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Fuzzy Multi-dimensional Analysis and Resolution Operation

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Abstract

In this paper a new original approach to the analysis of fuzzy multi-dimensional distributions is described. A uniform method for representing fuzzy multi-dimensional distributions by means of sectioned vectors and matrices is proposed. Sectioned matrix is interpreted as fuzzy conjunctive normal form, while its line vectors are interpreted as fuzzy disjunctions. Several useful characteristics of fuzzy distributions and disjunctions are defined and studied. The main operation for manipulating fuzzy multi-dimensional distributions is an original fuzzy resolution which is applied to any two disjunctions on some variable and results in a third disjunction called resolvent. The property of adjacency of two disjunctions is defined and the criterion of adjacency is formulated. It is shown that the proposed resolution operation is a generalization of the conventional resolution and the whole approach can be viewed as a generalization of propositional logic. Methods for finding prime disjunctions, projection on a variable (thus solving the satisfiability problem) and transforming into the dual form are proposed.

1 Introduction

Let us consider the following problem. Given an n-dimensional space called the universe of discourse which is equal to the Cartesian product of n variables. There is some (global) distribution over this space which is supposed to be represented by means of a combination of elementary (local) distributions over individual variables. A global characteristic of the distribution is said to be some quantity which depends on the values in all (or almost all) points of the universe of discourse. The problem consists in calculating a global characteristic of the distribution without the necessity to access values in all points of the universe of discourse, i.e., taking into account only local distributions over individual variables by means of which the multi-dimensional distribution is represented.

Such a formulation is obviously too general. Therefore, to obtain concrete results we have to reduce it to more concrete case by specifying more exactly types of variables, their sets of values, operations used to combine distributions, global characteristics to find.

One such particular but probably the most important case has been paid a lot of attention in Logic, Algebra, Switching Theory, Cybernetics, Artificial Intelligence, and other fields where it usually has its own name and is described in special terms depending on the problem being solved. The main assumptions for this case are as follows:

- all variables have only two values 0 and 1,
- distributions take values from the set $\{0, 1\}$,
- logical connectives $\wedge$ and $\vee$ are used to combine elementary distributions, and
- the maximal or minimal value is usually a global characteristic to search for.

There are only 4 different two-valued distributions over two-valued variables which are called elementary propositions in propositional logic (Fig. 1):

- truth constant 0,
- truth constant 1,
- proposition P, and
- proposition $\neg P$.

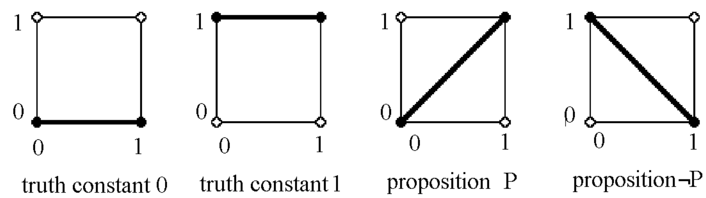

Fig. 1. Four Boolean propositions. Both the variable and the distribution are two-valued.

Combining different local distributions by logical connectives $\wedge$ and $\vee$ which are interpreted with the help of conventional truth tables, we can represent different global distributions over n-dimensional hypercube. One traditional problem that many other theoretical and applied problems are reduced to, is the problem of satisfiability which is obviously equivalent to finding the maximal value of the global distribution over the n-dimensional universe of discourse. There is a lot of different methods and their modifications for solving this problem, e.g., based on the operation of consensus (resolution in the Artificial Intelligence), transformation into the dual form, covering techniques, etc.

In this paper it is supposed that

- all variables take their values in finite sets,
- all distributions are fuzzy membership functions from the domain of definition (values of individual variables or their Cartesian product) to the unit interval [0,1], and
- logical connectives $\wedge$ and $\vee$ interpreted with the help of the minimum and maximum operations are used for combining distributions.

Maximal value of the global distribution over the universe of discourse is considered as a global characteristic to be found. We also consider a more general problem of finding a projection of the global distribution on some variable which allows us to solve more efficiently the problem of logical inference. Other useful problems can be also formulated, e.g., finding a global entropy of a multi-dimensional fuzzy distribution.

Currently exact methods for solving this problem do not exist. However, a lot of inexact methods have been proposed in the field of approximate reasoning (mainly for application to knowledge based systems). Perhaps the most well-known of them is the Zadeh's combination and projection principle [1, 2] which can formulated as follows:

- each statement is translated into a possibility distribution,
- all possibility distributions are conjunctively (with the help of minimum operation) *combined* into an overall possibility distribution π,
- the distribution π is *projected* on various variables of interest (e.g., using the generalized modus ponens).

Unfortunately, it is only a principle and it does not provide us a concrete procedure for finding projections. The main disadvantage of other approaches (see, e.g., [3]) is that they do not guarantee that the conclusion (projection) obtained is correct like similar methods in the Boolean fields (Switching Theory, Boolean functions, Propositional Logic etc.). In other words, we do not know whether the projection resulted from the procedure is equal to the real projection of our distribution.

In this paper we propose a new original operation of fuzzy resolution which can be used to solve this problem. We will suppose that the global distribution is represented by means of a number of fuzzy disjunctions combined with the connective $\wedge$ (minimum). Each fuzzy disjunction consists of n local distributions (possibly trivial) combined by the connective $\vee$. The operation of fuzzy resolution is applied to any two disjunctions on some variable and results in a third disjunction called resolvent (consensus). The resolvent possesses several useful properties (described below in the paper) which allow us to say that this operation is a generalization of the conventional resolution. Thus having this fuzzy resolution we can more or less easily transfer onto fuzzy case almost all resolution (consensus) based methods developed for the boolean case.

This paper originates from an original approach of A. Zakrevsky [4] called the logic of finite predicates where a new technique of sectioned boolean vectors for representing disjunctions and the corresponding consensus operation was proposed. On the basis of the technique of Boolean sectioned vectors and matrices an EDIP diagnostic expert system shell was implemented [6, 7, 8]. An inference process in the EDIP system is based on the procedure of finding all prime vector disjunctions by means of the operation of generalized consensus.

Later [9, 10, 11, 14] the formalism of A. Zakrevsky including the technique of sectioned vectors and the operation of consensus was generalized on fuzzy case where the components take their values from the unit interval [0,1]. In addition some new properties degenerated in the crisp case were studied, as well as new procedures of logical inference were developed which underlie an EDIP for Windows 3.x expert system shell [13]. In this generalization of the Zakrevsky's formalism instead of the term 'consensus' the term 'resolution' was used, which is conventional in the Artificial Intelligence.

This approach to fuzzy multi-dimensional analysis was reformulated in logical terms as a generalized fuzzy propositional logic [14, 15] and a logic of possibility distributions [16]. It was also applied to such fields as diagnosis [17], fuzzy control [18], aggregation of information [19, 20] and decision making [21].

2 Method of Sectioned Vectors and Matrices

Let $x_1, x_2, \ldots, x_n$ be elementary logical variables taking their values from the sets $X_1, X_2, \ldots, X_n$ called (elementary) domains respectively. In general, domains are supposed to be any continuos interval but in our examples we will only consider for simplicity the case of domains consisting of a finite number of values a_{ij}, where $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n_i$. The Cartesian product of all domains $X_1 \times X_2 \times \ldots \times X_n$ forms the universe of discourse Ω with the power $n_1 \times n_2 \times \ldots \times n_n$ in the finite case. Each element $\omega = \langle x_1, x_2, \ldots, x_n \rangle \in \Omega$ is an ordered

n-tuple of values of all variables.

