DYNAMIC INPUT-OUTPUT ANALYSIS WITH DISTRIBUTED ACTIVITIES

Thijs ten Raa*

Abstract—This paper offers a new approach to economic models in which activities take time. Departing from a standard economic model (Leontief's dynamic input-output model), we recast the activities from ordinary vectors into temporal distributions. In doing so, we preserve the formal structure and simplicity of the standard model. This is the secret of the power of our approach which asserts itself in the resolution of some open dynamic input-output problems. In particular, we are able to solve models with singular capital structures (i.e., singular derivatives coefficients matrices), unbalanced growth and different time profiles of investment or other production activities.

De cost gaet voor de baet uyt.
(The cost goes before the benefit.)

TEMPORALLY distributed activities are important. De Galan (1980, p. 217) ascribes labor market failure to, among other things, the sluggishness of certain adjustments which results from the fact that activities such as education take time. Furthermore, if one neglects the time used up in the production process, then one will generate too high growth rates as most dynamic economic models actually do. Yet little work has been done on modeling with distributed economic activities. Exceptions are input-output analysis with transit and production lags by Bródy (1965), or with investment lead times by Gladyshevskii and Belous (1978), Johansen (1978), and Zhuravlev (1982). But these studies are in the realm of balanced growth in which the structure of the problem is the same as in static input-output; Zhuravlev comes closest to our work by inclusion of turnpike results.

Temporally distributed activities will be considered single elements in a distribution space and be subjected to the calculus of distributions, which yields simple expressions for seemingly complicated equations involving lags and so on, and solutions to the distributed input-output models. It enables us to resolve open dynamic input-output problems, such as the solution of equations with singular capital structures and analysis of economies with different time profiles of investment or other production activities under conditions of unbalanced growth. It should be mentioned that the same approach is relevant for regional economic models as that of Leontief et al. (1977). Then economic activities are modeled as spatial distributions. This topic is the subject of ten Raa (1984). Similarly, our analysis of distributed activities may serve as a model for capital of circulation and the complete economic system as outlined by Foley (1982).

The organization of the present paper is as follows. Section I identifies the economic subjects of this study and develops the central theme: input-output profiles are considered single elements in a distribution space, a concept that is defined in the appendix. Section II analyzes the static input-output model with possibly continuously distributed activities. Section III widens the scope to the case of balanced growth. Section IV handles a pure dynamic model with a possibly singular capital structure. Section V solves the traditional dynamic input-output model. By synthesis of the treatment of continuity (section II) and invertibility (section IV), section VI analyzes the distributed dynamic input-output model. Section VII concludes the paper.

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*Tilburg University.
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I. Productive Capital

Productive capital is divided into circulating capital and fixed capital (Marx, 1974, p. 158). Circulating capital is absorbed in production and consists of flows of goods. Fixed capital must merely be present when production takes place and consists of stocks of goods. Circulating and fixed capital are represented by, respectively, the input-output flow coefficients matrix A and the input-output stock coefficients matrix B, both of Leontief.

Circulating capital (A) is fluid, but it can be like water or like syrup. Some circulating capital, such as electricity, is absorbed immediately, but other circulating capital, such as minerals, must be treated during some time. Electricity is (super) fluid capital; minerals are working capital. (Super) fluid capital is, by definition, processed instantaneously; working capital is defined to be capital in the pipe line.

The same distinction can be made with regard to fixed capital (B). Some fixed capital, such as a stapler, is ready for immediate use, but other fixed capital, such as a transport container, must be present some time in advance. A stapler is instant capital; the container is advanced capital. Instant capital is fixed capital which can produce instantaneously, while advanced capital must be installed in advance, all by definition.

A good starting point for the incorporation of the production times in the circulating and fixed capital matrices A and B is Marx (1974, p. 239). For example, if input i's production time in sector j equals τ_{ij}, then we can write interindustry demand for i at time 0 as $\Sigma_j a_{ij} x_j(\tau_{ij})$ where $x(t)$ is the output vector at time t.

In general, the i^{th} input requirement for one unit of sector j^{th} output is represented by an input profile on the past. We shall now introduce a powerful point of view. Giving up the idea of a_{ij} being some number altogether, we redefine an *input-output coefficient* as a nonnegative distribution on the nonpositive time axis. The width of its support (τ_{ij}) reflects the production time. This set-up obviously applies to capital stock coefficients as well. Then the width of the support of the distribution reflects the investment lead time.

II. Static Input-Output Analysis

An input-output flow coefficient a_{ij} is a *nonnegative* distribution with *nonpositive support*, where $i, j = 1, \ldots, n$ represent the sectors of the economy. The future and current flow requirements exercised by sector j—with *output* distribution x_j—on sector i at time t sum up to, heuristically,

$$\int_{s=-\infty}^{0} a_{ij}(s) x_j(t-s)\, ds,$$

abstracting from technical change. Summing over j, which may be done under the integral sign, we obtain interindustry demand for i at time t:

$$\int_{s=-\infty}^{0} \sum_{j=1}^{n} a_{ij}(s) x_j(t-s)\, ds.$$

The material balance for good i at time t between output and interindustry demand plus *final demand* z_i reads

$$x_i(t) = \int_{s=-\infty}^{0} \sum_{j=1}^{n} a_{ij}(s) x_j(t-s)\, ds + z_i(t).$$

Letting x, z and A denote the output and final demand vectors and the input-output flow coefficients matrix,

$$x(t) = \int_{s=-\infty}^{0} A(s) x(t-s)\, ds + z(t).$$

Invoking the notation for the convolution product (appendix), we obtain

$$x = A^* x + z. \tag{1}$$

Formulation (1) is free of integrability requirements. It holds for x and z n-dimensional vector distributions and A an $n \times n$-dimensional matrix distribution (nonnegative and with nonpositive support) over time in the sense of Schwartz (1957). The purpose of this section is to solve the *Leontief planning problem* of finding output x fulfilling (1) given final demand z.

Our input-output distribution a_{ij} is essentially the outgrowth of temporal disaggregation of a traditional input-output coefficient. Thus, summing over time we capture the traditional coefficient, now denoted $\int a_{ij}$. This expression is shorthand for $\langle a_{ij}, 1\rangle$ (where $\int$ generalizes the Lebesgue-Stieltjes integral). We see that the traditional input-output matrix corresponds to $\int A$. This matrix is defined component-wise. Therefore, the well-known conditions for the producibility of final demand now apply to $\int A$:

ASSUMPTION. *Nonnegative matrix $\int A$ fulfills the Hawkins-Simon (1949) conditions.*

There are some well-known equivalent conditions. One is the condition that the *spectral radius is less than one*:

$$\rho\left(\int A\right) < 1. \tag{2}$$

The other is the *convergence* of the power expansion of the inverse of $I - \int A$:

$$\sum_0^\infty \left(\int A\right)^k < \infty. \tag{3}$$

We wish to derive that $\Sigma_0^\infty A^{*k}$ itself converges ($A^{*2} = A^*A$ and so on) and is a continuous functional. For then it is the *inverse distribution* of $I\delta - A$ (where δ is the Dirac distribution or unit point mass at the origin defined in the appendix). The latter is the operator in our equation (1) which consequently can be solved. Essentially, standard input-output results are confirmed for the more general case of distributed inputs.

