Image Compression With Haar Wavelets

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March 3, 2021

1   Haar Wavelet

Wavelets are described by a mother function $\psi(t)$ accompanied by a father (or scaling) function $\phi(t)$. For the Haar wavelet, these functions are defined as

$$\psi(t) = \begin{cases} 1, & 0 \leq t < 1/2, \\ -1, & 1/2 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \phi(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

For $\psi(t)$ to be a wavelet, we require that it integrates to zero on its domain.

1.1   Child Wavelets

The Haar wavelet's support is the unit interval $[0, 1]$. We can contract the mother function to get a wavelet half the size on the interval $[0, 1/2]$, namely $\psi(2t)$. Also, translating to fit the interval $[1/2, 1]$, we get $\psi(2t-1)$. See Figure 1. Since these smaller wavelets are derived from the mother, we call them daughter wavelets distinguishing them by parameters in subscripts, $\psi_{1,0}(t) = \psi(2^1 t - 0) = \psi(2t)$ and $\psi_{1,1}(t) = \psi(2^1 t - 1) = \psi(2t - 1)$. Likewise, son wavelets are derived from the father function.

2   Wavelet Basis

Our primary goal is to approximate a signal $f(t)$ using wavelets. We will see that we are limited to signals with finite energy, that is $f(t)$ must be in the Hilbert space $L^2(\mathbb{R}) = \left\{ f \mid f : \mathbb{R} \to \mathbb{R}, \int_{-\infty}^{\infty} f^2 < \infty \right\}$, the set of all square integrable functions.
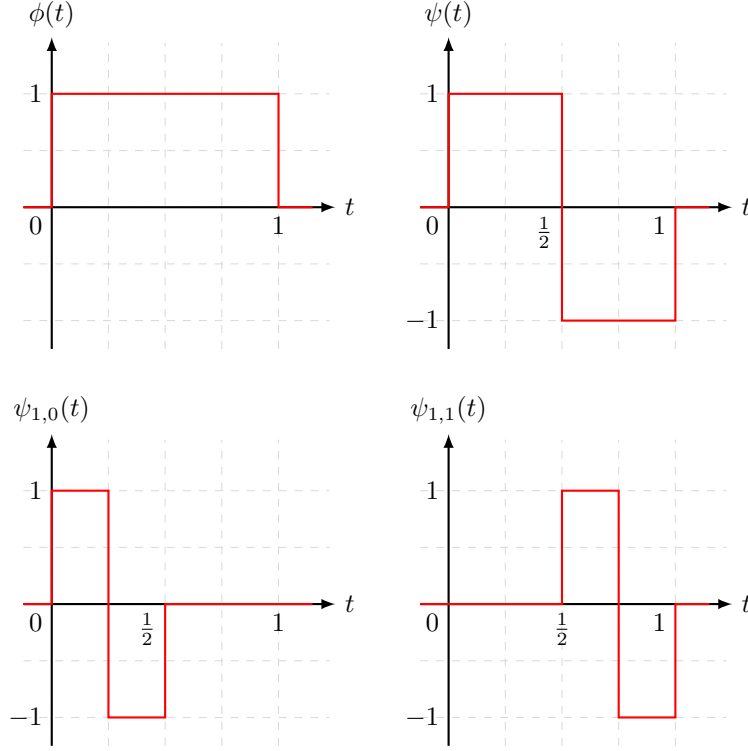
Figure 1: First generation of Haar wavelet functions.

2.1 Multiresolution Analysis

A nice feature of many wavelets, including the Haar, is that we can make multiresolution approximations for a signal, thereby increasing computational efficiency. First, consider $V_j = \text{span}\left\{\phi(2^j t - k) \mid k \in \mathbb{Z}\right\}$ describing the subspace of $L^2(\mathbb{R})$ generated by 2^j resolution father functions. A signal $f \in L^2(\mathbb{R})$ can be approximated by the projection of f onto V_j, that is

$$f_j(t) = \sum_{k \in \mathbb{Z}} h_k \phi(2^j t - k) \in V_j, \quad \text{where} \quad h_k = \langle f, \sqrt{2}\phi(2^j t - k)\rangle.$$

Notice that $\phi(t)$ lives in the reference space V_0. Then $\phi(t)$ can be represented as a linear combination of functions in V_1 and we have $V_0 \subset V_1$. In general, we have a sequence of nested subspaces such that