Below in this section we describe a technique which we use to write fuzzy disjunctions. This technique was originally proposed by Zakrevsky for the case of multi-valued variables and two-valued distributions. Later it was generalized by Savinov onto the case of fuzzy distributions. In the method of sectioned vectors we use the following terminology:

- *component* $\mathbf{v}_{ij}$ of the vector $\mathbf{v}$ is the number which is equal to the local distribution value in one point of the domain; in the case of crisp distributions it is equal either 0 or 1; in the fuzzy case it takes values from the interval [0,1]; in other words, the component $\mathbf{v}_{ij}$ is equal to the local distribution value in the point $x_i = a_{ij}$, where a_{ij} is the j-th value of the i-th variable;

- *section* $\mathbf{v}_i$ is a sequence of n_i components for all values of a local distribution, e.g., if $\mathbf{v}_i = \{0.7, 1, 0.2, 0\}$ then $\mathbf{v}_{i1} = 0.7$, $\mathbf{v}_{i2} = 1$, $\mathbf{v}_{i3} = 0.2$, $\mathbf{v}_{i4} = 0$;

- *vector* $\mathbf{v}$ consists of n sections $\mathbf{v}_i$ separated by points (when the operation is implicitely implied) or explicitely by the name of operation, e.g., $\{0.7, 1, 0.2, 0\}.\{1, 0.4, 0\}.\{1, 0\}$, where $n_1 = 4$, $n_2 = 3$, $n_3 = 2$;

- *matrix* is made up of a number of vectors each of them representing one line.

An interpretation of fuzzy vector is a rule by means of which we can compute the global distribution this vector defines over the universe of discourse proceeding from the local distributions the vector is made up. There are two interpretations of fuzzy vectors: as disjunctions and as conjunctions. If the vector $\mathbf{d}$ is interpreted as disjunction then the value of its global distribution in some point is equal to the maximum of n corresponding components (Fig. 2).

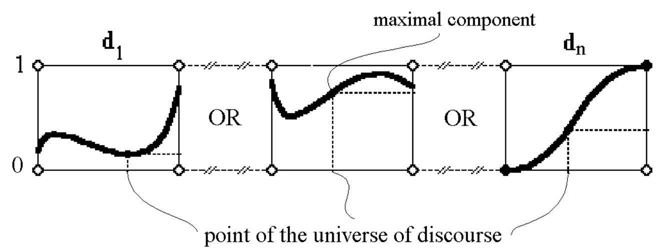

Fig. 2. Interpretation of the vector as disjunction by means of the maximum of n corresponding components.

For example, the disjunction

$$\{0.7, 1, 0.2, 0\} \quad \{1, 0.4, 0\} \quad \{1, 0\}$$

defines the distribution which in the point $\langle a_{13}, a_{22}, a_{32} \rangle$ is equal to $\max(0.2, 0.4, 0) = 0.4$.

The interpretation of sectioned vectors as conjunctions is dual, i.e., it uses the operation of minimum.

Sectioned matrices have two interpretations: as conjunctive normal form (CNF), and as disjunctive normal form (DNF). The interpretation as CNF means that the value in some point of the universe of discourse is equal to the minimum of the values which are assigned to this point by its lines. The lines of the CNF are interpreted as disjunctions.

It can be easily proved that any fuzzy distribution over the universe of discourse can be represented in the form of fuzzy CNF. Such a CNF is made up of $|\Omega| = n_1 \times n_2 \times \ldots \times n_n$ line disjunctions each of which represents a fuzzy distribution value in the corresponding point of the universe of discourse. In other words, one line of this matrix is responsible for representing the distribution value in some point and it does not influence any other points. Each section of the disjunction satisfying this condition has to consist of all 1's except of one component which is equal to the corresponding distribution value. We say that it pricks a hole down to the necessary level in the distribution surface. Of course, it is not a procedure for building and representing fuzzy multidimensional distributions – it demonstrates only that for any arbitrary fuzzy distribution there exists a sectioned matrix interpreted as CNF which represents it (i.e., with the same semantics).

3 Characteristics of Disjunctions

In this section we define three global characteristics of fuzzy distributions. Here 'global' means that to calculate the characteristic one needs to access all elements of the domain of definition and corresponding fuzzy distribution values.

The maximal value that the fuzzy distribution takes over the domain of definition is said to be the *consistency* (Fig. 3). For example, the local distribution $\{0, 0.1, 0.3, 0.7\}$ has the consistency 0.7, and the consistency of the disjunction $\{0, 0.5\}.\{1, 0, 0\}$ is equal to 1.

The minimal value that fuzzy distribution takes over the domain of definition is said to be the *constant* (Fig. 3). For example, the local distribution $\{0.1, 0.5, 1\}$ and the disjunction $\{0, 0.5\}.\{1, 0, 0\}$ have the constants 0.1 and 0 respectively.

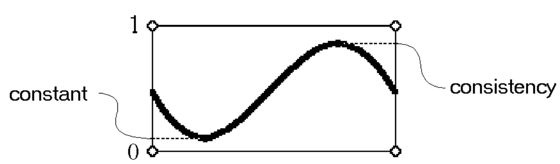

Fig. 3. Consistency and constant of the distribution.

We will need a quantity called a degree of *incomparability* of two distributions. Let us define at first a relative degree of incomparability. The degree of incomparability of the distribution P in relation to Q is equal to the maximal value of the distribution P which is exactly greater than the corresponding (i.e., in the same point of the universe) value of the proposition Q:

$$\mathrm{incomp}_Q(P) = \max_{P(x)>Q(x)}(P(x))$$

Thus in order to compute this quantity one at first needs to select in P all the values which are exactly greater than the corresponding values in Q, and then to choose among them the maximal value (Fig. 4).

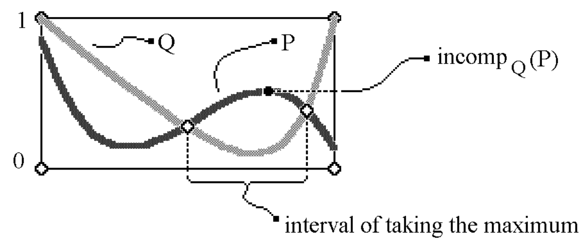

Fig. 4. Relative degree of incomparability.

If the condition $\forall x P(x) \leq Q(x)$ holds, i.e., there is nothing to choose the maximal value from (the distribution P is included into Q), it is supposed by definition that $\text{incomp}_Q(P) = 0$. For example, degree of incomparability of the proposition $P = \{0, 0.5, 0.7, 1\}$ in relation to the proposition $Q = \{0.2, 0.4, 0.6, 1\}$ is equal to $\text{incomp}_Q(P) = \max(0.5, 0.7) = 0.7$, whereas $\text{incomp}_P(Q) = \max(0.2) = 0.2$.

The (mutual) degree of incomparability is equal to the minimal of two relative degrees of incomparability:

$$\text{incomp}(P, Q) = \min(\text{incomp}_Q(P), \text{incomp}_P(Q))$$

Note that it is important that the degree of incomparability is defined not from informal interpretation of the word "incomparable" but from the formal requirements of the fuzzy resolution what will be shown below.

Consequence relation on fuzzy distributions is defined in a traditional way. Namely, the distribution Q is said to be a logical consequence of the distribution P iff the condition

$$\forall x P(x) \leq Q(x)$$

holds, i.e., P is included into Q.

Obviously, if Q is a logical consequence of P then Q can be removed from a set axioms or theorems. In particular, disjunction which is a consequence of another disjunction can be removed from the matrix. The process of removing such disjunctions is called absorption.