PROPOSITION 1. *Let the above assumption be fulfilled. Then $\Sigma_0^\infty A^{*k}$ exists and is continuous. And for every z which is nonnegative and agrees with a bounded function near infinity, there is a solution x to (1) which is similar.*

Proof. See the appendix.

Remark. The assumed boundedness can be relaxed to hold just almost surely.

Example. A typical fully distributed input-output flow coefficient is $A(t) = \frac{1}{4}e^t\check{H}(t)$. Here $\check{H}$ is the *Heaviside function on the negative reals*, defined by $\check{H}(t) = 1$ for $t < 0$ and zero elsewhere. The assumption is fulfilled as the total mass of A is $\frac{1}{4}$. Now let us calculate $\Sigma_0^\infty A_0^{*k}$. For $t < 0$,

$$A^{*2}(t) = \int_t^0 A(s)A(t-s)\,ds = \tfrac{1}{16}\int_t^0 e^s e^{t-s} ds = \tfrac{1}{16}e^t(-t).$$

And so on, for $t < 0$,

$$A^{*k}(t) = \tfrac{1}{4^k}e^t(-t)^{k-1}\Big/(k-1)! = \tfrac{1}{4}e^t\left(-\tfrac{1}{4}t\right)^{k-1}\Big/(k-1)!.$$

In sum, for $t < 0$,

$$\sum_0^\infty A^{*k}(t) = \delta + \sum_0^\infty A^{*k+1}(t) = \delta + \sum_0^\infty \tfrac{1}{4}e^t\left(-\tfrac{1}{4}t\right)^k\Big/k! = \delta + \tfrac{1}{4}e^t e^{-\frac{1}{4}t} = \delta + \tfrac{1}{4}e^{\frac{3}{4}t}.$$

For $t \geq 0$, $A^{*k}(t) = 0$. Thus the inverse operator is $\delta + \frac{1}{4}e^{\frac{3}{4}(\cdot)}\check{H}$.

III. Balanced Growth

An input stock coefficient b_{ij} is also a *nonnegative* distribution with *nonpositive support*. The future and current capacity expansion of sector j demands, heuristically,

$$\int_{s=-\infty}^0 b_{ij}(s)\dot{x}_j(t-s)\,ds$$

of i at time t, where the dot denotes differentiation (see the appendix). These *investment* terms are separated from final demand. The material balance for good i at time t becomes

$$x_i(t) = \int_{s=-\infty}^0 \sum_{j=1}^n \left[a_{ij}(s)x_j(t-s) + b_{ij}(s)\dot{x}_j(t-s)\right] ds + z_i(t)$$

or

$$x = A^*x + B^*\dot{x} + z \tag{4}$$

which is free of integrability requirements. The purpose of this paper is to solve the Leontief planning problem for equation (4).

The remainder of this section is confined to the case of *balanced growth*:

$$z(t) = z(0)e^{gt}$$

and also

$$x(t) = x(0)e^{gt}.$$

PROPOSITION 2. *Let the assumption of section II be fulfilled. Consider balanced growth. Then there is a maximum growth rate $g^* > 0$ such that the following holds. For every z with $z(0)$ nonnegative and $0 \leq g < g^*$ there is a solution x to (4) with similar $x(0)$ and g.*

Proof. See the appendix.

IV. Pure Dynamics

In dynamic input-output analysis, as well as control theory, it is assumed that the matrix of coefficients of the first (vector) derivative is invertible. This practice constitutes a problem which manifests itself most pointedly in equation (4) of the last section if we neglect the nonderivative term and concentrate the derivative coefficients in the origin by putting $A = 0$ and $B = B_0\delta$, B_0 an ordinary but possibly singular matrix. Then equation (4) becomes

$$x = B_0\dot{x} + z. \tag{5}$$

The standard technique is to write $\dot{x} = B_0^{-1}x - B_0^{-1}z$ and to proceed as usual. However, as Bródy (1974, p. 137) notes:

> Yet the presence of the matrix $[B_0^{-1}]$ alerts us to further theoretical problems. First, B_0 itself can be singular in practice. If, say, two sectors have the same capital structure, then B_0, having two columns equal, will be singular, and if the capital structures are very similar, B_0 will be severely ill-conditioned. Furthermore, $[B_0^{-1}]$, if it exists at all, will have the economically meaningful growth rate, λ, as its eigenvalue of minimal modulus. Actual computation, then, will be dominated by other eigenvalues, and therefore be clumsy and inexact.

Another source of trouble is that some goods need no fixed capital or only to a minor extent. In such a case B_0 has zero or almost zero rows and, therefore, is singular or "severely ill-conditioned."

We shall overcome the singularity shortcoming of dynamic input-output analysis and control theory by deriving a more general solution to equation (5) which does not hinge on the invertibility of B_0. In other words, B_0's with zero eigenvalues must be facilitated. We shall proceed gradually and first admit only zero eigenvalues with a *complete* system of eigenvectors, that is, zero eigenvalues with a number of eigenvectors equal to the multiplicity of the zero eigenvalues. For then the role of B_0^{-1} can be played by a generalized inverse reminiscent of the one of Rao (1974), that is, any A_0 satisfying $B_0 A_0 B_0 = B_0$. The *form* of the generalized inverse is chosen such that equation (5) *can be solved* explicitly which will be done after the presentation and discussion of the definition. Surprisingly, Moore-Penrose generalized inverses do not work, in spite of suggestions in the literature. For example, Kendrick (1972) and Livesey (1973, 1976) make a number of full rank assumptions that implicitly rule out conditions such as capital structure similarity across sectors.

