Calculus of Complex Zonotopes

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Models of hybrid systems, that describe the system dynamics as an interaction between continuous and discrete variables, pose a significant challenge for formal verification. Verification of some of the properties of these systems, like safety and stability, usually requires computing the set of reachable states of the system. For a general class of these models that have differential equations and discrete modes for switching, exact computation of the reachable set is undecidable [**?**], except in particular cases. Even in a simpler case like a discrete time affine hybrid system, the exact reachable set at any given time is a union of exponential number of convex sets in the number of time steps. Instead, we can find a sufficiently accurate over-approximation of the infinite set of reachable states. To do so, we need to represent an infinite set of states symbolically by a *set representation*, which can be efficiently manipulated for the computation of the desired over-approximation. Some examples of a set representation are *convex polyhedron*, represented by either linear inequalities [4, 10] or a convex hull of a finite set of points [8]; *ellipsoid* [3, 7], represented by a positive semi-definite matrix and a center; *polynomial sub-level sets* [2, 9] ; etc. In numerical abstract interpretation, a set representation is usually an *abstract domain*, which is a lattice with an abstraction and a concretization map [6].

1 Problem and related work

We consider the problem of efficiently computing a sufficiently accurate over-approximation of the unbounded time reachable set that is useful for verifying properties like invariance and stability. The unbounded time reachable set is usually approximated as a *positive invariant*, which is a set of states whose next set of reachable states is contained within itself. The potentiality of a set representation for efficient computation of a positive invariant of a hybrid system is affected by the following characteristics. (i) Closure of the set representation under set operations like linear transformation, Minkowski sum, intersection and inclusion-checking, etc, and efficient computation of the same. (ii) Existence of a positive invariant for the system that can be encoded by the set representation. (iii) Efficient encoding of the positive invariant by the set representation. Well known set representations for computing reachable sets have some of the above characteristics, but not all. For example, polytopes have the advantage that they are closed under linear transformation, Minkowski sum and intersection and for a stable linear transformation, there exists a polytopic invariant. However, the Minkowski sum operation is costly because it can lead to a quadratic increase in the number of vertices and an exponential increase in the number of faces. Also, the number of faces of a polytopic positive invariant for a stable linear system can be arbitrarily large. Ellipsoids have the advantage that they are closed under linear transformation and can also efficiently represent the positive invariant of a stable linear transformation. But ellipsoids are not closed under Minkowski sum. Recently work computes the over-approximation of the Minkowski sum of ellipsoids by a minimum volume ellipsoid []. Still, there can be significant approximation error in the ellipsoidal approximation of the Minkowski sum.

Real zonotope, its usefulness and drawback. Zonotope [5] is yet another set representation, which is specified as a linear combination of real vectors and a center, such that the absolute values of the combining coefficients are bounded within unity. Geometrically, a zonotope is the Minkowski sum of line segments and hence a type of polytope. Zonotopes have also been extended to the more general quadratic zonotopes [1], which is a quadratic function of real valued intervals. Zonotopes have the advantage that they are efficiently closed under linear transformation and Minkowski sum operations. Therefore, they have been successfully applied to the computation of bounded time reachable sets of uncertain continuous linear systems [**?**] and affine hybrid systems with large dwell time for switching [**?**]. But for over-approximation of the unbounded time reachable set by a positive invariant, real zonotopes have the have the following drawback. We can not guarantee the existence of a positively invariant real zonotope for the case of a stable linear transformation having complex eigenvalues. When the eigenvectors of a stable linear transformation are real valued, then collecting the eigenvectors of the matrix among the generators of the zonotope gives a positive invariant. However, this result does not hold when the eigenvectors have complex values, i.e., have non-zero imaginary and real parts.

2 Contribution

Complex zonotope. To address the above drawback of real zonotopes, we extend them to the complex valued domain to yeild a set representation called *complex zonotope*. A basic representation of the complex zonotope is a linear combination of complex valued vectors with complex combining coefficients, whose absolute values are bounded within unity. We show that a complex zonotope captures contraction along the complex valued eigenvectors of a linear transformation, which relates to existence of a positively

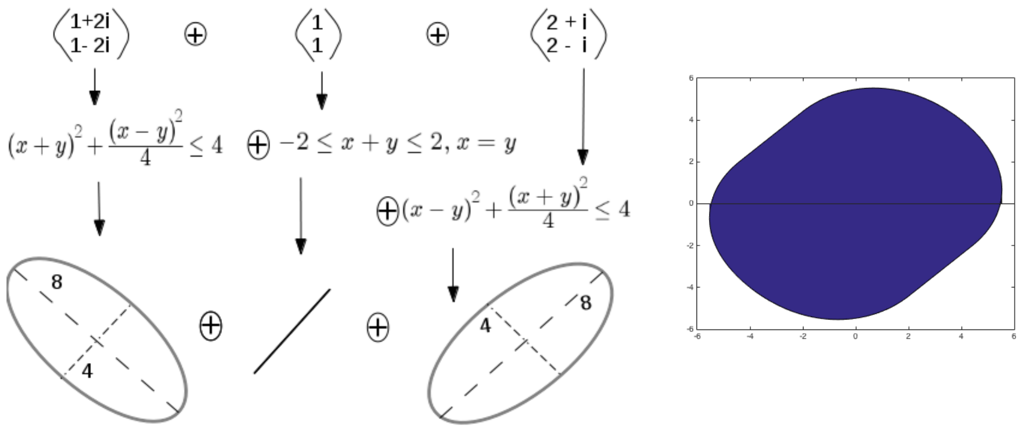

Figure 1: Figure of a complex zonotope

invariant complex zonotope. The real projection of a complex zonotope can represent not only Minkowski sums of line segments, but also some ellipsoids. Hence, they can represent non-polyhedral sets as well as polytopic zonotopes. They are different from quadratic zonotopes [**?**], because while the latter is a quadratic function of real valued intervals, a complex zonotope is a complex linear combination of circles in the complex plane.

