Ellipses

February 14, 2018

1 Class Ellipse

- r center of mass
- $u[0]$,$u[1]$ are the x,y axis
- a_x,a_y are the axis lengths

1.1 Useful Functions

The surface is described by

$$f\left(P\right)=\left[\frac{\left(P-r\right)\cdot u[0]}{a_x}\right]^2+\left[\frac{\left(P-r\right)\cdot u[1]}{a_y}\right]^2-1=0$$

can described implicitly in terms of a single parameter (angle) α:

$$P\left(\alpha\right)=a_x\,\cos\alpha\,u[0]+a_y\sin\alpha\,u[1]+r$$

$$\partial_\alpha P\left(\alpha\right)=-a_x\,\sin\alpha\,u[0]+a_y\cos\alpha\,u[1]$$

The gradient $\nabla f\left(P\right)$ is

$$\nabla f\left(P\right)=2\frac{\left(P-r\right)\cdot u[0]}{a_x^2}\,u[0]+2\frac{\left(P-r\right)\cdot u[1]}{a_y^2}\,u[1]$$

In the case $P\left(\alpha\right)$, we have $\left(P-r\right)\cdot u[0]=a_x\,\cos\alpha$

$$\nabla f\left(P\left(\alpha\right)\right)=2\cos\alpha\,u[0]/a_x+2\sin\alpha\,u[1]/a_y$$

that can be used to check that $\partial_\alpha f\left(P\left(\alpha\right)\right)=0$

$$\partial_\alpha f\left(P\left(\alpha\right)\right)=\nabla f\cdot\partial_\alpha P=0$$

and to calculate $\partial_\alpha\nabla f\left(P\left(\alpha\right)\right)$

$$\partial_\alpha\nabla f\left(P\left(\alpha\right)\right)=-2\sin\alpha\,u[0]/a_x+2\cos\alpha\,u[1]/a_y$$

2 Distance

We will look for the distance between two ellipses using Newton-Raphson. Given two ellipses E_i $(i = A, B)$, we will individuate two points P_i on the surfaces of E_i (i.e. corresponding to two angles θ_i) such that

$$\begin{cases} \nabla_A f_A \wedge \nabla_B f_B & = & 0 \\ P_B & = & P_A + \alpha \nabla_A f_A \end{cases}$$

i.e. we look for the 0's of 3 equations (as $P_i = (x_i, y_i)$) :

$$F = \begin{pmatrix} \nabla_A f_A \wedge \nabla_B f_B \\ x_A - x_B + \alpha \left.\nabla_A f_A\right|_x \\ y_A - y_B + \alpha \left.\nabla_A f_A\right|_y \end{pmatrix} = 0$$

with Hessian

$$H = \left\| \begin{array}{ccc} (\partial_{\theta_A} \nabla_A f_A) \wedge \nabla_B f_B & \nabla_A f_A \wedge (\partial_{\theta_B} \nabla_B f_B) & 0 \\ \partial_{\theta_A} x_A + \alpha \, (\partial_{\theta_A} \nabla_A f_A)_x & -\partial_{\theta_B} x_B & \left.\nabla_A f_A\right|_x \\ \partial_{\theta_A} y_A + \alpha \, (\partial_{\theta_A} \nabla_A f_A)_y & -\partial_{\theta_B} y_b & \left.\nabla_A f_A\right|_y \end{array} \right\|$$

Therefore, we have all the routines for the Hessian

3 Class Ellipse

- r center of mass
- $u[0]$,$u[1]$ are the x,y axis
- a_x,a_y are the axis lengths
- e_x,e_y are the x,y exponents

3.1 Useful Functions

The surface is described by

$$f(P) = \left[\frac{(P - r) \cdot u[0]}{a_x} \right]^{e_x} + \left[\frac{(P - r) \cdot u[1]}{a_y} \right]^{e_y} - 1 = 0$$

(where for a^e we really mean $sgn(a) |a|^e$) and can described implicitly in terms of a single parameter (angle) α:

$$P(\alpha) = a_x s_x |\cos \alpha|^{2/e_x} \; u[0] + a_y s_y |\sin \theta|^{2/e_y} \; u[1] + r$$

$$\partial_\alpha P(\alpha) = -(2a_x/e_x) \, s_x |\sin \alpha| \; |\cos \alpha|^{(2-e_x)/e_x} \; u[0] + (2a_y/e_y) \, s_y |\cos \alpha| \; (\sin \alpha)^{(2-e_x)/e_y} \; u[1]$$

where $s_x = sgn(\cos \alpha)$, $s_y = sgn(\sin \alpha)$

The gradient $\nabla f\,(P)$ is

$$\nabla f\,(P) = 2\frac{(P-r)\cdot u[0]}{a_x^2}\,u[0] + 2\frac{(P-r)\cdot u[1]}{a_y^2}\,u[1]$$

In the case $P\,(\alpha)$, we have $(P-r)\cdot u[0] = a_x\,\cos\alpha$

$$\nabla f\,(P\,(\alpha)) = 2\cos\alpha\,u[0]/a_x + 2\sin\alpha\,u[1]/a_y$$

that can be used to check that $\partial_\alpha f\,(P\,(\alpha)) = 0$

$$\partial_\alpha f\,(P\,(\alpha)) = \nabla f\cdot\partial_\alpha P = 0$$

and to calculate $\partial_\alpha \nabla f\,(P\,(\alpha))$

$$\partial_\alpha \nabla f\,(P\,(\alpha)) = -2\sin\alpha\,u[0]/a_x + 2\cos\alpha\,u[1]/a_y$$