



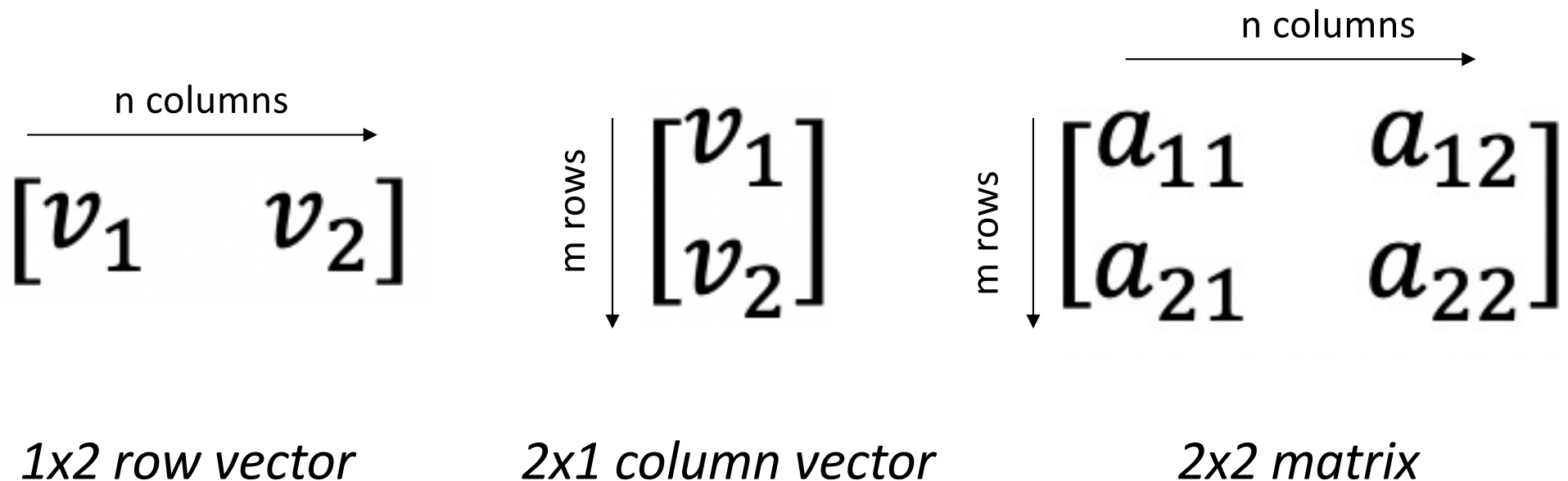
```
18  int main(void) {
19      Complex z;
20      Complex res;
21      cout << "Enter real and imag parts of z: ";
22      cin >> z; // Converts to operator>>(cin,z)
23      res = z*2.5; // Converts z*2.5 to operator*(z,2.5)
24      cout << "z*2.5 = " << res << endl; // Converts to operator<<(cout,res)
25      res = 2.5*z; // Converts 2.5*z to operator*(2.5,z)
26      return 0;
27  }
28
29  // New overload for Complex*float
30  Complex operator*(Complex foo, float f) {
31      Complex bas;
32      bas.a = foo.a * f;
33      bas.b = foo.b * f;
34      return bas;
35  }
36
37  Complex operator+(Complex foo, Complex bar) {
38      Complex bas;
39      // Implement rules of complex number addition
40      bas.a = foo.a + bar.a;
41      bas.b = foo.b + bar.b;
42      return bas;
43  }
```

ENME 202

Linear Algebra Review

Vectors and Matrices

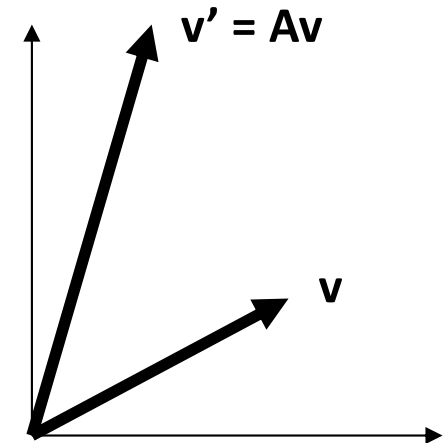
- A row vector is a 1 x n (rows x columns) list of values
- A column vector is an m x 1 (rows x columns) list of values
- A matrix is an m x n (rows x columns) array of values



Vectors and Matrices

- An $n \times 1$ column vector can be interpreted as a line in n -dimensional space
- An $m \times n$ matrix can be interpreted as a transformation acting on a column vector

$$\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



Dot Product

Dot product of two row vectors v and w :

$$v \cdot w = vw^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2$$

The dot product of a vector with itself is equal to the square of the magnitude (length) of the vector:

$$\|v\|^2 = v \cdot v = v_1^2 + v_2^2 + \dots + v_n^2$$

→ The dot product can provide a measure of vector length

The dot product is also defined geometrically as:

