ON JUSTIFICATION OF AMBARTSUMIAN`S PLATE THEORY VIA Γ-CONVERGENCE

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Abstract. Ambartsumian`s plate theory, formulated by S. A. Ambartsumian in 1950s based on improved hypotheses, is justified using the concept of Γ-convergence. Limiting energies of known plane models are summarized. It is shown, that Ambartsumian`s theory is of fourth order, equivalent to the linearized von Kármán theory. Moreover, up to a constant multiplier, the limiting energy in Ambartsumian`s and linearized von Kármán theories coincide.
Аннотация. Теория пластин Амбарцумяна, построенная Амбарцумяном в 1950-х годах на основе уточнённых гипотез, обосновывается понятием Γ-сходимости. Приводятся предельные энергии известных плоских моделей. Показано, что теория Амбарцумяна, как и линеаризированная теория фон Кармана, имеет четвёртый порядок. Более того, с точностью до постоянного множителя, предельная энергия теории Амбарцумяна и линеаризированной теории фон Кармана совпадают.

1. Introduction

Many applied problems are formulated as variational problems of minimization (maximization) of some specific functional $J:\mathcal{F}\to\mathbb{R}\cup\{\infty\}$, where $\mathcal{F}$ is some function space. It is required to find such $f\in\mathcal{F}$, that for all $\varphi\in\mathcal{F}$, $J(f)\le J(\varphi)$.

In such cases, the Euler-Lagrange equations provide necessary conditions for minimizers of J. However, before solving the Euler-Lagrange equations, the existence of a solution must be established. One of the general methods of the calculus of variations for proving the existence of a minimizer for a given functional is the so-called direct method [1]. One of the advantages of the direct method is that besides proving the existence, it provides a tool for computing the minimizer to desired accuracy.

The algorithm of the direct method goes as follows:

a) Choose a minimizing sequence $f_n\in\mathcal{F}$ for J.

b) Show, that there is a convergent subsequence $\{f_{n_k}\}\in\{f_n\}$ with $f_{n_k}\to f_0$ in $\mathcal{F}$.

c) Prove, that J is sequentially lower semi-continuous in $\mathcal{F}$.

Recall, that f_n is a minimizing sequence for J, if

$$\lim_{n\to\infty} J(f_n)=\inf_{f\in\mathcal{F}} J(f).$$

J is called sequentially lower semi-continuous if

$$\liminf_{n\to\infty} J(f_n) \geq J(f_0)$$

for any convergent sequence $f_n \to f_0$. The existence follows from the inequality

$$\inf_{f\in\mathcal{F}} J(f) = \lim_{n\to\infty} J(f_n) = \lim_{k\to\infty} J\left(f_{n_k}\right) \geq J(f_0)$$

and the trivial bound

$$J(f_0) \geq \inf_{f\in\mathcal{F}} J(f).$$

In various applied problems the corresponding functional is not sequentially lower semi-continuous, and thus the existence of a minimizer is impossible to prove using the direct method. See [1] for many such examples. To overcome this difficulty lower semi-continuous envelopes of functionals with lack of that property are considered. Even though there are different approaches on explicit construction of lower semi-continuous envelopes for various types of functionals (see, for instance, [1–4] and references therein), it is actually very complicated to apply in a particular problem.

Another way for derivation a lower semi-continuous functionals in a particular problem is to compute the Γ-limit of the model functional, reducing, for instance, the problem dimension [5–7]. Recently, dimension reduction via Γ-convergence is applied in elasticity theory to derive energy functionals of various theories of beams, plates and shells (see, for instance, [8–18]). A hierarchy of plate models, including membrane, Kirchhoff, von Kármán, Reissner–Mindlin, and of higher order, are justified via Γ-convergence. The aim of the paper is to find out the order of Ambartsumian`s plate theory in the hierarchy of other theories. It is motivated by wide applicability of Ambartsumian`s theory, considering pure bending and transverse shear of thick plates, being a ***linear*** theory. It could be interesting, for instance, from biomechanics point of view: soft tissues are subjected to in-plane compressive stresses, leading to bending of the tissue in space and compression (in transverse direction) in its plane [19].

