, \ldots, n_u$.

B. Sufficient Conditions for the Invariance Constraint

The invariance constraint in (3) is equivalent to:

$$(A_K x + B_w w)^T Q (A_K x + B_w w) \leq \alpha^2, \forall x \in \mathcal{E}_\alpha, \forall w \in W$$

where $A_K := A - B_u K$.

It can be verified that:

$$\begin{aligned}(A_K x + B_w w)^T Q (A_K x + B_w w) - \alpha^2 =& -\mu(\alpha^2 - x^T Q x) \\ & -(v+w)^T D_w (v-w) \\ & -y^T L_c(\mu, D_w, K, \alpha) y\end{aligned} \tag{8}$$

for any D_w and μ, where $y = [\ w^T \quad x^T \quad 1\ ]^T$, and

$$L_c(\mu, D_w, K, \alpha) :=$$

$$\begin{bmatrix} D_w - B_w^T Q B_w & -B_w^T Q A_K & 0 \\ \star & \mu Q - A_K^T Q A_K & 0 \\ \star & \star & \alpha^2 - \alpha^2\mu - v^T D_w v \end{bmatrix}$$

It follows from (8) that $(A - B_u K)\mathcal{E}_\alpha \oplus B_w W \subseteq \mathcal{E}_\alpha$ if there exist $\mu \in \mathbb{R}$ and $D_w \in \mathbb{D}^{n_w \times n_w}$ such that

$$\mu \ \geq \ 0, \quad D_w \ \geq \ 0, \quad L_c(\mu, D_w, K, \alpha) \geq 0.$$

Notice that the matrix $L_c(\mu, D_w, K, \alpha)$ is nonlinear due to its (2,2) and (3,3) entries. A simple schur complement argument, a subsequent pre- and post-multiplication by $\mathrm{diag}(\alpha^{-\frac{1}{2}}I, \alpha^{\frac{1}{2}}I, \alpha^{-\frac{1}{2}}I, \alpha^{\frac{1}{2}}Q)$, followed by redefinitions $D_w := \alpha^{-1}D_w$, $\mu := \alpha\mu$ and $\hat{K} := \alpha K$ results in the following sufficient conditions for invariance constraint.

$$\mu \ \geq \ 0, \quad D_w \geq \ 0, \quad \mathcal{L}(\mu, D_w, K, \alpha) \geq 0 \tag{9}$$

where

$$\mathcal{L}(\mu, D_w, K, \alpha) :=$$

$$\begin{bmatrix} D_w & 0 & 0 & B_w^T Q \\ \star & \mu Q & 0 & \alpha A^T Q - \hat{K}^T B_u^T Q \\ \star & \star & \alpha - \mu - v^T D_w v & 0 \\ \star & \star & \star & \alpha Q \end{bmatrix}$$

C. Complete Problem Formulation

The problem to compute an upper/lower bound on α_o, call it α_{oo}, can be summarized as followed.

$$\begin{aligned}\alpha_{oo} = \min/\max \{\, \alpha : & \ (6), (7), (9) \text{ are satisfied for some} \\ & \overline{\mu}_j, \underline{\mu}_j, \mu \in \mathbb{R}, j = 1, \ldots, n_u \text{ and diag-} \\ & \text{onal } D_w \geq 0\}.\end{aligned} \tag{10}$$

Remark 4: The optimization problem in (10) is a semidefinite program through which we can efficiently compute, for a given Q, the (near) optimal volume ellipsoidal RPI set and the corresponding K in one step.

Remark 5: An inspection of the (3,3) block of the matrix $\mathcal{L}(\mu, D_w, K, \alpha)$ shows that $\mu \leq \alpha$. Furthermore, an inspection of the matrix composed of the (2,2), (2,4), (4,2) and (4,4) blocks of $\mathcal{L}(\mu, D_w, K, \alpha)$ then shows that $\alpha(A - B_u K)^T Q (A - B_u K) \leq \mu Q \leq \alpha Q$. This illustrates that a necessary condition for the existence of $\mathcal{E}_\alpha$ is that there exists a K such that $Q - (A - B_u K)^T Q (A - B_u K) \geq 0$. This condition is equivalent to the existence of a feasible solution (K) to the LMI $\begin{bmatrix} Q & (A - B_u K)^T Q \\ \star & Q \end{bmatrix} \geq 0$.