4 Reduced Forms of Disjunctions

Let us consider the following example. Three disjunctions

$$\begin{array}{ll} \{1, 0, 1, 1\} & \{0.5, 1\} \\ \{1, 0.5, 1, 1\} & \{0, 1\} \\ \{1, 0.5, 1, 1\} & \{0.2, 1\} \end{array}$$

are semantically equivalent, i.e., they represent the same distribution over the universe of discourse. Thus in general case disjunctions represent semantics not uniquely, i.e., several different in the form disjunctions can represent the same in the meaning proposition about the universe. The uniqueness of representation takes place only for disjunctions with the constant equal to 0, when there is at least one element from the universe with the distribution value 0. If the disjunction constant (the minimal value of the corresponding distribution) is not equal to 0, then its representation is not unique because the components which are between 0 and the disjunction constant may vary in this interval (provided that this does not change the constant itself). So it is clear that in the disjunction

$$\{1, 0, 1, 1\} \quad \{0.5, 1\}$$

with the constant 0.5 the second component of the first section may be changed between 0 and 0.5, e.g.,

$$\{1, 0.27, 1, 1\} \quad \{0.5,\ 1\}$$

To overcome this non-uniqueness of representation let us introduce a so called *reduced forms* of disjunction. The disjunction $\mathbf{d}$ is said to be in k-th reduced form iff the constants of all its sections $\mathbf{d}_i$ except for the k-th section $\mathbf{d}_k$ are equal to 0

$$\mathrm{const}(\mathbf{d}_i) = \min(\mathbf{d}_{ij}) = 0$$

and all the rest of components are exactly greater than $\mathrm{const}(\mathbf{d}_k)$.

In other words, disjunction in k-th reduced form may not contain components in the interval $[0, \mathrm{const}(\mathbf{d}_k)]$, i.e., any component is either equal to 0 or is greater than $\mathrm{const}(\mathbf{d}_k)$, and, in addition, each proposition must include at least one component equal to 0 except for the k-th section. For example, the disjunction

$$\{1, 0.2, 0.9, 1\} \quad \{1,\ 0\}$$

is in its 1st reduced form, whereas the disjunctions

$$\begin{array}{ll} \{1, 0.2, 0.9, 1\} & \{1, 0.2\} \\ \{1, 0.2, 0.9, 1\} & \{1, 0.1\} \\ \{1, 0.2, 0.9, 1\} & \{0.2, 0.2\} \end{array}$$

are not reduced.

This definition does not say how to reduce disjunctions. Now we will propose a procedure for reducing disjunctions which is based on operations of subtraction/addition of the value p from/to the section $\mathbf{d}_i$. These operations result in a new local distribution $\mathbf{d}_i - p/\mathbf{d}_i + p$ such that

$$\mathbf{d}_i - p = \begin{cases} \mathbf{d}_i, & \text{if } \mathbf{d}_i > p \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathbf{d}_i + p = \begin{cases} \mathbf{d}_i, & \text{if } \mathbf{d}_i > p \\ p, & \text{otherwise} \end{cases}$$

Thus to compute $\mathbf{d}_i - p$ ($\mathbf{d}_i + p$) we have to change onto 0 (p) all the components which are less than or equal to p (Fig. 5).

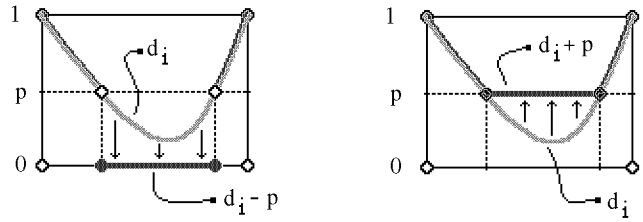

Fig. 5. Operation of subtraction/addition of the value from/to the section.

The whole procedure for reducing disjunctions is as follows:

- find the disjunction constant:

$$\mathrm{const}(\mathbf{d}) = \max(\min(\mathbf{d}_{1j}), \ldots, \min(\mathbf{d}_{nj}))$$

  (The disjunction constant is equal to the maximum of all local constants.)

- subtract the disjunction constant from all non-k-th propositions
- add the disjunction constant to the k-th proposition

For example, the constant of the disjunction

$$\{0, 0.2, 0.3, 1\} \quad \{0.3, 1\}$$

is equal to 0.3, therefore its 1st reduced form is the following:

$$\{0.3, 0.3, 0.3, 1\} \quad \{0, 1\}$$

According to this approach if a disjunction is in a reduced form then it involves a section which is responsible for storing the disjunction constant value. We can transfer the constant from one section to another but such a section will always exist and thus we have n different reduced forms.

Another approach [14, 15, 16] consists in introducing one special component which is responsible for storing the disjunction constant value and is also said to be the disjunction constant (or constant proposition in logical terms). For simplicity such a representation is not used in this paper but it is really useful in many situations (e.g., when representing disjunctions in knowledge base) since the reduced form is defined uniquely and the disjunction constant value is represented explicitly.

5 Resolution Principle

5.1 General Definition

Logical inference usually consists in building new disjunctions from the source disjunctions. Then they are added to the CNF and can serve as premises to continue the inference process. Here we will not touch the question what logical inference is needed for and what requirements it have to meet. We will only consider how disjunctions can be generated.

The main operation for generating disjunctions can be fuzzy resolution. It is applied to two disjunctions on some section (variable) and results in a third disjunction called a resolvent. If $\mathbf{u}$ and $\mathbf{v}$ are two disjunctions and $\mathbf{w}$ is their resolvent on k-th variable, then we write:

$$\mathbf{u}\langle x_k \rangle \mathbf{v} = \mathbf{w}$$

where $\langle x_k \rangle$ denotes the resolution on k-th variable.

Now let us consider how given two premises the resolvent is built. Each section of the resolvent depends on (is constructed from) only two corresponding sections of the premises. k-th proposition of the resolvent (which the resolution is applied to) is equal to the conjunction of the two corresponding propositions from the source disjunctions; every non-k-th proposition of the resolvent is equal to the disjunction of the two corresponding propositions:

$$\mathbf{w}_i = \begin{cases} \mathbf{u}_i \wedge \mathbf{v}_i, \ \mathbf{w}_{ij} = \min\left(\mathbf{u}_{ij}, \mathbf{v}_{ij}\right), & \text{if } i = k, \ j = 1, 2, \ldots, n_k \\ \mathbf{u}_i \vee \mathbf{v}_i, \ \mathbf{w}_{ij} = \max\left(\mathbf{u}_{ij}, \mathbf{v}_{ij}\right), & \text{otherwise} \end{cases}$$

Conjunction and disjunction of elementary propositions about the same variable are equal to the componentwise minimum and maximum, respectively.

The resolution operation can be represented in the form of the following pattern (Fig. 6):

	x_1 max		x_k min		x_n max
$\mathbf{u}$ $\mathbf{v}$	$\mathbf{u}_1$ $\mathbf{v}_1$	$\ldots$ $\ldots$	$\mathbf{u}_k$ $\mathbf{v}_k$	$\ldots$ $\ldots$	$\mathbf{u}_n$ $\mathbf{v}_n$
$\mathbf{w}$	$\mathbf{u}_1 \vee \mathbf{v}_1$	$\ldots$	$\mathbf{u}_k \wedge \mathbf{v}_k$	$\ldots$	$\mathbf{u}_n \vee \mathbf{v}_n$

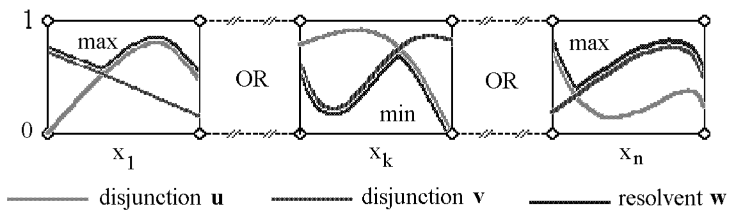

Fig. 6. Operation of fuzzy resolution.