The paper is organized as follows. In Section 2 the known Γ-limits of the three-dimensional nonlinear elasticity energy functional are summarized by the order. In Section 3 Ambartsumian`s plate theory is shown to be the Γ-limit of the three-dimensional functional of fourth order, equivalent to the linearized von Kármán theory with linearized isometry constraint.

2. Γ-limits of the three-dimensional nonlinear elasticity theory

Known Γ-limits of the three-dimensional nonlinear elasticity are summarized in this section. For more details and further developments see [14]. Denote the three-dimensional energy of elastic plate of constant thickness h, which occupies the domain $\Omega_h = \left\{(x,y,z),\ x\in[0,a], y\in[0,b], z\in\left[-\frac{h}{2},\frac{h}{2}\right]\right\}$ in Cartesian system, by

$$E_h[\boldsymbol{r}] = \int_{\Omega_h} W(\nabla \boldsymbol{r})\, d\boldsymbol{x}.$$

Here $\boldsymbol{r}_h : \Omega_h \to \mathbb{R}^3$ is the three-dimensional deformation of the plate, W is the energy density function (for particular energy density functions see, for instance, [21]). Rescaling the energy by h, allows to consider the energy per unit volume

$$I_h[\boldsymbol{r}] = \frac{1}{h} E_h[\boldsymbol{r}] = \int_{\Omega} W(\nabla_h \boldsymbol{r})\, d\boldsymbol{x}, \quad \Omega = \Omega_0 \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

where Ω_0 is the middle plane of the plate, $\nabla_h \boldsymbol{r} = \left(\nabla' \boldsymbol{r}_0, \frac{\partial r_3}{\partial x_3}\right)$, $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$, $\boldsymbol{r}_0 = (r_1, r_2)$ is the deformation of Ω_0. Then, a hierarchy of plate models can be justified using the notion of Γ-convergence. Here the known Γ-limits are shortly described, for full derivation see [12–14]. In what follows it is assumed, that

1. $W(\mathbf{Q}\mathbf{F}) = W(\mathbf{F})$ for all spatial rotations $\mathbf{Q} \in \mathrm{SO}(3)$ and deformation gradients $\mathbf{F} \in \mathbb{R}^{3\times 3}$ (rotationally invariance),
2. $W = 0$ on $\mathrm{SO}(3)$ (isotropy),
3. $W(\mathbf{F}) \geq c \ \mathrm{dist}^2(\mathbf{F}, \mathrm{SO}(3))$ for $c > 0$,
4. W is C^2 in the neighborhood of $\mathrm{SO}(3)$.

Here we summarize known Γ-limits of fully two-dimensional models. For one-dimensional models (string, rod, beam) see [8].

2.1. Membrane energy

The simplest two-dimensional model corresponds to the case of stretched/compressed membrane. In the limit when $h \to 0$, the membrane energy looks like [LDR]

$$E_{mem}[\boldsymbol{r}_0] = \lim_{h\to 0} I_h[\boldsymbol{r}] = \lim_{h\to 0} \frac{1}{h} E_h[\boldsymbol{r}] = \begin{cases} \int_{\Omega} W_q(\nabla' \boldsymbol{r}_0)\, d\boldsymbol{x}', & \boldsymbol{r}_0 \in \mathcal{A}_m, \\ \infty, & else. \end{cases}$$

Here $W_q(\mathbf{F}) = \min_{\boldsymbol{a}\in\mathbb{R}^3} W(\mathbf{F} + \boldsymbol{a} \otimes \boldsymbol{e}_3)$ is the quasiconvex envelope of W, $\mathcal{A}_m = \left\{\boldsymbol{r}_0 \in W_0^{1,p}(\Omega; \mathbb{R}^3),\ \frac{\partial \boldsymbol{r}_0}{\partial x_3} = 0\right\}$ is the set of admissible membrane deformations.

Therefore, the membrane theory is of ***first*** order in h. This expression is derived using the notion of Γ-convergence in [12] (see also other articles of those authors). Note, that the same expression is obtained earlier by Pipkin in a number of papers (see, for instance, [20]) using simpler considerations of tension field theory. Pipkin`s procedure for deriving quasiconvex energy functional is much simpler and in some sense more intuitive, than that from [12].

Actually, W_q depends on the deformation through the first fundamental form of the membrane. Since the principal stretches of the membrane are the square roots of its first fundamental form, W_q is expressed in term of those stretches: $W_q = \tilde{W}_q\left(\lambda_1, \lambda_2\right)$.

