Determinant of a Matrix

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

A Matrix is an array of numbers:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix
(This one has 2 Rows and 2 Columns)

The determinant of that matrix is (calculations are explained later):

$$3\times6 - 8\times4 = 18 - 32 = \mathbf{-14}$$

What is it for?

The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

Symbol

The **symbol** for determinant is two vertical lines either side.

Example:

|A| means the determinant of the matrix **A**

(Exactly the same symbol as absolute value.)

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

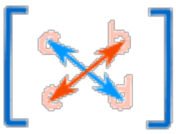
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:



- Blue means positive (+ad),
- Red means negative (-bc)

Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$\begin{aligned} |B| &= 4\times8 - 6\times3 \\ &= 32-18 \\ &= 14 \end{aligned}$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

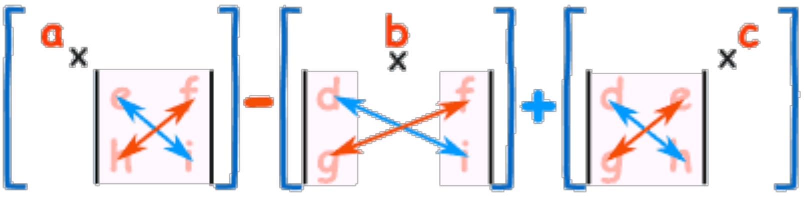
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:



To work out the determinant of a **3×3** matrix:

- Multiply **a** by the **determinant of the 2×2 matrix** that is **not** in **a**'s row or column.
- Likewise for **b**, and for **c**
- Add them up, but remember that **b** has a negative sign!

As a formula *(remember the vertical bars || mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"The determinant of A equals a times the determinant of ... etc"

Example:

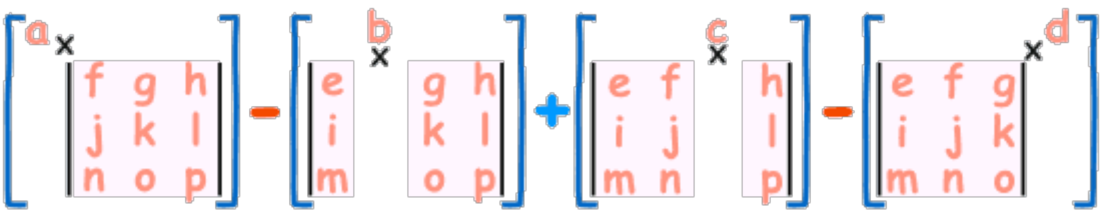
$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

|C| = 6×(-2×7 - 5×8) - 1×(4×7 - 5×2) + 1×(4×8 - -2×2)
= 6×(-54) - 1×(18) + 1×(36)
= **-306**

For 4×4 Matrices and Higher

The pattern continues for 4×4 matrices:

- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the + - + - pattern (+a... -b... +c... -d...). This is important to remember.

The pattern continues for 5×5 matrices and higher. Usually best to use a Matrix Calculator for those!

Not The Only Way

This method of calculation is called the "Laplace expansion" ... I like it because the pattern is easy to remember. But there are other methods (just so you know).

Summary

- For a 2×2 matrix the determinant is **ad - bc**
- For a 3×3 matrix multiply **a** by the **determinant of the 2×2 matrix** that is **not** in **a**'s row or column, likewise for **b** and **c**, but remember that **b** has a negative sign!
- The pattern continues for larger matrices: multiply **a** by the **determinant of the matrix** that is **not** in **a**'s row or column, continue like this across the whole row, but remember the + - + - pattern.

Question 1 Question 2 Question 3 Question 4
Question 5 Question 6 Question 7 Question 8
Question 9 Question 10