Here are two examples of applying the resolution:

$\mathbf{u}$	$\{0, 0.1, 0.2, 1\}$	$\{0, 1\}$
$\mathbf{v}$	$\{1, 0.3, 0.5, 0\}$	$\{0, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_1 \rangle \mathbf{v}$	$\{0, 0.1, 0.2, 0\}$	$\{0, 1\}$

$\mathbf{u}$	$\{0, 1\}$	$\{0, 1, 0.7\}$	$\{1, 0.2, 1\}$
$\mathbf{v}$	$\{1, 0\}$	$\{1, 1, 0.2\}$	$\{0, 0.1, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_2 \rangle \mathbf{v}$	$\{1, 1\}$	$\{0, 1, 0.2\}$	$\{1, 0.2, 1\}$

The main property of the resolvent is that it is a consequence of its two premises:

$$\text{If } \mathbf{w} = \mathbf{u}\langle x_k \rangle \mathbf{v}, \text{ then } \mathbf{u} \wedge \mathbf{v} \models \mathbf{w}.$$

It means that we can add any resolvent to the CNF containing its two premises and the whole semantics will not change.

If there is only one variable, then we obtain extensional case and the resolution is reduced to ordinary conjunction. Thus the resolution operation in some sense can be viewed as a generalization of conjunction onto multidimensional case.

In the case of two-valued variables and two-valued distributions the behavior of this resolution coincides with that for the classical case of boolean propositions except of the fact that our resolution can be applied to any two disjunctions even if their k-th propositions are not contrary and/or there exist non-k-th contrary propositions. It is obvious that in the first case the resolvent will be a consequence of one of its premises, while in the second case it will be valid, i.e., involve the constant proposition (truth constant 1).

Let us consider the following example.

$\mathbf{u}$	$\{0,1\} = x_1$	$\{0,1\} = x_2$	$\{0,0\} = 0$
$\mathbf{v}$	$\{0,0\} = 0$	$\{1,0\} = \neg x_2$	$\{0,1\} = x_3$
$\mathbf{w} = \mathbf{u}\langle x_2\rangle\mathbf{v}$	$\{0,1\} = x_1$	$\{0,0\} = 0$	$\{0,1\} = x_3$

In this classical example of the boolean resolution we obtain "good" resolvent since sections $\mathbf{u}_2$ and $\mathbf{v}_2$ are contrary, while other sections are not. Each section is written in sectioned form and conventional form with the help of boolean propositions.

In the following example we obtain the resolvent which is a consequence of (weaker than) both premises since the sections $\mathbf{u}_2$ and $\mathbf{v}_2$ are not contrary. Note that classical resolution is not applied to such disjunctions.

$\mathbf{u}$	$\{0,1\} = x_1$	$\{0,1\} = x_2$	$\{0,0\} = 0$
$\mathbf{v}$	$\{0,0\} = 0$	$\{0,1\} = x_2$	$\{0,1\} = x_3$
$\mathbf{w} = \mathbf{u}\langle x_2\rangle\mathbf{v}$	$\{0,1\} = x_1$	$\{0,1\} = x_2$	$\{0,1\} = x_3$

In this example we also obtain "bad" resolvent since two premises involve contrary non-k-th propositions which result in the constant proposition (truth value 1) in the resolvent.

$\mathbf{u}$	$\{0,1\} = x_1$	$\{0,1\} = x_2$	$\{0,0\} = 0$
$\mathbf{v}$	$\{1,0\} = \neg x_1$	$\{1,0\} = \neg x_2$	$\{0,1\} = x_3$
$\mathbf{w} = \mathbf{u}\langle x_2\rangle\mathbf{v}$	$\{1,1\} = 1$	$\{0,0\} = 0$	$\{0,1\} = x_3$

5.2 Resolution on Reduced Disjunctions

If the resolution is considered as a generalization of conjunction (multidimensional conjunction), then it is natural to suppose that its goal is to infer the disjunction which is equivalent to the conjunction of two source disjunctions. In that case the resolvent would represent just both source premises in one clause, and consequently the source disjunctions could be removed as superfluous. However, such ideal variant is impossible because the conjunction of two disjunctions in general case cannot be represented in the form of only one disjunction. Nevertheless, it is possible to formulate the criterion of "quality" of one or another resolution operation: the closer the semantics of two disjunctions is approximated by their resolvent, the better is the resolution rule.

It is a characteristic property of the general definition of the resolution formulated in the previous section that the resolvent content (semantics, i.e., the corresponding fuzzy distribution) depends on the premises form. In the following two examples the premises have different forms but the semantics is the same, while the resolvents have different semantics:

$\mathbf{u}$	$\{1, 0.3\}$	$\{0, 0\}$
$\mathbf{v}$	$\{0, 1\}$	$\{1, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_1 \rangle \mathbf{v}$	$\{0, 0.3\}$	$\{1, 0\}$

$\mathbf{u}$	$\{1, 0\}$	$\{0.3, 0.3\}$
$\mathbf{v}$	$\{0, 1\}$	$\{1, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_1 \rangle \mathbf{v}$	$\{0, 0\}$	$\{1, 0.3\}$

Thus a question arises: which form of premises is the best from the point of view of the above formulated criterion, or which form of premises generates the strongest resolvent.

The resolvent components are decreased only in k-th section when conjuncting two source sections (in the rest of sections the components can be only increased), i.e., it is exactly k-th section that is responsible for non-trivial semantical properties of the resolvent. The higher are component values in disjunctive (non-k-th) sections, the weaker is the resolvent. In the ideal case when all non-k-th sections consist only of zero components the resolvent is exactly equal to the conjunction of two premises, for example:

$\mathbf{u}$	$\{0, 0, 0\}$	$\{1, 0.5, 0.3\}$	$\{0, 0\}$
$\mathbf{v}$	$\{0, 0, 0\}$	$\{0, 0.5, 1\}$	$\{0, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_2\rangle\mathbf{v}$	$\{0, 0, 0\}$	$\{0, 0.5, 0.3\}$	$\{0, 0\}$

Here the constant of the first disjunction is equal to 0.3, and the constant of the second disjunction is equal to 0. If in this example the constant of the first disjunction is transferred from the second sections ($\mathbf{u}_2 = \mathbf{u}_2 - 0.3$) into any other disjunctive section (e.g., the third one, i.e., $\mathbf{u}_3 = \mathbf{u}_3 + 0.3$), then the resolvent becomes worse, i.e., weaker than that in the ideal case:

$\mathbf{u}$	$\{0, 0, 0\}$	$\{1, 0.5, 0\}$	$\{0.3, 0.3\}$
$\mathbf{v}$	$\{0, 0, 0\}$	$\{0, 0.5, 1\}$	$\{0, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_2\rangle\mathbf{v}$	$\{0, 0, 0\}$	$\{0, 0.5, 0\}$	$\{0.3, 0.3\}$

It becomes more clear if transform the obtained resolvent into the following equivalent form (2nd reduced form):

$\mathbf{w} = \mathbf{u}\langle x_2\rangle\mathbf{v}$	$\{0, 0, 0\}$	$\{0.3, 0.5, 0.3\}$	$\{0, 0\}$

Thus our conclusion is that in order to infer the strongest resolvent on k-th section, the premises have to be transformed into the k-th reduced form. However all properties of the resolution described below will be formulated for the general definition independent of the premises form.

5.3 Adjacency of Disjunctions

Although the goal of the application of resolution is to obtain a new non-trivial disjunction differing from both the first premise and the second premise, this requirement cannot always be satisfied (it is not satisfied for the majority of disjunction pairs and variables). More exactly, two disjunctions $\mathbf{u}$ and $\mathbf{v}$ are said to be *adjacent* on the variable

x_k if their resolvent on k-th variable is not a logical consequence of the disjunction $\mathbf{u}$ and disjunction $\mathbf{v}$, i.e., it cannot be absorbed by its premises.