DEFINITION. *Let B_0 be a square matrix of which the zero eigenvalue has a complete system of eigenvectors. A generalized inverse of B_0 is a square matrix B_0^- such that $B_0^- B_0^2 = B_0$.*

Justification. Bring B_0 on *triangular form*

$$T\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}T^{-1}$$

such that the *zero* eigenvalues are precisely arranged in the diagonal of J_1. J_1 and J_2 are upper triangular and all nonzero eigenvalues are displayed on the diagonal of J_2. T is the base transformation matrix. By the nature of B_0, J_1 may be assumed zero. This can be seen by taking the Jordan canonical form of B_0 which fulfills the described conditions on J_1 and J_2. Since the diagonal of upper triangular J_2 never vanishes, this matrix is invertible. The generalized inverse is now

$$B_0^- = T\begin{pmatrix} K & 0 \\ L & J_2^{-1} \end{pmatrix}T^{-1}$$

with K and L arbitrary matrices of appropriate size. This is easily verified:

$$\begin{aligned} B_0^- B_0^2 &= T\begin{pmatrix} K & 0 \\ L & J_2^{-1} \end{pmatrix}T^{-1}T\begin{pmatrix} 0 & 0 \\ 0 & J_2^2 \end{pmatrix}T^{-1} \\ &= T\begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix}T^{-1} = B_0. \end{aligned}$$

Thus, the generalized inverse of B_0 exists but is not unique. The justification is rounded off by the next example which shows that B_0^- generalizes the inverse.

Examples. (1) B_0 regular. Then $B_0^- = B_0^{-1}$. (2) B_0 zero. Then B_0^- is arbitrary. (3) $B_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then the Rao inverse defined in the introduction of this section is $\left(\begin{smallmatrix} a & b \\ 1-a & c \end{smallmatrix}\right)$. The Moore-Penrose inverse is obtained by putting $a = \frac{1}{2}$ and $b = c = 0$. Our B_0^- is obtained by putting $a = 1$. (4) $B_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $B_0^2 = 0$ and B_0^- is undefined. Indeed, B_0 does not fulfill the assumption: zero is the only eigenvalue, but there is no complete system of eigenvectors.

Equation (5) is now rewritten such that solving it amounts to inverting a distribution: $(I\delta - B_0\dot{\delta})^* x = z$. The next proposition inverts the operator. The solution features the *Heaviside function on the negatives* $\check{H}$ which was defined by $\check{H}(t) = 1$ for $t < 0$ and zero elsewhere (section II, the example).

PROPOSITION 3. *Let B_0 be a square matrix whose zero eigenvalue has a complete system of eigenvectors. Then*

$$\begin{aligned}(I\delta - B_0\dot{\delta})^{*-1} = \check{H}\exp(B_0^- t)B_0^{-2}B_0 \\ +\delta(I - B_0^- B_0).\end{aligned}$$

Proof. See the appendix.

Examples. (1) B_0 regular. Then $B_0^- = B_0^{-1}$. Hence the inverse operator becomes $\check{H}\exp(B_0^- t)B_0^{-1}$. (2) B_0 zero. Then the inverse operator is $0 + \delta(I - 0) = \delta I$, as should be. (3) $B_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then we may choose $B_0^- = I$. Hence the inverse operator becomes $\check{H}e^t\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + \delta\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}$.

Remarks. (1) In example 3 the arbitrary

$$B_0^- = \begin{pmatrix}1 & b\\ 0 & c\end{pmatrix}.$$

But in the inverse operator,

$$\begin{aligned}\exp(B_0^- t)B_0^{-2}B_0 &= \sum_0^\infty \frac{t^k}{k!}B_0^{-(k+2)}B_0\\ &= \sum_0^\infty \frac{t^k}{k!}B_0^{-(k+1)}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\\ &= \sum_0^\infty \frac{t^k}{k!}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\end{aligned}$$

and

$$B_0^- B_0 = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix},$$

i.e., the arbitrariness in B_0^- is immaterial in the sense that a and b do not show up in the inverse operator. In fact, it is easy to show that the inverse operator is unique. (2) The inverse in example 3 has a negative component. This means that there is *disinvestment*, in fact, of good 1 in sector 2. For further discussion see Leontief (1970).

Next we admit zero eigenvalues with an incomplete system of eigenvalues. B_0 can now be any square matrix. This complete generality has a price, however. It is no longer possible to express the inverse operator in sole terms of B_0. We now have to invoke its triangular form factors J_1, J_2 and T.

PROPOSITION 4. *Let $B_0 = T\begin{pmatrix}J_1 & 0\\ 0 & J_2\end{pmatrix}T^{-1}$ be the triangular form with diag J_1 zero and diag J_2 never vanishing. Then*

$$(I\delta - B_0\dot{\delta})^{*-1} = T\begin{pmatrix}\sum_0^{n-1}\overset{(k)}{\delta}\, J_1^k & 0\\ 0 & \check{H}\exp(J_2^{-1}t)J_2^{-1}\end{pmatrix}T^{-1},$$

where $\overset{(k)}{\delta}$ is the k^{th} derivative of δ, $k = 0, \dots, n$, and n is the size of B_0 or the number of sectors in the economy.

Proof. See the appendix.

Example. $B_0 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. Then $J_1 = B_0$, $J_2 = \varnothing$ and $T = I$. Hence the inverse becomes

$$\sum_0^1 \overset{(k)}{\delta}\, B_0^k = \delta I + \dot{\delta}B_0 = \begin{pmatrix}\delta & \dot{\delta}\\ 0 & \delta\end{pmatrix}.$$

V. Traditional Dynamics

In traditional input-output, production is instantaneous. The coefficients are concentrated in the origin: $A = A_0\delta$ and $B = B_0\delta$ with A_0 and B_0 matrices and δ the Dirac distribution (see the appendix). Equation (4) of section II reduces to the familiar dynamic input-output equation,

$$x = A_0x + B_0\dot{x} + z. \tag{6}$$

Although equation (6) is standard, its Leontief planning problem has not been solved yet, due to the B_0 singularity problems discussed in the last section. The next proposition does it. Recall that the Leontief planning problem was defined in section II as that of finding output x given final demand z. The connection with initial value problems will be discussed in section VI.

PROPOSITION 5. *Let the assumption of section II be fulfilled, i.e., A_0 fulfills the Hawkins-Simon (1949) conditions. Let B_0's zero eigenvalues have a complete system of eigenvectors. Then for every z the solution to equation* (6) *is*

$$\begin{aligned}x = \{\check{H}\exp[B_0^-(I - A_0)t]\}^*B_0^-(I - A_0)\\ \times B_0^-B_0\sum_0^\infty A_0^kz + (I - B_0^-B_0)\sum_0^\infty A_0^kz.\end{aligned}$$

Proof. See the appendix.

Remarks. (1) z may not grow too fast, for then x would become infinite. Formally, z must fulfill the convolution condition of Schwartz (1961). In fact, z must be tempered by a growth rate which is less than g^* of proposition 2. (2) The assumption on B_0 can be dropped. Then the solution is modified by applying proposition 4 instead of 3 in the derivation. (3) In the special case that B_0 is invertible, the solution reduces to

$$x = \check{H}\exp[B_0^-(I - A_0)t]^*B_0^{-1}z.$$

This agrees with the literature, e.g., Bródy (1974, p. 136). (4) A discrete time formulation and solution to the problem yields Leontief's (1970) "Dynamic Inverse." The relation between the formulations, such as the bias involved, is discussed in ten Raa (1986).