$$v_1 \cdot v_2 = \|v_1\| \|v_2\| \cos \theta$$

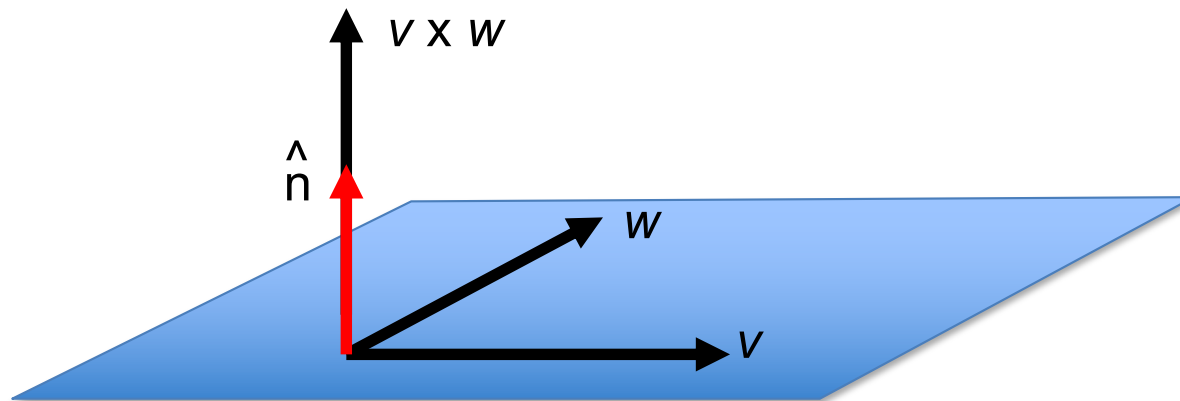
→ The dot product can provide a measure of the angle between two vectors

Cross Product

Cross product of two vectors v and w :

$$v \times w = \|v\| \|w\| \sin \theta \hat{n}$$

- \hat{n} is a unit vector perpendicular to v and w , with direction defined by the *right hand rule*



→ The cross product can be used to define the plane in which vectors v and w lie

Matrix Addition

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Matrices must have the same dimensions for addition to be valid

Matrix-Vector Multiplication



$$\begin{matrix} A_{m \times n} & & v_{n \times 1} \\ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n} & \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1} & = & \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}_{m \times 1} \end{matrix}$$

inner dimensions must match

outer dimensions are the same as the result

The diagram illustrates the dimension matching for matrix-vector multiplication. It shows a matrix of size m by n multiplied by a vector of size n by 1, resulting in a vector of size m by 1. Blue arrows and text labels highlight the matching inner dimensions (n) and the resulting outer dimensions (m).

Matrix-Vector Multiplication

We can instead think about A as a *column vector of row vectors* to evaluate the product:

$$\begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix} v = \begin{bmatrix} A_1^r v \\ A_2^r v \\ \vdots \\ A_m^r v \end{bmatrix}$$

Alternately, decompose A into a *row vector of column vectors* and take the weighted sum:

$$\begin{bmatrix} A_1^c & A_2^c & \cdots & A_n^c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = A_1^c v_1 + A_2^c v_2 + \cdots + A_n^c v_n$$

Matrix-Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}_{n \times 1}$$

← Recall matrix-vector multiplication

$$= \begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix} [B_1^c] = \begin{bmatrix} A_1^r B_1^c \\ A_2^r B_1^c \\ \vdots \\ A_m^r B_1^c \end{bmatrix}_{m \times 1}$$

Matrix-Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{21} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{bmatrix}_{n \times 2} \quad \leftarrow \text{Add a second column...}$$

$$= \begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix} [B_1^c \quad B_2^c] = \begin{bmatrix} A_1^r B_1^c & A_1^r B_2^c \\ A_2^r B_1^c & A_2^r B_2^c \\ \vdots & \vdots \\ A_m^r B_1^c & A_m^r B_2^c \end{bmatrix}_{m \times 2}$$

Matrix-Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}_{n \times p}$$

$$= \begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix} [B_1^c \quad B_2^c \quad \cdots \quad B_p^c] = \begin{bmatrix} A_1^r B_1^c & A_1^r B_2^c & \cdots & A_1^r B_p^c \\ A_2^r B_1^c & A_2^r B_2^c & \cdots & A_2^r B_p^c \\ \vdots & \vdots & \ddots & \vdots \\ A_m^r B_1^c & A_m^r B_2^c & \cdots & A_m^r B_p^c \end{bmatrix}_{m \times p}$$

- **2x2 example:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Note that matrix multiplication is not commutative

Identity Matrix

- Identity matrix (3x3):

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For any square matrix A:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

- Some square matrices have an inverse, such that:

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

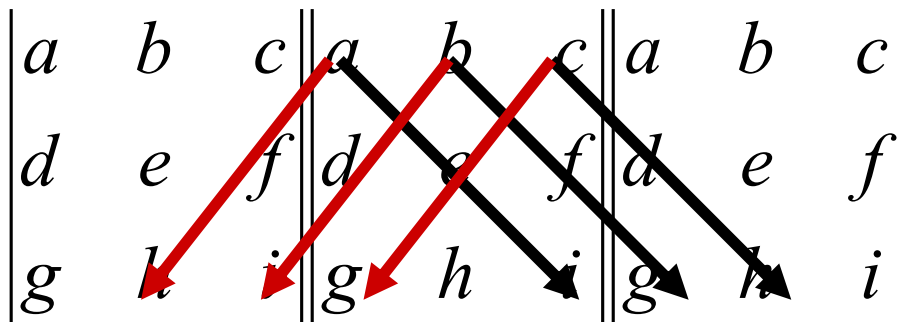
Determinant of a Matrix: $|A|$

- The determinant can be interpreted as a scaling factor for the linear transformation defined by the matrix
- If $|A| = 0$ then A has no inverse
- Determinant of a 2x2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A| = ad - bc$$

Determinant of a 3x3 Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$



- Take product along each line
- Sum products from left to right (black)
- Subtract products from right to left (red)

Matrix Inversion

$$A^{-1} = \frac{\overset{\text{adjoint matrix}}{adj(A)}}{|A|} = \frac{\overset{\text{cofactor matrix}}{cof(A)^T}}{|A|}$$

The cofactor matrix is a matrix formed by the determinants of the minors of A_{ij} multiplied by -1^{i+j} .

Each minor term is defined by the matrix A with the i^{th} row and j^{th} column removed.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$