Now it is clear that it makes sense to apply the resolution to adjacent disjunctions only, otherwise the resolvent is a consequence of one of two premises and it does not contain new information. For example, the following two disjunctions are adjacent on the first variable and they are not adjacent on the second variable:

$\mathbf{u}$	$\{0, 0.5, 0.2\}$	$\{1, 0\}$
$\mathbf{v}$	$\{1, 0.2, 1\}$	$\{0, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_1 \rangle \mathbf{v}$	$\{0, 0.2, 0.2\}$	$\{1, 0\}$

$\mathbf{u}$	$\{1, 0.5, 0\}$	$\{1, 0.2\}$
$\mathbf{v}$	$\{0, 0.2, 1\}$	$\{0, 0\}$
$\mathbf{w} = \mathbf{u}\langle x_2 \rangle \mathbf{v}$	$\{1, 0.5, 1\}$	$\{0, 0\}$

In this context the problem can be formulated as follows:

> *how to find out whether two disjunctions are adjacent or not, using only their form, and not constructing the resolvent*

This problem can be solved with the help of the following criterion.

Disjunctions $\mathbf{u}$ and $\mathbf{v}$ are adjacent on the variable x_k iff $\forall i = 1, \ldots, n$ except for $i = k$

$$\text{const}\,(\mathbf{u}_i \vee \mathbf{v}_i) < \text{incomp}\,(\mathbf{u}_k, \mathbf{v}_k)$$

In other words, the minimal value of the disjunction of two sections $\mathbf{u}_i$ and $\mathbf{v}_i$ has to be strictly less than the degree of incomparability of the sections $\mathbf{u}_k$ and $\mathbf{v}_k$. Informally, two disjunctions are adjacent iff the mutual degree of incomparability of two sections $\mathbf{u}_k$ and $\mathbf{v}_k$ (which the resolution is applied to) is high enough to compensate the validity resulted from the disjunction of non-k-th sections.

Thus the adjacency of two disjunctions is influenced by the following two factors:

1. too low degree of incomparability of the propositions about the k-th variable;

2. too high constant (degree of validity) in non-k-th sections.

The first factor is a generalization of the conventional condition (see, e.g., [22]) that the resolution is applied only to disjunctions involving contrary literals. It is natural that in fuzzy case the contrariety of two literals is also fuzzy (the degree of incomparability). Note that the contrariety and incomparability coincide only in two-valued non-fuzzy case (classical propositional calculus); in any other case the incomparability condition is weaker.

The second factor influencing the adjacency of two disjunctions is a generalization of the conventional condition which consists in the absence of the second pair of contrary literals in disjunctions.

When computing the value constant$(\mathbf{u}_i \vee \mathbf{v}_i)$ we have to construct the i-th section of the resolvent $\mathbf{w}_i = \mathbf{u}_i \vee \mathbf{v}_i$, i.e., in fact, to find out if two disjunctions are adjacent or not with the help of this criterion it is necessary to construct $n-1$ sections of the resolvent. Thus it could be easier to construct the resolvent and then to check if it is a consequence of one of its premises. However it is not so, since for the majority of disjunction pairs the condition incomp$(\mathbf{u}_k, \mathbf{v}_k) = 0$ holds, and therefore the criterion constant$(\mathbf{u}_i \vee \mathbf{v}_i) <$ incomp $(\mathbf{u}_k, \mathbf{v}_k)$ cannot be satisfied in any case. In addition, even if incomp $(\mathbf{u}_k, \mathbf{v}_k) > 0$ it makes sense to check the criterion for each new section of the resolvent rather than to check the adjacency after building all sections. Thus the whole procedure for generating resolvents is as follows:

1. build the k-the section of the resolvent: $\mathbf{w}_k = \mathbf{u}_k \wedge \mathbf{v}_k$;

2. find the value incomp $(\mathbf{u}_k, \mathbf{v}_k)$;

3. if incomp $(\mathbf{u}_k, \mathbf{v}_k) = 0$ then goto 9;

4. for $i = 1, 2, \ldots, n$ (except for $i = k$);

5. build the i-th section of the resolvent: $\mathbf{w}_i = \mathbf{u}_i \vee \mathbf{v}_i$;

6. if constant($\mathbf{w}_i$) $\geq$ incomp ($\mathbf{u}_k, \mathbf{v}_k$) then goto 9;
7. next i (goto 5);
8. the disjunctions $\mathbf{u}$ and $\mathbf{v}$ are adjacent and the resolvent $\mathbf{w}$ is built;
9. the disjunctions $\mathbf{u}$ and $\mathbf{v}$ are not adjacent.

6 Equivalent Transformations of Fuzzy Sectioned Matrices

6.1 Finding Prime Disjunctions

The problem of finding prime disjunctions has the same significance as that in the boolean case since once we have found prime disjunctions we can solve many other problems. In this section we consider a method for generating prime disjunctions which is based on the operation of fuzzy resolution.

Prime disjunctions are always defined in relation to some fuzzy distribution which is supposed to be represented by a fuzzy CNF. In other words, a disjunction may be prime in relation to one fuzzy distribution and it may be not prime in relation to another fuzzy distribution. Prime disjunction is a disjunction which is a consequence of the corresponding fuzzy distribution but is not a consequence of any other disjunction (among those which are a consequence of this distribution). Thus prime disjunction is in a certain sense the strongest disjunction among those which can represent the corresponding fuzzy distribution, i.e., those which can be added to the CNF not changing the distribution. If a prime disjunction is in a reduced form then it can be shown that if any its component is decreased then new disjunction is already not a consequence of the corresponding distribution, i.e., no one component of a prime disjunction in a reduced form can be decreased.

The method of finding prime disjunctions based on the resolution operation consists in applying the resolution to different disjunctions from the CNF on different variables, adding the obtained resolvents to the CNF and absorbing disjunctions which follow from other disjunctions in this CNF. The process is stopped when any new resolvent is absorbed, i.e., no new resolvent can be generated. The main problem of this method is an order of applying the resolution. According to the breadth first approach the resolution is applied to all disjunction pairs in the current CNF and new resolvents are added to the end of this CNF but the resolution is not applied to them. After the generation phase the process of absorption is carried out when all disjunctions which follow from any other are removed from the CNF. After that new generation-absorption pass is started until no new resolvents can be generated. Obviously, before generating each new resolvent we check if two disjunctions are adjacent and if yes then after the resolvent is built we check if it is not a consequence of some disjunction already in the CNF. The whole process is shown in Fig. 7.

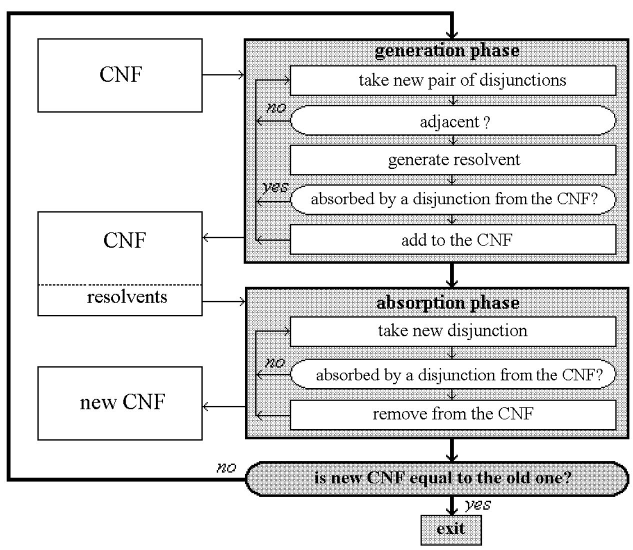

Fig. 7. Breadth first strategy of generating prime disjunctions.

