A Note on the Maximum Size of a Closed Epidemic

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A Note on the Maximum Size of a Closed Epidemic

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Summary

Daniels successfully approximated the distribution of the maximum number of infectives during an epidemic in a closed population, by reducing the problem to a curved boundary problem for Brownian motion. The purpose of this note is to present a simpler way of tackling the latter problem, using the invariance properties of Brownian motion.

Keywords: CLOSED EPIDEMIC; CURVED BOUNDARY; BROWNIAN MOTION

In Daniels (1974), the problem of approximating the maximum size of a closed epidemic is effectively reduced to that of estimating, for large N, the distribution of

$$Y_N \equiv \max_{0 \leqslant t \leqslant 2} \{W(t) - N^{1/2} c(t)\}, \tag{1}$$

where $W(t)$ is standard Brownian motion, and where $c(\cdot)$ is a continuous function on $[0, 2]$ satisfying

$$\begin{aligned} &c(1) = 0; \quad c(t) > 0 \quad \text{for } t \neq 1; \\ &c(t) = \tfrac{1}{2}(t-1)^2 + O(|t-1|^3) \quad \text{in some neighbourhood of } t = 1. \end{aligned} \tag{2}$$

The limit distribution of Y_N is easily seen to be standard Normal, but Y_N is stochastically larger than its limit, and Daniels found that the next asymptotic term was needed for a useful approximation. This he evaluated by a "scanning" technique, based on a renewal equation: the purpose of the present note is to show how his result can be obtained by a simpler argument.

The following conventions are adopted. Standard Brownian motions on $[0, \infty)$ are denoted by $W^{(j)}(\cdot)$, $j \geqslant 1$, and the symbol $\doteq$ denotes equality in distribution between random variables. If $\{X_n\}$ is a sequence of random variables, "$X_n = O(1)$ as $n \to \infty$" means that there exists a probability distribution function $F(x)$ on $[0, \infty)$ such that $P[|X_n| \geqslant x] \leqslant 1 - F(x)$ for all n and for all $x \geqslant 0$. The order estimates given below are not explicitly justified, since a detailed treatment, while involving only routine calculations, would obscure the main argument. The rapid decay of the normal tail probabilities ensures that the "obvious" estimates are indeed correct.

The basic approach is to write

$$\begin{aligned} Y_N &= W(1) + \max\left[\max_{0 \leqslant t \leqslant 1}\{W(t) - W(1) - N^{1/2} c(t)\}, \max_{1 \leqslant t \leqslant 2}\{W(t) - W(1) - N^{1/2} c(t)\}\right] \\ &= W^* + \max\{C_1(N), C_2(N)\}, \end{aligned} \tag{3}$$

and to evaluate the distributions of $C_1(N)$ and $C_2(N)$ in turn. $C_2(N)$ is independent of both W^* and $C_1(N)$ because, on $t \geqslant 1$, $W(t) - W(1) \doteq W^{(1)}(t-1)$, independent of the past; and we have

$$\begin{aligned}C_2(N) &\doteq \max_{0\leqslant u\leqslant 1}\{W^{(1)}(u)-N^{1/2}c(1+u)\}\\ &= \max_{0\leqslant v\leqslant N^{1/3}}\{W^{(1)}(vN^{-1/3})-N^{1/2}c(1+vN^{-1/3})\}\\ &\doteq N^{-1/6}\max_{0\leqslant v\leqslant N^{1/3}}\{W^{(2)}(v)-N^{2/3}c(1+vN^{-1/3})\}.\end{aligned} \tag{4}$$

Expanding $c(\cdot)$ according to (2) within some δ-neighbourhood of 1, and disregarding the negligible contribution from outside it, it follows that

$$\begin{aligned}N^{1/6}C_2(N) &\doteq \max_{0\leqslant v\leqslant \delta N^{1/3}}[W^{(2)}(v)-N^{2/3}\{\tfrac{1}{2}v^2N^{-2/3}+O(|v|^3N^{-1})\}]\\ &= \max_{v\geqslant 0}\{W^{(2)}(v)-\tfrac{1}{2}v^2\}+N^{-1/3}\varepsilon_N,\end{aligned} \tag{5}$$

where, for suitably chosen δ, $\varepsilon_N = O(1)$ as $N\to\infty$.

Next, on $0\leqslant t\leqslant 1$, observe that, conditional on $W^* = w$,

$$W(t)-W(1)\doteq W^{(3)}(1-t)-(1-t)\{w+W^{(3)}(1)\}, \tag{6}$$

so that

$$C_1(N)\doteq \max_{0\leqslant s\leqslant 1}[W^{(3)}(s)-s\{w+W^{(3)}(1)\}-N^{1/2}c(1-s)]. \tag{7}$$

Arguing as above, it follows that, conditional on $W^* = w$,

$$\begin{aligned}N^{1/6}C_1(N) &\doteq \max_{0\leqslant v\leqslant \delta N^{1/3}}[W^{(4)}(v)-\tfrac{1}{2}v^2-vN^{-1/6}\{w+N^{-1/6}W^{(4)}(N^{1/3})\}+O(N^{-1/3}|v|^3)]\\ &= \max_{v\geqslant 0}\{W^{(4)}(v)-\tfrac{1}{2}v^2\}+N^{-1/6}\eta_N(w),\end{aligned} \tag{8}$$

where, for each w, $\eta_N(w) = O(1)$ as $N\to\infty$; hence

$$N^{1/6}C_1(N)\doteq \max_{v\geqslant 0}\{W^{(4)}(v)-\tfrac{1}{2}v^2\}+N^{-1/6}\eta_N^*, \tag{9}$$

where $\eta_N^* = O(1)$ as $N\to\infty$, and where $C_1(N)$ depends on W^* only through η_N^*.

Combining (5) and (9) with (3), it thus follows that

$$Y_N\doteq W+N^{-1/6}\max(X_1,X_2)+O(N^{-1/3}),$$

where W, X_1 and X_2 are independent, W has a standard normal distribution, and

$$X_j\doteq \max_{t\geqslant 0}\{W(t)-\tfrac{1}{2}t^2\},\quad j=1,2.$$

With some further working, this can be expressed as

$$Y_N\doteq W+\lambda N^{-1/6}+O(N^{-1/3}),$$

where $\lambda = E\{\max(X_1,X_2)\}\simeq 1{\cdot}00$ is the "bias" referred to in Daniels (1974): this estimate of λ has been computed both by Daniels and by the author, using different methods. Note that the error term is of smaller order than the $O(N^{-1/4})$ estimated by Daniels.

REFERENCE

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