VI. Distributed Dynamics

Last but not least we shall solve the Leontief planning problem for the full equation with distributed input-output coefficients of section III, reproduced here for convenience:

$$x = A^*x + B^*\dot{x} + z. \tag{4}$$

The technique will be factorization. The matrix distributions will be split into parts which are concentrated in the origin and parts away from the origin. The first parts will be subjected to proposition 4, the latter ones to a standard device of the theory of distributions. The procedure works provided that there is an intermediate time span over which the coefficients matrices are regular. Therefore, it is assumed that on some open interval $(\epsilon, 0)$, however small, A agrees with an integrable function and B with an absolutely continuous function. (This means that for all positive γ, there is a positive Δ such that

$$\sum_{k=1}^{m} \|B(t_k) - B(s_k)\| < \gamma$$

for every finite system of pairwise disjoint subintervals (s_k, t_k) of $(\epsilon, 0)$ with total length $\Sigma_{k=1}^{m}(t_k - s_k)$ less than Δ.) The assumption is met in applied econometrics where A and B have finite supports. We now have

PROPOSITION 6. *Let the assumption of section II be fulfilled. On some interval* $(\epsilon, 0)$, *let* A *be integrable and* B *absolutely continuous. Then for every* z *the solution* x *to the above equation is the convolution product of a locally integrable function, a multiple of the Dirac distribution and a distribution with support in the negatives, all with* z. *The expression for* x *is specified in the proof.*

Proof. See the appendix.

Remarks. (1) Remark 1 of section V applies. (2) A discrete time formulation and solution to the problem is presented in ten Raa (1986) who also comments on the length of the industrial reporting period as compared to the time between purchase of input and production of output in terms of significance of distributed versus traditional input-output.

Example. Consider a simple economy with a fixed capital good 1 and a circulating capital good 2. For simplicity, let each sector require only one kind of capital. To keep the example interesting, let each sector require capital of the other. The circulating capital consumption by sector 1 is $1/3$ per unit of output, exponentially distributed. One unit of fixed capital is needed per unit of output in sector 2. Formally,

$$A(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tfrac{1}{3} e^{t} \check{H}(t)$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \delta. \tag{7}$$

A straightforward application of proposition 6 in the appendix shows that the solution to the Leontief planning problem for final demand z is given by the output path

$$x(t) = \begin{pmatrix} \frac{3}{4}z_1(t) + \frac{3}{4}\dot{z}_2(t) - \frac{3}{16}z_2(t) + \frac{3}{16}\int_{-\infty}^{0} e^{\frac{3}{4}s} z_1(t-s)\, ds + \frac{9}{64}\int_{-\infty}^{0} e^{\frac{3}{4}s} z_2(t-s)\, ds \\ \frac{3}{4}z_2(t) + \frac{1}{4}\int_{-\infty}^{0} e^{\frac{3}{4}s} z_1(t-s)\, ds + \frac{3}{16}\int_{-\infty}^{0} e^{\frac{3}{4}s} z_2(t-s)\, ds \end{pmatrix}. \tag{8}$$

Note that current and future values of final demand, z, determine output, x. Capital requirements are met by the appropriate past production levels, as indicated by the same formula for x. Should one desire a particular level of output and capital stock at, say, time zero, then the formula implicitly determines all feasible future paths of final demand, z, and their sustaining output paths, x. The selection of such a path of final demand z requires a behavioral rule or plan and goes beyond the scope of the present article. For an interesting discussion of this issue I refer to Leontief (1963).

VII. Conclusion

The new approach to economic models with temporally distributed activities, exemplified by distributed input-output analysis, consists of four steps. First, the standard, nondistributed economic model (Leontief's dynamic input-output model) is taken as the point of departure. Second, activities are reinterpreted as temporal distributions. Third, the ordinary product is replaced by the convolution product. Fourth, the consequent model is subjected to the calculus of distributions.

The approach offers a unifying and extending framework for the dynamic inverse of Leontief (1970) and also Bródy (1974), and for the distributed lag studies of Bródy (1965), Gladyshevskii and Belous (1978), Johansen (1978), and Zhuravlev (1982).

The application of the theory of distributions of Schwartz (1957) is novel and promising for economic science. This paper features the following results:

1. Solutions to dynamic economic models with singular capital structures.
2. Unbalanced growth solutions to the traditional dynamic input-output model.
3. Analysis of the dynamic input-output model with distributed activities.

APPENDIX

Distributions for Economists

Having reconsidered input-output coefficients, now being nonnegative distributions on the nonpositive time axis, it may be helpful to the nonmathematical reader to give a precise account of the concepts involved. Nonnegative distributions are essentially measures (Schwartz, 1957). Unsigned distributions are generalizations that cover basically all linear operators. So let us first recapitulate the concept of a measure and then generalize. Throughout this paper, time will be the underlying space.

A measure associates amounts of mass with subsets of the time axis. Thus, a measure can be viewed as a mapping from indicator functions to the reals. The indicator functions are "test" functions representing subsets of time. A measure is no arbitrary mapping defined on the test space of indicator functions, but must be nonnegative and additive, meaning that the measure of the sum of indicator functions that is still an indicator function equals the sum of the measures. It is possible to extend measures to the test space of continuous and bounded functions: First, the measure of a multiple of an indicator function is defined as the multiple of the measure of the indicator function itself. Second, the measure of a step function is the sum of the measures of the steps. And third, the measure of a continuous and bounded function is defined by a limit process of step functions. By the nonnegativity and additivity assumptions, a measure is a nonnegative linear operator on the test space of continuous and bounded functions.

A prime example is the *Dirac* measure, δ, that represents the unit point mass at the origin. Being a measure, it is defined by the value it associates with an *indicator function*, 1_I. Here I is a subset of the time axis and 1_I is defined by $1_I(t) = 1$ if $t \in I$ and $1_I(t) = 0$ if $t \notin I$. The value δ associates with 1_I could be denoted $\delta(1_I)$. However, since the argument itself is a function here, $\langle \delta, 1_I \rangle$ is more common notation. Measure δ is defined by $\langle \delta, 1_I \rangle = 1_I(0)$. In other words, if I contains the origin, then $\langle \delta, 1_I \rangle = 1$—$I$ embodies one unit of mass—but if I does not contain the origin, then $\langle \delta, 1_I \rangle = 0$—$I$ embodies no mass. The extension of δ to a continuous and bounded function, ϕ, is straightforward: $\langle \delta, \phi \rangle = \phi(0)$.