For example, if we have a fuzzy CNF represented by means of the sectioned matrix (**1**, **2** and **3** are disjunctions)

$$\mathbf{D} = \left| \begin{array}{lll} \{0.3, 0\} & \{1, 0.5, 0\} & \{0, 0, 0, 0\} \\ \{0, 0\} & \{0, 0.4, 1\} & \{0, 0.2, 0.7, 1\} \\ \{0, 1\} & \{0, 0, 0\} & \{1, 0.3, 0, 0\} \end{array} \right| \begin{array}{l} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array}$$

then the generation of resolvents at the first pass results in the matrix

$$\mathbf{D}_1 = \left| \begin{array}{lll} \{0.3, 0\} & \{1, 0.5, 0\} & \{0, 0, 0, 0\} \\ \{0, 0\} & \{0, 0.4, 1\} & \{0, 0.2, 0.7, 1\} \\ \{0, 1\} & \{0, 0, 0\} & \{1, 0.3, 0, 0\} \\ \{0.3, 0\} & \{0, 0.4, 0\} & \{0, 0.2, 0.7, 1\} \\ \{0, 0\} & \{1, 0.5, 0\} & \{1, 0.3, 0, 0\} \\ \{0, 1\} & \{0, 0.4, 1\} & \{0, 0.2, 0, 0\} \end{array} \right| \begin{array}{l} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} = \mathbf{1}\langle x_2 \rangle \mathbf{2} \\ \mathbf{5} = \mathbf{1}\langle x_1 \rangle \mathbf{3} \\ \mathbf{6} = \mathbf{2}\langle x_3 \rangle \mathbf{3} \end{array}$$

No one disjunction in this matrix can be absorbed therefore it is taken as an input of the pass 2. During the pass 2 we generate only one disjunction **7**; other generated disjunctions are absorbed by the disjunctions which are already in the matrix. For example, the resolvent $\mathbf{2}\langle x_2 \rangle \mathbf{5}$ is absorbed by the disjunction **4** and is not added to the matrix. The disjunction **7** does not absorbs previous disjunctions **1**–**6** and the pass 2 is finished with the matrix

$$\mathbf{D}_2 = \left| \begin{array}{lll} \{0.3, 0\} & \{1, 0.5, 0\} & \{0, 0, 0, 0\} \\ \{0, 0\} & \{0, 0.4, 1\} & \{0, 0.2, 0.7, 1\} \\ \{0, 1\} & \{0, 0, 0\} & \{1, 0.3, 0, 0\} \\ \{0.3, 0\} & \{0, 0.4, 0\} & \{0, 0.2, 0.7, 1\} \\ \{0, 0\} & \{1, 0.5, 0\} & \{1, 0.3, 0, 0\} \\ \{0, 1\} & \{0, 0.4, 1\} & \{0, 0.2, 0, 0\} \\ \{0.3, 1\} & \{0, 0.4, 0\} & \{0, 0.2, 0, 0\} \end{array} \right| \begin{array}{l} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} = \mathbf{1}\langle x_2 \rangle \mathbf{2} \\ \mathbf{5} = \mathbf{1}\langle x_1 \rangle \mathbf{3} \\ \mathbf{6} = \mathbf{2}\langle x_3 \rangle \mathbf{3} \\ \mathbf{7} = \mathbf{3}\langle x_3 \rangle \mathbf{4} \end{array}$$

During the pass 3 we generate only one disjunction (the disjunction **7** has to be transformed to its 2-nd reduced form)

$$\{0, 1\} \quad \{0, 0.4, 0.3\} \quad \{0, 0.2, 0, 0\} \quad \mathbf{8} = \mathbf{6}\langle x_2 \rangle \mathbf{7}$$

which absorbs its premises, disjunctions **6** and **7**. Thus we obtain the matrix

$$\mathbf{D}_2 = \left| \begin{array}{lll} \{0.3, 0\} & \{1, 0.5, 0\} & \{0, 0, 0, 0\} \\ \{0, 0\} & \{0, 0.4, 1\} & \{0, 0.2, 0.7, 1\} \\ \{0, 1\} & \{0, 0, 0\} & \{1, 0.3, 0, 0\} \\ \{0.3, 0\} & \{0, 0.4, 0\} & \{0, 0.2, 0.7, 1\} \\ \{0, 0\} & \{1, 0.5, 0\} & \{1, 0.3, 0, 0\} \\ \{0, 1\} & \{0, 0.4, 0.3\} & \{0, 0.2, 0, 0\} \end{array} \right| \begin{array}{l} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} = \mathbf{1}\langle x_2 \rangle \mathbf{2} \\ \mathbf{5} = \mathbf{1}\langle x_1 \rangle \mathbf{3} \\ \mathbf{8} = \mathbf{6}\langle x_2 \rangle \mathbf{7} \end{array}$$

Each pair of disjunctions from this matrix generates a resolvent which is absorbed by some other disjunction. So we cannot generate more disjunctions and the process is stopped. Thus we obtain the final matrix $\mathbf{D}_3$ which consists of 6 prime disjunctions.

6.2 Transforming Matrix into the Dual Form

Let us consider only transformation of DNF into CNF. The backward transformation is carried out in the dual way. This problem is formulated as follows. There is a sectioned matrix $\mathbf{C}$ consisting of fuzzy conjunctions and representing a fuzzy DNF. It is necessary to transform it into the matrix of disjunctions $\mathbf{D}$ interpreted as a CNF and characterized by the same fuzzy distribution.

This procedure is based on the operation of adding the conjunction $\mathbf{c}$ to the disjunction $\mathbf{d}$ which results in the CNF $\mathbf{D}$:

$$(\mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \ldots \wedge \mathbf{c}_n) \vee (\mathbf{d}_1 \vee \mathbf{d}_2 \vee \ldots \vee \mathbf{d}_n) = \mathbf{D}$$

It can be proved that this operation results in n disjunctions:

$$\begin{array}{l} ((\mathbf{d}_1 \vee \mathbf{c}_1) \vee \mathbf{d}_2 \vee \ldots \vee \mathbf{d}_n) \wedge \\ (\mathbf{d}_1 \vee (\mathbf{d}_2 \vee \mathbf{c}_1) \vee \ldots \vee \mathbf{d}_n) \wedge \\ \ldots\ldots\ldots \\ (\mathbf{d}_1 \vee \mathbf{d}_2 \vee \ldots \vee (\mathbf{d}_n \vee \mathbf{c}_1)) \end{array}$$

or in the form of sectioned matrix

$$\mathbf{D} = \left| \begin{array}{llll} (\mathbf{d}_1 \vee \mathbf{c}_1) & \mathbf{d}_2 & \ldots & \mathbf{d}_n \\ \mathbf{d}_1 & (\mathbf{d}_2 \vee \mathbf{c}_1) & \ldots & \mathbf{d}_n \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{d}_1 & \mathbf{d}_2 & \ldots & (\mathbf{d}_n \vee \mathbf{c}_1) \end{array} \right|$$

Thus to add the conjunction $\mathbf{c}$ to the disjunction $\mathbf{d}$ we have to copy $\mathbf{d}$ n times and for each i-th copy $\mathbf{d}^i$ fulfill the transformation

$$\mathbf{d}_i^i = \mathbf{d}_i^i \vee \mathbf{c}_i$$

Note that if the disjunction $\mathbf{d}$ consists of only 0 (complete inconsistency) then we obtain the procedure for transforming the DNF consisting from a single conjunction into the CNF.

For example the conjunction

$$\{1,1\} \quad \{0,0.5,1\} \quad \{1,0.7,0.2,0\}$$

being added to the disjunction

$$\{1,0\} \quad \{0,0.8,0\} \quad \{0,0.4,0.9,1\}$$

results in the matrix

$$\mathbf{D} = \left| \begin{array}{lll} \{1,1\} & \{0,0.8,0\} & \{0,0.4,0.9,1\} \\ \{1,0\} & \{0,0.8,1\} & \{0,0.4,0.9,1\} \\ \{1,0\} & \{0,0.8,0\} & \{1,0.7,0.9,1\} \end{array} \right|$$

Obviously the first disjunction can be removed from the matrix since it is absolutely valid (its constant is equal to 1).