Remark 6: Note that any problem-defined Quadratic state constraints can easily be incorporated into the above formulation by including - in problem (10) - constraints of the form: $\alpha \leq \overline{x}$.

IV. Polytopic Characterization

In this section, we consider Z to be of the form:

$$Z = P_z := \{x \in \mathbb{R}^n : \ -z \leq x \leq z\}$$

where $z > 0$.

Remark 7: Note that it is reasonable to consider Z to be symmetric (i.e. $x \in Z \Leftrightarrow -x \in Z$) since we have assumed the disturbance set W to be symmetric. Note also that the inequalities given in the definition of the set P_z need to be satisfied element-wise.

In order to optimize P_z, we seek the smallest/largest sum of elements of z i.e. $\sum_{i=1}^n z_i$, call it γ_o, along with the corresponding controller gain K such that:

$$\begin{aligned}\gamma_o =: \quad & \min/\max \quad \sum_{i=1}^n z_i \\ & \underline{u} \leq -K P_z \leq \overline{u} \\ & (A - B_u K) P_z \oplus B_w W \subseteq P_z\end{aligned} \tag{11}$$

Remark 8: The constraint $(A - B_u K) P_z \oplus B_w W \subseteq P_z$ ensures that the polytopic set is invariant by construction.

In order to compute γ_o exactly, we next derive the necessary and sufficient conditions for both the input and invariance constraints in (11).

IEEE Catalog Number: CFP10827-ART

ISBN: 978-1-4244-7314-4

A. Necessary and Sufficient Conditions for the Input Constraints

The input constraints in (11) can be written as:

$$-e_{uj}^T K x \;\leq\; e_{uj}^T \overline{u} \quad , \qquad \forall x \in P_z \tag{12}$$
$$-e_{uj}^T K x \;\geq\; e_{uj}^T \underline{u} \quad , \qquad \forall x \in P_z \tag{13}$$

where $j = 1, 2, \ldots, n_u$ and e_{uj} denotes the jth column of the $n_u \times n_u$ identity matrix.

Through the application of Lemma 1, it can be shown that (12) is satisfied if and only if there exist $\rho_x^j,\ \overline{\rho}_x^j \in \mathbb{R}^n$ such that

$$\begin{aligned}&\rho_x^j \geq 0,\ \overline{\rho}_x^j \geq 0,\ \rho_x^j = \overline{\rho}_x^j - K^T e_{uj},\\ &e_{uj}^T \overline{u} - z^T \overline{\rho}_x^j - z^T \rho_x^j \geq 0.\end{aligned}$$

By eliminating ρ_x^j from the above, we obtain the following equivalent conditions

$$\begin{aligned}&\overline{\rho}_x^j \geq 0,\ \overline{\rho}_x^j - K^T e_{uj} \geq 0,\\ &e_{uj}^T \overline{u} - 2z^T \overline{\rho}_x^j + z^T K^T e_{uj} \geq 0.\end{aligned} \tag{14}$$

In order the linearize the conditions in (14), we first define $Z_d = \mathrm{diag}(z)$ so that $z = Z_d e$ where e is the n dimensional vectors of ones. Then, pre-multiplying the first and second inequality by Z_d and redefining $\overline{\rho}_x^j := Z_d \overline{\rho}_x^j$ and $\hat{K} := K Z_d$ shows that the constraint in (12) is satisfied if and only if, for $j = 1, 2, \ldots, n_u$, there exist $\overline{\rho}_x^j \in \mathbb{R}^n$ such that

$$\begin{aligned}&\overline{\rho}_x^j \geq 0,\ \overline{\rho}_x^j - \hat{K}^T e_{uj} \geq 0,\\ &e_{uj}^T \overline{u} - 2e^T \overline{\rho}_x^j + e^T \hat{K}^T e_{uj} \geq 0.\end{aligned} \tag{15}$$

Analogous to the above procedure, using Lemma 1 followed by linearization, it can be shown that (13) is satisfied if and only if, for $j = 1, 2, \ldots, n_u$, there exist $\underline{\rho}_x^j \in \mathbb{R}^n$ such that

$$\begin{aligned}&\underline{\rho}_x^j \geq 0,\ \underline{\rho}_x^j + \hat{K}^T e_{uj} \geq 0,\\ &-e_{uj}^T \underline{u} - 2e^T \underline{\rho}_x^j - e^T \hat{K}^T e_{uj} \geq 0.\end{aligned} \tag{16}$$

B. Necessary and Sufficient Conditions for the Invariance Constraint

Since the sets P_z and W are symmetric about the origin, the constraint in (11) is equivalent to

$$e_i^T (A_K x + B_w w) \;\leq\; e_i^T z \quad , \quad \forall\ x \in P_z,\ \forall\ w \in W \tag{17}$$

for $i = 1, 2, \ldots, n$ where e_i denotes the ith column of the $n \times n$ identity matrix.

