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Note

Removal of Infinite Eigenvalues in the Generalized Matrix Eigenvalue Problem

1. INTRODUCTION

We consider here the generalized matrix eigenvalue problem

$$[\mathbf{A}(p) - \lambda \mathbf{B}(p)]\,\mathbf{x} = \mathbf{0}, \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are complex $n \times n$ matrices which depend on a parameter p.

We are primarily concerned with the case where $\mathbf{B}$ is singular. In many physical problems, Eq. (1) with $|\mathbf{B}| = 0$ arises from discretization of an eigenvalue problem for an ordinary (or partial) differential equation system in which one or more of the boundary conditions (or differential equations) do not involve the eigenvalue λ. The most widely used methods for solving (1) are the QZ and LZ algorithms [1, 2].

Although our focus is on the general case of (1), we will introduce the topic in terms of the Orr–Sommerfeld equation governing the linear stability of parallel shear flows. In the case of plane Poiseuille flow between parallel plates, the Orr–Sommerfeld equation is

$$\{(D^2 - a^2)^2 + ia\,\mathrm{Re}[(c - U(z))(D^2 - a^2) + D^2 U(z)]\}\; Y(z) = 0 \tag{2a}$$

$$Y(-1) = DY(-1) = Y(1) = DY(1) = 0 \tag{2b–2e}$$

where $D = d/dz$, c is the eigenvalue which is to be computed, Re is the Reynolds number, a is the wavenumber, and $U(z) = 1 - z^2$ is the basic velocity profile whose stability is being studied.

When (2a)–(2e) are discretized, a homogeneous set of linear algebraic equations of the form (1) is obtained. If a finite difference approximation has been used, then four identically zero rows of $\mathbf{B}$ will be associated with the finite difference approximations to the boundary conditions (2b)–(2e) [3]. If a spectral technique has been employed, in which the expansion functions do not individually satisfy (2b)–(2e), then four equations appearing in (1) must be used to satisfy these boundary conditions [4, 5], and $\mathbf{B}$ will again have four rows of zeros.

In the sequel, we assume that the rank of $\mathbf{B}$ is $n - k$, where k is the number of zero rows of $\mathbf{B}$. As a result, the determinantal polynomial of (1) will be of degree $m \leqslant n - k$. Hence, (1) will have fewer than n finite eigenvalues. The number of

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infinite eigenvalues of (1) is generally k, but may be less, according to the dependence of $\mathbf{A}$ and $\mathbf{B}$ on the parameter p [6]. In what follows, we assume that the number of infinite eigenvalues is k.

In the Orr–Sommerfeld (2a)–(2e) and other hydrodynamic and convective stability problems, solution of the matrix eigenvalue problem (1) frequently gives a number of "spurious" eigenvalues in addition to the eigenvalues of interest. These spurious eigenvalues are frequently of very large magnitude, do not converge as the order of the discretization is increased, and do not correspond to eigenvalues of the differential equation eigenvalue problem [7].

As an example we cite the spurious modes found in a Chebyshev-tau discretization of the Orr–Sommerfeld problem [7], in which the magnitude of the spurious eigenvalues grows like n^2. In this case, roundoff and other numerical errors may cause the truly infinite eigenvalues of (1) to appear large and finite, thus leading to confusion between infinite and "spurious" eigenvalues.

We note that Zebib [8] has recently described a spectral method for the Orr–Sommerfeld problem which, by employing functions which individually satisfy the boundary conditions in which the eigenvalue is absent, yields a nonsingular $\mathbf{B}$. The resulting eigenvalue problem is free of infinite or spurious eigenvalues. This method can be generalized to a broad class of ordinary differential equation eigenvalue problems.

There are, however, many other situations in which (1) arises for which it is not possible or practical to discretize the underlying problem so that $\mathbf{B}$ is nonsingular. As examples, we mention the solution of Helmholtz's equation (frequently on an irregular domain) by finite difference or finite element techniques [9], and the solution of structural dynamics problems in which rigid body constraints or other (linear) algebraic (i.e., nondifferential) equations are included in the discretized equations. In these cases, it would be highly desirable to have available a method which can remove the infinite eigenvalues of (1) without modifying [9] the finite ones.

2. Current Methods for Removing Infinite Eigenvalues

If $\mathbf{A}$ is nonsingular, a simple way to handle (1) is to consider the reciprocal problem [10]:

$$(\mathbf{B} - \mu\mathbf{A})\mathbf{x} = \mathbf{0}, \tag{3}$$

where $\mu = 1/\lambda$. It is clear that the "missing" solutions of the characteristic equation of (1) correspond to zero eigenvalues of (3) which in turn may be referred to as infinite eigenvalues of (1). This method, although simple, has a disadvantage in the case where some of the finite eigenvalues of (1) are very large. The reason is that due to roundoff error, infinite and very large eigenvalues of (1) (or zero and very small eigenvalues of (3)), may be indistinguishable.

An alternative method is to reduce the order of (1) to the rank of $\mathbf{B}$ in a manner such that the finite eigenvalues of the system are preserved [3, 11]. Known as the "reduced" method, this technique, which is in principle capable of removing k of the infinite eigenvalues from the problem altogether, employs a sequence of elementary row and column operations on $\mathbf{A}$ and $\mathbf{B}$. It requires $O(kn^2)$ multiplications.

