```
int main(void) {
    Complex z;
    Complex res;
    cout << "Enter real and imag parts of z: ";
    cin >> z;  // Converts to operator>>(cin,z)
    res = z*2.5; // Converts z*2.5 to operator*(z,2.5)
    cout << "z*2.5 = " << res << endl; // Converts to operator<<(cout,res)
    res = 2.5*z;  // Converts 2.5*z to operator*(2.5,z)
    return 0;
}

// New overload for Complex*float
Complex operator*(Complex foo, float f) {
    Complex bas;
    bas.a = foo.a * f;
    bas.b = foo.b * f;
    return bas;
}

Complex operator+(Complex foo, Complex bar) {
    Complex bas;
    // Implement rules of complex number addition
    bas.a = foo.a + bar.a;
    bas.b = foo.b + bar.b;
    return bas;
}
```

ENME 202

Linear Algebra Review

Vectors and Matrices

- A row vector is a 1 x n (rows x columns) list of values
- A column vector is an m x 1 (rows x columns) list of values
- A matrix is an m x n (rows x columns) array of values

$$\begin{array}{c} \text{n columns} \\ \begin{bmatrix} v_1 & v_2 \end{bmatrix} \end{array}$$

1x2 row vector

$$\begin{array}{cc} \text{m rows} & \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{array}$$

2x1 column vector

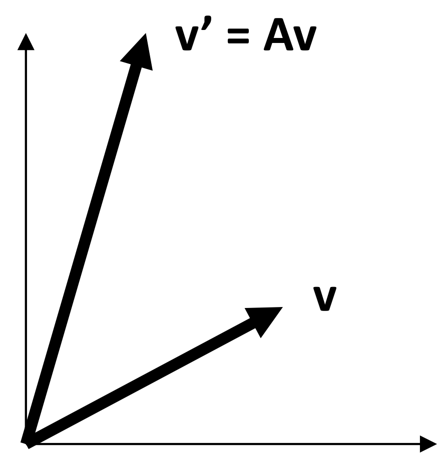
$$\begin{array}{cc} & \text{n columns} \\ \text{m rows} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

2x2 matrix

Vectors and Matrices

- A 1x*n* vector can be interpreted as a line in n-dimensional space
- An *mxn* matrix can be interpreted as a transformation acting on a vector

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



Dot Product

Dot product of two row vectors *v* and *w*:

$$v \cdot w = vw^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2$$

The dot product of a vector with itself is equal to the square of the magnitude (length) of the vector:

$$\|v\|^2 = v \cdot v = v_1^2 + v_2^2 + \ldots + v_n^2$$

→ The dot product can provides a measure of vector length

The dot product is also defined geometrically as:

$$v_1 \cdot v_2 = \|v_1\| \|v_2\| \cos\theta$$

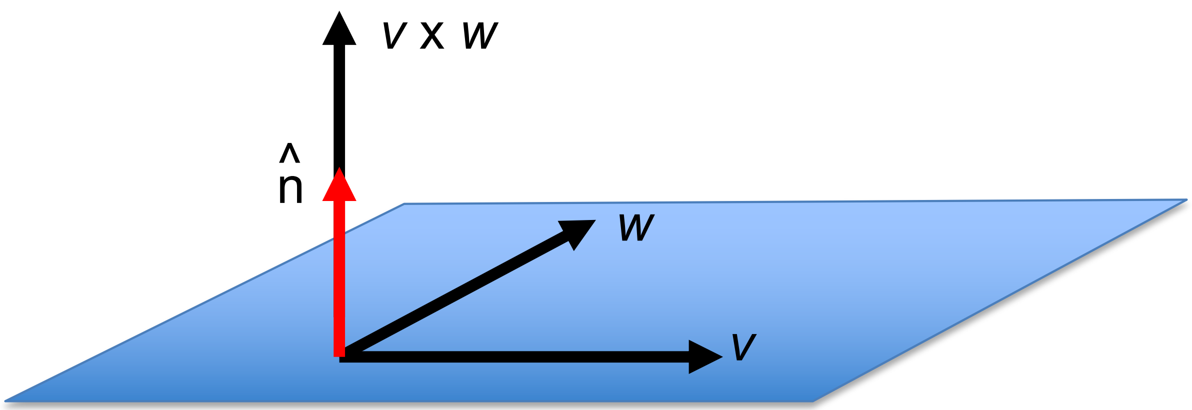
→ The dot product can provide a measure of the angle between two vectors

Cross Product

Cross product of two vectors *v* and *w*:

$$v \times w = \|v\| \|w\| \sin\theta \hat{n}$$

- $\hat{n}$ is a unit vector perpendicular to *v* and *w*, with direction defined by the *right hand rule*



→ The cross product can be used to define the plane in which vectors *v* and *w* lie

Matrix Addition

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Matrices must have the same dimensions for addition to be valid

Matrix-Vector Multiplication

$$\underset{A_{\text{mxn}}}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{mxn}} \underset{v_{\text{nx1}}}{\begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix}_{nx1}} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}_{mx1}$$

inner dimensions must match

outer dimensions are the same as the result

Matrix-Vector Multiplication

We can instead think about *A* as a *column vector of row vectors* to evaluate the product:

$$\begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix} v = \begin{bmatrix} A_1^r v \\ A_2^r v \\ \vdots \\ A_m^r v \end{bmatrix}$$

Alternately, decompose *A* into a *row vector of column vectors* and take the weighted sum:

$$[A_1^c \;\; A_2^c \;\; \cdots \;\; A_n^r] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = A_1^c v_1 + A_2^c v_2 + \cdots + A_n^c v_n$$

Matrix-Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mxn} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{21} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}_{nxp}$$

$$= \begin{bmatrix} A_1^r \\ A_2^r \\ \vdots \\ A_m^r \end{bmatrix} \begin{bmatrix} B_1^c & B_2^c & \cdots & B_p^c \end{bmatrix} = \begin{bmatrix} A_1^r B_1^c & A_1^r B_2^c & \cdots & A_1^r B_p^c \\ A_2^r B_1^c & A_2^r B_2^c & \cdots & A_2^r B_p^c \\ \vdots & \vdots & \ddots & \vdots \\ A_m^r B_1^c & A_m^r B_2^c & \cdots & A_m^r B_p^c \end{bmatrix}_{mxp}$$

- **2x2 example:**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Note that matrix multiplication is not commutative

Identity Matrix

- Identity matrix (3x3):

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For any <u>square</u> matrix A:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

- Some square matrices have an <u>inverse</u>, such that:

$$\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$$

Determinant of a Matrix: |A|

- The determinant can be interpreted as a scaling factor for the linear transformation defined by the matrix

- If |A| = 0, then A has no inverse

- Determinant of a 2x2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A| = ad - bc$$

Determinant of a 3x3 Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

- Take product along each line
- Sum products from left to right (black)
- Subtract products from right to left (red)

Matrix Inversion

$$A^{-1} = \frac{\overbrace{adj(A)}^{\text{adjoint matrix}}}{|A|} = \frac{\overbrace{cof(A)}^{\text{cofactor matrix}}{}^T}{|A|}$$

The <u>cofactor</u> matrix is a matrix formed by the determinants of the minors of A_{ij} multiplied by -1^{i+j}.

Each <u>minor</u> term is defined by the matrix A with the i^{th} row and j^{th} column removed.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$