Measures have been defined on the wide class of continuous and bounded functions. A distribution is a generalization of a measure. In other words, there are more distributions than measures. This is obtained by defining distributions on a narrower class of test functions. At first sight this procedure seems paradoxical, but it is right. By requiring that operators are defined *only* on a smaller set of functions, one admits more of them, in other words, generalizes. Distributions are defined on the *test space* of infinitely differentiable functions with compact support. (The support of a function is defined as the closure of the set of points where the function is nonzero.) The test space is endowed with a natural topology that corresponds with uniform convergence of all derivatives. A *distribution* is formally defined as a continuous linear mapping from this test space to the reals (or sometimes the complex numbers).

Since measures are defined a fortiori on the narrow test space of infinitely differentiable functions with compact supports, they are distributions. Distributions also generalize locally integrable functions f. For such an f one can define the distribution T_f by $\langle T_f, \phi \rangle = \int f(t)\phi(t)\,dt$. A first manifestation of the flexibility of distributions is the possibility to define their *derivatives* no matter what. The definition of the derivative of a distribution T, $\dot{T}$, should generalize the derivative of, say, a continuously differentiable function, f. In other words, we want $\dot{T}_f = T_{\dot{f}}$. Now $T_{\dot{f}}$ is defined by

$$\langle T_{\dot{f}}, \phi \rangle = \int \dot{f}(t)\phi(t)\,dt = -\int f(t)\dot{\phi}(t)\,dt = -\langle T_f, \dot{\phi} \rangle.$$

(The integration by parts produced no residual term as ϕ has compact support.) This motivates the following definition of $\dot{T}$: $\langle \dot{T}, \phi \rangle = -\langle T, \dot{\phi} \rangle$.

The *convolution product* of two continuous functions, f and g, with compact supports is defined by

$$(f*g)(t) = \int f(s)g(t-s)\,ds.$$

The definition of the convolution product of a distribution, T, and a test function, ϕ, should generalize; in other words, we want

$$T_f*\phi = f*\phi = \int f(s)\phi(t-s)\,ds = \langle T_f, \phi(t-\cdot) \rangle.$$

This motivates the definition of $T*\phi$ as an infinitely differentiable function by

$$(T*\phi)(t) = \langle T, \phi(t-\cdot) \rangle.$$

The definition of the convolution product can be generalized further to apply to two distributions, provided that a certain condition is fulfilled (Schwartz, 1961, p. 123). Elementary facts are $\delta*T = T$ (in other words, δ is the unit element) and $\dot{S}*T = S*\dot{T}$. It is easy to check this for $T = T_\phi$. Then

$$(\delta*\phi)(t) = \langle \delta, \phi(t-\cdot) \rangle = \phi(t-0) = \phi(t),$$

while

$$\begin{aligned}(\dot{S}*\phi)(t) &= \langle \dot{S}, \phi(t-\cdot) \rangle = -\langle S, [\phi(t-\cdot)]^{\cdot} \rangle \\ &= -\langle S, \dot{\phi}(t-\cdot)\cdot(-1) \rangle = \langle S, \dot{\phi}(t-\cdot) \rangle \\ &= (S*\dot{\phi})(t).\end{aligned}$$

Distributions even generalize operations such as differentiation. The convolution product of $\dot{\delta}$ and any distribution T yields $\dot{\delta}*T = \delta*\dot{T} = \dot{T}$. This device will take care of the investment term in the dynamic input-output equation. $\dot{\delta}$ is a distribution, but not a measure (which must be nonnegative). This is why distributions are more convenient tools for dynamic input-output than measures. Moreover, in some cases distributions along with the convolution product form an algebra and equations can be solved by finding inverse distributions. This observation is the clue to the resolution of distributed input-output problems.

Before starting the main analysis, let me disclaim any rigor or comprehensiveness in this mathematical section. A referee suggested a better introduction, namely Lighthill (1964), as well as a more advanced and encyclopedic treatment: Gel'fand and Shilov (1967).

Proof of Proposition 1. To demonstrate existence of $\Sigma_0^\infty A^{*k}$ and its continuity in test functions, ϕ, we estimate, using nonnegativity,

$$\left\langle \sum_0^\infty A^{*k}, \phi \right\rangle \le \left\langle \sum_0^\infty A^{*k}, \|\phi\|_\infty \right\rangle = \left\langle \sum_0^\infty A^{*k}, 1 \right\rangle \|\phi\|_\infty$$
$$= \int\left(\sum_0^\infty A^{*k}\right)\|\phi\|_\infty = \sum_0^\infty \left(\int A\right)^k \|\phi\|_\infty \tag{9}$$

where the last equality rests on the fact $\int(A^{*k}) = (\int A)^k$. (This fact will be established for $k = 2$, the further cases going by induction. The $(i, j)^{\text{th}}$ element of the left hand side matrix equals

$$\left[\int(A^{*2})\right]_{ij} = \int[(A^{*2})_{ij}] = \langle (A^{*2})_{ij}, 1\rangle$$
$$= \left\langle \sum_m a_{im}*a_{mj}, 1\right\rangle$$
$$= \sum_m \langle a_{im}, \langle a_{mj}, 1\rangle\rangle = \sum_m \left\langle a_{im}, \int a_{mj}\right\rangle$$
$$= \sum_m \langle a_{im}, 1\rangle \int a_{mj} = \sum_m \int a_{im} \int a_{mj}$$
$$= \left(\int A \int A\right)_{ij} = \left[\left(\int A\right)^2\right]_{ij},$$

the $(i, j)^{\text{th}}$ element of the right hand side.)

By the convergence consequence, (3), of the assumption, the right hand side of (9) is finite. Consequently, the distribution on the left hand side, $\Sigma_0^\infty A^{*k}$ exists and is continuous in ϕ. To demonstrate the second part of the proposition, consider a distribution z which is nonnegative and near infinity agrees with a bounded function. Then $z = z' + z''$ with z' nonnegative and agreeing with a bounded function and with z'' nonnegative and support bounded from above. We shall show that, first,

$$x' = \left(\sum_0^\infty A^{*k}\right)*z'$$

is nonnegative and agrees with a bounded function, and second,

$$x'' = \left(\sum_0^\infty A^{*k}\right)*z''$$

is a bounded (i.e., finite total mass) nonnegative distribution with support bounded from above. Since $x = x' + x''$, this is enough. Nonnegativity is obvious. To demonstrate the boundedness of x', choose nonnegative locally integrable functions A_m that approximate $\Sigma_0^\infty A^{*k}$ from below and define $x_m = A_m*z'$. Then, by nonnegativity, $x_m \uparrow x'$, and, defining $\| \ \|_\infty$ of a vector or matrix component wise as a vector or matrix of the same order,

$$\|x_m\|_\infty \le \|A_m\|_1\|z'\|_\infty = \langle A_m, 1\rangle\|z'\|_\infty$$
$$\le \left\langle \sum_0^\infty A^{*k}, 1\right\rangle\|z'\|_\infty = \int\left(\sum_0^\infty A^{*k}\right)\|z'\|_\infty$$
$$= \sum_0^\infty \left(\int A\right)^k \|z'\|_\infty$$

by the first part of the proof, (9). By the assumption and the principle of monotone convergence, x_m converges in the $\|\cdot\|_\infty$-norm. In fact, $\|x_m\|_\infty \uparrow \|x\|_\infty$ for our $x_m \uparrow x$. Taking the limit in our inequality we obtain $\|x\|_\infty \le \Sigma_0^\infty(\int A)^k\|z'\|_\infty$. x'' is as desired since the supports of both $\Sigma_0^\infty A^{*k}$ and z'' are bounded from above so that these distributions fulfill the convolution condition of Schwartz (1961) and their convolution also has support bounded from above. Q.E.D.