In general, before generating i-th disjunction $\mathbf{d}^i$ at the i-th step it is natural to check whether the section $\mathbf{c}_i$ is present (i.e., it is not equal to the constant 1) and whether the disjunction $\mathbf{d}_i \vee \mathbf{c}_i$ is not valid.

The procedure of transforming DNF into CNF is based on the operation of adding a conjunction to a disjunction. At each step of this procedure we add new conjunction from the DNF to all disjunctions from the CNF. At the beginning we add the first conjunction from the DNF to the empty CNF which involves only one disjunction consisting of all 0's. The number of disjunctions in the matrix $\mathbf{D}$ grows very quickly. Therefore it is necessary to carry out periodically the procedure of absorption.

For example, let us suppose that we have to transform the matrix of DNF

$$\mathbf{C} = \left| \begin{array}{lll} \{0.3,1\} & \{1,0.5,0\} & \{0,1,1,0\} \\ \{0,1\} & \{0,0.4,1\} & \{0,0,0,1\} \\ \{1,0\} & \{1,1,0\} & \{1,0.3,0,0\} \end{array} \right| \begin{array}{l} \mathbf{c}^1 \\ \mathbf{c}^2 \\ \mathbf{c}^3 \end{array}$$

consisting of three conjunctions $\mathbf{c}^1$, $\mathbf{c}^2$ and $\mathbf{c}^3$ into the matrix of CNF. The whole procedure consists of 3 steps. At each step we add to the DNF one conjunction from $\mathbf{C}$. Thus at the first step we transform the conjunction $\mathbf{c}^1$ into the matrix of DNF, i.e., we add this conjunction to the initial disjunction consisting of all 0. This results in the matrix

$$\mathbf{D} = \left| \begin{array}{lll} \{0.3,1\} & \{0,0,0\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} & \{0,0,0,0\} \\ \{0,0\} & \{0,0,0\} & \{0,1,1,0\} \end{array} \right| \begin{array}{l} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array}$$

At the second step we have to add the conjunction $\mathbf{c}^2$ to the matrix $\mathbf{D}_1$, i.e., this conjunction have to be added to each disjunction from $\mathbf{D}_1$. First, we make 3 copies of each of our disjunctions

$$\mathbf{D}_1 = \left|\begin{array}{lll} \{0.3,1\} & \{0,0,0\} & \{0,0,0,0\} \\ \{0.3,1\} & \{0,0,0\} & \{0,0,0,0\} \\ \{0.3,1\} & \{0,0,0\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} & \{0,0,0,0\} \\ \{0,0\} & \{0,0,0\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0,0\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0,0\} & \{0,1,1,0\} \end{array}\right| \begin{array}{l} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{3} \\ \mathbf{3} \end{array}$$

Then we impose on them the corresponding section of the conjunction $\mathbf{c}^2$ by the operation $\vee$

$$\mathbf{D}_2 = \left|\begin{array}{lll} \{0.3,1\} \vee \{0,1\} & \{0,0,0\} & \{0,0,0,0\} \\ \{0.3,1\} & \{0,0,0\} \vee \{0,0.4,1\} & \{0,0,0,0\} \\ \{0.3,1\} & \{0,0,0\} & \{0,0,0,0\} \vee \{0,0,0,1\} \\ \{0,0\} \vee \{0,1\} & \{1,0.5,0\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} \vee \{0,0.4,1\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} & \{0,0,0,0\} \vee \{0,0,0,1\} \\ \{0,0\} \vee \{0,1\} & \{0,0,0\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0,0\} \vee \{0,0.4,1\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0,0\} & \{0,1,1,0\} \vee \{0,0,0,1\} \end{array}\right| \begin{array}{l} \mathbf{1.1} \\ \mathbf{1.2} \\ \mathbf{1.3} \\ \mathbf{2.1} \\ \mathbf{2.2} \\ \mathbf{2.3} \\ \mathbf{3.1} \\ \mathbf{3.2} \\ \mathbf{3.3} \end{array}$$

and finally we obtain the following matrix (disjunctions **1.2**, **1.3** are absorbed by the disjunction **1.1**, i.e., the disjunction **1** has not changed)

$$\mathbf{D}_2 = \left|\begin{array}{lll} \{0.3,1\} & \{0,0,0\} & \{0,0,0,0\} \\ \{0,1\} & \{1,0.5,0\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,1\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,0\} & \{0,0,0,1\} \\ \{0,1\} & \{0,0,0\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0.4,1\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0,0\} & \{0,1,1,1\} \end{array}\right| \begin{array}{l} \mathbf{1.1} \\ \mathbf{2.1} \\ \mathbf{2.2} \\ \mathbf{2.3} \\ \mathbf{3.1} \\ \mathbf{3.2} \\ \mathbf{3.3} \end{array}$$

After adding to the matrix $\mathbf{D}_2$ the third conjunction $\mathbf{c}^3$ we obtain the final matrix $\mathbf{D}_3$ which is equivalent to the source matrix of conjunctions $\mathbf{C}$:

$$\mathbf{D}_3 = \left|\begin{array}{lll} \{0.3,1\} & \{1,1,0\} & \{0,0,0,0\} \\ \{0.3,1\} & \{0,0,0\} & \{1,0.3,0,0\} \\ \{0,1\} & \{1,1,0\} & \{0,0,0,0\} \\ \{0,1\} & \{1,0.5,0\} & \{1,0.3,0,0\} \\ \{1,0\} & \{1,0.5,1\} & \{0,0,0,0\} \\ \{0,0\} & \{1,0.5,1\} & \{1,0.3,0,0\} \\ \{1,0\} & \{1,0.5,0\} & \{0,0,0,1\} \\ \{0,0\} & \{1,1,0\} & \{0,0,0,1\} \\ \{0,0\} & \{1,0.5,0\} & \{1,0.3,0,1\} \\ \{0,1\} & \{1,1,0\} & \{0,1,1,0\} \\ \{0,1\} & \{0,0,0\} & \{1,1,1,0\} \\ \{1,0\} & \{0,0.4,1\} & \{0,1,1,0\} \\ \{0,0\} & \{0,0.4,1\} & \{1,1,1,0\} \\ \{1,0\} & \{0,0,0\} & \{0,1,1,1\} \\ \{0,0\} & \{1,1,0\} & \{0,1,1,1\} \end{array}\right| \begin{array}{l} \mathbf{1.1.2} \\ \mathbf{1.1.3} \\ \mathbf{2.1.2} \\ \mathbf{2.1.3} \\ \mathbf{2.2.1} \\ \mathbf{2.2.3} \\ \mathbf{2.3.1} \\ \mathbf{2.3.2} \\ \mathbf{2.3.3} \\ \mathbf{3.1.2} \\ \mathbf{3.1.3} \\ \mathbf{3.2.1} \\ \mathbf{3.2.3} \\ \mathbf{3.3.1} \\ \mathbf{3.3.2} \end{array}$$

This matrix can be further transformed, e.g., disjunction **1.1.2** is absobed by **2.1.2** and should be removed.