Since we assume $v > 0$ and $z > 0$, it follows from Lemma 1 that (17) is satisfied if and only if there exist $\mu_w^i,\ \bar{\mu}_w^i \in \mathbb{R}^{n_w}$ and $\mu_x^i,\ \bar{\mu}_x^i \in \mathbb{R}^n$ such that

$$\begin{aligned}&\mu_w^i \geq 0,\ \bar{\mu}_w^i \geq 0,\ \mu_x^i \geq 0,\ \bar{\mu}_x^i \geq 0,\\ &\bar{\mu}_w^i = \mu_w^i + B_w^T e_i,\ \bar{\mu}_x^i = \mu_x^i + A_K^T e_i,\\ &e_i^T z - v^T(\mu_w^i + \bar{\mu}_w^i) - z^T(\mu_x^i + \bar{\mu}_x^i) \geq 0.\end{aligned}$$

By eliminating $\bar{\mu}_w^i$ and $\bar{\mu}_x^i$, it follows that (17) is satisfied if and only if there exist $\mu_w^i \in \mathbb{R}^{n_w}$ and $\mu_x^i \in \mathbb{R}^n$ such that

$$\begin{aligned}&\mu_w^i \geq 0,\ \mu_w^i + B_w^T e_i \geq 0,\ \mu_x^i \geq 0,\ \mu_x^i + A_K^T e_i \geq 0,\\ &e_i^T z - v^T(2\mu_w^i + B_w^T e_i) - z^T(2\mu_x^i + A_K^T e_i) \geq 0.\end{aligned} \tag{18}$$

Using $e_i^T z = e_i^T Z_d e$ and $A_K = A - B_u K$, pre-multiplying the third and fourth inequalities in (18) by Z_d and redefining $\mu_x^i := Z_d \mu_x^i$ and $\hat{K} := K Z_d$ shows that the invariance constraint in (11) is satisfied if and only if, for $i = 1, 2, \ldots, n$, there exist $\mu_w^i \in \mathbb{R}^{n_w}$ and $\mu_x^i \in \mathbb{R}^n$ such that

$$\mu_w^i \geq 0,\ \mu_w^i + B_w^T e_i \geq 0,\ \mu_x^i \geq 0, \tag{19}$$
$$\mu_x^i + (A Z_d - B_u \hat{K})^T e_i \geq 0, \tag{20}$$
$$\begin{aligned}&e_i^T Z_d e - v^T(2\mu_w^i + B_w^T e_i)\\ &\quad - e^T(2\mu_x^i + (A Z_d - B_u \hat{K})^T e_i) \geq 0\end{aligned} \tag{21}$$

C. Complete Problem Formulation

The problems to compute the smallest and largest P_z, along with the corresponding optimal controller K, can be summarized as follows.

1) Smallest P_z

In order to compute the smallest P_z, we consider

$$\begin{aligned}\gamma \;\geq\; \sum_{i=1}^{n} z_i \quad &\Leftrightarrow \quad \gamma - \sum_{i=1}^{n} z_i \geq 0\\ &\Leftrightarrow \quad \gamma - \sum_{i=1}^{n} e_i^T Z_d e_i \geq 0\end{aligned} \tag{22}$$

Therefore, the complete problem becomes

$$\begin{aligned}\gamma_o = \min\{\,\gamma : &(15), (16), (19-21), (22) \text{ are satisfied for}\\ &\text{some } \mu_w^i \in \mathbb{R}^{n_w}, \overline{\rho}_x^j, \underline{\rho}_x^j, \mu_x^i \in \mathbb{R}^n, j = 1, \ldots, n_u,\\ &i = 1, \ldots, n, \text{ and diagonal } Z_d > 0\}.\end{aligned} \tag{23}$$