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R}),$$

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forming a *multiresolution analysis* (MRA). In the case of the Haar wavelet, the MRA allows us to construct the mother function from our choice of the father function $\phi(t) = 1$ on the unit interval and zero otherwise. Then W_j is generated by $\{\psi(2^j t - k) \mid k \in \mathbb{Z}\}$ such that $V_{j+1} = V_j \oplus W_j$. In other words, W_j contains the information that is lost when a signal is approximated at a lower resolution. For $f_j \in V_j$ and $w_j \in W_j$, the n-level decomposition of a signal $f_n \in V_n$ is given by

$$f_n = f_{n-1} + w_{n-1} = f_{n-2} + w_{n-2} + w_{n-1} = \cdots = f_0 + w_0 + w_1 + \cdots + w_{n-1}$$

and we call f_j the *average detail* and w_j the *difference detail*.

2.2 Orthogonality of the Haar Wavelet

In the previous section, we constructed a spanning set using $\phi(t)$ to approximate a signal $f(t) \in L^2(\mathbb{R})$. Using the relationship between the Haar wavelet functions, we see that $\psi(t) = \phi(2t) - \phi(2t - 1)$. Then it follows that we can construct a spanning set using $\psi(t)$ such that

$$f(t) = \sum_{k \in \mathbb{Z}} g_k \psi(2^j t - k),$$

where g_k are some scaling coefficients. Notice that as j increases, the spaces spanned by these sets approach $L^2(\mathbb{R})$.

By orthogonality, we can show that these spanning sets are linearly independent and, therefore, form an orthogonal basis. First, we see that the functions in V_j are disjoint and the functions in W_j are disjoint, so the pairwise inner products in each space are always zero. Also, if $\phi_{j,k} \in V_j$ is translated by k and $\psi_{j,k'} \in W_j$ is translated by k' with $k \neq k'$, then $\phi_{j,k}$ and $\psi_{j,k'}$ are disjoint with an inner product of zero. So, suppose $\phi_{j,k}$ and $\psi_{j,k}$ have the same support, for example, consider $\phi = \phi_{0,0}$ and $\psi = \psi_{0,0}$ on the unit interval. Then

$$\langle \phi, \psi \rangle = \int_{-\infty}^{\infty} \overline{\phi(t)} \psi(t) \, dt = \int_{-\infty}^{\infty} \phi(t)\psi(t) \, dt = \int_{0}^{1/2} 1 \, dt - \int_{1/2}^{1} 1 \, dt = 0.$$

Other cases are similar, so we can conclude $\langle \phi_{j,k}, \psi_{j,k} \rangle = 0$ for all $j, k \in \mathbb{Z}$. Hence, the Haar wavelet functions ϕ and ψ form an orthogonal basis under $t \mapsto (2^j t - k)$ mappings.

2.3 Vector Representation

For computations, it is convenient to represent wavelets as vectors. For example, we can define the first generation of wavelet vectors as

$$\phi \equiv (1,1,1,1), \qquad\qquad \psi \equiv (1,1,-1,-1),$$
$$\psi_{1,0} \equiv (1,-1,0,0), \qquad\qquad \psi_{1,1} \equiv (0,0,1,-1),$$

which span $\mathbb{R}^4$. Further, we can demonstrate that these vectors are mutually orthogonal by computing each of the inner products

$$\langle \psi, \phi \rangle = (1,1,-1,-1)(1,1,1,1)^T = 1+1-1-1 = 0,$$
$$\langle \psi_{1,0}, \phi \rangle = (1,-1,0,0)(1,1,1,1)^T = 1-1+0+0 = 0,$$
$$\langle \psi_{1,1}, \phi \rangle = (0,0,1,-1)(1,1,1,1)^T = 0+0+1-1 = 0,$$
$$\langle \psi_{1,0}, \psi \rangle = (1,-1,0,0)(1,1,-1,-1)^T = 1-1+0+0 = 0,$$
$$\langle \psi_{1,1}, \psi \rangle = (0,0,1,-1)(1,1,-1,-1)^T = 0+0-1+1 = 0,$$
$$\langle \psi_{1,0}, \psi_{1,1} \rangle = (1,-1,0,0)(0,0,1,-1)^T = 0+0+0+0 = 0.$$

As in the continuous case, we see that the wavelet vectors form an orthogonal basis. It will be especially useful to normalize these vectors so that the total energy of a signal is not changed. Thus, when we divide by the norms $\|\phi\| = 2$, $\|\psi\| = 2$, $\|\psi_{1,0}\| = \sqrt{2}$, and $\|\psi_{1,1}\| = \sqrt{2}$, we obtain an orthonormal wavelet basis.