Proof of Proposition 2. Substituting the balanced growth expressions, equation (4) becomes

$$x(0)e^{gt} = (A + Bg)*x(0)e^{gt} + z(0)e^{gt}$$

or, by definition of the convolution product (see the first section of this appendix),

$$x(0)e^{gt} = \langle A + Bg, x(0)e^{g(t-\cdot)}\rangle + z(0)e^{gt}$$

or, by linearity of distribution $A + Bg$,

$$x(0)e^{gt} = e^{gt}\langle (A + Bg)e^{-g(\cdot)}, x(0)\rangle + z(0)e^{gt}.$$

Dividing through we obtain

$$x(0) = A(g)*x(0) + z(0) \tag{10}$$

with

$$A(g) = (A + Bg)e^{-g(\cdot)}. \tag{11}$$

(11) shows that for $g \ge 0$, $A(g)$ is a nonnegative distribution. It has nonpositive support. Thus $A(g)$ is an input-output matrix distribution. By the nonpositivity of the support, $\int A(g)$ is an increasing function of g. By a standard result on nonnegative matrices, spectral radius $\rho[\int A(g)]$ is a continuous function of g. By spectral radius consequence (2) of the assumption, $\rho[\int A(0)] < 1$. Ruling out the trivial case $A = B = 0$ (for which $g^* = \infty$ fulfills the proposition), $\rho[\int A(\infty)] = \infty$. By the intermediate value theorem there is a $g^* > 0$ such that $\rho[\int A(g^*)] = 1$. It follows that for $0 \le g < g^*$, $\rho[\int A(g)] < 1$. By (2), $A(g)$ fulfills the Hawkins-Simon conditions and we may subject it to proposition 1. Hence for every $z(0)$ which is nonnegative there is a nonnegative solution to (10). The solution can be taken constant and denoted $x(0)$. It remains to justify that g^* is the maximum growth rate. This rests on the fact that for $g = g^*$, $\rho[\int A(g)] = 1$, which implies, by the theory of nonnegative matrices, that the condition—for every nonnegative $z(0)$ there is a solution $x(0)$ of

$$x(0) = \left[\int A(g)\right]x(0) + z(0) = A(g)*x(0) + z(0)$$

—is violated. Consequently, in the statement of the proposition, g has to be strictly less than g^* indeed and any higher critical growth rate would invalidate the proposition. Q.E.D.

Proof of Proposition 3. To prove that the expression in the statement of the Proposition is truly the inverse distribution, we multiply through by the operator, $I\delta - B_0\dot{\delta}$, to check that it yields the unit distribution, δI. Thus,

$$\begin{aligned}
&[\check{H}\exp(B_0^- t)B_0^{-2}B_0 + \delta(I - B_0^- B_0)]*(I\delta - B_0\dot{\delta})\\
&\quad = \check{H}\exp(B_0^- t)B_0^{-2}B_0 + \delta(I - B_0^- B_0)\\
&\qquad -[\check{H}\exp(B_0^- t)]^{\cdot}B_0^{-2}B_0^2 - (I - B_0^- B_0)B_0\dot{\delta}\\
&\quad = \check{H}\exp(B_0^- t)B_0^{-2}B_0 + \delta(I - B_0^- B_0)\\
&\qquad -\check{H}\exp(B_0^- t)B_0^{-3}B_0^2 + \delta B_0^{-2}B_0^2 - (B_0 - B_0^- B_0^2)\dot{\delta}\\
&\quad = \check{H}\exp(B_0^- t)B_0^{-2}(B_0 - B_0^- B_0^2)\\
&\qquad +\delta[I - B_0^-(B_0 - B_0^- B_0^2)] - (B_0 - B_0^- B_0^2)\dot{\delta}
\end{aligned}$$

which equals δI for $B_0 - B_0^- B_0^2 = 0$. Q.E.D.

Proof of Proposition 4. The left hand side of the statement of the Proposition equals

$$\begin{aligned}
&(I\delta - B_0\dot{\delta})^{*-1}\\
&\quad = \left(I\delta - T\begin{pmatrix} J_1 & 0\\ 0 & J_2\end{pmatrix}T^{-1}\dot{\delta}\right)^{*-1}\\
&\quad = \left[T\left(I\delta - \begin{pmatrix} J_1 & 0\\ 0 & J_2\end{pmatrix}\dot{\delta}\right)T^{-1}\right]^{*-1}\\
&\quad = T\begin{pmatrix} I_1\delta - J_1\dot{\delta} & 0\\ 0 & I_2\delta - J_2\dot{\delta}\end{pmatrix}^{*-1}T^{-1}\\
&\quad = T\begin{pmatrix} (I_1\delta - J_1\dot{\delta})^{*-1} & 0\\ 0 & (I_2\delta - J_2\dot{\delta})^{*-1}\end{pmatrix}T^{-1}.
\end{aligned}$$

Since the diagonal of upper triangular J_2 never vanishes, this matrix is regular, and by proposition 3 (example 1),

$$(I_2\delta - J_2\dot{\delta})^{*-1} = \check{H}\exp(J_2^{-1}t)J_2^{-1}.$$

Since the diagonal of upper triangular J_1 is zero, the n^{th} power of this matrix is zero, and therefore,

$$\begin{aligned}
\sum_0^{n-1}\overset{(k)}{\delta}\,J_1^k*(I_1\delta - J_1\dot{\delta}) &= \sum_0^{n-1}\overset{(k)}{\delta}\,J_1^k - \sum_0^{n-1}\overset{(k+1)}{\delta}\,J_1^{k+1}\\
&= \overset{(0)}{\delta}\,J_1^0 - \overset{(n)}{\delta}\,J_1^n = \delta I_1
\end{aligned}$$

or

$$(I_1\delta - J_1\dot{\delta})^{*-1} = \sum_0^{n-1}\overset{(k)}{\delta}\,J_1^k.$$

Substitution of the two derived inverses yields the right hand side of the statement of the Proposition. Q.E.D.