2) Largest P_z

In order to compute the largest P_z, we consider

$$\begin{aligned}\gamma \;\leq\; \sum_{i=1}^{n} z_i \quad &\Leftrightarrow \quad \gamma - \sum_{i=1}^{n} z_i \leq 0\\ &\Leftrightarrow \quad \gamma - \sum_{i=1}^{n} e_i^T Z_d e_i \leq 0\end{aligned} \tag{24}$$

Therefore, the complete problem becomes

$$\begin{aligned}\gamma_o = \max\{\,\gamma : &(15), (16), (19-21), (24) \text{ are satisfied for}\\ &\text{some } \mu_w^i \in \mathbb{R}^{n_w}, \overline{\rho}_x^j, \underline{\rho}_x^j, \mu_x^i \in \mathbb{R}^n, j = 1, \ldots, n_u,\\ &i = 1, \ldots, n, \text{ and diagonal } Z_d > 0\}.\end{aligned} \tag{25}$$

IEEE Catalog Number: CFP10827-ART

ISBN: 978-1-4244-7314-4

Remark 9: Note that since our algorithm computes γ_o exactly (instead of upper or lower bound), the optimizations in (23) and (25) yield the smallest and largest P_z respectively, along with the correpsonding optimal gain K.

Remark 10: The optimization problems (23) and (25) are semidefinite programs. This is significant since it enables us to efficiently compute the optimal RPI set P_z along with the corresponding linear controller K in a non-iterative way.

Remark 11: In the proposed algorithm, we can easily incorporate any problem-defined state constraints by including, in problems (23) and (25), an LMI condition of the form: $Z_d e \leq z_c$.

Remark 12: Let $y = [z^T \;\; \mu_w^{1T} \cdots \mu_w^{nT} \;\; \mu_x^{1T} \cdots \mu_x^{nT} \; \overline{\rho}_x^{1T} \cdots \overline{\rho}_x^{n_uT} \; \underline{\rho}_x^{1T} \cdots \underline{\rho}_x^{n_uT}]^T \in \mathbb{R}^{n(n_w+n+2n_u+1)}$ and define a column vector c whose first n elements are 1 while the rest are 0 i.e. $c := [1, \ldots, 1, 0, \ldots, 0]^T \in \mathbb{R}^{n(n_w+n+2n_u+1)}$. Then, it is easy to verify that for the case with a fixed K, problems (23) and (25) can be transformed into Linear Programs (LP) which optimize the objective function $c^T y$ subject to constraints of the form $Ey \leq G$.

V. Numerical Examples

In this section, we present two numerical examples to demonstrate the effectiveness of the proposed algorithm. The algorithm was implemented on a Core™2 Duo Laptop with 2GHz processor and 2GB RAM.

A. Ellipsoidal RPI Set

We consider the following unstable system:

$$x_{k+1} = \begin{bmatrix} 0.9 & 0.75 & 1 \\ 0.7 & 1.25 & 0.4 \\ 0.3 & 0.4 & 0.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix} w_k \quad (26)$$

The (non-symmetric) input constraints are given by: $-3 \leq u_k \leq 4$. The disturbance is assumed to belong to the set: $-1 \leq w_k \leq 1$. We consider two distinct RPI set structures as follows:

Case 1: $Q = I$.

With the RPI set specified to be a unit ball, solving (10) as a minimization problem yields the following Z and K:

$$Z = \left\{x \in \mathbb{R}^3 : \; x^T x \;\leq\; 2.74\right\}, \quad K = [0.7 \;\; 0.788 \;\; 0.725]$$

Similarly, solving (10) as a maximization problem yields the following Z and K:

$$Z = \left\{x \in \mathbb{R}^3 : \; x^T x \;\leq\; 8.44\right\}, \quad K = [0.585 \;\; 0.675 \;\; 0.519]$$

Case 2: $Q = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -1 & 6 \end{bmatrix}$.

For this case, with the RPI set specified to be an ellipse with $\lambda(Q^{-1}) = \{0.13, 0.21, 0.41\}$, solving (10) as a minimization problem yields the following Z and K:

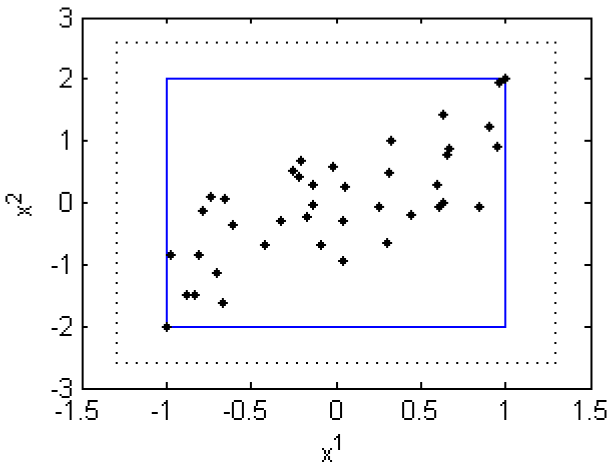

Fig. 1. Simulation results with w_k randomly distributed between ± 1

$$Z = \left\{x \in \mathbb{R}^3 : x^T Q x \leq 18.21\right\}, \quad K = [0.73 \;\; 0.898 \;\; 0.685]$$

Similarly, solving (10) as a maximization problem yields the following Z and K:

$$Z = \left\{x \in \mathbb{R}^3 : x^T Q x \leq 30.85\right\}, \quad K = [0.707 \;\; 0.854 \;\; 0.53]$$

Remark 13: Using MATLAB code, the proposed algorithm took approximately 0.28 sec to compute the RPI sets and their corresponding optimal controllers.

B. Polytopic RPI Set

We consider an example taken from [11]. It consists of the following system:

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_k \qquad (27)$$

where $w_k \in W := \left\{ w \in \mathbb{R}^2 : \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq w \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

To remain consistent with the original example, we leave the input unconstrained.

For the system in (27), we solve (23) to obtain the following optimal P_z and K:

$$P_z = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -1 \\ -2 \end{bmatrix} \leq x \leq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \qquad K = [1 \quad 1]$$

Fig. 1 shows the simulation results with the initial state $x_0 = [1 \quad 2]^T$ and disturbance w_k randomly distributed between -1 and $+1$. We see that the computed controller K keeps the disturbed system state within the optimal polytopic RPI set (solid rectangle).

Remark 14: Using MATLAB code, the proposed algorithm took approximately 0.2 sec to compute the smallest RPI set and the corresponding optimal controller.

Remark 15: The minimal RPI (mRPI) set is given by: $F_\infty = \oplus_{i=0}^{\infty}(A - B_u K)^i B_w W$ [7]. Since F_∞ invloves Minkowski's sum of infinite many terms, therefore it is

IEEE Catalog Number: CFP10827-ART

ISBN: 978-1-4244-7314-4

generally impossible to compute unless the system dynamics are nilpotent [5]. It is interesting to note that for the system in (27), our algorithm yields a K for which, the closed-loop dynamics are nilpotent i.e. $(A - B_u K)^i = 0 \quad \forall\ i \geq 2$. In fact, for the considered example, the set F_∞ (computed through the evaluation of the Minkowski sum) comes out to be exactly the same as the P_z obtained through our algorithm. In other words, for the example in (27), our algorithm yields the mRPI set *exactly*.

As highlighted in Remark 12, for the case with a fixed K, the proposed algorithm computes the optimal RPI set by solving a single Linear Program. For the example in (27), using the pre-determined controller considered in [11] i.e. $K = [1.17 \quad 1.03]$, the proposed algorithm computes exactly the same P_z as the mRPI set approximation given in [11] (shown as the dotted rectangle in Fig. 1). Note, however, that even for this case with a pre-determined K, our proposed algorithm is computationally more efficient since, unlike the iterative algorithm presented in [11], it is based on a single Linear Program.

VI. Conclusion

In this paper, we have proposed a novel algorithm for the efficient computation of RPI sets along with their corresponding linear state-feedback gain. Ellipsoidal as well as polytopic RPI set characterizations have been derived.

In the literature, the problem of computing a suitable RPI set generally requires iterative computations. Furthermore, most existing methods do not allow for the optimization of the control law K. The proposed algorithm, however, avoids any such iterations and computes both an optimal (smallest or largest) RPI set and the corresponding control law K in one step through a single semidefinite program. Furthermore, the proposed scheme incorporates constraints on both the input and state in a non-conservative manner. Also, for the case with a fixed K, the optimal polytopic RPI set can be computed through a single Linear Program.

It is worth noting that other polytopic set structures can also be considered. However, in those cases, the required formulation (and the resulting algorithm) is likely to be very different from the one presented in this paper.

Acknowledgment

The author wishes to thank Dr. Imad M Jaimoukha for useful discussions.

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IEEE Catalog Number: CFP10827-ART

ISBN: 978-1-4244-7314-4