Template complex zonotope. In order to easily refine a complex zonotope by adding arbitrary generators while approximating a given set, we introduce the *template complex zonotope* representation. In a template complex zonotope, the absolute values of combining coefficients have variable bounds called scaling factors. The scaling factors can be adjusted to find better over-approximations while adding arbitrary generators in a *template*. Just like simple zonotopes, template complex zonotopes are also efficiently closed under linear transformation and Minkowski sum operations. But in addition, template complex zonotope can easily represent a positive invariant for a linear transformation based on the eigenstructure of the matrix, as described previously. We propose a sufficient condition for checking inclusion between template complex zonotopes, expressed as a set of second-order conic (convex) constraints on the scaling factors and other auxiliary variables. This condition is useful in practice, which we demonstrate through our experiments in safety and stability verification on some benchmark examples.

Augmented complex zonotope. Template complex zonotopes are not closed under intersection with sub-level sets of linear inequalities. This is also a drawback of real zonotopes. Previous work addressed this drawback for real zonotopes by allowing linear constraints on the combining coefficients, in a more general representation called *constrained zonotopes* [11]. But if we emulate this approach for complex zonotopes, we get linear constraints on the complex valued combining co-coefficients in addition to the quadratic absolute value bounds. However, for the resultant set representation having quadratic absolute value constraints as well as linear constraints, the problem of finding a reasonable convex-relaxation for inclusion-checking, that works well in practice, seems intractable. Instead, we define a different representation called *augmented complex zonotope*, which is geometrically equivalent to a complex zonotope, but whose representation allows efficient computation of a non-trivial over-approximation of the intersection with a class of linear constraints. The over-approximation is computed as a simple algebraic expression of the parameters of the augmented complex zonotope. In the course of the derivation of this result, we also find a general mathematical result about intersection of Minkowski sum of convex sets with another convex set.

Unbounded time safety verification. We express operations on augmented complex zonotopes like linear transformation, Minkowski sum, over-approximation of intersection with constraints and support of a vector as affine expressions of some of the specification parameters. We also derive a second order conic program for checking inclusion between two augmented complex zonotopes. Henceforth, we derive a second order conic program for checking the safety of an affine hybrid system. The fact that a complex zonotope can both capture the contraction along complex eigenvectors and also efficiently represent linear transformation and Minkowski sum makes it an efficient set representation for unbounded time safety verification. We demonstrate the efficiency of the safety verification approach by implementing on some benchmark examples.

Stability verification of nearly periodic linear impulsive systems. Our analysis of template complex zonotopes for the case of discrete time switched linear systems is extended to verify stability of *nearly periodic linear impulsive systems*. We first derive bounds on the amount of contraction of a complex zonotope after a linear transformation. Then we derive a second order conic program to synthesize a contractive complex zonotope for a collection of linear transformations. We apply these results to verify the exponential stability of nearly periodic linear impulsive systems. We implement the procedure on some benchmark examples to demonstrate its efficiency.

3 Organization

The dissertation contains five main chapters and a conclusive chapter. In Chapter 1, we briefly discuss the use of set representation for computing reachable sets of hybrid systems and the desirable characteristics of a good set representation. In light of these characteristics, we provide a short review of some set representations that discusses their advantages and drawbacks. Then we review simple/real zonotopes and the commonly used operations on them in reach set computation. We also review the definitions polynomial and constrained zonotopes, which are previously known extensions of the simple zonotope. Then we shall describe the shortcoming of the real zonotope in computing a positive invariant.

In Chapter 2, we introduce the complex zonotope set representation that addresses the drawback of real zonotope described in the previous chapter. We define the basic representation of a complex zonotope and state a result about how a complex zonotope can efficiently represent a positive invariant for a stable linear transformation based on the eigenstructure. Next we define the more general template complex zonotope representation, which permits a way of refining a complex zonotope while approximating a given set. We discuss operations on template complex zonotopes that are used in reach set computation and derive a second order conic program for checking inclusion between template complex zonotopes. Using the results, we derive a second order conic program for unbounded time safety verification of linear systems and illustrate it with an example.

In Chapter 3, we introduce augmented complex zonotopes for computing non-trivial intersection with a class of sub-level sets of linear inequalities called sub-parallelotopes. Apart from the intersection, we discuss other operations on augmented complex zonotope used in the computation of reachable sets. Also, we state a second order conic program for checking inclusion between augmented complex zonotopes.

In Chapter 4, we apply augmented complex zonotopes for unbounded time safety verification of affine hybrid systems. We derive a second order conic program for checking safety, which is based on computing a suitable positively invariant augmented complex zonotope. We implement the approach on some benchmark examples to demonstrate its efficiency.

In Chapter 5, we apply template complex zonotopes for verifying exponential stability of linear impulsive systems with sampling time uncertainty. We discuss a procedure for stability verification, which consists of synthesizing a possibly contractive template complex zonotopes and verifying their contraction. We implement this procedure on some benchmark examples to demonstrate its efficiency.

The concluding remarks are stated in Chapter 6.

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