AN EXPLORATION OF SIGNAL PROCESSING

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1. Introduction

1.1. **Who is this course for?** This mini-course was originally designed for the *2014 Math Mentorship Program*, in which ambitious high school students are selected to complete a three or four month research project with a graduate student or faculty member at the University of Toronto.

1.2. **Intention.** With the aim of developing familiarity with elements of signal analysis and data compression, this mini-course is designed to introduce students to the basic tenets of Fourier series, as well as the intuition behind the Fourier transform. The student will learn the requisite elementary background in linear algebra, calculus and coding through a combination of independent exercise and mentor-led discussion. With the foundations set, the student will achieve the goal of the course through the development of their very own implementation of spectrogram software. The student will develop the code and tools necessary to write software that allows the student to analyze the frequency data of a song file, by visualizing the intensity of all musical notes at each point in the song.

2. Motivation

2.1. **What's the point?** Whether the students of this mini-course end up in a faculty of engineering, mathematics, computer science, statistics or physics, it is likely that there is some aspect of this mini-course that will serve to expand their perspective on their studies, or aid in their understanding of future material. The mathematics of this course comprise foundational tools which permeate a significant percentage of undergraduate mathematics. Moreover, Fourier series is a fundamental tool of reasoning in computer science, physics and mathematics. Lastly, coding facility is seen as important tool in the "world of today", where success in most academic and industrial research hinges on familiarity with at least one coding language. Thus, the student will hopefully learn elements of signal analysis that will be unique to what will be encountered in future academics, but will also complement the student's understanding of concepts they will undoubtedly encounter in the future.

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2.2. **But why Fourier series?** Historically, Fourier series represents a contentious moment of Mathematics — one in which the "right" answer was not obvious (go figure!). While Fourier, himself, was adamant that a "large number" of functions could be represented as a sum waves, the tools available at the time didn't allow him to establish his theory in full rigour. However, with the development of the necessary tools came a very beautiful area of mathematics, which ties together many other areas of mathematics that the student may have previously considered to be distinct (*e.g.,* calculus and linear algebra). Moreover, the content of this mini-course is primed to elucidate the multitude of avenues that emerge from Fourier methods, leading into new, ever more beautiful enclaves of mathematics. Indeed, by supplementing a primarily mathematical foundation of signal analysis with elements of coding, the student is primed for introduction to the rich area of numerical analysis, harmonic analysis and complex analysis. Fourier series is then an optimal tool for introduction to partial differential equations, or as the foundation for an extension to wavelet methods, integral transforms and Hilbert spaces (cf. Stein & Shakarchi).

2.3. **A rejection of existing resources.** Where many existing introductory courses on Fourier series are targeted merely at having students compute Fourier coefficients and Fourier transforms (perhaps with the purpose of solving simple PDEs), we instead, with greater ambition, rely on an intuitionistic development of integral calculus, linear algebra (basis transforms) and properties of convolutions to achieve the aims outlined above. By supplementing proof-based learning with coding exercises in the student's favourite coding language we allow abstract results to be verified in a tangible way — in particular, one that allows a second vessel for learning, and visualization of results.

3. Syllabus

Week 1: Introduction, promises of fame, fortune and endless possibilities

Week 2: Review of integration and differentiation with heavy emphasis on what it means to integrate a function, what it means to integrate a product of two functions, *etc.*Definition of a norm and relation of integral norm to linear algebra.

Week 3: Review of matrix multiplication. Derivation of $O(2)$, the orthogonal group of 2×2 matrices. Introduction to Matlab — vector and matrix multiplication, plotting, the `fft` function.

Week 4: Derivation of the heat equation and Fourier series solutions to the heat equation. Approximating solutions to the heat equation and approximation of piecewise continuous functions in Matlab.

Week 5: Further development of integration concepts. Introduction to numerical integration.

Week 6: TBA