7 Finding Projections on Variables

The problem of finding projections on variables is one of the most important in multi-dimensional analysis. With the help of this operation we obtain a local distribution over the values of one variable from a multi-dimensional distribution. Let us suppose that it is required to find a projection of the distribution represented by the matrix of CNF $\mathbf{D}$ onto the variable x_k. Generally, projection can be defined in different ways but we will suppose that it is defined by means of the operation of maximum. Namely, j-th value of the projection on the variable x_k, i.e., its value in the point a_{kj} is equal to

$$\max\left(\mathbf{D}\left(x_1,\ldots,x_k=a_{kj},\ldots,x_n\right)\right)$$

on all values of all variables except for $x_k = a_{kj}$. In other words, we take maximum in all points which have the k-th component (k-th dimension) equal to a_{kj}. Thus to calculate the whole projection in n_k

points of k-th domain we have to take maximum in all points of the universe of discourse. Obviously, the maximum of the projection on any variable is equal to the maximum of the whole multi-dimensional distribution and therefore we can solve the satisfiability problem (finding the consistency) by finding any projection.

We can also redefine the projection by means of the logical consequence relation. Let us suppose that n projections on all variables are represented by the conjunction $\mathbf{c}$, i.e., each section $\mathbf{c}_i$ of this conjunction represents a projection on i-th variable. Then all its components $\mathbf{c}_{ij}$ must be minimal provided that it is still a consequence of $\mathbf{D}$. In other words, to find the conjunction of projections $\mathbf{c}$ we have to take the trivial conjunction consisting of all 1 and gradually decrease its components untill we reach the border where it ceases to be a consequence of $\mathbf{D}$.

One procedure for finding projections is based on a theorem [10] which affirms that disjunction $\mathbf{u}$ is a consequence of the matrix of CNF $\mathbf{D}$ iff there exists such a prime disjunction $\mathbf{p}$ of this matrix that the disjunction $\mathbf{d}$ is its consequence:

$$\mathbf{D} \models \mathbf{u} \Leftrightarrow \text{ there exists prime } \mathbf{p} : \mathbf{p} \models \mathbf{u}$$

Using this criterion, we can check whether a disjunction follows from the matrix, and we can also find minimal disjunctions which satisfy this condition. It can be shown that the projection on the k-th variable of the distribution represented by the matrix of CNF $\mathbf{D}$ is equal to the minimum of projections of all prime disjunctions on this variable. Note that we have to have *all* prime disjunctions to carry out this procedure.

Projection on the k-th variable of one disjunction is equal to the k-th section of this disjunction plus constant M which is equal to the maximal value in all remaining sections (i.e., M is equal to the maximum of all non-k-th section components). For example, projection of the disjunction

$$\{0.3, 0\} \quad \{0, 0.4, 0\} \quad \{0, 0.2, 0.7, 1\}$$

on the 3rd variable is equal to $\{0, 0.2, 0.7, 1\} + 0.4 = \{0.4, 0.4, 0.7, 1\}$

while its projections on the variables x_1 and x_2 is equal to the constant 1 (we say that the projection is absent).

Let us consider an example of the matrix consisting of all prime disjunctions from the previous section

$$\mathbf{D} = \left| \begin{array}{lll} \{0.3, 0\} & \{1, 0.5, 0\} & \{0, 0, 0, 0\} \\ \{0, 0\} & \{0, 0.4, 1\} & \{0, 0.2, 0.7, 1\} \\ \{0, 1\} & \{0, 0, 0\} & \{1, 0.3, 0, 0\} \\ \{0.3, 0\} & \{0, 0.4, 0\} & \{0, 0.2, 0.7, 1\} \\ \{0, 0\} & \{1, 0.5, 0\} & \{1, 0.3, 0, 0\} \\ \{0, 1\} & \{0, 0.4, 0.3\} & \{0, 0.2, 0, 0\} \end{array} \right| \begin{array}{l} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{8} \end{array}$$

The projection of this matrix on the variable x_1 is equal to $\{0.4, 1\}$ (disjunction **8**), on variable x_2 — $\{1, 0.5, 0.3\}$ (disjunction **1**), and on variable x_3 — $\{0.4, 0.4, 0.7, 1\}$ (disjunction **4**).

8 Conclusion

Distribution as logical proposition

The notion of fuzzy relation or fuzzy multi-dimensional distribution which has been studied in the paper is certainly not new and has been paid a lot of attention in fuzzy literature. On the other hand, one traditional direction of fuzzy research has consisted in fuzzifying classical logics. These two approaches have been developed in great extent independently. Fuzzy relations are usually described in algebraic terms (e.g., as fuzzy relational algebra) while fuzzy logics are usually obtained from some classical logic by introducing fuzzy parameters (in fact, there are two big approaches to fuzzifying classical logics: (i) fuzzifying interpretations (e.g., [23, 24]) and fuzzifying formulas themselves, e.g., introducing weights to propositions (e.g., [25])). One general result of this paper is that we have established a connection between these two directions. Now we know that any local (fuzzy) distribution can be viewed as a proposition in logical sense and we can combine them just as ordinary propositions by means of connections $\wedge$ and $\vee$ to build more complex propositions [14], particularly, fuzzy CNF and fuzzy DNF. Of course, it is not enough to declare that the distribution (relation) is a proposition and in the paper we have shown that the whole behavior of our formal system is analogous to and even more general then that of the propositional logic.

Inference as equivalent transformation

Traditionally, logical inference has been thought of as applying inference rules to axioms and theorems which have been already proved and obtaining new theorems (deduction process). As a result we could infer logical statements which express in an explicit form different properties of the formal system hidden in the original representation by means of axioms. Although we have showed in the paper that our approach to logical inference is analogous to this one, we also give another interpretation for it. According to this view inference process is considered as an equivalent transformation of our representation of the semantics by means of axioms to some other representation which is more appropriate in the sense of explicit representation of necessary properties. In this case a consequence relation is only one of many possible equivalent transformations which allows us to remove unnecessary statements. This interpretation of logical inference seems more general especially when considering non-minimax operations for composing distributions.

Values of variables and values of distributions

One general result of the approach described in the paper is that we clearly distinguish two notions, values of variables and values of distributions, which are often mixed in traditional formalisms. The values of individual logical variables can be associated with the syntax or objective part of the problem domain. They define the matter of propositions, i.e., what the proposition is about, e.g., it can be a state space. On the other hand, values of distributions are associated with the semantics or subjective part of the problem domain. They define the proposition itself, e.g., what we think or know about possible states. However in the case of superpositions the same set of values can represent both syntactic and semantic values. For example, when we negate some proposition we in essence apply the proposition (negation) to the set of values which are semantical for the negated proposition but syntactic for the negation.

Inference as finding projections

Suppose we know that inference process is an equivalent transformation of our representation to some form, i.e., to infer something we have to change the form of representation in such a way that the semantics remain the same. Then a question arises: What form of representation we have to seek for, and why it is better than other forms, i.e., what is the goal of inference process? An answer is the following. Our general goal is to reveal interesting in some sense (global) properties of a multi-dimensional distribution, i.e., to find the form of representation where these properties would be explicit. Usually explicit form of representation assumes that the property is expressed in one statement. The property which is looked for in most cases is the projection of the whole distribution on some variable(s) or the proposition about one variable. Although there may be also other properties (e.g., correlations between individual variables) this one is supposed to be the most important and is considered to be the goal of general inference process.

Fuzzy resolution operation and its properties

Perhaps the most important result described in the paper is a new fuzzy resolution operation. It generalizes traditional consensus operation and resolution in logic in two directions: (i) values of variables are supposed to be many-valued [4] and even continuos, and (ii) distributions are supposed to take values from the interval [0,1] [9]. The criterion of adjacency formulated in the paper enforces the analogy with crisp case since it allows us to determine when the resolvent is not trivial. To define correctly the resolution operation and the criterion of adjacency we had to introduce such new notions as reduced forms, degree of incomparability, and constant of disjunction.

Negation as proposition

Although it is not described in this paper, it can be easily shown that we do not need an operation of negation [14]. Instead of it we can use more general operation of superposition (proposition about proposition), a particular case of which represents negation.

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