3 Decomposition and Reconstruction

In general, a wavelet decomposition is obtained using wavelet transforms. A series of inner products are computed using the signal and wavelet for each level of the decomposition. However, thanks to MRA, these computations can be largely reduced for the Haar wavelet using a much more efficient algorithm. Moreover, we saw that the difference detail contains the lost information. Thus, the original signal can be perfectly reconstructed.

Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ be a signal in $\mathbb{R}^n$ for n even[1]. The average of the

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[1]If this is not the case, then a zero entry is appended to the signal.

first two entries in $\mathbf{f}$ is computed by

$$a_1^1 = \frac{1}{\sqrt{2}}(1,1,0,0,\ldots,0)\mathbf{f}^T = \frac{f_1 + f_2}{\sqrt{2}}.$$

In general, the average detail of the mth pair of entries in $\mathbf{f}$ is found using the formula

$$a_m = \frac{f_{2m-1} + f_{2m}}{\sqrt{2}}, \tag{2}$$

for $m = 1, 2, \ldots, n/2$. Similarly, the difference detail is computed by

$$d_m = \frac{f_{2m-1} - f_{2m}}{\sqrt{2}}. \tag{3}$$

Example 3.1. Let $\mathbf{f} = (-1, 1, 9, 4, 7, 6, -4, 10)$. We find the first average details using Equation (2)

$$a_1^1 = \frac{-1+1}{\sqrt{2}} = \frac{0}{\sqrt{2}}, \qquad\qquad a_2^1 = \frac{9+4}{\sqrt{2}} = \frac{13}{\sqrt{2}},$$
$$a_3^1 = \frac{7+6}{\sqrt{2}} = \frac{13}{\sqrt{2}}, \qquad\qquad a_4^1 = \frac{-4+10}{\sqrt{2}} = \frac{6}{\sqrt{2}},$$

and the first difference details using Equation (3)

$$d_1^1 = \frac{-1-1}{\sqrt{2}} = \frac{-2}{\sqrt{2}}, \qquad\qquad d_2^1 = \frac{9-4}{\sqrt{2}} = \frac{5}{\sqrt{2}},$$
$$d_3^1 = \frac{7-6}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \qquad\qquad d_4^1 = \frac{-4-10}{\sqrt{2}} = \frac{-14}{\sqrt{2}}.$$

Then we have

$$\mathbf{a}^1 = \frac{1}{\sqrt{2}}(0, 13, 13, 6), \qquad\qquad \mathbf{d}^1 = \frac{1}{\sqrt{2}}(-2, 5, 1, -14).$$

Notice that the size of the first average and difference details is half of the original signal.

4 Haar Wavelet Transform

We represent the process of decomposing a signal as the transformation

$$\mathbf{f} \xrightarrow{\mathcal{H}_1} (\mathbf{a}^1 \mid \mathbf{d}^1), \tag{4}$$

where $\mathcal{H}_1$ is the first level Haar wavelet transform. Depending on the length n of a given signal vector $\mathbf{f}$, we use the son wavelets to get

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \cdots & 1 & 1 \end{pmatrix} \tag{5}$$

and the daughter wavelets to get

$$G = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \cdots & 1 & -1 \end{pmatrix}, \tag{6}$$

where H and G each have $n/2$ rows and n columns. Now, we can write the Haar wavelet decomposition as $\mathbf{a}^1 = H\mathbf{f}$ and $\mathbf{d}^1 = G\mathbf{f}$ to get the matrix equation

$$\begin{pmatrix} \mathbf{a}^1 \\ \mathbf{d}^1 \end{pmatrix} = \begin{pmatrix} H\mathbf{f} \\ G\mathbf{f} \end{pmatrix} = \begin{pmatrix} H \\ G \end{pmatrix} \mathbf{f}. \tag{7}$$