Proof of Proposition 5. The proof is organized as follows. First we rewrite equation (6) such that on the left hand side an operator applies to unknown x and on the right hand side there is known z. Then we find the inverse of the operator. Convoluting through with the inverse, the left hand side becomes x and the right hand side the convolution product of the inverse and z. This is the solution. Simplification finishes the proof. We differentiate (6) by parts (see the first section of this appendix) which, incidentally, makes the treatment of circulating and fixed capital uniform:

$$[(I - A_0)\delta - B_0\dot{\delta}]*x = z.$$

We factorize the operator:

$$(I - A_0)\delta - B_0\dot{\delta} = (I - A_0)\left[I\delta - (I - A_0)^{-1}B_0\dot{\delta}\right].$$

Here we used the Hawkins-Simon conditions. These also yield that the B_0 property carries over to $(I - A_0)^{-1}B_0$. Proposition 3 inverts the operator:

$$\begin{aligned}
&\left[I\delta - (I - A_0)^{-1}B_0\dot{\delta}\right]^{*-1}(I - A_0)^{-1}\\
&\quad = \Big\{\check{H}\exp\left[\left[(I - A_0)^{-1}B_0\right]^- t\right]\\
&\qquad\quad \times\left[(I - A_0)^{-1}B_0\right]^{-2}(I - A_0)^{-1}B_0\\
&\qquad\quad + \delta\Big(I - \left[(I - A_0)^{-1}B_0\right]^-\\
&\qquad\qquad (I - A_0)^{-1}B_0\Big)\Big\}(I - A_0)^{-1}
\end{aligned}$$

where $[(I - A_0)^{-1}B_0]^-$ is the generalized inverse of $(I - A_0)^{-1}B_0$ which can also be written $B_0^-(I - A_0)$, by which the inverse operator becomes

$$\begin{aligned}
&[\check{H}\exp[B_0^-(I - A_0)t]B_0^-(I - A_0)B_0^-B_0\\
&\qquad + \delta(I - B_0^-B_0)]\sum_0^{\infty}A_0^k.
\end{aligned}$$

Convoluting with the right hand side, z, we obtain the solution,

$$\begin{aligned}
x &= \{\check{H}\exp[B_0^-(I - A_0)t]B_0^-(I - A_0)B_0^-B_0\\
&\qquad + \delta(I - B_0^-B)\}\sum_0^{\infty}A_0^k*z\\
&= \left\{\check{H}\exp[B_0^-(I - A_0)t]B_0^-(I - A_0)B_0^-B_0\sum_0^{\infty}A_0^k\right\}*z\\
&\qquad + \delta(I - B_0^-B)\sum_0^{\infty}A_0^k*z\\
&= \{\check{H}\exp[B_0^-(I - A_0)t]\}*B_0^-(I - A_0)B_0^-B_0\sum_0^{\infty}A_0^k z\\
&\qquad + (I - B_0^-B)\sum_0^{\infty}A_0^k z.
\end{aligned}$$

Q.E.D.

Proof of Proposition 6. The organization of the proof is just as of the previous one. Thus, refer to the first paragraph of the proof of Proposition 5. By the assumption that A is integrable on $(\epsilon, 0)$, $A = A_0\delta + A_1 + A_2$ with A_0 a matrix, A_1 a locally summable function on $(-\infty, 0)$, and A_2's support in $(-\infty, \epsilon)$ for some negative ϵ. Here we use the fact that a nonnegative distribution concentrated in the origin must be a multiple of the Dirac distribution according to Schwartz (1957).

Similarly, $B = B_0\delta + B_1 + B_2$ with B_0 a matrix, B_1 absolutely continuous and B_2's support in $(-\infty, \epsilon)$. Through differentiation by parts (see the first section of this appendix), the equation becomes

$$(I\delta - A - \dot{B})*x = z. \tag{12}$$

By substitution, the operator in (12) becomes

$$I\delta - A - \dot{B} = C_0 - C_1 - C_2 \tag{13}$$

with

$$C_0 = (I - A_0)\delta - B_0\dot{\delta} \tag{14}$$

the traditional dynamic input-output operator,

$$C_1 = A_1 + \dot{B}_1 \tag{15}$$

a locally integrable function on $(-\infty, 0)$, and

$$C_2 = A_2 + \dot{B}_2 \tag{16}$$

whose support is in $(-\infty, \epsilon)$. We factorize the operator, (13):

$$C_0 - C_1 - C_2 = C_0 * \left(I\delta - C_0^{*-1} * C_1 - C_0^{*-1} * C_2 \right). \tag{17}$$

Here

$$C_0^{*-1} = \left\{ \check{H} \exp\left[B_0^- (I - A_0) t \right] B_0^- (I - A_0) B_0^- B_0 + \delta(I - B_0^- B_0) \right\} \sum_0^\infty A_0^k \tag{18}$$

(or a straightforward modification) by (14) and proposition 5 (remark 2); A_0 fulfills the Hawkins-Simon conditions by the assumption of section III and the fact $0 \leq A_0\delta \leq A$. In further factorizing, the operator, (17), becomes

$$C_0 * \left(I\delta - C_0^{*-1} * C_1 \right) * \left[I\delta - \left(I\delta - C_0^{*-1} * C_1 \right)^{*-1} * C_0^{*-1} * C_2 \right]. \tag{19}$$

This makes sense and is invertible provided that the factors

$$I\delta - C_0^{*-1} * C_1,$$

and

$$I\delta - \left(I\delta - C_0^{*-1} * C_1 \right)^{*-1} * C_0^{*-1} * C_2$$

are invertible. By (18), C_0^{*-1} is the sum of an infinitely differentiable function on $(-\infty, 0)$ and a multiple of the Dirac distribution. Consequently its convolution product, with C_1 which is locally summable by (14), exists and is a locally summable function on $(-\infty, 0)$. By Schwartz (1961, p. 143) $I\delta - C_0^{*-1} * C_1$, has an inverse which is the sum of the Dirac distribution and a locally integrable function C_1^- on $(-\infty, 0)$. The last factor of (19) now is

$$I\delta - \left(I\delta + C_1^- \right) * C_0^{*-1} * C_2,$$

where the remainder has support in $(-\infty, \epsilon)$ by virtue of C_2 as specified in (16). Consequently this factor has a power expansion inverse which is the sum of the Dirac distribution and a distribution C_2^- with support in $(-\infty, \epsilon)$. Thus the inverse of (19) becomes

$$\left(I\delta + C_2^- \right) * \left(I\delta + C_1^- \right) * C_0^{*-1}$$

with C_0^{*-1}, C_1^- and C_2^- given by (18) and the above text. Convoluting through this distribution with z yields the specific expression for x. To determine the nature of this solution, recall that C_0^{*-1} is, by (18), the sum of an infinitely differentiable function on $(-\infty, 0)$ and a multiple of the Dirac distribution, C_1^- is a locally integrable function on $(-\infty, 0)$, and C_2^- is a distribution with support in $(-\infty, \epsilon)$. Let us summarize this symbolically as

$$C_0^{*-1} \in C_-^\infty + D_0', \quad C_1^- \in L^1_{\mathrm{loc},-},$$

and

$$C_2 \in D'_{(-\infty, \epsilon)}.$$

It follows that the inverse belongs to

$$\left(D_0' + D'_{(-\infty, \epsilon)} \right) * \left(D_0' + L^1_{\mathrm{loc},-} \right) * \left(C_-^\infty + D_0' \right).$$