2.2. Plate energy

Various plate theories for describing as bending regimes, as well as bending and stretching/compression regimes, exist. The simplest theory lies on Kirchhoff`s hypotheses and is often referred to as Kirchhoff (Kirchhoff-Love) plate theory. The theory lies on the assumptions, that the in-plane displacements of the plate depend on x_3 linearly, and the normal displacement is independent of x_3. In other words, it is assumed, that $\boldsymbol{r}\left(\boldsymbol{x}\right) = \boldsymbol{r}_0\left(\boldsymbol{x}'\right) + x_3\boldsymbol{n}\left(\boldsymbol{x}'\right)$.

In [13, 14] it is established, that

$$E_K\left[\boldsymbol{r}_0\right] = \lim_{h\to 0}\frac{1}{h^2}I_h\left[\boldsymbol{r}\right] = \lim_{h\to 0}\frac{1}{h^3}E_h\left[\boldsymbol{r}\right] = \begin{cases} \dfrac{1}{24}\displaystyle\int_\Omega Q_2\left(\mathbf{II}\right)d\boldsymbol{x}', & \boldsymbol{r}_0 \in \mathcal{A}_p, \\ \infty, & else. \end{cases}$$

Here $$Q_2\left(\mathbf{G}\right) = \min_{\boldsymbol{a}\in\mathbb{R}^3} Q_3\left(\mathbf{G} + \boldsymbol{a}\otimes\boldsymbol{e}_3 + \boldsymbol{e}_3\otimes\boldsymbol{a}\right),$$ and

$Q_3\left(\mathbf{F}\right) = \left.\dfrac{\partial^2 W}{\partial \mathbf{F}^2}\right|_{\mathbf{F}=\mathrm{Id}}\left(\mathbf{F},\mathbf{F}\right) = \displaystyle\sum_{i,j,k,l=1}^{n}\left.\dfrac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}\right|_{\mathbf{F}=\mathrm{Id}} F_{ij}F_{kl}$ is twice the linearised energy,

$\mathbf{II} = -\dfrac{\partial^2 \boldsymbol{r}_0}{\partial x_1 \partial x_2}\cdot\boldsymbol{n}$ is the second fundamental form, and $\boldsymbol{n}$ is the unit normal to the mid-surface of the plate, $\mathcal{A}_p = \left\{\boldsymbol{r}_0 \in W_0^{1,p}\left(\Omega;\mathbb{R}^3\right), \left(\nabla'\boldsymbol{r}_0\right)^T\nabla'\boldsymbol{r}_0 = \mathrm{Id}\right\}$ is the set of admissible plate deformations, consisting of isometric immersions, i.e. the mid-surface of the plate is not stretched or compressed. Therefore, Kirchhoff`s theory is of ***third*** order in h.

Since the limiting energy density function depends on the second fundamental form, the eigenvalues of which are the principal curvatures, then it can be represented as a function of those: $Q_2 = \tilde{Q}_2(\kappa_1, \kappa_2)$.

Particularly, for isotropic materials [13, 14]

$$Q_2(\mathbf{G}) = 2\mu\left|\frac{\mathbf{G}+\mathbf{G}^T}{2}\right|^2 + \lambda \mathrm{tr}^2\mathbf{G},$$

$$Q_3(\mathbf{F}) = 2\mu\left|\frac{\mathbf{F}+\mathbf{F}^T}{2}\right|^2 + \frac{2\mu\lambda}{2\mu+\lambda}\mathrm{tr}^2\mathbf{F}.$$

The next order plate theory is von Kármán plate theory. The former relies on Kirchhoff`s hypotheses, but uses nonlinear strain-displacement relations [22]. It is established in [14], that von Kármán theory is of fifth order in h, moreover, the limiting energy reads as

$$E_{vK}[\boldsymbol{r}_0] = \lim_{h\to 0}\frac{1}{h^4} I_h[\boldsymbol{r}] = \lim_{h\to 0}\frac{1}{h^5} E_h[\boldsymbol{r}] =$$
$$= \frac{1}{2}\int_{\Omega}\left[Q_2\left(\frac{1}{2}\left[\nabla'\boldsymbol{u} + (\nabla'\boldsymbol{u})^T + \nabla'\boldsymbol{w}\otimes\nabla'\boldsymbol{w}\right]\right) + \frac{1}{24}Q_2\left((\nabla')^2\boldsymbol{w}\right)\right]d\boldsymbol{x}'.$$