Since H and G were constructed from an orthonormal Haar wavelet basis, the matrix $\left(\begin{smallmatrix} H \\ G \end{smallmatrix} \right)$ is orthonormal as well. Then left-hand multiplication by $\left(\begin{smallmatrix} H \\ G \end{smallmatrix} \right)^T$ will allow us to solve for $\mathbf{f}$, and we get

$$\mathbf{f} = \begin{pmatrix} H \\ G \end{pmatrix}^T \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{d}^1 \end{pmatrix} = \left(H^T \mid G^T \right) \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{d}^1 \end{pmatrix} = H^T\mathbf{a}^1 + G^T\mathbf{d}^1. \tag{8}$$

Therefore, we say that the signal is reconstructed by the inverse Haar wavelet transform

$$(\mathbf{a}^1 \mid \mathbf{d}^1) \xrightarrow{\mathcal{H}_1^{-1}} \mathbf{f}. \tag{9}$$

4.1 Multi-Level Decomposition

After obtaining a decomposed signal $(\mathbf{a}^1 \mid \mathbf{d}^1)$, we can continue breaking the average detail into smaller components. Using the level-two Haar wavelet transform, we have

$$\mathbf{f} \xrightarrow{\mathcal{H}_1} (\mathbf{a}^1 \mid \mathbf{d}^1) \xrightarrow{\mathcal{H}_2} (\mathbf{a}^2 \mid \mathbf{d}^2 \mid \mathbf{d}^1).$$

Using appropriately-sized matrices H and G, we decompose the first-level average detail $\mathbf{a}^1$ into $(\mathbf{a}^2 \mid \mathbf{d}^2)$.

4.2 Two-Dimensional Haar Wavelet Transform

The idea of signal decomposition using the Haar wavelet can be extended so that we can transform a matrix $\mathbf{f}$ into first-level details. Then we have

$$\mathbf{f} \xrightarrow{\mathcal{H}_1} \left(\begin{array}{c|c} \mathbf{a}^1 & \mathbf{h}^1 \\ \hline \mathbf{v}^1 & \mathbf{d}^1 \end{array} \right), \tag{10}$$

where $\mathbf{h}^1$ is the horizontal difference detail, $\mathbf{v}^1$ is the vertical difference detail, and $\mathbf{d}^1$ is the diagonal difference detail. If $\mathbf{f}$ is an $m \times n$ signal matrix, for m and n even[2], then the first-level details can be computed as

$$\mathbf{a}^1 = H\mathbf{f}H^T, \qquad\qquad \mathbf{h}^1 = H\mathbf{f}G^T,$$
$$\mathbf{v}^1 = G\mathbf{f}H^T, \qquad\qquad \mathbf{d}^1 = G\mathbf{f}G^T,$$

where H and G on the left are sized $m/2 \times m$ and H^T and G^T on the right are sized $n \times n/2$ so that multiplication is defined.

5 Image Compression

When applying k multi-level Haar transforms, the dimensions of the decomposed matrix are the same as the original signal matrix. Since no information has been lost, the signal can always be perfectly reconstructed. This also means that the file size stored in memory has not been reduced.

5.1 Thresholding

Wavelet transforms like the Haar are well-suited for image compression because they preserve low-frequency information, such as general shapes and boundaries, while the high-frequency noise is mapped into small difference details. If the magnitude of the difference is small enough, say less than some parameter λ, then we can set the value to zero. This is known as thresholding, which handles the actual compression. A larger threshold value λ gives better compression, however, if λ is too large, then much of the image quality will be lost. We

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[2]If this is not the case, then add a row and/or column of zeros to $\mathbf{f}$ to make the dimensions even.

represent thresholding as the transform

$$T(x; \lambda) = \begin{cases} 0, & |x| \leq \lambda, \\ x, & |x| > \lambda, \end{cases} \tag{11}$$

where x is an element of the k-level wavelet decomposition of a signal $\mathbf{f}$. Since T maps a range of values $-\lambda \leq x \leq \lambda$ to zero, the function is not one-to-one. Thus, when applying a threshold transform, we no longer have perfect reconstruction, but sparsity of the image matrix increases. This corresponds to a reduction in file size as desired.

In particular, Equation (11) is known as hard thresholding. Other types of thresholding (e.g. soft) are common and use neighboring pixels to blend the image. This leads to a variety of wavelet applications in denoising images and data.

5.2   Method

In Matlab, we can import a color image as a 3-dimensional array in RGB or YCbCr color space. For wavelet decomposition, YCbCr is preferred since RGB layers are usually correlated.