This space can be written out as

$$D'_- * C_-^\infty + \left(D_0' + D'_{(-\infty, \epsilon)} \right) * \left(D_0' + L^1_{\mathrm{loc},-} \right)$$

or

$$C_-^\infty + \left(D_0' + L^1_{\mathrm{loc},-} \right) + D'_{(-\infty, \epsilon)} * \left(D_0' + L^1_{\mathrm{loc},-} \right)$$

which is simply

$$L^1_{\mathrm{loc},-} + D_0' + D'_{(-\infty, \epsilon)}.$$ Q.E.D.

Example. Consider the full input-output equation, (4), with A and B given by (7) of section VI. Then, in the proof of proposition 6,

$$A = A_0\delta + A_1 + A_2 = 0 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} 1/3 e^{(\cdot)} \check{H} + 0$$

and

$$B = B_0\delta + B_1 + B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \delta + 0 + 0,$$

so that (14), (15), and (16) reduce to

$$C_0 = I\delta - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{\delta},$$
$$C_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tfrac{1}{3} e^{(\cdot)} \check{H},$$
$$C_2 = 0.$$

Consequently, the operator, (19), reduces to

$$\left[I\delta - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{\delta} \right] * \left[I\delta - C_0^{*-1} * \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tfrac{1}{3} e^{(\cdot)} \check{H} \right]. \tag{19'}$$

The second factor of this operator, (19′), becomes by the example to proposition 4,

$$\begin{aligned} I\delta - \begin{pmatrix} \delta & \dot{\delta} \\ 0 & \delta \end{pmatrix} * \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tfrac{1}{3} e^{(\cdot)} \check{H} &= I\delta - \tfrac{1}{3} \begin{pmatrix} \dot{\delta} & 0 \\ \delta & 0 \end{pmatrix} * e^{(\cdot)} \check{H} \\ &= I\delta - \tfrac{1}{3} \begin{pmatrix} e^{(\cdot)}\check{H} - \delta & 0 \\ e^{(\cdot)}\check{H} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \tfrac{4}{3}\delta - \tfrac{1}{3} e^{(\cdot)}\check{H} & 0 \\ -\tfrac{1}{3} e^{(\cdot)}\check{H} & \delta \end{pmatrix}. \end{aligned} \tag{20}$$

The inverse of this second factor, (20), is

$$\begin{aligned} &\begin{pmatrix} \left[\tfrac{4}{3}\delta - \tfrac{1}{3} e^{(\cdot)}\check{H} \right]^{*-1} & 0 \\ \tfrac{1}{3} e^{(\cdot)}\check{H} * \left[\tfrac{4}{3}\delta - \tfrac{1}{3} e^{(\cdot)}\check{H} \right]^{*-1} & \delta \end{pmatrix} \\ &= \begin{pmatrix} \tfrac{3}{4} \left[\delta - \tfrac{1}{4} e(\cdot)\check{H} \right]^{*-1} & 0 \\ \tfrac{1}{4} e^{(\cdot)}\check{H} * \left[\delta - \tfrac{1}{4} e^{(\cdot)}\check{H} \right]^{*-1} & \delta \end{pmatrix} \\ &= \begin{pmatrix} \tfrac{3}{4} \left[\delta + \tfrac{1}{4} e^{\frac{3}{4}(\cdot)}\check{H} \right] & 0 \\ \tfrac{1}{4} e^{(\cdot)}\check{H} * \left[\delta + \tfrac{1}{4} e^{\frac{3}{4}(\cdot)}\check{H} \right] & \delta \end{pmatrix} \\ &= \begin{pmatrix} \tfrac{3}{4}\delta + \tfrac{3}{16} e^{\frac{3}{4}(\cdot)}\check{H} & 0 \\ \tfrac{1}{4} e^{\frac{3}{4}(\cdot)}\check{H} & \delta \end{pmatrix}, \end{aligned}$$

by the example to proposition 1 and the easily verifiable elementary fact, $e^{(\cdot)}\check{H}*e^{\frac{3}{4}(\cdot)}\check{H} = 4e^{\frac{3}{4}(\cdot)}\check{H} - 4e^{(\cdot)}\check{H}$. Convoluting through with the inverse of the first factor of (19′) which is given by the example to proposition 4, we obtain the inverse operator related to (19′),

$$\begin{pmatrix} \frac{3}{4}\delta + \frac{3}{16}e^{\frac{3}{4}(\cdot)}\check{H} & 0 \\ \frac{1}{4}e^{\frac{3}{4}(\cdot)}\check{H} & \delta \end{pmatrix} * \begin{pmatrix} \delta & \dot{\delta} \\ 0 & \delta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4}\delta + \frac{3}{16}e^{\frac{3}{4}(\cdot)}\check{H} & \left[\frac{3}{4}\delta + \frac{3}{16}e^{\frac{3}{4}(\cdot)}\check{H}\right]*\dot{\delta} \\ \frac{1}{4}e^{\frac{3}{4}(\cdot)}\check{H} & \left[\frac{1}{4}e^{\frac{3}{4}(\cdot)}\check{H}\right]*\dot{\delta} + \delta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4}\delta + \frac{3}{16}e^{\frac{3}{4}(\cdot)}\check{H} & \frac{3}{4}\dot{\delta} + \frac{9}{64}e^{\frac{3}{4}(\cdot)}\check{H} - \frac{3}{16}\delta \\ \frac{1}{4}e^{\frac{3}{4}(\cdot)}\check{H} & \frac{3}{16}e^{\frac{3}{4}(\cdot)}\check{H} - \frac{1}{4}\delta + \delta \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{3}{4}\delta & \frac{3}{4}\dot{\delta} - \frac{3}{16}\delta \\ 0 & \frac{3}{4}\delta \end{pmatrix} + \begin{pmatrix} \frac{3}{16} & \frac{9}{64} \\ \frac{1}{4} & \frac{3}{16} \end{pmatrix} e^{\frac{3}{4}(\cdot)}\check{H}.$$

Convoluting through this distribution with z yields the specific expression for x, (8), posted in the example to proposition 6 in section VI. Q.E.D.

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