It is clear that, if high accuracy (large n) is desired, use of the "reciprocal" method [10] will not permit a clear distinction between the infinite eigenvalues of (1) and those which grow like, say n^4. For that reason, the "reduced" method [3, 11] was used in the calculations of Gary and Helgason [3] and Orszag [4]. Here, we present an alternative to the "reciprocal" and "reduced" methods which avoids the disadvantages of the former and requires fewer operations than the latter.

3. The Mapping of the Infinite Eigenvalues

We describe below a transformation of equation (1) which preserves the finite eigenvalues of (1), but which maps k of the infinite eigenvalues to one or more specified points in the complex plane. This allows the infinite eigenvalues of (1) to be easily distinguished from any that are of large but finite modulus, such as "spurious" eigenvalues which grow like n^4.

Assuming that (1) has been arranged so that the k zero rows of $\mathbf{B}$ are at the bottom, we have

$$\mathbf{B} = [\mathbf{B}', \mathbf{O}_2]^T$$

where $\mathbf{B}'$ is an $(n-k)\times n$ matrix and $\mathbf{O}_1$ is a $k\times n$ matrix of zeros. We then premultiply (1) by

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{O}_2 \\ \mathbf{O}_2^T & \lambda \mathbf{I}_k + \mathbf{E}_k \end{bmatrix},$$

where $\mathbf{I}_r$ is the identity matrix of order r, $\mathbf{O}_2$ is a zero matrix of order $(n-k)\times k$, and $\mathbf{E}_k$ is a diagonal matrix of order k with elements

$$e_{ii} = -\sigma_i,$$

where the constants σ_i are arbitrary. Thus, (1) yields

$$(\mathbf{CA} - \lambda \mathbf{CB})\mathbf{x} = \mathbf{0},$$

from which it follows that

$$0 = |\mathbf{C}(\mathbf{A} - \lambda \mathbf{B})| = |\mathbf{C}|\,|\mathbf{A} - \lambda \mathbf{B}| = |\mathbf{A} - \lambda \mathbf{B}| \prod_{j=1}^{k} (\lambda - \sigma_j). \tag{4}$$

Thus, the new eigenvalue problem defined by the leftmost equality in (4) has all of the finite eigenvalues of (1) but none of the infinite ones associated with the zero rows of **B**, as the latter have been mapped to specified points in the finite plane. Now, any "spurious" eigenvalues of large modulus are not associated with the zero rows of **B**, but must instead be due to the discretization technique employed. In the case where the original eigenvalue problem has eigenvalues near one of the σ_j's, it may be desirable to select other values of σ_j in order to avoid bothersome root clustering.

In practice, the transformation of Eq. (1) to Eq. (4) requires only kn multiplications, because

$$\mathbf{CB} = \mathbf{B},$$

and

$$\mathbf{CA} = \mathbf{F} + \lambda\mathbf{G},$$

where **F** and **G** are square matrices of order n which do not depend on λ. Thus, the leftmost equality in (4) can be written as

$$0 = |\mathbf{F} + \lambda(\mathbf{G} - \mathbf{B})|, \tag{5}$$

where $\mathbf{G} - \mathbf{B}$ has no zero rows. Thus, all of the infinite eigenvalues of (1) will have been removed, unless $|\mathbf{G} - \mathbf{B}| = 0$ (e.g., if some of the rows of $\mathbf{B}'$ are linearly dependent on each other or on the lower k rows of **A**). Checking the rank of $\mathbf{G} - \mathbf{B}$ provides an easy test of whether the number of infinite eigenvalues of (1) exceeds the number of zero rows of **B**.

The computation of **F** and **G** is especially easy if we partition **A** as

$$\mathbf{A} = \begin{bmatrix} \mathbf{H} \\ \hline \mathbf{P} \end{bmatrix},$$

where **H** and **P** are $(n-k)\times n$ and $k \times n$ matrices, respectively. We then have

$$\mathbf{F} = \begin{bmatrix} \mathbf{H} \\ \hline \mathbf{EP} \end{bmatrix}, \qquad \mathbf{G} - \mathbf{B} = \begin{bmatrix} -\mathbf{B}' \\ \hline \mathbf{P} \end{bmatrix}.$$

In some applications preservation of the sparsity of the original problem may be of concern. In this regard, we note that (a) the sparseness pattern of **F** is exactly the same as that of **A** (because **E** is diagonal) and (b) the sparseness pattern of $\mathbf{G} - \mathbf{B}$ is that of **B** in the upper $n-k$ rows and that of **A** in the lower k rows. This lack of fill-in, combined with the very low operations count (nk multiplications which can be easily vectorized with a vector length of n) make the method especially attractive for large sparse versions of (1) arising from problems in fluid mechanics, structural dynamics, and plasma physics, in which **A** and **B** are typically sparse.

The method described above preserves the order of the eigenvalue problem (as the "reciprocal" method does) and at the same time clearly distinguishes the truly infinite eigenvalues from any large but finite ones (as achieved by the "reduction of order" method). This is accomplished with a lower operations count kn than the $O(kn^2)$ of existing methods [3, 11].

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DIMITRIOS A. GOUSSIS

Department of Mechanical and Aerospace Engineering
Princeton University
Princeton, NJ 08544

ARNE J. PEARLSTEIN

Department of Aerospace and Mechanical Engineering
University of Arizona
Tucson, Arizona 85721