Here $\boldsymbol{u}$ and $\boldsymbol{w}$ are the averaged in-plane and out of plane displacements [14]

$$\boldsymbol{u} = \lim_{h\to 0}\frac{1}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\begin{pmatrix}\boldsymbol{r}_1\\ \boldsymbol{r}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{x}' & x_3\end{pmatrix} - \begin{pmatrix}x_1\\ x_2\end{pmatrix}\right]dx_3,\quad \boldsymbol{v} = \lim_{h\to 0}\frac{1}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\boldsymbol{r}_3 dx_3.$$

On the other hand, linearized von Kármán plate theory is of ***fourth*** order. The limiting energy is

$$E_{linvK}[\boldsymbol{r}_0] = \lim_{h\to 0}\frac{1}{h^3} I_h[\boldsymbol{r}] = \lim_{h\to 0}\frac{1}{h^4} E_h[\boldsymbol{r}] = \begin{cases}\dfrac{1}{24}\displaystyle\int_{\Omega} Q_2\left((\nabla')^2\boldsymbol{w}\right)d\boldsymbol{x}', & \det(\nabla')^2\boldsymbol{w} = 0,\\ \infty, & \textit{else.}\end{cases}$$

3. Ambartsumian`s theory of plates as Γ-limit of three dimensional nonlinear elasticity theory

Ambartsumian`s theory of anisotropic plates was developed in 1950s by prominent scientist Sergey A. Ambartsumian and his colleagues. It is mainly based on the following assumptions [23]:

a) the displacement w normal to the mid-surface of the plate does not depend on x_3,

b) the shear stresses τ_{13} and τ_{23} (or corresponding strains ε_{13} and ε_{23}) have a given distribution over the thickness of the plate:

$$\tau_{13} = \text{l.t.} + f_1(z)\varphi(x,y), \quad \tau_{23} = \text{l.t.} + f_2(z)\psi(x,y),$$

where l.t. denotes the loading terms, f_1, f_2 denote the given distribution, satisfying $f_i\left(\pm\frac{h}{2}\right) = 0,\ i = 1,2,$, φ, ψ are the behavior of the stresses over the mid-surface of the plate. The choice of characteristic functions f_1, f_2 defines the order in h of the corresponding theory. In [23] and in what follows it is taken $f_1(z) = f_2(z) = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)$. In this case $\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{r}_0(\boldsymbol{x}') + f(x_3)\boldsymbol{n}(\boldsymbol{x}')$. Using the procedure, applied in [24], it is shown, that Ambartsumian`s plate theory is of fourth order in h. Moreover for $W(\nabla\boldsymbol{r}) = \frac{1}{2}\left|(\nabla\boldsymbol{r})^T\nabla\boldsymbol{r} - \mathrm{Id}\right|^2$ the limiting energy is derived in the form of Wilmore functional (the derivation simply repeats the procedure described in [14, 24], that is why is not brought here)

$$E_{Amb}[\boldsymbol{r}_0] = \lim_{h\to 0}\frac{1}{h^3}I_h[\boldsymbol{r}] = \lim_{h\to 0}\frac{1}{h^4}E_h[\boldsymbol{r}] = \begin{cases} \frac{\sqrt{6}}{5!}\int_\Omega |\mathbf{II}|^2\,d\boldsymbol{x}', & \boldsymbol{r}_0 \in \mathcal{A}_p, \\ \infty, & else. \end{cases}$$

Thus, in this case the deformation of the plate mid-surface is isometric as well.

Note, that up to a constant multiplier, Ambartsumian`s limiting energy coincides with the Kirchhoff limiting energy for the same energy density, being of higher order. There exists also the non-zero limit $E_{Amb}[\boldsymbol{r}_0] = \lim_{h\to 0}\frac{1}{h^5}E_h[\boldsymbol{r}]$, the physical meaning of which is under current study and will be reported elsewhere.

4.Conclusion

Ambartsumian`s plate theory is of ***fourth*** order in the thickness of the plate. The limiting energy is the known Wilmore functional, coinciding with the limiting energy within the linearized von Kármán plate theory.

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